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AN APPROACH TO A GENERAL THEORY OF PRIORITIES:

AN OUTLINE OF PROBLEMS AND METHODS

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PREFACE

A part of this memorandum was read to the Tenth Annual Logistics Conference of the Office of Naval Research, held at The George Washington University in January, 1960, and appears in the Conference Proceedings in the Naval Research Logistics Quarterly of December, 1960, under the title, "Priority Indicators in More Than One Dimension." Another portion appeared in the same journal under the title, "Comments on 'A General Theory of Measurement-Applications to Utility'," in September, 1960. A summary of some of the literature involved had previously appeared in the same journal under the title, "A Selected, Descriptive Bibliography of References on Priority Systems and Related Non-Price Allocators," in March, 1958.

The author is deeply indebted to Professor Oskar Morgenstern for his advice and encouragement during the preparation of this study.

W. Giles Mellon

April 25, 1962.

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CHAPTER I

INTRODUCTION

". . . without a pricing mechanism there is no economic calculation"

L. von Mises

"Prices as such have nothing to do with the problem, although — like Voltaire's God — it may be desirable to invent them if they do not exist!"

Paul Samuelson

Introductory Remarks

Although the efficient allocation of resources is Economics in the sense of the usual textbook definition, and is assuredly the core of economic theory, in many situations the system of money prices is insufficient in itself as a guide to optimum behavior. One of the most important of these situations is that of the allocation of resources within the military establishment, where money costs are indeed a significant factor, but yet must be combined with some "measure of military worth" in arriving at many decisions. It is this "combinatorial" case which poses the greatest theoretical problems, and it is precisely in this case that our theoretical analysis is currently most deficient.[1]

[1] cf. ". . . there exists no real theory of priorities; especially the working of priorities in systems where also prices prevail and localized decisions are frequently possible, is only imperfectly understood." Oskar Morgenstern, "Consistency Problems in the Military Supply System," Naval Research Logistics Quarterly, Vol. 1, No. 4 (December, 1954), p. 271; ". . . there is a need for imaginative conceptual and theoretical work in the general problem of promoting efficient use of resources in an environment in which the market pattern is incomplete. While the Defense Departments usually purchase factors and products on markets, . . . there are no markets for valuing the outputs (military capabilities or worths) which they produce . . . . There is a lot of ready-made economic theory which is applicable to this problem to some extent, but it requires development and adaptation." Charles Hitch, "National Security Policy as a Field for Economics Research," RAND Paper P-1776 (August 19, 1959).

This memorandum concerns itself specifically with the illustration of how measures of military worth — or what we have chosen to call "priorities" — can be developed and applied to particular military decision problems. At the same time, however, it is important to remember that such measures of worth are of vital importance in those other areas where a system of money prices is not sufficient in and of itself to guide the allocation of resources. The most important of these cases, as well as the most well known to economists, is, of course, the theory of consumer behavior, where an optimal expenditure program depends on the consumer's evaluation (utility function) of the various combinations of goods available, as well as on their relative and absolute prices and his total income. But there are other well-known cases which are of great importance. As examples, we list the following which are arranged in reverse order of the importance of money prices, and in direct order of the importance of priority or non-price indicators, in each.

1. Central planning in the theoretical "socialist state."
2. Central planning and control in Soviet-type economies.
3. Evaluation of economic development programs for underdeveloped economies.
4. Administration of war production and rationing in a capitalist economy in time of war.
5. Evaluation of public investment in a capitalist economy.
6. Evaluation of alternative choices of action within a business organization.

The desirability — indeed the necessity — of making efficient economic calculations in each of the above is beyond question. Yet, a system of money prices may not exist at all (1); money prices may exist purely as an accounting device, or on only a limited scale in one or two sectors of the economy (2); money prices may be an important consideration, but the natural operation of the price system must be constrained to avoid socially undesirable allocation patterns (3, 4); money costs must be weighed against non-monetary factors, as for example, the probability of reducing loss of life by flood control, to reach a decision (5)[2]; and finally, while the overall motivation may be one of profit maximization, certain factors — as for example the value of good public relations — cannot be easily expressed in dollar terms.[3]

Thus, while this discussion is confined to problems related to military decision, it is hoped that the reader can see how much of the discussion is applicable to other economic situations as well.

Definitions

Throughout this discussion, we have chosen to use the term "priority" as opposed to the terms "utility" or "military worth." To a certain extent, this choice represents a personal whim, but as so many particular connotations have been attached to "utility," and even to "military worth," it was desired to use a terminology which would begin its theoretical life with a nearly clean slate.[4]

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[2] cf. D. von Dantzig, "Economic Decision Problems for Flood Prevention," Econometrica, Vol. 24, No. 3 (July, 1956), 276-287. There is, of course, extensive treatment of this problem in the literature on the benefits-cost criterion.

[3] A particularly good discussion of the advantages of a price system is to be found in F. A. von Hayek, "Pricing Versus Rationing," The Banker, Vol. 51, No. 164 (September, 1939), 242-249, though Professor von Hayek takes a rather extreme view on the effectiveness of the money price system in wartime. See also, Tjalling C. Koopmans, "Uses of Prices," Cowles Commission Papers, Special Paper No. 3, 1954.

[4] Thus, we would maintain that there is, at the moment, no commonly agreed upon meaning of "priority" in economic theory. An assertion to the contrary, however, can be found in the "Letter to the Editor" by Joseph B. Kruskal, The Naval Research Logistics Quarterly, Vol. 6, No. 3 (September, 1959), 261.

We define "priority" to mean "any indicator which may be associated with an 'alternative' so as to permit the alternative to be 'evaluated' with respect to other alternatives," where the exact nature of an "alternative" and the degree of "evaluation" involved are, for the moment, left undefined.

"Priority systems" are defined as "the method or scheme of classification by which the priority indicators are associated with the set of alternatives."

Priority systems are sometimes sufficient in themselves to establish a "priority program." By a "priority program" we mean "a plan of action which, on the basis of the priority indicators assigned to the various alternatives under consideration and whatever other information is relevant, informs the responsible decision-making authority which alternative or combination of alternatives, and in what order and degree, should be adopted." Thus, if the alternatives were, for example, a set of development projects, and the objective of the priority program was simply to select the four which have the highest priority ranking, then clearly the priority system is sufficient by itself to determine the program. But, as we have indicated above, the more usual — and more interesting situation — is where this is not the case.

Consider the following example. It is desired to maximize the value of a set of alternatives, each of these alternatives has a money cost associated with it, and total expenditures for the operation are limited, so that only a limited number of combinations or alternatives are feasible. If it were possible to derive a single-valued, order-preserving function, which represents the value of these alternatives up to a linear transformation (as, for example, the well-known von Neumann-Morgenstern index of utility), then the computation of the optimal "priority program" in this case is an example of an assignment-transportation linear programming problem. Assignment of personnel to jobs on the basis of test scores, or the assignment of plants to locations are two other examples of this type of problem.[5] Furthermore, if we solve the program for alternative values of the budget restraint, we obtain a function which relates the changes in payoff to changes in costs, on the basis of which the administering authorities should be able to reach better decisions as to whether additional expenditures on a program are justified. In the case where the alternatives are "lumpy," as for expensive, discrete projects, this latter information is especially valuable, since it can be true that a slight increase in the amount of funds available for expenditure can allow the inclusion of an additional project with a relatively high payoff, thus sharply increasing the payoff from the total program. This is particularly applicable to the case of financing in the military services, where the budget restraints are seldom completely rigid, but are always subject to a degree of flexibility resulting from the bargaining process inherent in their establishment.[6] It is the obvious similarity of problems of this type

[5] cf. Tjalling C. Koopmans and Martin J. Beckman, "Assignment Problems and the Location of Economic Activity," Econometrica, Vol. 25, No. 1 (January, 1957), 53-72; reprinted as Cowles Foundation Paper No. 8, 1957.

[6] A. C. Enthoven and H. S. Rowen ("An Analysis of Defense Organization," RAND Paper P-1640 (March, 1959), 5-6) make this point in the following way: "The defense economy is more like the economy of a college student being supported by his father than like the economy of a household trying to allocate optimally within a fixed set of income possibilities . . . . The military departments, like the college boy, know full well that how much they get depends very much on what and how much they ask for." See also Richard B. Maffei, "Simulation, Sensitivity, and Management Decision Rules," The Journal of Business of the University of Chicago, Vol. 21, No. 3 (July, 1958), 177-186.

to the theory of consumer behavior which is the basis for the majority of the attempts to develop a theory of "military worth" for application to problems of military decision.

Assumptions of the Analysis

The discussion of the development of priority systems and their use in the solution of priority programs is based on five major assumptions. As each of these has considerable interest in its own right, they are reviewed briefly below.

1. It is assumed that the priorities discussed are obtained by the direct questioning of expert personnel. This method of approach is well established, and is the basis of several recognized studies in military decision making.[7] It is also assumed that the answers so obtained obey certain well recognized axioms of consistent behavior. Thus, if an "authority" prefers A to B and B to C, we assume that he is transitive — i.e., prefers A to C — as a matter of course, and without directly soliciting that information.

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[7] cf. R. J. Aumann and J. B. Kruskal, "Assigning Quantitative Values by Qualitative Methods in an Allocation-Procurement Problem," Technical Report No. 21, Analytical Research Group, James Forrestal Research Center, Princeton University (July, 1956). A revised version under the title "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem," appeared in the Naval Research Logistics Quarterly, Vol. 6, No. 1 (March, 1959, 1-16. Also, Henry Solomon, Joseph P. Fennell, and Marvin Denicoff, "A Method for Determining the Military Worth of Spare Parts," Serial T-82/58, George Washington University Logistics Research Project (April, 1958), of which a revised version under the title "Summary of a Method for Determining the Military Worth of Spare Parts" appeared in the Naval Research Logistics Quarterly, Vol. 7, No. 3 (September, 1960), 221-34. According to their investigations, substantial agreement as to relative worth was obtained for 92.4% of the total number of components ranked by three judges.

There are two major difficulties involved in this approach. The first is the difficulty of determining which comparisons should be made by questioning expert authority, and which should be made by a more direct method. Thus, if one wishes to determine whether or not one piece of equipment is superior to another, it may be preferable to subject them both to a series of mechanical tests to determine relative performance, rather than to ask a number of experienced military personnel which piece of equipment they consider as superior.[8] The second difficulty is that the answers obtained by such questioning may not, in fact, obey the rules of consistency.

Certainly, the first objection raises a major problem in the use of the direct-questioning approach. We shall assume for the purposes of this discussion that military planners possess a satisfactory method of deciding which approach is most applicable in a particular case, but we do so in full knowledge of the strength of such an assumption. We would add, however, that in the very common case where there are several dimensions of comparison, the direct testing approach will usually need to be supplemented by obtaining the opinions of expert military personnel. Thus, one might discover by testing that radar set A has a greater range than radar set B, but is also more difficult to maintain. In this situation, it is necessary to obtain a weighing of the importance of greater range against greater dependability to decide which set to purchase — a weighing which can only be obtained by an individual's decision. Therefore, in many situations, the methods of

[8] I am indebted to Professor Ansley J. Coale of Princeton University for several conversations on this point.

direct questioning and empirical testing are necessary complements to one another, rather than direct substitutes.

The second objection is less damaging. While it is true that under questioning expert personnel might give inconsistent comparisons, the method which we utilize in this paper assumes that all comparisons which are implied by the usual rules of consistency do, in fact, hold, and, therefore, that the expert personnel are only questioned as to relationships which do not follow from these rules. We would add, however, that it is a fairly simple matter to determine whether or not a set of orderings is internally consistent,[9] and that, therefore, even if we did not follow a policy of automatically excluding the possibility of inconsistent comparisons, it would be possible to locate quickly the inconsistent relations, and ask the expert authorities being questioned to make new comparisons eliminating their inconsistencies.

2. It is assumed for the purposes of this discussion that the various rankings obtained from expert authorities are either the views of a single individual, or represent the unanimous views of a group of individuals. In practice, in many situations the priority indicators must be based on a consensus of the opinions of a group of experts: we may face an institutional arrangement where a group, such as a board of directors or the Joint Chiefs of Staff, must be consulted and arrive at a joint decision; we may deliberately choose to ask several persons their opinions in the hope that the average of these opinions is more likely to be nearer the true answer than the estimate of a single person;

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[9] cf. Harold W. Kuhn, "Solvability and Consistency for Systems of Linear Equations and Inequalities," American Mathematical Monthly, Vol. 63, No. 4 (April, 1956), 217-232.

or, finally, we may need to consult several persons on the rankings, either because it would be impossible for any one person to answer the large number of questions which may be necessary to construct the scale which measures the alternatives — as may be the case in the von Neumann-Morgenstern approach to measurable utility or if we were attempting to construct an indifference map — or because any one person may not be sufficiently familiar with all of the alternatives to be able to judge their relative importance with any degree of competence. In practice, it is most unlikely that a group will be unanimous in its opinions. Thus, in most realistic situations, the military analyst will be faced with devising a method of aggregating the individual rankings into a single composite ranking. While we, for this discussion, assume in effect that such a method for devising a composite ranking exists, it should be noted that a very considerable portion of the literature of welfare economics is devoted to demonstrating the difficulties of obtaining such a ranking. An important question, therefore, is whether the problem of aggregation of expert opinion is closer to the problem of obtaining a community welfare function, or is, in fact, merely a statistical regression problem of estimating a function on the basis of a number of points which are randomly distributed about it,[10] but we shall

[10]The latter is the position adopted by Kenneth J. Arrow in his discussion of a paper by Commander Walter H. Keen ("A Note on the Construction of Numerical Utilities for a Finite Set of Economic Items," George Washington University Logistics Papers, Appendix 1 to the Quarterly Program Report No. 1, (November, 1949-February, 1950)) in Social Choice and Individual Values, Wiley (New York, 1951), where he does not question Keen's method of obtaining an ordering of objectives based on combining a set of individual orderings. Keen's approach to the priority problem is first to obtain an ordering of the set of objectives according to the method given by M. G. Kendall and modified by Garret Schuyler ("The Ordering of N Items Assigned to k Rank Categories by Votes of m Individuals," Journal of the American Statistical Association, Vol. 43, No. 244 (December, 1948), 559-563; see also M. G. Kendall, The Advanced

not consider this question further in our discussion.

3. We restrict ourselves to problems which may be classified as "simple maximum," that is, where all relevant variables, while perhaps only imperfectly known, are under the control of the military decision maker and not partially under the control of other intelligent entities which are in opposition to him. This latter situation is the domain of the Theory of Games of Strategy. As our discussion of the degrees of measurement which result from the questioning of expert personnel has considerable relevance for Game problems, however, we briefly note the connection.

Theory of Statistics (London, 1943), p. 410.) In the case in which we have only 3 categories of ranking, Schuyler suggests the following criterion for establishing an ordering of the alternatives. If an alternative j receives aj number of first place votes, bj number of second place votes, and cj number of second place votes, then the value of the indicator to be attached to the alternative

$$C_j = \frac{3aj + 2bj + cj}{aj + bj + cj} .$$

When this ordering is established, the method of ranking is reused to establish the preference of the group for various probability combinations of the alternatives which, when a zero point and a unit of measure are determined which can be done arbitrarily, establishes the N-M utility index. The validity of the method thus depends on two factors: whether the problem of expert opinion is exempt from the critique which Arrow makes of group welfare functions; and the suitability of the N-M utility index method in this context. In a letter to the author, Professor Arrow expresses a similar position:

> As you quite correctly observe, I feel that the question of pooling experts' opinions is different from that of a consensus of welfare judgments because presumably they are all judging the same underlying reality. I must admit that this gets into complicated epistemological questions if pushed far enough. However, if we postulate the existence of an objectively valid scale of some kind in the universe and assume that each expert's opinion is a random function with the objective scale or a parameter and with a probability distribution which has some degree of specification a priori, then we have a manageable statistical problem -- that of estimating the underlying scale.

The necessity for, and the utilization of, numeral indicators of priority or utility in the theory of games is well known. It was the purpose of the theory of utility devised by von Neumann and Morgenstern[11] to provide a measure of the subjective value of the outcomes of alternative courses of action to the players involved; and it is the inherent inability of this measure to overcome, other than by assumption, the problems of interpersonal comparison of value which constitutes one of the most serious gaps in the theory's applicability to major problems. In game theory, as in the study of military operations research, the concentration has been on the way in which the measures of worth could be utilized, once they have been developed.

When numbers for the game matrix are available, the solution to the game can be used to generate other priority indicators which can, in turn, be used to resolve other non-price allocation problems. This point has been well demonstrated by Professor Thomson Whitin.[12] Whitin begins his discussion with a variation on the well-known "Colonel Blotto Problem."[13] From an example, he shows that it is possible to compute a function which gives the marginal value of each unit of military strength which is added to the forces of each player, in terms of its contribution to the value of the game for that player;

[11] John von Neumann and Oskar Morgenstern, The Theory of Games and Economic Behavior, 3rd ed. rev. (Princeton, 1953).

[12] The Theory of Inventory Management, 2nd ed. (Princeton, 1957), Chapter 9, "The Problem of Military Value," pp. 189-207.

[13] cf. D. W. Blackett, "Some Blotto Games," Naval Research Logistics Quarterly, Vol. 1, No. 1 (March, 1954), 55-60; J. W. Tukey, "A Problem in Strategy," Econometrica, Vol. 17, No. 1 (January, 1949), 73; John McDonald, "A Theory of Strategy," Fortune (June, 1949), 102.

and also the changes in this value which will result if these additional forces become available to the player after he has committed himself to a strategy. From the first of these outcomes, we obtain a numerical value which enables us to decide whether a unit should be utilized on one front or another; and from the second, a numerical measure of the value of holding units in reserve, rather than committing them all at once. The relevance of this to the main line of discussion is quite clear. One way to determine the military worth of an alternative is in terms of its contribution to the game payoff, a "derived" value, rather than one which is determined directly.

Thus, one method of obtaining numerical indicators of value (priority, utility) is through this approach, which we may call "derivation." No attempt is made to measure the value of an alternative directly, but only in terms of its marginal contribution to the outcome of a conflict situation. The approach could be generalized to include situations of uncertainty, "games against nature."

In a second section Whitin applies game theory to the problem of choosing the optimum point on the production possibility curve — the derivation of the appropriate ratio of production between two types of military equipment, given technical limits on production. He derives the following: a description of a game which gives the alternative courses of action for all players in the face of all possible combinations of weapons and disposition of these forces by the other player; a solution to the problem of which strategy to employ if one should discover the productive plans of the enemy; and the optimal strategy to be used if the enemy discovers one's own production plans. Finally, Whitin illustrates the manner in which these strategies will be altered if the relative payoff values of the objectives of the game are changed.

To approach the problem in this manner is, however, simply to push the basic problem back one step, for although the derived indicators will be useful in many non-price allocation problems, their existence rests on the establishment of a real-valued function which measures the value of the alternatives in the conflict situation to each of the players in comparable terms. (This problem is, of course, simply by-passed in the Blotto formulation.) A somewhat more promising approach is suggested by our discussion of priority indicators where the measurement is not "numerical," but of the form of orderings or orderings-on-distances between alternatives. (See Chapter III.) Especially for opposing players, it is usually highly unrealistic to suppose that we could ever obtain numerical measures of their evaluation of alternatives. Whitin suggests, however, that it may not be necessary to have absolutely precise measures of the relative importance of these objectives, since the choice of optimal strategies for each situation can remain invariant through substantial changes in their value. Thus, we may attach a value of 2 to objective A and 1 to B, even though we know that military worth can never be determined exactly, since variations between 2.5 and 1.5 in the value of A, or between 1.5 and .5 in the value of B might not alter the choice of strategy. This suggests two lines of research which should be pursued if game theory is to be applied to concrete decision problems. First, the possibility of using a less-than-numerical scale should be investigated from a theoretical point of view.[14] And second, the relative stability of the solutions of

[14] Strangely enough, this important line of research has received comparatively little attention. In their original formulation

particular games when the players' evaluations of the outcome are changed should be examined. The following approach could be used. From qualified sources we obtain an estimate of the values which are attached to the set of "alternatives" by "our side," together with an estimate of the degree of variance these estimates involve. We also obtain such information on our opponent's evaluation of objectives, again based on

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(The Theory of Games and Economic Behavior, op. cit., section 66, pp. 603 ff.) von Neumann and Morgenstern discuss the possible generalization of the utility concept to avoid the use of numerical and transferable utility. These remarks have been extended by Lloyd S. Shapley and Martin Shubik, "Solution of n-Person Games with Ordinal Utilities," Econometrica (abstract), Vol. 21, No. 2 (April, 1953), 197. Their remarks may be summarized as follows: It is assumed that every player can order every possible outcome of the game. If we form the Cartesian product of these ordered sets of outcomes, then each possible outcome will be represented by a vector in this product. Coalitions can be formed. Any coalition is "effective" with respect to any of the set of vectors if there exists a strategy for the coalition whose outcome is as desirable as any of the vectors, despite any actions which may be taken by the players who are outside the coalition. Within the coalition there may be some provision for side-payments. Among the sets of vectors there will be some which will be dominated by the particular vector we have chosen, in the sense that the set of players which prefer this vector to all others is an effective set. There may exist a set of outcomes where each player receives at least as much as he would playing alone: clearly the solution set will be chosen from this set. In fact, the solution set will be that subset which includes every dominating vector and which excludes every dominated one of the vectors contained in the set which gave each player at least as much as he could obtain by playing alone. As the authors point out, this provides no systematic method for finding the solutions to particular games, nor does it guarantee that a solution does exist. That we can do nothing more than describe the properties that a solution would possess in this case indicates the fact that if we have only an ordering of objectives, we can say almost nothing about the solution to the game.

In another approach to this problem, R. Duncan Luce and Robert W. Adams ("The Determination of Subjective Characteristic Functions in Games with Misperceived Payoff Functions," Econometrica, Vol. 24, No. 2 (April, 1956), 158-171) assume that players operate on the basis of incorrect estimates of an opponent's utility function. By obtaining a function which measures a player's valuation of the coalition which might be formed, the authors show that a player's subjective characteristic function may be derived from this information observations by our own experts: we would here expect that the degree of variance in these estimates would be substantially greater and the degree of measurement less than in the first case. We then compute the solution to the game and the sensitivity of the solution to changes in the values attached to the alternatives and compare the latter with our own estimates of the accuracy of the values we have assigned to the alternatives. By a suitable statistical decision criterion, we then attempt to decide whether or not we are justified in taking a decision based on our information.

4. We assume throughout that expressed preferences are determinate and not probabilistic, i.e., that the military experts questioned will not, under the same set of exterior circumstances, sometimes assert that

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if certain plausible axioms hold. While this approach may be useful in deriving utility functions under experimental conditions, it sheds no light on our basic problem of poverty of information about an opponent's ranking of objectives, since in this case we would probably not be in possession of his ranking of the coalitions. John C. Harsanyi ("Bargaining in Ignorance of the Opponent's Utility Function," Cowles Foundation Discussion Paper No. 46, December 11, 1957) considers the question of how to bargain when one has only limited information concerning the numerical qualities of an opponent's utility function, but again does not utilize intermediate stages of measurement.

Finally, Martin Shubik ("Some Experimental Non-Zero Sum Games with Lack of Information About the Rules," Management Science, Vol. 8, No. 2 (January, 1962), 215-234; reprinted as Cowles Foundation Paper No. 170) has discussed games where a player is ignorant of his opponent's payoff function, and shows how after a series of experimental games players were able to rank the entries in their opponent's matrices with fair accuracy.

As we indicate below, there are several degrees of measurement which may exist between the numerical comparable utility which game theory assumes and the simple orderings of alternatives by all parties. The nature of the solution to games where one or more of the evaluations of alternatives by one of the competing parties is in these forms has never been investigated, and deserves some future research.

A is preferred to B, and sometimes that B is preferred to A.[15]

5. We deal throughout with situations where the set of alternatives to be measured is specified in advance — that is, we do not consider situations where the priority ratings must be continuously modified to take account of the fact that new and unforeseen alternatives must be considered.

Method of Approach

The method of approach utilized in the remainder of the discussion is as follows: Each degree of measurement — simple ordering through uniqueness to a linear transformation in one or many dimensions — which is to be obtained by the questioning of expert authorities is examined in its relation to particular problems in military decision making. We emphasize the latter, because, in general, it will be found that there exists a certain probability that an initial degree of measurment will enable us to solve the particular type of priority program at hand. We then move to the next level of measurement, and repeat the process. Finally, the utilization of numerical scales is discussed briefly.

This way of approaching the question — that a certain level of measurement will resolve a specific priority program with a certain

[15] For a discussion of a theory of utility based on the opposite assumption, see R. Duncan Luce, Individual Choice Behavior (New York, John Wiley, 1959). Such probabilistic utilities should, however, not be confused with the von Neumann-Morgenstern index of measurable utility — discussed at a later point in the study — which, though based on choices between combinations of probabilities and alternatives, is nevertheless a deterministic utility. Cf. Richard G. Davis and Walter G. Mellon, "Majumdar on 'Behaviourist Cardinalism'," Economica, Vol. 27, No. 107 (August, 1960), 253-255.

level of probability (when we are dealing with less-than-numerical scales of measurement) — is in direct contrast to the more usual axiomatic formulations. It is suggested that for practical purposes, however, it is less useful to know that it is strictly necessary to have a very high level of measurement (which implies a very strong set of axioms) to resolve all cases of a particular priority program, than to know that we may get satisfactory results in most or even some cases with a considerably weaker degree of measurement.

Naturally, in any specific case, a certain degree of measurement either will or will not resolve the priority program, but from any particular level of measurement we can deduce precisely which additional comparisons are necessary to move closer to a resolution. Thus, it will not be necessary to develop a higher scale of measurement which covers all possible alternatives, but only those relevant to the particular problem. The examples in the three chapters which follow are designed to bring out the relative strength of this general method.[16]

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[16] We abstract from the important consideration that in practice, the gains in computing optimal priority programs should be weighed against the costs of obtaining additional degrees of measurement. One simple method to take it into account would be to present expert personnel with the following question: To move to the next level of measurement will involve a certain effort and dollar cost, but it will also improve the chances of solving the problem by x %. Is the expected result worth the additional outlay?

The emphasis on the axiomatic method in utility theory as opposed to the approach followed in this paper has its parallel in the fact that theoretical economics has placed such great attention on the nature of optimal solutions under restrictive assumptions, and yet has taken very little notice of the pathbreaking paper by R. G. Lipsey and R. K. Lancaster ("The General Theory of Second Best," Review of Economic Studies, Vol. XXIV, No. 63 (1956-1957), 11-32), which examines the problem of rational behavior, when some of the conditions for a theoretical optimum are not met.

CHAPTER II

SIMPLE ORDERS

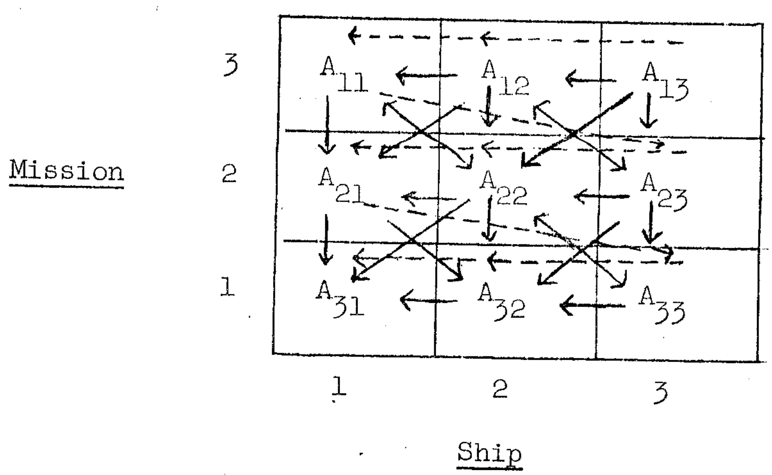
Section 1. Simple Orders in One Dimension

Example 1.

Consider the following situation. A base commander has three ships available, and has been ordered by his Theatre Command to carry out three missions. His problem is to assign these ships to the designated missions in such a way as to maximize the total military worth of the entire operation. The resulting assignment matrix is illustrated by Figure 1.

Fig. 1

Assignment Matrix for
3 x 3 Example



In the case where we possess numerical values for each of the particular assignments, the A_{ij} of the illustration, the problem is mathematically trivial — unless, of course, the values of the assignments vary according to which ship is assigned to which mission. All possible sets of assignments are compared, and the one which contains the A_{ij} summing to the highest value is selected. It is true that the number of possible assignment sets increases as the prime of the number of items to be assigned — to allocate 25 items to 25 positions involves the comparison of 1.55×10^{25} separate assignment sets — so that the actual solution of any reasonably sized problem by enumeration is highly impractical. But, despite the considerable mathematical ingenuity which has been displayed in devising efficient methods of determining the optimal assignment, the problem in this form is inherently uninteresting.

Let it be supposed, however, that we possess only a complete order on the A_{ij} . Then, our information is limited to the following propositions. If A_1, A_2, A_3, etc., stand for the various elements in the assignment matrix, then:

1. $A_1 \; P \; A_2 \; P \; A_3 \longrightarrow A_1 \; P \; A_3$
2. If $A_1 \; P \; A_2$, then $(A_1, A_3) \; P \; (A_2, A_3)$.
3. If $A_1 \; P \; A_2$, then (A_1) in (A_2, A_3) .
4. If $A_1 \; P \; A_2$, and $A_3 \; P \; A_4$, then

   $(A_1, A_3) \; P \; (A_2, A_4)$.
5. If $A_1 \; P \; A_2$, and $A_3 \; P \; A_4$, then

   (A_1, A_4) in (A_2, A_3) .

Let the particular order on the A_{ij} of Fig. 1 obtained by questioning expert personnel be of the following form:

$$A_{13} \; P \; A_{23} \; P \; A_{12} \; P \; A_{11} \; P \; A_{22} \; P \; A_{33} \; P \; A_{21} \; P \; A_{32} \; P \; A_{31} \; .$$

There are 6 (3!) possible assignment plans:

(A_{31}, A_{12}, A_{23})	(A_{21}, A_{32}, A_{13})
(A_{31}, A_{22}, A_{13})	(A_{11}, A_{22}, A_{33})
(A_{21}, A_{12}, A_{33})	(A_{11}, A_{32}, A_{23}) .

By application of [4.], we see that none of these assignments is revealed as preferred to any other. This, however, is a result of the particular order which we have chosen on the A_{ij} — an order, however, which has a special significance as is shown below. Because the result is a function of a particular order on the A_{ij} , we would expect that in certain cases an order established by the questioning dominates all others. And, in fact, of the 362,880 (9!) possible ways in which the A_{ij} might be ordered, in 10,080 $(\frac{1}{36})$ of these cases, the first three A_{ij} in the order correspond to one of the possible assignment plans, so that one plan is definitely revealed as preferred to all others.

Thus, in an "average" case the results will fall somewhere between these two extremes. On the one hand, there is a high probability that one plan will not be revealed as preferred to all others, but, on the other, there is a high probability that some of the plans will be revealed as inferior to some other plans. If it should be the case that the set of undominated plans is fairly small, then the most preferred plan can be efficiently selected by soliciting additional comparisons from qualified personnel.

In the above example, we have been assuming that each of the many thousand orders on the A_{ij} is equally possible. Let us assume, however, for the moment that it is possible to order separately the relative importance of each ship, and the relative importance of the mission to which it is assigned. Let mission 3 and ship 3 be the most important (have the highest priority indicator attached), mission 2 and ship 2, the next most important, and mission 1 and ship 1 the least important. It then follows that assignment A_{13} has the highest priority attached to it, followed by either A_{12} or A_{23}. More generally, we may say that an assignment has a higher priority than another assignment if it consists of assigning a higher ranked ship to a higher ranked mission, a higher ranked ship to an equivalent mission, or an equivalent ship to a higher ranked mission. The resulting relationships between the assignments are given by the direction of the arrows in Fig. 1. (This is equivalent to the assumption of Pareto optimality discussed in Section 2.)

If these relationships hold between the assignments, then it follows that no simple order on the assignments can ever yield one assignment plan which is preferred to <u>all</u> other plans, and indeed — more strongly — no assignment plan which is preferred to <u>any</u> other assignment plan. The proof of this proposition is fairly obvious. Compare the positions in the assignment matrix to which the lowest ranked ship can be assigned, that is, the column in the assignment matrix which is nearest the origin. If in assignment plan 1, ship 1 is assigned to mission 2, then in any other assignment plan, ship 1 can be assigned to missions 1, 2, or 3. The assignment of ship 1 to any of the three missions, of course, eliminates the row in the assignment matrix corresponding to the mission selected from ever being used again in the same plan. Let it be assumed that in plan 1, ship 1 is assigned to mission 3 and ship 2 to mission 2, and that in plan 2, ship 1 is assigned to mission 2 and ship 2 to mission 1. By [4.], plan 1 is clearly preferred to plan 2. But, the last two elements of the two plans are now determined, as there is only one degree of freedom left in each complete assignment. In plan 1, ship 3 must be assigned to mission 1, and in plan 3, ship 3 must be assigned to mission 3. As A_{13} P A_{33}, by our assumption, then the two plans are incomparable by [5.]. By extension, it follows that any two assignment plans, where the elements of the assignment matrix are so ordered, will be incomparable. And, in turn, if we make the (major) assumption that it is possible to conceive of the absolute worth of the assigned object independently of the mission to which it is assigned, then — since it is always possible to arrange the rankings along each axis as in the above example — a simple order on the elements of the assignment matrix will never reveal one assignment plan as preferred to any other. The question of the independence of object and mission is, naturally, one which would be expected to hold in some situations, and not in others, but we shall adhere to this assumption in this discussion in order to illustrate how higher orders of measurement can resolve the resulting incomparabilities which it of necessity involves. This is justified since, if higher degrees of measurement can accomplish such resolution in a situation where lower degrees of measurement <u>always</u> lead to incomparabilities, it follows that they can do so where lower degrees of measurement <u>sometimes</u> lead to incomparabilities — i,e., where other orders on the A_{ij} are present than the type we assume above.

<u>Example 2.</u>

The director of a military research program, operating under a limited budget, has a number of research projects which he would like to undertake: the dollar cost of each of these projects, we assume, is known in advance, but the most information which the director can obtain from his staff on the relative expected value of the projects is a complete order. The problem — a very common one in military decision — is to find the best combination of projects which can be purchased within the available budget.[17]

For illustrative purposes, assume that there are five projects under consideration, all of which involve a positive expected payoff, which are ordered:

$$A_1 \; P \; A_2 \; P \; A_3 \; P \; A_4 \; P \; A_5 \; ;$$

and which have expected dollar costs of:

A_1,	\$10,000	A_4,	\$30,000
A_2,	\$20,000	A_5,	\$50,000
A_3,	\$60,000		

The total budget is assumed at \$100,000.

There are 20 possible combinations which are feasible — that is, which do not violate the budget restraint. These are:

1. A_1	8. A_1, A_4	15. A_1, A_2, A_3
2. A_2	9. A_1, A_5	16. A_1, A_2, A_4
3. A_3	10. A_2, A_3	17. A_1, A_2, A_5
4. A_4	11. A_2, A_4	18. A_1, A_3, A_4
5. A_5	12. A_2, A_5	19. A_1, A_4, A_5
6. A_1, A_2	13. A_3, A_4	20. A_2, A_4, A_5
7. A_1, A_3	14. A_4, A_5	

[17] Our example is, of course, simple, as it is chosen to illustrate the effectiveness of gradually increasing the degree of measurement, rather than to provide insight into actual problems. For a discussion of how budget constraints interact with the values of the projects selected, see Scott C. Daubin, "The Allocation of Development Funds: An Analytic Approach," Naval Research Logistics Quarterly, Vol. 5, No. 3 (September, 1958), 263-76. The general question of the nature of various types of budget restraints is discussed in G. Hadley and T. M. Whitin, "Budget Constraints in Logistics Models," Naval Research Logistics Quarterly, Vol. 8, No. 3 (September, 1961), 215-220.

In this trivial example, it is readily seen that combination 15 (A_1, A_2, A_3) is preferred to combinations 16-20 by [4.], and by extension, to programs 1-14. This is entirely fortuitous, as the result is in most cases a function of the combination of the particular order chosen, and of the size of the budget restraint. Thus, if the budget of the Program Director were raised to $110,000, combination ($A_1$, A_2, A_4, A_5) would become feasible, which by [5.] is incomparable to combination 15. It is interesting to note, however, that either the original order on the alternatives or the interaction of the costs of the projects within the budget restraint can be the controlling factor in any particular case. In this example, it is entirely the latter, i.e., no one of the 120 (5!) possible orders on the projects could ever reveal any combination of projects superior to combination 15.

Because of the fact that the projects have discrete costs attached to them, the interaction of the budget restraint with the orderings of the expected payoffs from the projects means that there is no systematic way, as in our first example, to determine the probability that a simple order on the projects will establish a priority program. If we specify the size of the budget restraint and the costs of each project, however, then, if all orderings on the projects are regarded, in the abstract, as being equally probable, such a probability can be determined. The assumption that all conceivable orders on the alternatives are equally possible is not as strong as it would first appear. As is shown in Example 1, it is possible to impose in particular cases reasonable restrictions on the set of all possible orders, and then consider as equally probable only that subset of orders which conform to the restrictions.

As in Example 1, the normal (more probable) case would be that some combinations are revealed by the orderings as dominating some other feasible combinations, but that the set of these dominant combinations will (most probably) contain more than a single element. As we show in Chapter III, if higher levels of measurement can be obtained from questioning expert personnel, then it is possible to reduce this set of dominant combinations by a certain proportion with each new degree of measurement. It may, of course, be true that in problems of this type we arrive at the highest degree of measurement which it is in practice possible to obtain without having reduced the non-dominated set of combinations to a single element, but even in this case, there is a considerable value in having eliminated a number of clearly inferior choices. While this does not show up clearly in our illustrations because of their small size, in any normal-sized problem which would involve thousands of comparisons, the fact that even moderate increases in the degree of measurement will reduce the non-dominated set of choices very quickly to a manageable size becomes of great importance.

Example 3.

A third common type of priority program is an extension of both of the above examples, where it is necessary to rank the complete set of all assignment plans, or the set of all possible combinations of projects. Such a ranking might be useful in cases where, for example, additional resources to carry out less than the top-rated projects are expected to become available in the future, and it is desired to have all possible projects rated in advance.

We shall not examine this example in detail at each stage of measurement, for its relationship to our two basic examples is clear. In the context of our probabilistic approach, it is evident that the chances that the orderings on the alternatives will be such as to completely order all the many answers to a priority program are much smaller than the chances that such an order will select one answer as optimal, but will not resolve the incomparabilities between the set of dominated answers.[18]

Section 2. Orders in More Than One Dimension

Of more interest than the single-valued examples discussed in the section above are those cases where the priority indicators exist in more than one dimension of measurement — that is, given a number of points which exist in a space whose dimensions are the evaluation of their alternatives with respect to more than one attribute by qualified experts, how are these alternatives to be ranked? Indeed, on closer examination, almost every single dimensional case can be logically extended into a multi-dimensional one. Thus, in Example 2 in the preceding section, it is to be noted that combination 15 is superior to combinations 18, 19, and 20 in an absolute sense — that is, it is not

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[18]An analogous situation has been discussed where the problem is the assignment of employees to positions on the basis of job aptitude ratings. H. C. Taylor and J. T. Russell ("The Relationship of Validity Coefficients to the Practical Effectiveness of Tests in Selection: Tables and Discussion," Journal of Applied Psychology, Vol. 23, No. 5 (October, 1939), 565-578) note that a major consideration is the proportion of the individuals tested which are to be hired. They show that even a selection procedure which has a low correlation with the criterion of eventual job success is still much better than hiring without the test, provided that only a small proportion of those tested are eventually hired. Conversely, they show that even a test which predicts eventual job success with great accuracy has little value when nearly all applicants tested are hired.

only preferred by [4.], but it has also a smaller total cost. This, however, is not true for combinations 16 and 17, which are inferior by the assumed order on the alternatives and by [4.], but which have a smaller total cost than 15. Of course, if there is really nothing worthwhile on which to spend funds other than the projects listed, then it is better to proceed with combination of projects 15, rather than to proceed with combination 16, which involves a lower expected return and the waste of $40,000. But in a realistic situation, new promising projects may suddenly be suggested on which this extra $40,000 might be spent in such a way as to more than make up the difference between 15 and 16. Again, the research program, as in the defense report, may be part of a much larger program in which the $40,000 could be used to bring a total payoff to the overall program, though at a reduced payoff to the research sector.[19] If these considerations are admitted, then the problem is at once transformed into a two-dimensional one, where an optimum solution depends on the ability to weigh expected payoffs against payoffs from spending excess funds in other ways. Again, we can add another dimension of measurement by specifying in addition to a ranking or higher degree of measurement on the expected payoff from the projects, a measure of the expected variance in these payoffs. Thus, project 1 could be expected to be far more valuable than project 2, but the chances of its success are less.[20] It is clear, therefore, that the

[19]Cf. Charles Hitch, "Sub-Optimization in Operations Problems," Journal of the Operations Research Society of America, Vol. 1, No. 3 (May, 1953), 87-99.

[20]Cf. the author's review of: Harry M. Markowitz, Portfolio Selection (New York, Wiley, 1959) in the Naval Research Logistics Quarterly, Vol. 7, No. 1 (March, 1960), 91-92.

logic of priority indicators in more than one dimension is of considerably greater relevance to most decisions, military and otherwise, than is the single-dimensional case. For illustrative purposes, we confine ourselves to examples of two-dimensional cases, though in practice most decisions would involve several dimensions — that is, the decision space would be the positive hyperoctant.

Example.

Assume that there exists a central supply depot which supports a number of outlying bases. Each of three bases submits from time to time requests for material to which it attaches a priority classification which represents its own appraisal of the relative urgency of the request. The central supply depot, in turn, must appraise the relative importance of each base as it contributes to overall theatre operation, and on the basis of these two sets of priorities must rank the requests in order of importance, with those requests having a lower composite rating either not having their requests filled at all, or else receiving the requested material at some other date.[21] For our illustrative purposes, we have omitted consideration of the problem of the time phasing of requests, although, as several authorities have pointed out, this is a vital consideration in the design of military priority systems.[22]

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[21] Cf. Morgenstern, op. cit. n.1, for a discussion of this same general problem. Much of the following section is based on the author's paper, "Priority Indicators in More Than One Dimension," Naval Research Logistics Quarterly, Vol. 7, No. 4 (December, 1960), 513-527, as are the sections on the multi-dimensional cases in Chapters III and IV.

[22] Cf. Morgenstern, op. cit., also J. C. Busby, "Comments on the Morgenstern Model," Naval Research Logistics Quarterly, Vol. 2, No. 4 (December, 1955), 225-236. Considerable attention has been paid to one phase of the time-phasing problem--priorities in waiting-line situations. With respect to priorities of this particular type, see Frederick F. Stephan, "Two Queues Under Preemptive Priority with Poisson Arrival and Service Rates," Operations Research, Vol. 6, No. 3 (May-June, 1958), 399-418. A comparison of these studies with this report will show, however, that the priorities dealt with are of a very simple type.

Let it be further assumed that there are five priority classes into which the bases can place their requests (so that the class of all requests is only semi- and not completely ordered), and that there are five priority classes to describe the relative importance of each base as appraised by the central supply depot. Thus, there are 25 possible composite priority rankings which can be attached to any particular request, and the problem is to rank these composite rankings so as to obtain a semi-order on the set of all requests submitted.[23] Since even in this simple example there are 25! (1.55×10^{25}) possible orders on these composite rankings, the approach must be to search for acceptable *a priori* criteria which can be applied to reduce the problem to manageable size.

Criteria of Ranking.

The first criterion which we discuss has the very desirable property of completely ordering all of the 25 points in and of itself, but has, as would be expected, the undesirable property of being too strong to be generally acceptable. This is to assume that the ordering on the combinations is "lexicographic." In such orders, one vector (priority combination) is considered preferred to another if a designated component of the first vector is greater than the corresponding component of the second vector. Only if the two vectors have identical values of this particular component, will the determination of which vector is preferred rest on the comparison of the value of other components in both vectors. In terms of the example given above, this type

[23] Once the 25 combinations have been ordered, it might be found that the distribution of the requests within these classes is such that it is possible to combine the 25 combinations into several larger groupings. Cf. Keen, *op. cit.*, n. 10.

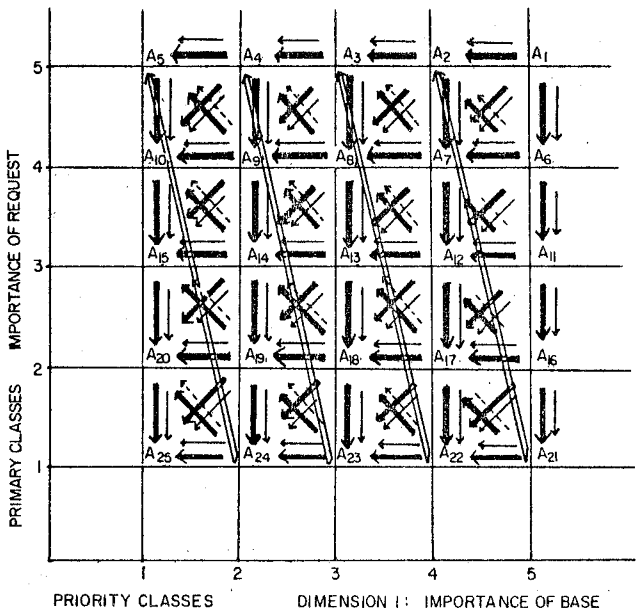
of ordering would imply that it could be decided which one of the two dimensions was the most important. Assume this was the relative importance (as given by the priority class in which it was placed) of the base from which a request originates. Then, for example, all requests of priority 5 bases would be ranked above all requests from priority 4 bases, no matter what degree of priority was attached to their requests by the originating bases. Only if two requests came from bases of equal priority ranking, would the ranking of the requests be determined by the priority attached to the requests by the bases. (The reader is referred to Figure 2, page 31, which illustrates each of the orderings we shall discuss.) Assuming that the ordering is lexicographic and that 1 is the dominant vector, the final ordering of the 25 points in the priority space will be of the form $A_1 \; P \; A_6 \; P \; A_{11} \; P \; A_{16} \; P \; A_{21} \; P \; A_2 \; \ldots \ldots \; A_{25}$, as shown in Figure 2.

The assumption that we are dealing with a lexicographic ordering is not an acceptable criterion in most cases of military supply. We would not ordinarily say, for example, that a set of ping-pong paddles for an important combatant ship is more important than the engine for a supply ship. In most cases, an infinitesimal movement along one of our dimensional axes does not outweigh very large movement(s) along the other(s), which the assumption of lexicographicity indicates.[24] This

[24] For a discussion of the difficulties in military supply that can develop when priorities of this "pre-emptive" type are applied, see Henry C. Eccles, Logistics in the National Defense (Harrisburg, 1959), p. 150. To assume that the higher priority is "overriding" or "preemptive" in the lexicographic sense is to deny the existence of an Archimedean property in the system, and to assert that no number of lower priority items can take precedence over a higher priority item, which is clearly implausible in many situations. In addition, we must consider the problem of "diminishing marginal worth." In general, as more and more items of supply are acquired, the marginal contribution

Fig. 2

Orders Under Alternative Criteria



Legend

1. Arrows ➡ and ⇒ indicate the preference relations established by the assumption that the ordering is lexicographic; relations between alternatives which are not shown are ensured through the property of transitivity.

2. Arrows ⟶ indicate the preference relations established by the assumption of Pareto Optimality.

3. Arrows ---> indicate the preference relationships established by the simple ordering on the dimensions.

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of each additional item will diminish. A carrier or a troop combat unit may be assigned a higher priority than a supply ship or a headquarters unit. Yet, it would be implausible to assert that a fifth spare part for the carrier is more important than the supply ship's engine. This insistence on the over-riding nature of priorities without recognition of the diminishing utility factor has been responsible for a great deal of the difficulty that priority systems have run into in military operations.

means that we must look for a weaker, but more acceptable criterion of ordering.

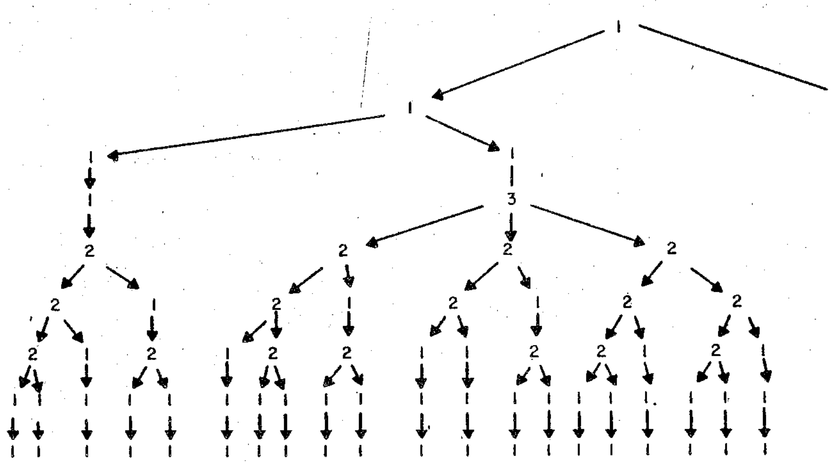
One criterion exists which is intuitively perfectly acceptable: this is the rather obvious concept known to economists as the Principle of Pareto Optimality.[25] Under this criterion one alternative is considered superior to another if it exceeds the latter in at least one dimension, and is no worse than the second in any other dimension. In terms of our example, this would mean that more important requests from more important bases are considered as more important than lower priority requests from lower priority bases; that requests from equally important bases take precedence according to the priority indicator attached by the requesting bases; and that requests from bases of different priority rankings which have the same priority assigned to them by the requesting base take precedence according to the priority classification of the requesting base. Figure 2 illustrates the effect of this criterion in establishing an order among the 25 points in the priority space. From the light arrows we see that Pareto Optimality establishes point A_1 as the most highly preferred, and point A_{25} as the least preferred, and sets up certain relationships among the other alternatives. This criterion does not, of course, like that of lexicographicity, establish a complete order, but it does reduce the number of possible orders to a small

[25]As the reader will have noted, nearly all of the material inherent in the problem of military worth has a certain relationship to welfare economics. Considered in this light, many of the difficulties and mistakes of the military worth theorist appear fairly obvious. In particular, while the "results" obtained by the "new" welfare economics are nearly all negative in that they point up what cannot be said concerning the solution of certain problems, the study of these limited results may at least keep researchers in related problems like military worth from wandering down some well-worn methodological blind alleys. The reader is recommended to Jerome Rothenberg, <u>The Measurement of Social</u>

fraction of the total number of orders possible if no criterion were imposed.[26] Assuming transitivity, then a series of additional pairwise comparisons between points will be able to establish a complete order. We know that A_1 must be the highest ranked point. The next point in the order must be either A_2 or A_6. If A_2 is selected, then the next point can only be A_3 or A_6, and so on. This process of choice generates a tree diagram of the form shown in Figure 3 (reduced to a 3x3 example and only one-half of the symmetrical tree illustrated for reasons of space).

Fig. 3

Ordering Tree for 3 x 3 Example



Welfare (Prentice Hall, Englewood Cliffs, New Jersey, 1961), for an extended discussion of ways in which welfare concepts — which we utilize only in relation to military decision problems — can be used in coming to social decisions.

[26]Thus, in the case where we have three priority classes in each of three dimensions, the application of the principle reduces the number of possible orders from 362,880 to 42. More generally, the total number of orders which obey the restriction of Pareto Optimality is given by $2! \cdot 3! \cdot 2! + 2! \cdot 2! + 2! \cdot 2!$ (number of orders beginning other than $A_1 \; P \; A_2 \; P \; A_3$ or $A_1 \; P \; A_4 \; P \; A_7$) $+ 2! \cdot 2! + 1 + 2! \cdot 2! + 1$ (number of orders beginning $A_1 \; P \; A_2 \; P \; A_3$ and $A_1 \; P \; A_4 \; P \; A_7$) . See also Garrett Birkhoff, Lattice Theory, Rev. Ed. (Providence, 1948).

By following down the diagram and obtaining the necessary directions as to which way to turn each time that a branch is reached, a complete order on the alternatives is obtained by a limited number of pairwise comparisons. The exact number of such comparisons which must be made depends on the particular path which is chosen. Thus, if we should have chosen the order A_1 P A_4 P A_7 P A_2 P A_5 P A_8 P A_3 P A_6 P A_9 , it could have been determined by six pairwise comparisons, while if the order began A_1 P A_2 P A_4 ... , it would be necessary to have more than six such comparisons, since we would be following a path where the ordering tree divides into three parts. By setting up the problem in the form we have just discussed, it is easy to determine which comparisons are relevant, and to have the experts questioned only with respect to these.

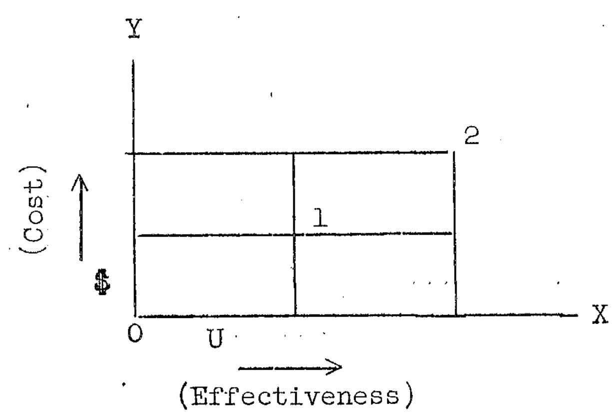
The alternative approach to obtaining the needed additional comparisons to establish the complete order by direct questioning of expert personnel is to search for a stronger criterion of ordering than Pareto Optimality. In the multi-dimensional context, the logical first step is to inquire whether this can be accomplished by ordering the dimensions of measurement themselves — that is, in the context of the supply base example, can anything additional be learned by obtaining the information that the importance of the base from which a request originates, as measured by the priority attached to that base by the central supply depot, is more important than the importance of the request itself, as measured by the priority indicator attached by the requesting base.

Taken simply in this form, such information is worthless, in the sense that it adds no additional information beyond that given by

Pareto Optimality. A statement that one attribute (dimension) is more important than another — i.e., that cleanliness is less preferable than Godliness — can mean only one of two things. Either we mean that Godliness is so much more important that any amount of it, no matter how small, outweighs any amount of cleanliness (denial of the Archimedean property), or we mean that some amount of it (unspecified) outweighs some amount of cleanliness (also unspecified), but presumably if we had enough of the latter, the direction of the inequality would be reversed. The first assertion is, of course, our previous one of lexicographicity. In the second meaning, if we take any two points, neither of which is Pareto-dominant with respect to the other, this type of ordering on the dimensions of measurement tells us nothing as to which point is "better." Let the points (alternatives) be as in Figure 4.

Fig. 4

Costs Versus Effectiveness

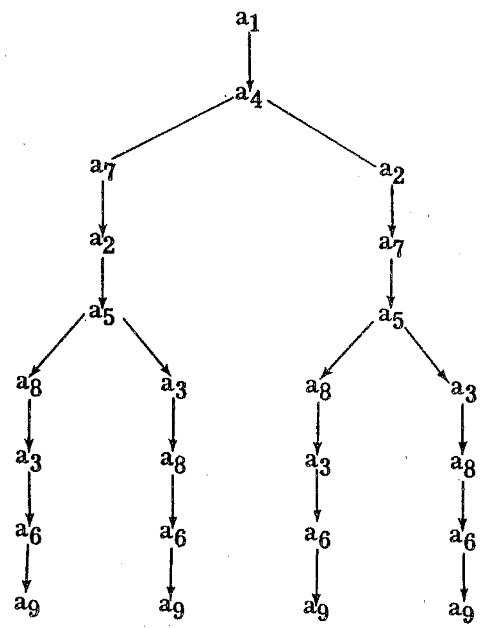


They may represent, for example, two weapons systems. Movement along the Y axis is taken to represent money costs, and movement along the X axis, military effectiveness. Suppose that we are in agreement that the military effectiveness of a weapons system is more important than the money cost. We cannot say that system 1 is preferred to system 2, since we do not know whether increased costs of system 2 are not more than offset by its increased military effectiveness from the statement that "military effectiveness is more important than costs." Of course, we can say that if two weapons systems have equal costs, we will choose the one with the greatest military effectiveness, but this is merely Pareto Optimality once again.

We can, however, make another assumption in the light of our example, providing that we realize its strength. This is to assume that a movement of one priority class in one dimension is more important than a movement of one priority class in another. Thus, in our example, we obtain an opinion from qualified personnel that, other things equal, we should consider the importance of the base from which a request originates as more important than the urgency attached to the request by the sending base, to the extent that a request from a base of a one-higher priority class should take precedence over a request from a base in a priority class one below it, even if the second request has a priority attached to it by the issuing base which is one class above that attached by the base issuing the first request. This assumption provides us with, in this example, 16 additional bits of information (indicated by - - > in Figure 2). Correspondingly, the maximum number of additional bits of information needed to establish a complete order is then once again reduced. For our 3x3 example, there are only 4 complete orders which obey this criterion plus Pareto Optimality: we therefore need to ask only 2 direct comparisons to establish the final complete order. (See diagram, Figure 5.)

Fig. 5

Ordering Tree Under
Stronger Assumptions



Again, we may either stop at this point and solicit the additional necessary comparisons directly which are needed to establish a complete order, or we may impose further over-all conditions.[27]

A next step might be as follows: an alternative is considered preferable to another if it is one priority group ranking higher in the more important dimension even if it is two groups lower in the less important dimension. Thus, in our example, this would mean that a request of urgency 3 from a base of priority class 5 would take precedence over a request of urgency 5 from a class 4 priority base. The reader can establish for himself in Figure 2 the new set of

[27]And, of course, conversely, just as the number of pieces of information necessary to obtain a complete order increases rapidly or the number of dimensions of measurement and|or the number of degrees of measurement (number of priority classes) in each dimension rises, so does the strength of the criterion necessary for a complete order to be established on the alternatives.

relationships which this assumption establishes among the alternatives. It should also be evident that a further extension of this line of relationships — i.e., that urgency 2 requests from class 5 bases take precedence over urgency 5 requests from class 4 bases, and so on — will eventually lead back to the assumption of lexicographic orders. The point is that this assumption can be approached to any degree which is thought warranted by the relationships between the alternatives as determined by the questioning of the expert personnel involved. As the criterion approaches closer to lexicographicity, the number of additional pieces of information needed to establish a complete order will, of course, diminish.

A more restrictive method of choosing a criterion would be to choose one which is "closed" rather than "open." Thus, it might be agreed upon that a difference of one priority class in the more important dimension of measurement is more important than a movement of one priority class in the less important dimension, <u>and also</u> that a movement of two priority classes in the less important priority class is more important than a movement of one priority class in the more important dimension — that is, a request of urgency 4 from a base of importance 5 takes precedence over a request of urgency 5 from a base of importance 5, but a request of urgency 5 from a base of importance 4 takes precedence over a request of urgency 3 from a base of importance 5. The criterion, together with transitivity and Pareto Optimality, will yield a complete order of the form shown in Figure 6.[28]

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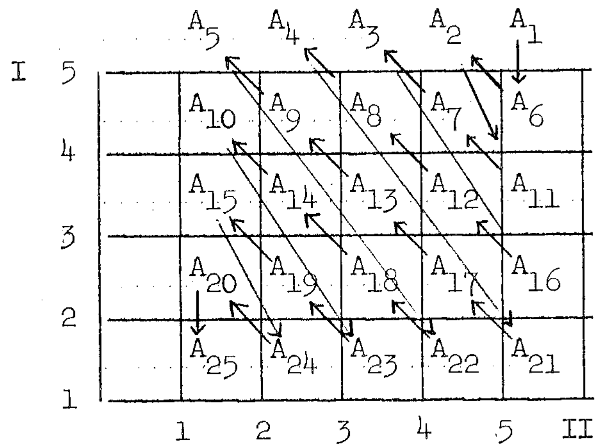
[28] This is equivalent to the following criterion: let the alternatives be ordered by their Cartesian distance from the origin — where we treat the numbers of the priority classes as if they were "real" numbers; where ties occur among alternatives, these are broken by referring to the ordering of the dimensions.

Fig. 6

Order Under "Closed" Criterion

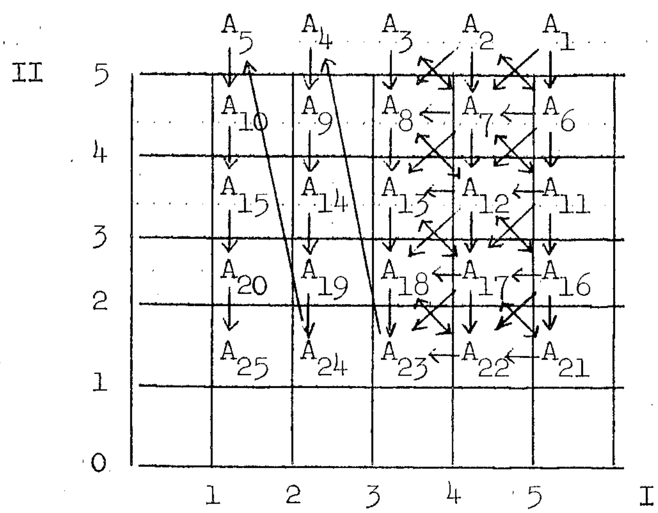


Two Reservations.

To the above remarks, two comments should be added. The first is that there is no reason that the approaches given above should be considered as mutually exclusive. For example, the assumption of a lexicographic ordering might be applied to a portion of a total problem, and some other of the criteria to the remainder. Thus, in the supply base illustration, we might make the assumption — based on the questioning of expert personnel — that above the minimum level of priority class 3 for the importance of the base, we are prepared to evaluate requests on the basis of the importance of the base and the urgency of the request as designated by the priority classification attached by the requesting base, with the former being rated as a more important consideration: below this priority class of importance of base, the priority of the base has an overruling importance. This semi-lexicographic case is illustrated in Figure 7: the complete order on the combinations can be determined by obtaining the answer to eleven direct pairwise comparisons.

Fig. 7

Semi-Lexicographic Order



Secondly, we should again emphasize that, no matter how plausible they may seem in the abstract, none of these criteria, except that of Pareto Optimality, must hold in any particular situation. In particular, they all imply a constancy of relationship within the preference structure which need not be present. In any application, this should be made explicitly clear to the expert personnel who are being questioned as to whether or not they would accept one of these more general criteria.

Upgrading of Priorities.

There is one additional problem inherent in these methods, or any other method, which is concerned with setting priorities within the particular multi-dimensional problem which relates to different levels of an organization. This is that as soon as the method of setting priorities for the organization as a whole is understood by the subordinate divisions of the organization, these divisions may attempt to inflate their own priority requests to assure that they are filled before those of the other divisions of the organization, thus upsetting the overall optimization procedure for the organization as a whole. This practice apparently occurs in military supply, despite the fairly tight control which can be exerted over subordinate activities: it certainly occurs in Government, and in other organizations. In another context we have dealt with the design of an allocation plan which makes misrepresentation of the true preference structure unprofitable.[29] Here, we note only the following.

First, in the context of the supply base example, the possibility of an upgrading of requests on the part of the bases is one strong argument for the adoption of a lexicographic ordering, in which the importance of the base from which the request originates is the dominant vector. For, in this case, the fact that a base starts attaching a priority 5 classification to all its requests can never mean that any of the requests are processed further ahead of those of another base than would otherwise be the case. In fact, this inflating of its requests could only harm the base (subordinate division of the organization) itself, since if it alters the real structure of its preferences by crowding all its requests into the higher priority classification, it may find "its toothbrushes arriving before its ammunition." Because

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[29] A lecture delivered to the Econometric Research Program, Princeton University, by the author and Professor Clive Granger on "Two Sets of Problems in the Theory of Bargaining," (December, 1959).

this method of processing requests avoids any advantages of a competitive race to inflate priorities, this may be an offsetting factor to some of the disadvantages of lexicographic orders which we have mentioned before. At the very least, it points up the fact that it would be doubly disadvantageous to establish a lexicographic method in which the dominant vector was the urgency attached to requests by the bases (subordinate division of the organization) since, in this case, it would clearly be to their advantage to always inflate the priorities attached to their requests. Any base (division) which did not enter the race would find that its requests were poorly treated, yet if all inflated the priorities of their requests, the system would clearly break down. (Obviously, certain elements of this problem were found in the breakdown of the priority system during World War II.) Any system which <u>compels</u> the division of an organization to take action which is clearly disadvantageous to the organization as a whole is most certainly undesirable.

There remains the entire family of non-lexicographic orders, which indeed the solicitation of expert opinions — in the absence of the explicit consideration of other factors like the one we are touching upon here — is more likely to produce. In the case of non-lexicographic orders, in the example, the degree to which each base can improve the treatment of its own requests will depend on the method of combined ordering adopted by the central supply depot (top level of organization), but in any case (except that of lexicographic orders with the X-axis dominant) such improvement will be possible. This is a necessary consequence of the application of Pareto Optimality, which guarantees that if a higher priority is attached by the requesting base, this will at the least not lower the request with respect to the order in which it will be treated. Thus, for example, if the order of base priority-request priority combination should be that shown in Figure 6, bases 1, 2, 3, or 4 would have an incentive to inflate their requests, since if they did not, they would be put at a disadvantage with respect to those which did: base 5 could divide up its requests as it saw fit between priority classifications 4 and 5, since A_1 and A_6 rank all the other points.

This problem — which is found in all organizational situations where the opinions of subordinate divisions of the organization, in competition with one another, must be taken into account by the upper levels of management — is a difficult one to approach. The simple answer — to keep the subordinate activities ignorant of the manner in which their opinions on the urgency of certain decisions are incorporated into a combined plan of decision covering all subordinate activities by the upper level of management — will not do, since, as we noted above, any non-lexicographic method of developing a combined plan would not lower in order a request for which a higher priority is requested. Thus, the subordinate activity would attach a higher priority to its requests even if it were totally ignorant of exactly the weight given this factor. It is true that in the case of lexicographic methods of weighting (which is one reason for adopting such a method, as we have already noted), such a procedure might rebound to the disadvantage of the subordinate activity, but this is offset by the undesirable implications of lexicographicity, and the fact that for the organization as a whole, it is certainly not desirable to have one subdivision operating inefficiently. If a lexicographic order, with importance of the sub-activity as the dominant dimension of rating, were to be adopted, then this should at least be made explicit to all sub-activities.

In our discussion of the problem of bargaining with arbitration,[30] we developed the following device to make misrepresentation of a true structure of preferences unprofitable. An arbiter obtains from each player an evaluation of what the latter feels to be his "fair share" of a certain prize — for example, a set of inherited objects, points at issue in the matter of a work contract, etc. The arbiter has at his disposal two methods of determining the way in which the "prizes" are to be distributed among the players: one of these rewards the player by giving him more than his true "fair share" based on his actual structure of preferences if he states these "true" preferences, and causes him to get less than his "fair" share if he misrepresents them; the second exactly the opposite. Let us suppose that a player by judicious misrepresentation could have made an extra $100, if a method of dividing the prizes had been used which assumed that each player told the truth about his preferences among the prizes. The arbiter, however, chooses the method of division by flipping a coin. Now, the player knows that if he misrepresents his preference and the method which allows misrepresentation to be rewarded should happen to be used, then he will gain $100 above his fair share; on the other hand, it is just as likely that the method which punishes misrepresentation by subtracting $100 from his fair share will be used. The expected profits of

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[30] *Ibid.*

cheating are then $ 0 , which means that the player might as well correctly represent his preference in the first place.[31]

Can the fact that misrepresentation of the preferences can be made unprofitable be applied to our present problem? The answer is yes, but at a cost. Presumably the best combined plan of allocation for the organization as a whole would involve a true statement of each sub-activity's relative appraisal of its needs, or at least for the moment we will so assume. A system designed to make misrepresentation unprofitable would have to imply a chance that some other method would be used, so that the expected value of the results generated by such a system would be lower than optimal. There would, then, be a question as to whether the losses to the organization as a whole by having a chance that a non-optimal method of allocation might be adopted as a device to make misrepresentation unprofitable, would be less or greater than the loss of efficiency which would result from a systematic misrepresentation of the importance of their requests by sub-activities of the organization. However, no matter how it is dealt with, we should stress that this problem would be encountered in such situations no matter which method of guiding allocation were used.

Numerical Representations.

If number relationships are preferred to order relationships, and also to simplify calculations in the more complicated situations, we may always replace the order relations with their numerical equivalents, provided we are careful to remember that the resulting figures and

[31] Cf. the discussion on "Problems of Fair Division" in R. Duncan Luce and Howard Raiffa, Games and Decisions (New York, Wiley, 1958), 363-367.

combinations of figures obey only the rules of order-arithmetic and not the rules of ordinary arithmetic. Thus, in our supply base example, we might assign the following weights to each of the priority classes on each axis of measurement. The importance of each of the dimensions can then be assigned a weight which when multiplied times the product of the weights assigned to each priority class will generate a series of numbers

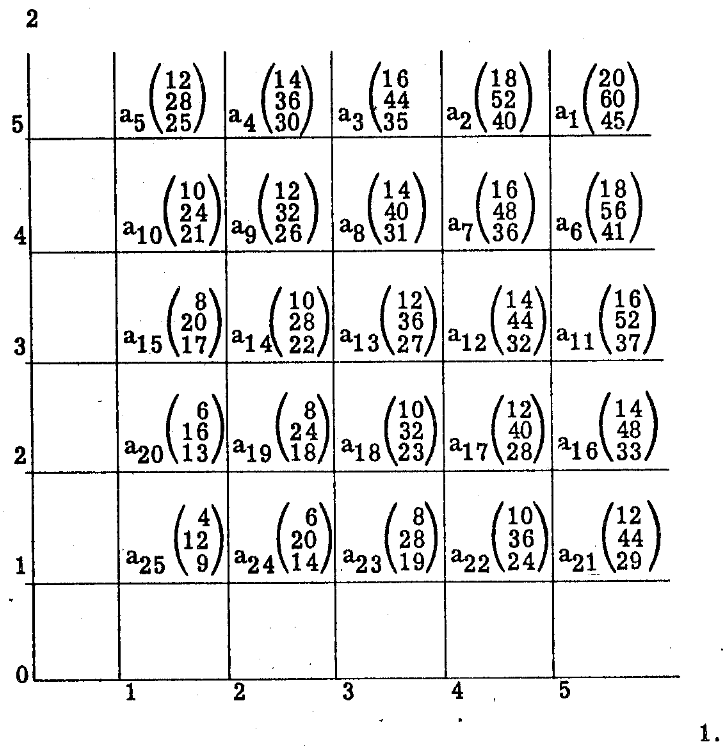
Table 1

Priority Class	Weight
1	2
2	4
3	6
4	8
5	10

to be assigned to each position which will order these alternative positions in the same way that the order relations which we have discussed above would. Thus, referring to Figure 8, we can see that if the two dimensions are given equal weight, a series of numbers is generated which corresponds to the relationship established by the condition of Pareto Optimality. Begin with the top number in bracket: if dimension 1 is assigned a weight twice that of 2 , the series of numbers generated obeys the criterion given on page 37, and so on. The "scales" which result can then be designed to obey any criterion we may choose, but only in the sense that they are unique up to a monotone transformation of the criterion. This property should be kept constantly in mind. For example, the "scales" so obtained cannot, in the "open-end" case, be used to establish the complete order among the alternatives, or, at least,

Fig. 8

Weightings Under Alternate Criteria



may very well contradict that which could be established by a direct questioning of the personnel involved. Thus, in Figure 8, the figures we have established to represent Pareto Optimality would indicate a_6 as preferred to a_{25} if they are read as real numbers, a relation which is not at all implied by Pareto Optimality.

This difficulty is not found in the "closed-end" case of ordering on the dimensions discussed on p. 38. Thus if dimension one is given the weight of 2.5 and two the weight of two, a complete order of the form of Figure 6 is determined. In such orders, the "scales" which are formed are invariant with respect to final order within the monotone transformation of the condition of the order on the dimensions.

Table 2

Dimensions 1	Scores 2	3	4
1	2	4	2.5
2	2	2	2.0
3	2	.	.
4	2	.	.
.	.	.	.
.	.		.
.	.		
.			

Summary.

What then can be learned from the simple order case? We have seen that, in the one-dimensional examples, the establishment of simple orders is sufficient to resolve only a rather limited number of priority programs. In the multi-dimensional example, we have seen that an order on alternatives can sometimes be established by questioning of expert personnel. This order can, in turn, be used in the computation of priority programs, as the examples in the one-dimensional case have indicated. On the one hand then, in contrast to many discussions of military worth which imply that a numerical scale is a necessity, simple orders are more effective than might be generally realized. On the other, simple orders can resolve only a proportion of the type of problems we have discussed, and for other problems of military decision — for example, the relative payoffs of allotting funds to two alternative research projects where we have payoffs in both as an increasing function of expenditures — they are virtually useless.

We are thus compelled to seek to obtain a higher degree of measurement by asking more sophisticated questions of our military experts — realizing, of course, that as the questions become more complex, the chances of obtaining meaningful answers are correspondingly reduced. Economists are accustomed to deal with only two types of measurement — ordinal and cardinal. But, as we see in the following chapter, there are a number of intermediate stages.

CHAPTER III

ORDERS-ON-DISTANCE

Section 1. Orders-on-Distance in One Dimension

The second major general class of measurement consists of "orders-on-distance," or "metric" scales. This general class may, in turn, be subdivided into two main sub-classes: "Orders-on-simple-distances" and "Orders-on-compound-distances," with the latter class capable of being subdivided still further. We discuss each of these various degrees of measurement in turn, and relate them to the examples used in Chapter II to show how a greater probability of resolution of these particular priority program problems is generated at each additional level of measurement.

A preliminary digression will serve to place this new level of measurement in its proper perspective. It is to be remembered that a number of order-on-distance relationships are predictable from a simple order on a set of alternatives. Assume that there are five alternatives, ordered as follows:

$$V\ P\ W\ P\ X\ P\ Y\ P\ Z\ .$$

The number of possible relationships which can exist between all continuous simple and compound distances in a set of alternatives is given by the expression:

$$\sum_{j=1}^{K} K - j\ , \text{ where } K = \sum_{i=1}^{N} N - i\ ,$$

where N is the number of alternatives, or alternatively by the expression

$$2\left(\binom{N}{3} + \binom{N}{4}\right) + \binom{N+1}{4}$$

written in the usual notation for ordered partitions. For our set of five ordered alternatives, there are thus 45 different possible comparisons between the continuous simple and compound distances between alternatives. For later reference, these are listed below.

1. VZ P VY	16. WZ P WX	31. VW ? WX *
2. VZ P VX	17. WZ P XY	32. VW ? XY *
3. VZ P VW	18. WZ P XZ	33. VW ? YZ *
4. VZ P WZ	19. WZ P YZ	34. VW ? WY +
5. VZ P WY	20. VX P VW	35. VW ? XZ +
6. VZ P WX	21. VX P WX	36. WY ? YZ **
7. VZ P XZ	22. WY P WX	37. WX ? XY *
8. VZ P XY	23. WY P XY	38. WX ? YZ *
9. VZ P YZ	24. XZ P XY	39. WX ? YZ +
10. VY P VX	25. XZ P YZ	40. XY ? YZ *
11. VY P VW	26. VY ? YZ **	41. VY ? WZ **
12. VY P WX	27. VX ? XZ **	42. VY ? XZ **
13. VY P WY	28. VW ? WZ +	43. VX ? WZ +
14. VY P XY	29. VX ? XY **	44. VX ? WY **
15. WZ P WY	30. VX ? YZ **	45. WY ? XZ **

The first 25 combinations are those where the relationship of the distances between alternatives is fixed by the establishment of a simple order on the alternatives of the form given above. The relationship in this case is simply that the whole is greater than any of its parts. For a set of N ordered alternatives, the number of comparisons between distances between alternatives which are fixed by the simple order is given by

$$D_1 = 2\binom{N}{3} + \binom{N}{4} .$$

Since these relationships are immediately established by the simple order, when we move to measurement of the form of orders-on-distance, the first step must be to make sure that the method of soliciting such information from expert opinions is efficiently designed to avoid asking for the comparisons which we already know from the order-on-the-alternatives, and is concentrated on asking for comparisons which will increase our degree of measurement. Alternatively, of course, the information which we gain from the simple order on alternatives can be used to check the answers to comparisons of distances between alternatives for consistency.

In realistic situations, however, the informational effectiveness of the order on the alternatives themselves will prove limited. Just as was the case in our previous discussion of the effectiveness of the criterion of Pareto Optimality in establishing a complete order on a set of alternatives, the amount of order-on-distance (metric) information conveyed by the order on the alternatives diminishes rapidly as the number of ordered alternatives increases. Thus, for three ordered alternatives, all (2) metric relationships are determined, while for ten ordered alternatives, a complete order determines only 450 out of the possible 990 metric relationships.

An examination of the 20 metric comparisons in our example where the direction of preference was not determined by the order on the alternatives shows that these fall into two classes. First, there are those comparisons which are between entirely disjoint distance intervals, such as number 33, the relationship of distance VW to distance YZ . For a set of N ordered alternatives, the number of such comparisons is given by

$$D_2 = \binom{N+1}{4} ,$$

and constitute numbers 26 through 40 in the listing on page 51.

Second, there are those combinations which are between distances which partially but not completely overlap one another.[32] For a set of N ordered alternatives, the number of such combinations is given by

$$D_3 = \binom{N}{4} .$$

These combinations are those which constitute numbers 41 through 45 in the listing on page 51.

The next question is, what additional ordering information is necessary to establish the direction of the preference relationship in these 20 combinations, and, therefore, a complete order on the entire set of all (in our example) 45 combinations. From the fact that the original order on the five alternatives enabled us to derive 25 bits of metric information, we may assume that it is not necessary to obtain all of the other 20 comparisons by direct questioning, but rather that the determination of some lesser number of comparisons will be sufficient to determine the remainder. This is true, and in fact, the "informational effectiveness" of certain additional directly-obtained order information — which we may define as the ratio of the number of bits of metric information indirectly determined to the number of those established by direct questioning of experts — is quite high.[33]

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[32] More exactly, those combinations where the number of simple distances (distances between alternatives which are adjacent in a particular order) are equal in both sides of the combination when an overlap occurs.

[33] Cf. Robert F. Fagot, "An Ordered Metric Model of Individual Choice Behavior," Technical Report No. 13, Applied Mathematics and Statistics Laboratory, Stanford University (September 12, 1957), Section 5. Also, C. G. Hempel and P. Oppenheim, "Studies in the Logic of Explanation," Philosophy of Science, Vol. 15, No. 2 (April, 1948), 135-175; John Kemeny and P. Oppenheim, "Systematic Power," Philosophy of

Let a preference order be established between the simple distances between the five alternatives, and let this order be of the form

VW P WX P XY P YZ .

From this order, we obtain directly the six pieces of metric information indicated with a single * in the listing on page 51. Using in addition our rules of order arithmetic, we obtain the additional 9 preference relationships which are indicated in the listing as ** .

There remain those five combinations marked with a + where the order on non-compound distances between alternatives is not sufficient to establish the direction of the preference relationship. These, in turn, are of three different types. The first is what may be termed as completely dependent. The listing on page 51 contains only one example of this type, combination 43 (VX ? WZ) . It is clear that the direction of the preference relationship in this combination depends on the direction of the relationship in combination 35 (VW ? XZ) . This in turn shows that the difficulty in determining a complete order on all combinations when possessed of an order on alternatives and an order on the simple distances between alternatives lies entirely with a subset of the set of non-overlapping compound distances.

The second type is what may be termed semi-dependent. Consider combinations 28, 34, and 35. Let the direction of the preference relationship in 28 be solicited directly. If (VW P WZ) , then the direction of preference in the other two combinations is fixed at once.

Science, Vol. 22, No. 1 (January, 1955), 27-33; and Sidney Siegel, "Individual Decision-Making Under Uncertainty: An Ordered Metric Model," unpublished manuscript — all of which make use of a similar concept.

If (WZ P VW) , then we must solicit the preference relationship in the other two combinations directly. Finally, the third or completely dependent type, is given by combination 39. Again, the direction of the preference relationship within the combination must be obtained by direct question.

Thus, a complete order is efficiently established between all simple and compound continuous distances between the ordered alternatives. There remain the relationships between non-continuous distances. For a set of five ordered alternatives, there are 9 such alternatives. These are:

1. VW ? WX + YZ	6. WX + YZ P XY
2. VW + XY P WX	7. VW + XY P YZ
3. VW + YZ P WX	8. VW + XY P YZ + WY
4. VW + YZ P XY	9. VX + YZ P XY
5. VW + XZ P WX	

In this example, the ordering on the simple distances is sufficient to establish the direction of the preference relation in all but one case. (This is, of course, merely fortuitous, in that we have chosen to order the simple distances in the same descending order as the alternatives — which is clearly a special case.) In the case of comparison 1, the direction of preference may or may not be determined by the order on compound continuous distances. Clearly, if VW P WZ , then it is; if WZ P VW , it is not.

<u>Orders on Simple Distances.</u>

Let it be assumed that we have four alternatives to be ranked in a single dimension of measurement. By the questioning of expert personnel, the following relationships are obtained: alternative W is preferred to alternative X , X to Y and Y to Z ; and also that alternative W is preferred to alternative X by more than alternative X is preferred to Y , and in turn, that alternative Y is preferred to Z by less than X is preferred to Y . These relationships are shown pictorially in Figure 9.[34]

W X Y Z

1 0

Fig. 9

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[34] The concept of metric relationships has been most thoroughly explored by: C. H. Coombs, "Psychological Scaling Without a Unit of Measurement," Psychological Review, Vol. 57, No. 3 (May, 1950), 145-158; "Mathematical Models in Psychological Scaling," Journal of the American Statistical Association, Vol. 46, No. 256 (December, 1951), 480-489; A Theory of Psychological Scaling (Ann Arbor, University of Michigan Engineering Research Institute, Bulletin No. 34, 1952); and Sidney Siegel, "A Method for Obtaining an Ordered Metric Scale," Psychometrika, Vol. 21, No. 2 (June, 1956), 207-216. Professor Siegel has used his method for obtaining the metric relations — which is in fact a weakened version of the von Neumann-Morgenstern utility theory — in a number of applications. Cf. "Level of Aspiration and Decision Making," Psychological Review, Vol. 64, No. 4 (1957), 253-262; "Utility of Grades: Level of Aspiration in a Decision Theory Context," Journal of Experimental Psychology, Vol. 55, No. 1 (January, 1958), 81-85 (with Selwyn W. Becker); "Prediction of Decisions from a Higher Ordered Metric Scale of Utility," Journal of Experimental Psychology, Vol. 52, No. 2 (August, 1956), 138-144 (with Paul M. Hurst). Professor Siegel does not base his metric scales on the answers to questions of the form: "Do you prefer A to B by more than you prefer B to C?" Instead, these relationships are deduced by the subject's choices between various probability combinations.

> ". . .suppose that an individual must choose between $(B,B;\frac{1}{2})$ and $(A,C;\frac{1}{2})$. That is, he must choose between getting B for sure or getting a 50-50 chance at A or C. By making the choice he yields new information. If he chooses the combination which gives him B for sure, his choice indicates that B is closer to A than it is to C. If he chooses $(A,C;\frac{1}{2})$, then B must be closer to C than it is to A. This is fundamentally new information because the statement $A > B > C$ tells us nothing about the distances (differences) between the entities on the utility scale." ("A Method for Obtaining an Ordered Metric Scale," op. cit.)

As has been pointed out, however, in other discussions (cf. Luce and Raiffa, op. cit., n. 31, p. 32; or Davis and Mellon, op. cit., n. 15), information of this type cannot be logically deduced from such

It will be readily observed that the establishment of these simple orders on the distances between alternatives establishes certain ranges of relationship which these distances can bear to one another, and, therefore, fixes certain ranges within which the alternatives themselves are confined over the entire dimension of measurement. Thus, let us arbitrarily assign the number 1 to the highest ranking alternative W and the number 0 to the lowest ranking alternative Z . (Any other numbers would, of course, do quite as well.) Governed by the rankings on the simple distances between the alternatives, the following ranges of values can be assigned to the alternatives. The logic behind the determination of the possible ranges which the alternatives may take is

Table 3

Alternative	Priority Indicator	Midpoint of Range
W	1	1
X	$\rightarrow$ 0 - $\rightarrow$ .666 ...	$\rightarrow$ .333 ..
Y	$\rightarrow$ 0 - $\rightarrow$ .333 ...	$\rightarrow$ .166 ..
Z	0	0

easily explained. At the extreme upper range, the distance XY is very

probability comparisons because of the possibility that the subject may not be maximizing expected utility due either to a positive adversion to risk or a positive utility of gambling. As we demonstrate below, however, measurement of the form of orders on distances between alternatives establishes a series of ranges of values which the alternatives may assume. It is, therefore, possible that comparisons of the nature used by Professor Siegel may in many instances yield information of the type which he asserts they do, because the deviation from the expected utility hypothesis on the part of the subject is sufficiently small and does not, therefore, alter the evaluation of the alternatives by more than an amount which would exceed the ranges established on the alternatives by an order on the distances between them which was not influenced by the subjects' attitudes toward risk. Thus, the Siegel method has a finite probability — impossible to specify in a non-concrete example — of escaping the objections raised to similar interpretations of the von Neumann-Morgenstern method of utility as yielding higher degrees of measurement, e.g., that an alternative is twice another.

slightly larger than the distance YZ , and since these both must be less than the distance WX , they both must be slightly smaller than $\frac{1}{3}$ of the total range separating all alternatives. At their extreme lower ranges, the distances XY and YZ approach zero, while the distance WX occupies nearly the entire range between all the alternatives. This then is the special property of all orderings on distances: that the largest ranked distance can never be reduced below slightly more than $\frac{1}{N}$th the total distance between alternatives, while the other alternatives, no matter what their number, can all be reduced to nearly zero,[35] and that at the upper limit of the range for all distances between alternatives but the highest ranked one, the alternatives will approach an even spacing over the entire range of measurement. (Naturally, an alternative may not be assigned to any point within the ranges <u>independently</u> of the assignment of other alternatives to points within their respective ranges in such a way which would violate the order on the alternatives themselves.)

Since the establishment of the ranking on the distances between alternatives fixes the position of the alternatives within certain ranges, it is clear that we can assign any values to the alternatives which are unique up to the transformation fixed by the ranges. The question is, what numbers are to be assigned. If we wish the numbers to stand only for the order-on-distance relations, as in the previous section, then any number within the ranges will do just as well as any other — provided that the order conditions are not violated. If we wish, however, to use the numbers so assigned as a substitute for an actual numerical scale on the alternatives, then the logic of how these numbers are to be selected

[35] As the reader can easily verify, this does not result from the fact that, for convenience, we have ranked the distances between alternatives in the same order as the alternatives themselves.

within the ranges established by the orders-on-distance becomes important. We consider the latter case here — which in any event includes the first case within it.

Digressing slightly, if we were to add several more alternatives to the example given above, then it is clear that the range over which those alternatives, which are neither first nor last in the ranking, can vary is decreased. Given a fixed origin and end point, no matter whether these be designated as 0 and 1 or 0 and 1,000,000, each addition of an alternative will decrease the range which non-fixed alternatives can take by $\frac{1}{N}$th the distance between the first and last of the ranked alternatives. Let the process be continued indefinitely: As the number of alternatives approaches ∞, the range which ~~each non-~~ fixed alternative can move within approaches $\frac{1}{\infty}$ — i.e., at ~~the~~ limit, each alternative has no range within which it can move. Or, the scale which relates the position of the alternatives has become invariant, except with respect to origin and unit of measurement — that is, unique to a linear transformation. With an infinite number of alternatives, then, an order on the distances between alternatives is equivalent to a numerical scale.[36] Since, however, in reality one never will have to deal with an infinite number of alternatives, it is then proper to consider orders-on-distance as representing a type of measurement which is intermediate between simple orderings and uniqueness to a linear transformation, approaching the latter as a limit, and always providing considerably more information than the former. Since the increase in the number of alternatives to be evaluated is generally represented as a

[36] Intuitively obvious, this proposition was first given a formal proof in Aumann and Kruskal, op. cit., n. 7.

factor tending to complicate the computation involved in any decision problem, it is somewhat reassuring to note that in this case such an increase improves the reliability of the measure of value of the alternatives as well.

If we were to pick a single number to assign to each alternative to represent the information conveyed by the orders-on-distance between alternatives, the first logical choice would be to pick the midpoint of the range fixed by the orders-on-distance. Thus, in our simple example above, the alternatives would be assigned the values shown in the last column of Table 3 on page 57.

Is this then the "best" way in which to assign numbers to the alternatives? Assuming that we are trying to construct some reasonable substitute for a true numerical scale, there is some question as to whether this question has any real meaning. If we assume that the orderings-on-distance are the highest form of measurement possible in the minds of the expert personnel from which they are obtained, then it would seem to make no difference how the numbers were selected within the ranges — if we use such numbers in any other way than as a convenient representation of the order-on-distance relationships, we are simply using arbitrary and non-existent relations as if they were meaningful. On the other hand, we may conceive of the orders-on-distance as the only obtainable reflections of a "true" numerical scale relating the alternatives to one another, which exists in the minds of the questioned expert personnel. In this case, since we have no information about the numerical scale other than the relationships implied by the orders-on-distance, we rationalize that the true subjective values which the individuals attach to alternatives may lie anywhere within the ranges established by the distance orderings. Thus, by fixing the value to be assigned to each alternative at the midpoint of the range determined by the orders-on-distance, the total deviation which could possibly occur between the values we assign to each alternative and the "true," but unknown, subjective values of the alternatives would be minimized. If, however, one does not believe that any such "true" values exist, or, indeed, that any such concept has any meaning, then, as stated above, any set of numbers which are unique up to the intervals established by the orders-on-distance would do precisely as well, and any argument in favor of the midpoint rule must be one merely of simplicity and convenience.

An interesting treatment of this problem is given in an important paper by Robert P. Abelson and John W. Tukey,[37] which we summarize below.

In assigning numerical values to our alternatives, subject to whatever order restrictions we have — that is, whether we have a simple order, an order-on-simple-distances, or some more restrictive measure — we produce a series of values: $x_1, x_2, \ldots, x_n$. Let us imagine the ideal or actual values to be assigned to these alternatives, as the series of values: $y_1, y_2, \ldots, y_n$. In a normal problem, the success or failure of our assignment could well be measured by the square of the formal product-moment correlation between the x's and y's , i.e., by

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[37] "Efficient Conversion of Non-Metric Information into Metric Information," a paper delivered at the annual meeting of the American Statistical Association, Washington, D. C., December, 1959. Summary in the Journal of the American Statistical Association, Vol. 55, No. 270 (June, 1960), 353; also "Efficient Utilization of Non-Numerical Information in Quantitative Analysis," mimeographed (Princeton University, 1958).

$$r^2 = \frac{\left[\sum_{i=1}^{N} (x_i-\bar{x})(y_i-\bar{y})\right]^2}{\sum_{i=1}^{N} (x_i-\bar{x})^2 \sum_{i=1}^{N} (y_i-\bar{y})^2} .$$

But, of course, we do not know the y sequence. All we know is that it must obey the same set of inequalities as the x system, the inequalities determined by our initial order restrictions. Tukey and Abelson suggest the following. Assume that Nature is playing a game in which it attempts to select a y sequence which obeys the inequalities, but which gives a minimum value of r^2 for any chosen x sequence. In turn, we play a game in which we choose the x sequence in such a manner that the minimum r^2 which can be attained by Nature is as high as possible. This, of course, is a "conservative" strategy.

The scales which result when this method is applied when any of a number of order relationships hold between a set of alternatives are given in some detail in the paper cited. In the context of this section, it can easily be seen that the midpoint rule is the result of applying this "maximin" criterion when the order relation is an order-on-distance between alternatives. Thus, from the relationships given in Fig. 9 and Table 3, if we were to make the assignment of the value .4 to alternative X , we are assuming that Nature would assign a value as close as possible to 0; and conversely, if we were to assign a value of say .2 , Nature would assign a value as near as possible to $\frac{2}{3}$. It It is to be noted that whatever value assignments we may choose to make, Nature will "inflict the most damage" by picking a value in one "corner" of the range. Thus, obviously, the way to minimize this "damage" is to assign a value to an alternative which is in the exact center of the range to which it is confined by the order-on-distance relations.

The concept of limits does not lend itself readily to a mechanical method for estimating the value of the priority indicators to be attached to any particular set of alternatives. This, however, need not prove troublesome, since these values can be easily approximated to any desired degree of "accuracy." Let the range of measurement between the highest and the lowest ranked alternative be divided into any number of equal intervals, and assume that an order on the distance between alternatives, for example, that $WX > XY$, means that WX is at least one unit of distance larger than the distance XY. Suppose that there are six alternatives and the distances between alternatives are ranked: $U > V > W > X > Y > Z$; and $UV > VW > WX > XY > YZ$. Then applying this variation of the midpoint rule, assign the following values to the alternatives: U = 1,000, V = 404, W = 295.5, X = 200, Y = 98.5, and Z = 0 (if the range of measurement is assumed divided into 1,000 equal parts).

These propositions may perhaps be better visualized by remembering that the ranges within which the values of the ranked alternatives are confined by the establishment of orders on the distances between them, define the limits of a polynomial in $N - 2$ dimension, where N is the number of ranked alternatives. Assume that we have five ranked alternatives: $V > W > X > Y > Z$; and an order on the distances between them: $VW > WX > XY > YZ$. If we assign V the value of 1,002 and Z the value of 0, then by the approximation rule given in the preceding section, the other alternatives will be confined within the following ranges: W, 750-6; X, 499-3; and Y, 249-1. This establishes

Figure 10, and point P's projection on all three axes establishes the values to be assigned to the three alternatives.

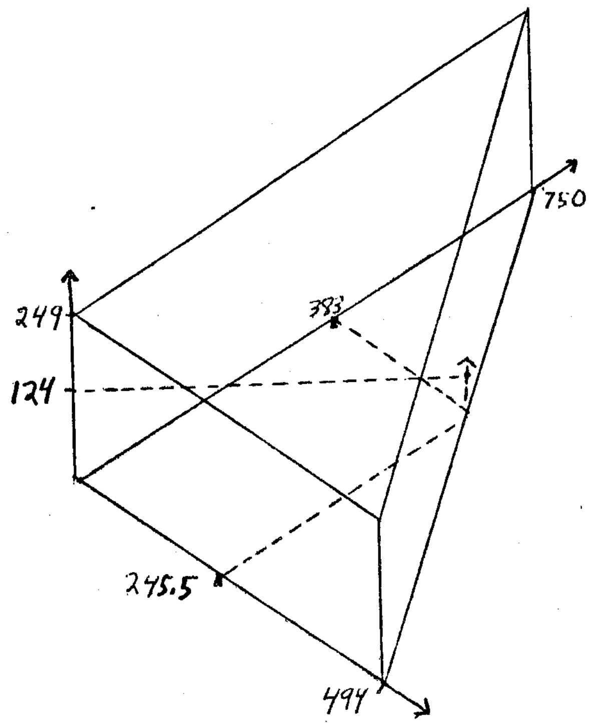

Fig. 10.

The rule of using the midpoint of the ranges established by the orders on distances between alternatives is, as the previous discussion implies, not as innocuous as it first appears, but implies, or results in, the assignment of values to the alternatives, which in turn implies a degree of measurement much stronger than that given by the orders on distances between alternatives. For three ranked alternatives, the smallest number for which there is any significance to ordering the distances between alternatives, the midpoint rule assigns a value to the middle ranked alternative in such a way that the distance between it and the highest ranked alternative is approximately three times the distance between the middle ranked alternative and the lowest ranked alternative. For four alternatives, W, X, Y, Z ranked in that order, and with the distances between alternatives ordered WX > XY > YZ , the midpoint method will assign a value to X , so that the distance WX is indicated as something greater than twice the sum of the distances XY + YZ . As the number of alternatives increases, this disparity in size, like the ranges within which alternatives are confined by the orders-on-distances between them, diminishes. Still, for any finite number of alternatives, the use of the midpoint rule will imply that the highest ranked distance between alternatives is greater than the sum of all the other distances between alternatives combined. This is easily shown. Let the total distance between the highest and lowest ranked of a set of alternatives be 1 . Then, the highest ranked ordering distance between alternatives will have a possible range of values ranging from $\longrightarrow 1$ to $\longrightarrow 1 - \frac{1}{N}$ where N is the number of ordered distances between alternatives. The midpoint of this range, which is the value which will be assigned to the second highest ranked alternative (assuming the usual pattern of ordering of our previous examples) is then

$$\longrightarrow \frac{1 - \frac{1}{N}}{2} .$$

Now, as N increases, the value of the expression approaches closer and closer to $\frac{1}{2}$, and, therefore, the size of the highest ranked distance between alternatives approaches closer and closer to $\frac{1}{2}$ the total distance between the highest and lowest ranked alternatives from above.[38]

It was clearly realized that in using the midpoint rule to establish priority indicators and then using these numbers as if they really represented measurement unique up to a linear transformation, one

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[38] As noted, op. cit., n. 37, this is the result of protecting yourself against Nature playing extreme values of the ranges which the alternatives may take.

was going far beyond the degree of measurement which could be legitimately implied by orders-on-distance between alternatives. But, on first inspection, the midpoint method seems both a logical and a "neutral" method of making this jump — or as logical and neutral a method as can be devised, given that the transition from one type of measurement to the other is unjustified by the information contained in the first. But the unusual properties of the scale which inevitably result from the application of the rule — that is, the fact that the largest ranked distance between alternatives is as large as all the others combined, and that the other distances are set approximately equal — casts some doubt on this point. This difficulty is not serious if the method is merely a convenient way of assigning values to the alternatives, but the increased degree of measurement which is implied over that given by the orders-on-distance is without operational significance. Thus, if one used the midpoint rule to assign values to a set of alternatives, used these values in the computation of a priority program, and then by testing determined that the solution of the program was invariant with changes in the values of the alternatives within the ranges determined by the orders-on-distances between the alternatives, then the midpoint rule is reduced to merely a computational convenience. If this is not the case, however, the question then remains as to whether we wish to substitute another rule for that of the midpoint in view of the rather peculiar scale which the application of that rule produces. The difficulty with making another choice is, of course, that if we really have no other information other than orders-on-distance between alternatives, we have no reason to assert that Nature will not play the corner sequence. To protest at the nature of the resulting scale then is really to assert that the degree of measurement we possess is of a higher — though unspecified — order than order-on-distance.[39]

It is, of course, possible that whatever method is used to obtain the information on individual rankings of the distances between alternatives may yield the result that some of these differences are, at least approximately, equal. Such information may, or may not, be of substantial value in refining the scale of measurement.

Suppose that we obtain the following relationships between a set of alternatives:

$$W \; P \; X \; P \; Y \; P \; Z \; ; \; WX \; P \; XY \sim YZ \; .$$

By comparison with the discussion above, it is easily seen that the use of the midpoint rule results in a scale which is only infinitesimally different from that obtained when the relation of strict preference held between the three orders on distance.

But let this relation be $WX \sim XY \; P \; YZ$. Then, the values which the alternatives may take are constrained as follows:

Table 4

Alternative	Range	Midpoint
W	1	1
X	⟶ .5 to ⟶ .666..	.583 ...
Y	⟶ .5 to ⟶ .333..	.4166..
Z	0	0

Thus, the effect of a tie involving the highest-ranked distance between alternatives is to restrict substantially the ranges through which the alternatives might vary. As Tukey and Abelson note — in relation to the

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[39] *Ibid.*, p. 4.

remarks earlier in this section, "Restrictions on the fraction of the total range allotted the biggest interval result in powerful increases in r^2.[40]

Example 1.

To illustrate the way in which orders-on-distance will yield additional information which may be of assistance in the computation of optimal programs, we return to the simple examples given in Chapter II.

Of the six possible assignment plans given in our Example 1, beginning on p. 18:

1. A_{11} A_{22} A_{33}
2. A_{11} A_{32} A_{23}
3. A_{21} A_{12} A_{33}
4. A_{21} A_{32} A_{13}
5. A_{31} A_{12} A_{23}
6. A_{31} A_{22} A_{13}

it was shown that if the ordering on the assignment matrix corresponds to that implied by Pareto Optimality (in the special sense described in the pages beginning on p. 18), then none of the six assignment plans can be revealed as preferred to any other. If we have an ordering on contiguous distances between the assignments, this situation will be altered as follows.

As we know from Example 1 in Chapter II, there are 42 possible complete orders which obey Pareto Optimality in the 3 x 3 example. Let us consider one of these orders, the lexicographic order shown by ---> in Figure 1. If, within this order, the distances between adjacent

[40] Ibid., p. 4.

alternatives are completely ordered, then one of the assignment plans may be determined to be preferred to all the remaining ones. This is very easily seen. Consider assignment plans 1 and 2. Since both contain A_{11} in common, we need compare only the remaining two elements in each plan. Previously, with only the simple orders to guide us, the two plans were incomparable, since A_{22} P A_{32} but A_{23} P A_{33} , or A_{33} P A_{32} but A_{23} P A_{22} . Since, however, the distances A_{23}, A_{22} and A_{33}, A_{32} are assumed to be completely ordered, either assignment plan 1 is preferred to plan 2 or vice versa — that is, if $(A_{33}A_{32})$ P $(A_{23}A_{22})$, A_{33} (of plan 1) is greater than A_{32} (of plan 2) by more than A_{23} exceeds A_{22} , and therefore plan 1 is superior. By similar reasoning, plan 1 is preferred to plan 2 according to the complete order on simple distances — one of the possible complete orders — which we have assumed below:

$$(A_{33}A_{32}) \text{ P } (A_{22}A_{21}) \text{ P } (A_{23}A_{22}) \text{ P } (A_{13}A_{12}) \text{ P } (A_{11}A_{23}) \text{ P } (A_{32}A_{31}) \text{ P } (A_{12}A_{11}) \text{ P } (A_{21}A_{33}) .$$

Now plan 1 is incomparable to the remaining plans. But by similar reasoning, we can establish that

Plan 2 P Plan 5

Plan 3 P Plan 4

Plan 5 P Plan 6

and, therefore, that plan 1 is preferred to all others.

This result is strictly fortuitous — that is, it depends on the particular ordering of the distances between alternatives which was established. Suppose that this order had been as follows:

$$(A_{33}A_{32}) \text{ P } (A_{22}A_{21}) \text{ P } (A_{13}A_{12}) \text{ P } (A_{23}A_{22}) \text{ P } (A_{11}A_{23}) \text{ P } (A_{12}A_{11}) \text{ P } (A_{32}A_{31}) \text{ P } (A_{21}A_{33}) .$$

Then an order of preference among the possible assignment plans would be established as follows:

1 P 2 ; 1 P 3

5 P 2

3 P 4

6 P 5 .

This order does not tell us whether plan 1 is preferred to or considered inferior to plan 6. To determine this, we compare $(A_{33}A_{31})$ and $(A_{13}A_{11})$. Although the distances to be compared are compound rather than simple distances, this comparison could be made in the present case because of the particular nature of the ordering. The rules of order-arithmetic apply to orderings-on-distances between alternatives as well as the ordering on the alternatives themselves. In the present case, by the application of assumption [5.] on page 16, we can say that $(A_{33}A_{31})$ P $(A_{13}A_{11})$ if $(A_{33}A_{32})$ P $(A_{13}A_{12})$ and $(A_{32}A_{31})$ P $(A_{12}A_{11})$. By the second ordering, however, we see that this is not the case, and, therefore, that plans 1 and 6 are incomparable.

The reader may question, why choose the comparison between $(A_{33}A_{31})$ and $(A_{13}A_{11})$ and not that between $(A_{31}A_{11})$ and $(A_{33}A_{13})$? The answer is that any single comparison is sufficient. If a single such comparison reveals that a plan is incomparable to another plan, then all possible comparisons (6 in the case where we are comparing two three-element plans) will show incomparability — and the same is true of preference.

This fact is extremely convenient, for in any realistically sized project there would be a considerable number of comparisons which would have to be made, if it were not true. The reason why it is true can most easily be seen in the context of another simple example.

Assume the following order on the elements of the assignment matrix (assume matrix not rearranged so as to be Pareto-optimal):

$$\text{(i)}\quad \overleftrightarrow{A_{21}\ P\ A_{31}}\ P\ \overleftrightarrow{A_{22}\ P\ A_{12}}\ P\ \overleftrightarrow{A_{33}\ P\ A_{13}}\ P\ A_{32}\ P\ A_{23}\ P\ A_{11}\ ,$$

and the following order-on-distance for the simple distances between these elements:

$$\text{(ii)}\quad (A_{22}A_{12})\ P\ (A_{12}A_{33})\ P\ (A_{21}A_{31})\ P\ (A_{31}A_{22})\ P\ (A_{33}A_{13})\ P$$
$$(A_{13}A_{32})\ P\ (A_{32}A_{23})\ P\ (A_{23}A_{11})\ .$$

Plans 1, 2, 4, and 5 are eliminated as inferior to plan 3 $(A_{21}A_{12}A_{33})$ and plan 6 $(A_{31}A_{22}A_{13})$. There are 6 possible different comparisons which can be made between these two remaining plans. These are:

$$1.\ \left\{\begin{matrix} A_{21}, A_{31} \\ A_{12}, A_{22} \\ A_{33}, A_{13} \end{matrix}\right. \qquad 2.\ \left\{\begin{matrix} A_{21}, A_{31} \\ A_{12}, A_{13} \\ A_{33}, A_{22} \end{matrix}\right. \qquad 3.\ \left\{\begin{matrix} A_{21}, A_{22} \\ A_{12}, A_{13} \\ A_{33}, A_{13} \end{matrix}\right.$$

$$4.\ \left\{\begin{matrix} A_{21}, A_{22} \\ A_{12}, A_{13} \\ A_{33}, A_{31} \end{matrix}\right. \qquad 5.\ \left\{\begin{matrix} A_{21}, A_{13} \\ A_{12}, A_{22} \\ A_{33}, A_{31} \end{matrix}\right. \qquad 6.\ \left\{\begin{matrix} A_{21}, A_{13} \\ A_{12}, A_{31} \\ A_{33}, A_{22} \end{matrix}\right.$$

But these comparisons, in fact, only appear to be different. In the final analysis, the relationship between the two plans will rest on the nature of the same relationship between elements of the plan. Thus, consider comparison group 1. We must determine in this comparison whether or not the sum of the distances $(A_{21}A_{31})$ and $(A_{33}A_{13})$ is greater than, less than, or incomparable to the distance $(A_{12}A_{22})$. While the latter is greater than any one of the two previous distances, we do not know whether or not it is greater than their sum, so the two plans are incomparable. The reader can verify that all the other comparison groups will involve making exactly the same comparison, and, thus, that it was sufficient to make only the first comparison to establish that the plans are incomparable.

Example 2.

The second example of computation of an optimal program was the selection of a "best" set of alternative projects. In the example given in Chapter II, let the budget restraint be raised to $110,000. The sole decision in this case is whether the preferred set of projects is (A_1, A_2, A_3) or (A_1, A_2, A_4, A_5), both of which are feasible.

In this case, to obtain information of the form that A_3 is preferred to A_4 by more than A_4 is preferred to A_5 does not, of necessity, decide between the two sets. One possibility of settling this impasse, short of a numerical scale, would be to obtain orderings on various combinations of projects — in this case, by questioning expert personnel as to whether they would rather have A_4 and A_5, or A_3 alone. If the number of combinations which have not been excluded from consideration by the budget restraint, simple ordering on the alternatives and ordering-on-the-distances between alternatives, is not overly large, then this direct method would be feasible.

It is possible, however, the the order-on-the-distances between alternatives is of a form which will allow a comparison between the combinations. Assume a ranking on the projects of (project A_6 has been added, so that project A_5 is assigned a value other than 0):

$$A_1 \; P \; A_2 \; P \; A_3 \; P \; A_4 \; P \; A_5 \; P \; A_6$$

and an order on the distances between projects of the form:

$$A_1A_2 \; P \; A_2A_3 \; P \; A_3A_4 \; P \; A_4A_5 \; P \; A_5A_6 \; .$$

Assigning the value of 1 to A_1 and 0 to A_6, the values which each project can assume are so constrained that A_3 is preferred to the sum of projects A_4 and A_5 (as in this special ranking each alternative will assume values greater than the sum of the values assumed by all the alternatives which follow it in the order).

Orders-on-Compound-Distances

We have seen that a (limited) number of the possible continuous distance relationships which hold between a set of ordered alternatives cannot be deduced, either from the order on the alternatives itself, or from an order on the distances between adjacent alternatives. In the limited number of cases, therefore, where the weaker levels of measurement take a form where the resolution of a priority program depends on one of these undetermined relationships, it is necessary to move to a higher level of measurement — to obtain orders on continuous compound, as well as simple, distances between alternatives.

Measurement in the form of orders-on-continuous-compound-distances will reduce the ranges within which the ordered alternatives are restricted, or — in the terms of the remarks on p. 62 — will raise the maxmin value of r^2.[41] As in the case of indifference between simple orders-on-distance discussed in the section beginning on p. 55, the extent of additional measurement will be dependent on which form the compound distance order relationship takes. Thus, for the four

[41] Ibid.

alternatives ordered:

W P X P Y P Z ,

with an order on the simple distances between alternatives of the form

WX P XY P YZ ,

the only comparison between compound distances which is not automatically determined by these relationships is whether WX P XZ or XZ P WX . If the former is true, then we establish a set of limited ranges on the alternatives which is almost identical to that given in Table 4 — i.e., the additional order information leads to only minor increase in the degree of measurement. On the other hand, if we establish that XZ P WX , then the ranges on the alternatives will be as in Table 5.

Table 5

Alternative	Range	Midpoint
W	1	1
X	$\rightarrow 0$ - $\rightarrow .5$	$\rightarrow .25$
Y	$\rightarrow 0$ - $\rightarrow 2.5$	$\rightarrow .125$
Z	0	0

If we are dealing with five or more alternatives, then orders on compound distances will always yield us an appreciably higher degree of measurement, for they will then begin to limit the ranges of distance between other than the two highest ranked alternatives. Nevertheless, as has been pointed out above, the increase in the degree of measurement will be greatest when other distances are established as preferred to that between the two highest ranked alternatives.

Example 1.

It follows that if we obtain a limited number of additional bits of order information on compound distance, we have a higher probability of resolving the problems which we have been using as examples. Thus, in the assignment problem example, it was noted on p. 70 that the two plans,

$$[1.]\quad (A_{11}, A_{22}, A_{33})$$

$$[2.]\quad (A_{31}, A_{22}, A_{13}) ,$$

were incomparable on the basis of the order information there assumed. This incomparability can be resolved by obtaining the appropriate relationship between the continuous compound distances $(A_{13}A_{11})$ and $(A_{33}A_{31})$. On the other hand, the example of p. 69 remains unresolved. Nor, indeed, can the determination of all possible relationships between compound distances possibly resolve this comparison. Again, this result is fortuitous. Had, for example, our order on the elements of the assignment problem been

$$A_{22} \text{ P } A_{12} \text{ P } A_{21} \text{ P } A_{31} \text{ P } A_{33} \text{ P } A_{13} \cdots$$

we could have certainly resolved the comparison by obtaining an order on compound distances. Had it been,

$$A_{22} \text{ P } A_{12} \text{ P } A_{21} \text{ P } A_{31} \text{ P } A_{32} \text{ P } A_{33} \text{ P } A_{13} \cdots ,$$

we might have been able to resolve the comparison with such an order.

Example 2.

In the example on p. 72, let the order on the projects be

$$A_1 \text{ P } A_2 \text{ P } A_3 \text{ P } A_4 \text{ P } A_5$$

and the order on the distances between alternatives

$$A_1A_2 \text{ P } A_2A_3 \text{ P } A_4A_5 \text{ P } A_3A_4 \text{ .}$$

Then an order on the continuous compound distances — in particular, the relation:

$$A_2A_3 \text{ P } A_2A_5 \text{ ; or } A_2A_5 \text{ P } A_2A_3$$

would be sufficient to establish one combination as preferred. The result is again fortuitous, depending in this case on the fact that the costs of the projects interacted with the budget restraint in such a way that the two project sets compared had two elements in common.

Orders on Non-Continuous Compound Distances

There remains our final higher gradation of measurement obtained by soliciting orders on distances between alternatives which are non-continuous — as in the examples given on p. 55.

Let five alternatives be ordered as follows:

(1) V P W P X P Y P Z

Let the order on the distances between alternatives be:

(2) WX P YZ P XY P VW ;

Let the order on the distances between continuous compound distances (not otherwise determined in (1) and (2) be:

(3) XZ P VX ;

and let the order on the compound non-continuous distances between alternatives (not otherwise already determined be:

(4) (VW, XY) P YZ

Alternative	Resulting Values: Range	Midpoint
		(1)
V	1	1
W	→ 1 - → 0	$\frac{1}{2}$
X	→ 1 - → 0	$\frac{1}{2}$
Y	→ 1 - → 0	$\frac{1}{2}$
Z	0	0
		(2)
V	1	1
W	→ 1 - → .75	.875
X	→ .67 → 0	.335
Y	→ .50 → 0	.250
Z	0	0
		(3)
V	1	1
W	→ 1 - → .75	.875
X	→ .67 → .50	.585
Y	→ .33 → .25	.290
Z	0	0

Alternative	Resulting Values Range	(4) Midpoint
V	1	1
W	→ 1 → .75	.875
X	→ .67 → .50	.585
Y	→ .33 → .25	.290
Z	0	0

The increasing power of each stage of measurement, and the great power of this final stage, are illustrated in the decreasing ranges within which set of orders confine the alternatives. This is evident even in our simple example, where only three examples fall in the field of measurement, and becomes more pronounced, of course, as the number of alternatives increases. As Tukey and Abelson note: ". . . higher-ordered metric scales can come very close to numerical scales."[42]

Yet, it must be remembered that coming "very close" in measurement to a numerical scale may still be sufficiently far away in many cases to make the necessary comparisons to resolve a priority program. Thus, if to decide which of two combinations is preferred it were necessary to determine whether W is preferred to a combination of X and Y , it is evident that the various degrees of ordering listed above are of little use. Using the values assigned by the midpoint rule in the case of simple orders, it is found that W P (X, Y) ; in the case of orders-on-simple distances that W P (X, Y) ; in the case of orders-on-compound-continuous distances between alternatives that (X, Y) P W ; and in the case of orders-on-compound non-continuous distances between alternatives that (X, Y) = W . From this we can deduce the general proposition that order-on-distance relationships cannot always resolve problems of the type discussed in our Example 2; they are, however, possible to resolve comparisons similar to those in Example 1 where the comparison sets have always an equal number of elements in each.

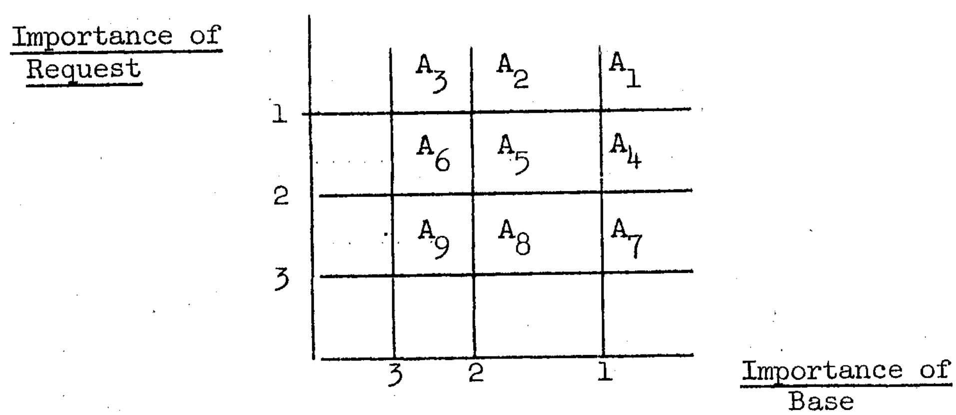
[42] *Ibid.*, p. 4.

Section 2. Orders-on-Distance in More Than One Dimension

The term "orders-on-distance" in more than one dimension can be used in one of two senses — either that the dimensions themselves are so ordered, or that an order-on-distance exists along each axis of measurement. We have already dealt with the first case to some extent in Chapter II, and so will confine the discussion to the second.

The simplest case is where there exists a series of priority classes along each axis; and an order-on-the distance between classes on one axis. Returning to the supply base example, let it be assumed that there exist three priority classes in each dimension: in the dimension of the importance of the base from which the request originates, it is known that the difference between a number 1 priority and a number 2 priority is greater than that between a number 2 and a number 3 priority; in the dimension of the importance of the request attached to a request by one of the outlying bases, there exists only a simple order. (See Figure 11.)

Fig. 11



The addition of the increased information on the X axis indicates nothing concerning the order of the points in the diagram, which must be obtained by means of one of the methods discussed in Chapter II. Let it be assumed that we have determined an order on the points of the form:

$$A_1 \; P \; A_2 \; P \; A_3 \; P \; A_4 \; P \; A_5 \; P \; A_6 \; P \; A_7 \; P \; A_8 \; P \; A_9 \; .$$

Then the orders-on-the-distances between priority classes on the X-axis add the following information on orders on simple distances:

$$A_1A_2 \; P \; A_2A_3 \; ; \; A_4A_5 \; P \; A_5A_6 \; ; \; A_7A_8 \; P \; A_8A_9 \; .$$

$$A_1A_2 = A_4A_5 = A_7A_8 \; ; \; A_2A_3 = A_5A_6 = A_8A_9 \; .$$ [43]

As would be expected, the lexicographic order on the points causes the field of measurement in this instance to fall apart into three distinct parts. Unless additional information is obtained, the order information at hand will confine the values to be assigned to the points within ranges no more constrained than those established by the simple order on the points — i.e., hardly at all. And we have here used an order on the points which yields a maximum of information. Thus, had the complete order been of the form:

$$A_1 \; P \; A_2 \; P \; A_4 \; P \; A_3 \; P \; A_5 \; P \; A_7 \; P \; A_6 \; P \; A_8 \; P \; A_9 \; ,$$

then only the first of the 9 orders-on-simple-distances appears in the ordering (and it is illegitimate to assume that the relation $A_1A_2 \; P \; A_2A_3$ holds for the distance A_2A_3 in the above order). And, in fact, no matter what degree of measurement is possessed along each axis in a multi-dimensional problem, it can do no better than a simple order in helping

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[43] Which involves the assumption that the distance between the priority classes on an axis are unaffected by the rankings on the other axis.

us to move closer to a numerical scale over each point in the decision space. But, once we obtain any additional information which relates the relative ranking of movement along one dimension to movement along another, then these higher degrees of measurement along each of the axes become increasingly significant.

Thus, in our first order let us assume that we are able to obtain the following additional piece of order information:

$$A_1A_2 \ P \ A_3A_4 \ ; \quad A_1A_2 \ P \ A_6A_7 \ .$$

It then follows that the range over which A_2 may move in the field of measurement without violating the order conditions is reduced by 13/15, with a similar restriction being imposed on the range of values which may be assigned to A_4, A_5, A_7, and A_8. This, however, it must be remembered, was a "powerful" piece of order information. Thus, the relation

$$A_1A_2 \ P \ A_3A_4 \ P \ A_6A_7$$

adds only slightly to our degree of measurement, and the relation

$$A_3A_4 \ P \ A_1A_2 \ ; \quad A_6A_7 \ P \ A_1A_2$$

adds nothing at all. Following through the same logic, we can see that if we can obtain the information

$$A_2A_3 \ P \ A_3A_4 \ ; \quad A_2A_3 \ P \ A_6A_7 \ ,$$

then the ranges which each alternative can occupy are restricted further, and so on.

The second case is where an order on the distance between priority classes is obtained for both (all) dimensions of measurement. Let it be assumed that in Figure 11 we obtain information that the difference between a priority 1 request from an outlying base and a priority 2 request is less than the difference between a priority 2 request and a priority 3 request. We then possess the following information:

$$A_1 \; P \; A_2 \; P \; A_3 \; P \; A_4 \; P \; A_5 \; P \; A_6 \; P \; A_7 \; P \; A_8 \; P \; A_9$$

$$A_1A_2 \; P \; A_2A_3 \; ; \; A_4A_5 \; P \; A_5A_6 \; ; \; A_7A_8 \; P \; A_8A_9$$

$$A_1A_2 = A_4A_5 = A_7A_8 \; ; \; A_2A_3 = A_5A_6 = A_8A_9$$

$$A_4A_7 \; P \; A_1A_4 \; ; \; A_8A_5 \; P \; A_2A_5 \; ; \; A_6A_9 \; P \; A_3A_6$$

$$A_1A_4 = A_2A_5 = A_3A_6 \; ; \; A_4A_7 = A_5A_8 = A_6A_9 \; .$$

As above, even the addition of what appears to be a considerable additional degree of measurement is still insufficient to enable us to appreciably limit the range of values which can be assigned to the points without violation of the order conditions. Also, as above, however, once the critical relationships are established which prevent the field of measurement from breaking into incomparable parts, then the order-on-distance relations are quite powerful in restricting the range of values which the alternatives can occupy.

Similar relationships hold as we obtain higher and higher degrees of measurement on one or both (all) of the dimensions of measurement. We may thus generalize the problem of orders-on-distance in more than one dimension in the following manner.

[1.] In each such problem, there will be critical order relationships, which, when established, will "tie" the various portions of the field of measurement together. In our example above, these relationships are the comparisons of the distances A_3A_4 and A_6A_7 and their relationship to the other distances between alternatives.

[2.] The number of such critical relationships — which we shall call "bridges" for convenience — depends on the nature of the order on the points in the decision space. It is minimized when the order is a lexicographic one — and in those cases is N - 1 in an N x N problem. If the order is not lexicographic — and in our 3 x 3 example, there is only one chance in 21 that it will be so (assuming that all orders which obey the rule of Pareto Optimality are equally possible) — then the number of "bridges" will be increased. In the 3 x 3 case, this number could reach a maximum of 8 in our first example.

[3.] The "bridge" relationships must <u>all</u> be established in order that the order-on-distance relationships may effectively limit the ranges over which the alternatives may vary.[44] One unestablished "bridge" relationship can defeat the entire measurement which would otherwise be established.

[4.] Moreover, the <u>direction</u> of the comparison between a bridging distance and other distances is critically important. As we have seen above, the information that a bridging distance is greater than an "ordinary" distance adds nothing to our degree of measurement. In order to obtain significant measurement information, the direction must run the other way for <u>all</u> bridging distances — that is, each of these must be judged as inferior to one of the "regular" distances.[45]

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[44]Though these relationships need not be established by direct comparison, but can in certain instances be deduced from the order on the "bridges." In our example, if A_1A_2 P A_3A_4, and A_3A_4 P A_6A_7, then A_1A_2 P A_6A_7.

[45]It will be recognized that the difficulty is precisely of the same nature as that which is encountered in the single dimensional case where there is no meaningful way of restricting the size of the largest distance between alternatives, though considerably more troublesome since there can be only one highest ranked interval in any simple order on alternatives, but any multi-dimensional problem will contain several bridge relationships. As in that case, we can assign each alternative a value equal to the midpoint of the range to which it is confined, provided that we have an order on the bridge distances themselves.

[5.] The effectiveness of the amount of measurement obtained by relationships between the "bridge" relationships themselves depends on the number of such relationships, as compared to the number of "ordinary" relationships, as well as on the type of order on the bridges. In the lexicographic example used above, the information A_3A_4 P A_6A_7 has the effect of imposing the following ranges on alternatives A_3, A_4, A_6, and A_7:

Alternative	Range	Midpoint
A_3	→ 1.0 → .0	.50
A_4	→ 1.0 → .0	.50
A_6	→ .5 → .0	.25
A_7	→ .5 → .0	.25 .

That is, it divides the range of values which can be assumed by A_6 and A_9 almost squarely in half, and in so doing imposes a bound on the values which can be assumed by the alternatives not involved in the comparison. As the number of bridges increases, reaching at a limit the entire number of simple distance comparisons in the order (a chance again of 1 in 21 in the 3 x 3 case), then the amount of measurement information which their comparison contributes rises proportionately.

What then can we say concerning the probably effectiveness of the degrees of measurement discussed in this section in the resolution of priority programs? It would most certainly appear that this effectiveness is limited. There is first the necessity for the strong assumptions as to the nature of the "bridges" — assumptions which, if we regard all orderings as equally possible, are unlikely to be met in an actual problem. Moreover, even if these assumptions are met in a particular case, we are still only at the point of Section 1, for the resulting degree of measurement — depending on its strength — may or may not be able to resolve the program. Thus, while measurement of the order-on-distance form can sometimes be useful in a multi-dimensional case, the introduction of additional degrees of measurement serves to reduce markedly the probabilities that such will be the case.[46]

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[46] This statement may be qualified somewhat. It is true that we must obtain a "proper" direction in the bridge relations with ordinary distances in order to have effective measurement over the entire range of measurement, but it is also true that the very simplest form of order-on-distances will suffice. Thus, one could attain a fairly powerful degree of measurement by having compound orders-on-distances along each axis of measurement and a simple order-on-distance tieing each bridge into the scale.

Appendix I. Ordered-Ratio Measurement[47]

We have seen from the above that the determination of all possible combinations of orders-on-distances between alternatives results — in the single dimension case — in a powerful scale of measurement which is sufficient to resolve a number of priority programs, where on a priori grounds it might have been thought that a numerical scale would have been necessary. There is, however, another type of measurement which overlaps the degree of measurement given by orders-on-distance, and which — assuming always that the necessary information can be obtained by questioning of expert personnel — can be used to supplement order-on-distance measurement. We discuss this alternative type below, first in relation to single dimension cases, and then in relation to the multi-dimensional examples.

Single Dimension Case.

This type of measurement — which we have called "ordered-ratio" — is obtained as follows. Expert personnel are asked to rank the distances between alternatives, not in relation to the distances between other alternatives, but as to whether or not they are greater or less than a certain fraction of the entire range of measurement, or or a certain fraction of another distance between alternatives. Thus, assume that four alternatives are ordered

W P X P Y P Z ,

and the distances between alternatives are ordered

XY P WX P YZ .

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[47]Abelson and Tukey, op. cit., n. 37, is the only other source where the author has seen the properties of this particular type of measurement; in contrast, metric (order-on-distance) scales have received considerable attention.

Let it be determined by questioning that the distance XY is not more than $\frac{1}{2}$ the distance XZ — i.e., of the entire range of measurement. This limitation on the largest interval, as we have previously noted, has the effect of sharply curtailing the range of values which can be assumed by the alternatives without violating the order conditions. Before X might take any value between $\rightarrow 1.0$ and $\rightarrow .50$, and Y any value between $\rightarrow .333 \ldots$ and $\rightarrow 0$; with the limitation, X is confined to the range .75 to $\rightarrow .50$ and Y to the range $\rightarrow .25$ to $\rightarrow 0$.

In this example, the only additional piece of order-on-distance information which would have been significant is the relationship

$$WX + YZ \; P \; XY ,$$

which establishes the same limitation on the ranges of the alternatives. Thus, any piece of ordered-ratio information may yield the same (or less) degree of measurement as order-on-distance relationships. But, what is more important, ratio information can be used to give a higher degree of measurement than order-on-distance relationships can ever yield. Thus, the information that the distance XY cannot exceed $\frac{2}{5}$ of the entire range of measurement will, with the assumed simple order on the distances, confine alternative X to a range of values $\rightarrow .70$ to $\rightarrow .50$ and alternative Y to a range $\rightarrow 20$ to $\rightarrow 0$; and the information that XY cannot exceed $\frac{1}{3}$ of the entire range of measurement would fix the value of X at $\frac{2}{3}$ and of Y at $\frac{1}{3}$.

The latter result is obtained because our initial example dealt with only four alternatives. Moreover, even if we were able by a series of questions to determine exactly what proportion of the total range of measurement is occupied by the largest distance between alternatives, this information need not interact with the ranges imposed by the available orders on distance in such a way as to be effective in severely curtailing the ranges of the alternatives — e.g., to know that XY is exactly $\frac{1}{2}$ of XZ tells us very little more than it is not more than $\frac{1}{2}$, and little more than the relation XY + YZ P WX .

The ranges to which the alternatives are confined can be further limited by soliciting ratio information concerning the smallest ranked distance as well. If we have, for example, the information that $YZ < \frac{1}{10} XZ$, this together with the information, $XY < \frac{1}{2} YZ$, and XY P WX P YZ , limits alternative X to a range of values between $\rightarrow .60$ and $\rightarrow .50$, and Y to a range of values between $\rightarrow .10$ and $\rightarrow 0$.

The relationship of ratio scales to those obtained from an order-on-distances between alternatives is thus one of possible substitutability or supplementarity. Possessing only a ranking on alternatives, it might be decided to obtain ratio information instead of orders-on-distances. The choice here is between which of the methods is judged the easiest to use in practice, as we could on *a priori* grounds not determine in advance which would yield the greatest degree of measurement. Even if order-on-distance relationships are obtained, we have seen that ratio information need not in *all* cases increase the degree of measurement. To this, however, two qualifications should be made.

First, while ratio information will not of necessity increase the amount of measurement beyond that which would be obtained from orders-on-distances, there is a very high probability that it will do so, especially if such information is obtained for the smallest ranked distance as well as the largest. Second, following from this, it would be expected that the use of ratio scales would be effective in many particular decision problems. As has been seen in this chapter, in practical decision situations there is always the possibility that the order-on-distance relation is unable to resolve a priority program. Where this occurs, the logical approach would be to isolate the key comparisons which must be made to resolve the program, but, where order-on-distance information has only led to incomparabilities, also solicit ratio information which, in a certain number of cases, will be able to lead to a resolution of the program.

Priority Indicators in More Than One Dimension

The use of ratio relationships can be particularly helpful in dealing with the problems involved in multi-dimensional cases. As we have seen in section 2 of this chapter, the critical relationships in the multi-dimensional case are those which we have called "bridges." Furthermore, it is necessary that all bridge relationships be dominated by non-bridge relationships in order to obtain a meaningful restriction of the range of values to be assigned to the alternatives associated with each bridge relationship. In the example given in [5.] (p. 83), the information that $A_3A_4 \longleftarrow .6\ A_1A_9$, for instance, has the effect of confining the alternatives involved in the two bridge relationships to the ranges

$$\begin{array}{lll} A_3 & 1.0 \rightarrow & .6 \\ A_4 & 1.0 \rightarrow & .0 \\ A_6 & .5 \rightarrow & 0 \\ A_9 & .5 \rightarrow & 0\ . \end{array}$$

Again, as the bridge relationships each occupy the same theoretical position as the highest ranked interval between alternatives in the single dimensional case, as in that case, ratio information will prove particularly powerful. As ratio information, however, may be less powerful than orders-on-compound distances which would "tie" the bridges to other intervals between alternatives, the choice between the two approaches will rest on which is the easiest to use in practice.

CHAPTER IV

NUMERICAL MEASURES OF PRIORITY

Section 1. Numerical Priorities in One Dimension

We may, in the computation of an optimal priority program, be unable to obtain measurement of a higher order than that represented by one of the orders-on-distances discussed in Chapter III, or by the combination order-ratio scales discussed in Appendix I. In this case, numerical values can be assigned to the midpoints of the ranges determined by these orders, and the values used in the computation of the program. From our previous discussion, it is possible to compute — in specific instances — the chances that this method will lead to a solution which dominates all others.

There are, however, three alternative methods by which measurement of a still higher order can be obtained — though these, of course, involve stronger assumptions than the measurements discussed in the previous two chapters. Each of these is discussed in turn below, together with examples of the types of problems where they can be most effectively employed, and the difficulties involved.[48]

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[48] It is, in fact, not always possible theoretically to move from a complete order on a set of alternatives to a numerical order, but the conditions which must be met are not restrictive. The major exception — when the initial order is lexicographic — is discussed in section 2 on multi-dimensional priorities.

See I. N. Herstein, "Some Mathematical Methods and Techniques in Economics," Quarterly of Applied Mathematics, Vol. XI, No. 3 (October, 1953) 249-262, reprinted as Cowles Commission Paper, New Series, No. 78 (1953); Gerard Debreu, "Representation of a Preference Ordering by a Numerical Function," in R. M. Thrall, C. H. Coombs, and R. L. Davis, eds., Decision Processes (New York, Wiley, 1954), 159-165, reprinted as Cowles Commission Paper, New Series, No. 97 (1954); Herman Wold, "A Synthesis of Pure Demand Analysis," Part II, Skandinavisk Aktuarietidskrift, Vol. 26, Nos. 3 and 4, 220-263; Tjalling C. Koopmans, Three Essays on the State of

Indifference

One of the more important developments of modern economic theory has been the indifference curve approach, which permits the derivation of most of the important propositions concerning consumer behavior without direct reliance on the concept of numerical utility. This approach, as yet, has not been utilized in any military operations research study. Yet, its application to the problem of military worth is clear. For example, if several types of equipment are to be allocated to various bases, one way to approach the problem is to devise a real-valued function representing the utility, or military worth, of each assignment, and then to maximize this function, subject to the restraints imposed by the quantity of each type of equipment on hand and|or their market prices and the amount of funds available to be spent on procurement. As we have noted previously, it is easy enough to conceive of the problem's solution in this form; the difficulty is in the determination of the worth function.

The question is, then, whether the indifference map approach could be used to bypass the necessity of establishing a numerical worth function. In certain limited military decision problems, this possibility would appear to be present. Suppose that one were charged with the responsibility of providing supplies for a warship. By appropriate questioning, let it be assumed that an indifference map of the form of

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Economic Science, (New York, McGraw-Hill, 1957), 19; Samuel Eilenberg, "Ordered Topological Spaces," American Journal of Mathematics, Vol. LXIII, No. 1 (January, 1941), 39-45. One aspect of the problem of moving from orderings to functions has been thoroughly explored: this is the case of the relationship of indifference maps to demand functions. Cf. Nicholas Georgescu-Roegen, "The Pure Theory of Consumers' Behavior," Quarterly Journal of Economics, Vol. L, No. 4 (August, 1936), 545-593.

Figure 12 can be established. The indifference curves indicate the

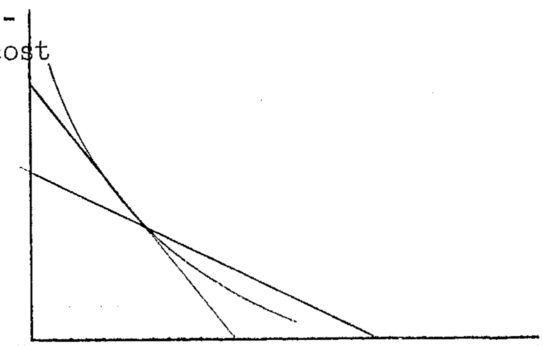

Fig. 12

various combinations of equipment which in the opinion of the questioned experts would contribute equally to the military effectiveness of the vessel, with the higher curves representing the military worth of having more of both items. The price line can represent whatever restrictions are imposed on the amount which can be placed aboard the ship. These restrictions may be in the form of limited storage space, in which case the slope of the price line will be the ratio of the area occupied by the two types of equipment, and its distance from the origin will represent the total amount of storage space available. Or the restrictions may be on the total amount of funds which are available to purchase equipment, so that this total is given by the distance of the price line from the origin, and its slope by the ratio of the prices of the two types of equipment. Or, both such restrictions may be present, so that the price line may have the broken shape of the line enclosing the shaded area on Figure 12.

What are the advantages of this method of approach? In addition to the fact that one does not have to determine a numerical worth function for the value of these different types of equipment, and for different amounts of each type of equipment, in order to take into account the phenomenon of diminishing marginal worth, the indifference curve method explicitly provides even more information than a numerical function of this form could. This is because it explicitly takes into account the fact that the value of a piece of equipment on board the ship may depend on whether or not another, complementary or supplementary, piece of equipment is also on board. Hicks' well-known objections to Pareto and Edgeworth do not appear applicable here.

Secondly, indifference curves provide for an adjustment to changes in the conditions which restrict the amounts to be allocated just as well as a numerical position would, and demonstrate clearly the relationship between changes in these restrictions and the pattern of allocation. Suppose that the Navy's budget is limited, and the costs of equipment A should rise — perhaps because a newer model is introduced. A natural inclination might be simply to cutback on total procurement of both items, perhaps by equal amounts. (It will be recalled that Congress recently considered reducing defense costs by cutting expenditures for military hardware by a flat percentage.) Yet this may be far from the most effective way of cutting back, which could be clearly determined if the relevant indifference map of military worth is before the relevant authorities. (Or, indeed, if they were only more aware of the type of thinking that it involves.)

There are, however, several basic objections to utilizing this method of approach in a practical problem solution situation, whatever may be its merits as a way of visualizing the theoretical properties of a solution.

The basic loss of information which occurs when we use indifference maps rather than numerical utility in a welfare economic context is not overly important in our present context. In the absence of numerical utility, one has no idea of how much better or worse one utility surface is than another. In our example, we would know that if the budget restraint is eased so that we can move to a higher utility curve, total military worth is increased, but we would not know by how much. This loss of information could be important if we were to possess a function which converted numerical utility into money costs. Since, however, it is unlikely that we would possess such a function, and since the resolution of such a problem would most generally be to submit both gain and costs of a program to a superior authority which decides if the gain in military worth merits the increased costs, this information loss may not be overly important.

There is the more important question of the logical restrictions which are necessarily imposed on an indifference map composed on the basis of the judgment of a number of persons, for in a realistic situation, it would not appear desirable, or even possible, to construct a map on the basis of the preference structure of a single individual.

Finally, the indifference map approach usually considers only two types of items at a single time. However, it would be possible to construct a series of maps, one for each pair of items to be rated or assigned, thus tracing out the multi-dimensional indifference surface two dimensions at a time. For any particular limitation, price or spatial or both, on the amount of items which can be carried or purchased, it is then possible to establish the point on the surface which constitutes the feasible optimum. Since it is impossible to assign fractions of an item, the problem is one in integer programming, where the valuing function is the rather peculiar one defined by the points of indifference.[49]

von Neumann-Morgenstern Utility.

The most intellectually prepossessing of the various methods for reaching a degree of measurement of uniqueness to a linear transformation, von Neumann-Morgenstern utility, has, to the author's knowledge, never been applied in a problem of military decision making. One fundamental difficulty is the large number of questions which a subject must answer in order that his complete utility function may be determined. A recent paper by Professor Johann Pfanzagl[50] attempts, however, to introduce an axiom which would reduce the number of questions necessary to determine such a scale. Unfortunately, this axiom seems intuitively unacceptable. In commenting on this axiom, the author wrote:

> The article by Professor Pfanzagl represents a significant contribution to measurement theory in general and to the theory of the measurement of utility in particular. I would like to comment briefly on the section of the paper which deals with von Neumann-Morgenstern utility, and more explicitly, on the axiom of "consistency" proposed by the author.
>
> The axiom is given as:
>
> (1) $M(a + b)\ P\ (b + c) = M(a\ P\ b) + c$.

[49]Cf. Stephen W. Rousseas and Albert G. Hart, "Empirical Verification of a Composite Indifference Map," The Journal of Political Economy, Vol. LIX, No. 4(August, 1951), 288-318; also W. Allen Wallis and Milton Friedman, "The Empirical Derivation of Indifference Functions," in Studies in Mathematical Economics and Econometrics in Memory of Henry Schultz, ed. Oscar Lange, Francis McIntyre, and Theodore O. Yntema (Chicago, University of Chicago Press, 1942).

[50]Johann Pfanzagl, "A General Theory of Measurement — Applications to Utility," Research Memorandum No. 5 (Econometric Research Program, Princeton University (December, 1958); reprinted in the Naval Research Logistics Quarterly, Vol. 6, No. 4 (December, 1959), 283-294.

I interpret its meaning as follows. Suppose that we consider a lottery ticket of the usual form, as, for example, ($10, $\frac{1}{2}$; $, $\frac{1}{2}$). The von Neumann-Morgenstern method asserts that, by offering various sums of money we can determine some sum of money, let us say $5.50, so that the subject is indifferent between this amount and the ticket, that is, so that they have equal utilities. The axiom of consistency appears to assert the following:

(2) If utility of ($10, $\frac{1}{2}$; $1, $\frac{1}{2}$) and $5.50 are equal, then utility of ($12, $\frac{1}{2}$; $3, $\frac{1}{2}$) and $7.50 are also equal,

where the c from (1) is $2, the proposition to hold for any value of c.

The axiom seems at first glance to be inherently plausible. It must, for example, hold for anyone who, like the subject in the example above, values alternatives on the basis of comparing expected money values. Even if this is not the case, if, for example, the subject were indifferent between the first ticket and $4, it is true that he would pay $6 for the second ticket since he is <u>sure</u> of at least $2 more on it. Professor Pfanzagl shows that if the consistency axiom is accepted, then the utility of money function of the subject is determined to within one parameter which is fixed by establishing a single point of indifference. The importance of this is obvious. The main disadvantage of the von Neumann-Morgenstern method as a device for establishing scales of utility in real situations has been the very large number of questions which the subject would have to answer. If the consistency axiom can be accepted, then much of this difficulty is avoided.

Both for the reason of applicability and for the reason of acceptability, the axiom should not, I believe, be accepted into our thinking. In constructing a scale to measure the utility of money, we must determine how the subject feels about $1 million as compared with $10,000, $100, $1, and so on. Suppose that a person is indifferent between the ticket ($10, $\frac{1}{2}$; $1, $\frac{1}{2}$) and $4. Let us, for the moment, agree to accept the consistency axiom. Then the subject will also be indifferent between the ticket ($1,000,010, $\frac{1}{2}$; $1,000,001, $\frac{1}{2}$) and an amount $1,000,004. The question is, what does this tell us about the shape of the utility function in the neighborhood of $1,000,000? I suggest that it may very well tell us nothing at all. The subject, whether he chooses as a gift the money or the ticket, is absolutely sure in the latter case of receiving at least $1,000,000 more than in the first case and will naturally be willing to pay $1,000,000 more for the ticket since he is sure to get it back anyway. The only real decision, therefore, in the second case, is whether he wants the ticket ($10, $\frac{1}{2}$; $1, $\frac{1}{2}$) or $4 sure, which we already know from the first comparison. We have no additional real information about how the subject feels about having $1 million as compared with having $1 or $4 or $10. If we really wish to determine the entire utility function from the first comparison plus the consistency axiom, we can only assume that the slope of the function is everywhere the same as in the neighborhood of the first comparison. Reduced to this form, the axiom seems far less plausible.

Next we may question whether or not the axiom might be expected to hold over considerable amounts, especially if we admit negative payoffs. Suppose that I offer as a gift to a person in moderate circumstances a choice of one of the following: \$100,000 or a ticket (\$1,000,000, $\frac{1}{2}$; \$0, $\frac{1}{2}$). He may very well prefer the sure \$100,000, especially if, for example, he needs \$50,000 for his wife's operation. Suppose that we subtract an equal amount c , say \$200,000, from each of the two alternatives. By the axiom of consistency he should still prefer the sure amount, which now has become - \$100,000. Yet he may very likely choose the ticket (\$800,000, $\frac{1}{2}$; - \$200,000, $\frac{1}{2}$) since this offers him his only chance of escaping from an overwhelming burden of debt.

Professor Pfanzagl makes the following point. If an individual is willing to pay \$2 for a ticket (\$10, $\frac{1}{2}$; \$1, $\frac{1}{2}$), it does not mean that the utility of the ticket is equal to the utility of \$2 but that the "status quo" has the same utility as a ticket of the form (status quo + \$8, $\frac{1}{2}$; status quo - \$1, $\frac{1}{2}$) since the cost of the ticket must be subtracted from the value of the prize. The two will be equivalent only if the axiom of consistency is valid. This is quite true. But there is no reason why the two should be equivalent. If we turn our previous example around, our subject might have chosen the ticket (\$1,000,000, $\frac{1}{2}$) \$0, $\frac{1}{2}$) rather than \$100,000 had he been in comfortable circumstances and wanted a chance at becoming rich: this does not mean that he would give you \$100,000 for the ticket. It is true that this distinction is not always made by writers on utility and that the failure to make it is an unwitting assumption of the consistency axiom, but this does not excuse the failure to make the distinction; it is probable that the "lottery ticket" terminology used in the von Neumann-Morgenstern construction has confused this point somewhat.

If we have interpreted the axiom correctly, and if it is true, as Professor Pfanzagl indicates, that the axiom has been tacitly assumed in the well-known studies of utility measurement that he cites,[51] then it is certainly true, as he states,

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[51]M. Friedman and L. J. Savage, "The Utility Analysis of Choices Involving Risk," Journal of Political Economy, Vol. LVI, No. 4 (April, 1948), 279-304; F. Mosteller and P. Nogee, "An Experimental Measurement of Utility," Journal of Political Economy, Vol. LIX, No. 5 (October, 1951), 371-404.

that much of the recent experimental work on utility measurement will need critical reappraisal, since it seems questionable that we shall want to accept the axiom.[52]

We have already touched on the second major objection to the use of von Neumann-Morgenstern utility in an actual decision situation[53]—that the attitude toward uncertainty of the personnel being questioned may distort their appraisal of the alternatives as alternative sure prospects. A method for separating such attitudes from the "sure-thing" appraisals remains a sine qua non if this method is to be used in the types of problems which we have been discussing.

Limited Conditions Scale.

Numerical scales of this type are obtained by soliciting certain restrictions on the nature of the scale from qualified personnel. Assume that the problem is to evaluate the expected payoff of a weapons system, as more and more funds are expended on it. The first logical condition which we can expect to hold is that

(1) $P(0) = 0$; i.e., zero expenditure results in a zero payoff.

Secondly, we can safely assume that

(2) $P(E)$ = non-negative; i.e., that as we increase expenditures we do not, at least, lower expected payoffs.

At this point, our experts provide us with a general appraisal of the nature of the payoff function. Several types of possible payoff functions are noted below.

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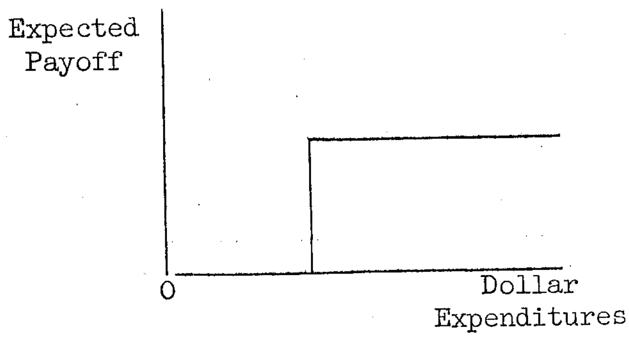
[52] W. G. Mellon, "Comments on 'A General Theory of Measurement-Applications to Utility'," Naval Research Logistics Quarterly, Vol. 7, No. 3 (September, 1960), 285-287.

[53] Cf. n. 34, page 56.

1. Step Function.[54]

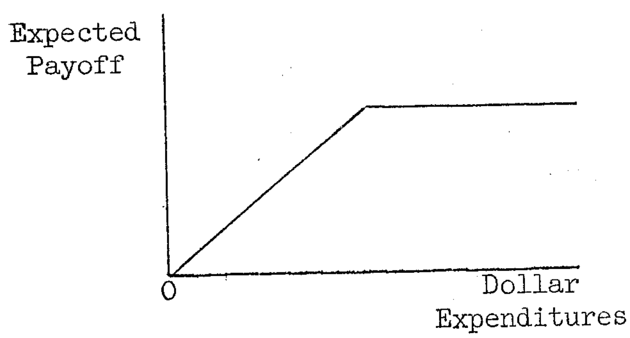
Fig. 13



In this type of payoff function, no return is gained until a certain amount is expended; a payoff is then reached which cannot be increased by the expenditure of additional funds.

2. Proportional Returns to Limit.[55]

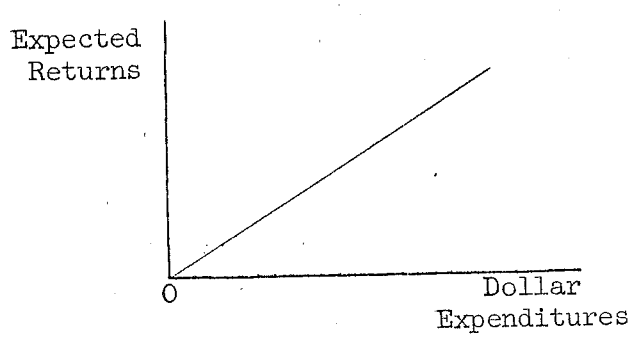
Fig. 14



In this type of payoff function, expenditures result in a proportional payoff up to a cutoff point; beyond that point additional expenditures add nothing to the payoff from the system.

3. Continuous Proportional Returns.

Fig. 15



In this type of return, expenditures continue to result in a proportional addition to payoff.

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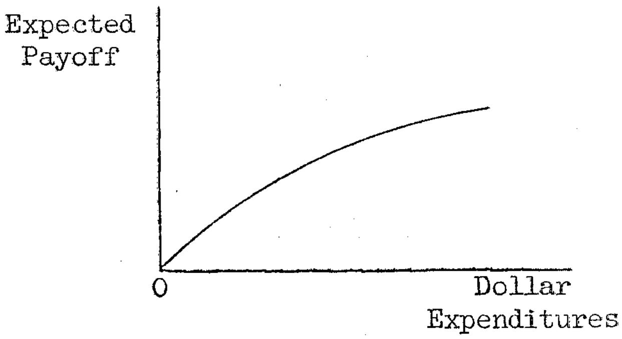
[54]Cf. Herbert A. Simon, "A Behavioral Model of Rational Choice," Quarterly Journal of Economics, Vol. LXIX, No. 1 (February, 1955), 99-118.

[55]Cf. Jacob Marschak and M. R. Mickey, "Optimal Weapons Systems," Naval Research Logistics Quarterly, Vol. 1, No. 2 (June, 1954), 116-140.

4. Saturation Function.[56]

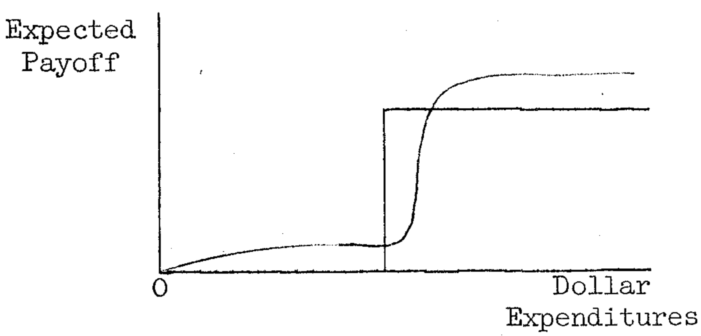
Fig. 16



In this type of payoff function, returns begin to decrease as expenditures rise beyond a certain point.

In addition, these four basic types of payoff function can be used as approximations of more complicated relations. Thus, the Step Function (1.) serves as an approximation of the payoff illustrated in Figure 17,[57] and so on.

Fig. 17



The use of such scales, once derived, is amply illustrated in the references attached to the various types of payoffs, and is not of great interest in the context of this paper. It is to be noted, however, that scales of these types can be dealt with in the same way as the degrees

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[56] Cf. T. E. Phipps, "The Balance of Peacetime Procurement and R and D Spending," Memorandum, April 25, 1957 (unpublished), Naval Warfare Analysis Group.

[57] Cf. n. 54, page 99.

of measurement considered in Chapters II and III — i.e., it may not be necessary to obtain information from expert personnel sufficient to establish exact mathematical values for the functions. Thus, the knowledge that the payoff function resulting from expenditures on weapons system A is of the form of Figure 13, while that resulting from continued expenditures on weapons system B is of the form of Figure 15, can be useful in arriving at the decision as to how to divide a given budget between the two systems — even if we have only a very rough idea as to where (at what level of expenditure) the break in Figure 13 takes place. On the other hand, if the payoff functions from the two weapons systems are of the form shown in Figure 16 — and it would be expected that this would be the more normal case — it is necessary to possess fairly exact information as to the numerical parameters of the functions in order to divide expenditures optimally between them.

The numerical parameters of these continuous payoff functions are derived by asking expert personnel questions of the form — what percent increase (decrease) in payoff will result from a given percent increase (decrease) in expenditures — and from the resulting answers fitting a smooth curve of one of the general forms discussed above. While — despite their apparent numerical properties — scales of this type may actually contain no higher a degree of measurement than the higher order-on-distance scales discussed in Chapter III, they are much more suited to the resolution of continuous payoff problems. In contrast, our earlier scales are better suited to problems where it is necessary to evaluate discrete alternatives.

Section 2. Numerical Priorities in More Than One Dimension

As in the case of lower degrees of measurement, the main theoretical interest in numerical scales lies in the multi-dimensional case. We treat first the case in which alternatives are measured in two or more dimensions and are lexicographically ordered, then the case where the alternatives are not so ordered.

Example 1. Lexicographic (Non-Archimedean) Orders.

Assume the following situation. The Congress wishes to establish a rule which will govern the appointment of individuals to civil service positions and which will take into account the following two factors: the length of military service of the applicant, and his suitability for the position as measured by his score on a standard examination. Let us assume that the following method is adopted: Persons with more military service are always preferred to persons with less service, and only among veterans with equal time of service is preference given to the applicant with the higher test score.[58] (This special type of lexicographic order we may call the case of overriding veteran's preference.)

We thus have established a lexicographic ordering in two dimensions: in each dimension the alternatives are measured on a numerical scale, but these scales are in noncomparable terms. From Figure 18, we can see that the five applicants illustrated would be ranked in the order:

$$x_1 \; P \; x_2 \; P \; x_3 \; P \; x_4 \; P \; x_5 \; .$$

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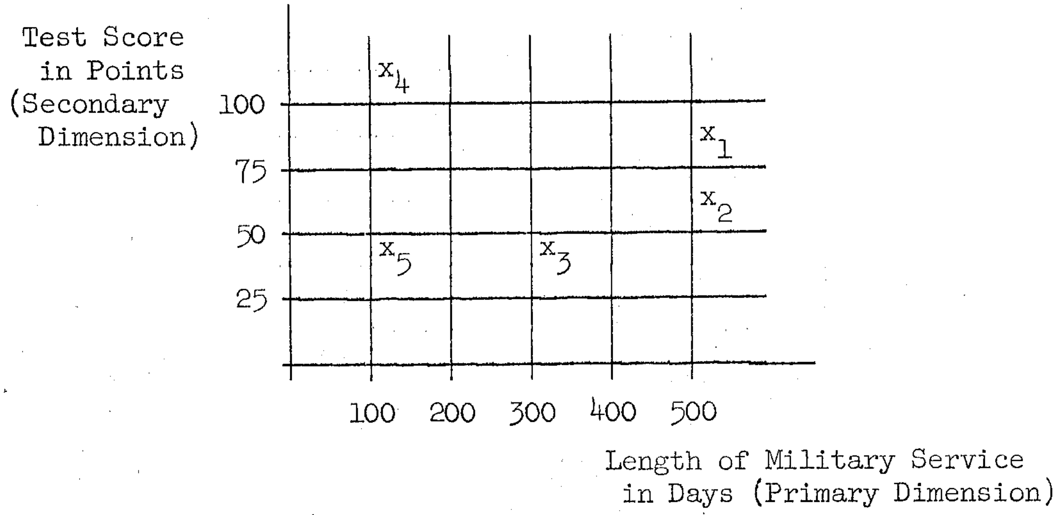
[58] This is similar to the systems which determine the seniority of officers in the armed forces.

Fig. 18



While this rule is sufficient to establish a semi-order on all candidates, let it be assumed that it is necessary to have a numerical scale which relates each candidate to one another — perhaps to accomplish an optimal assignment of the rated personnel to a set of positions. It is, however, a particular feature of lexicographic orders that — strictly speaking — while they will completely order all points in N-dimensions, which no other general ordering rule will accomplish, it is impossible to establish a continuous real-valued function over all the ordered points. However, as we shall see, this limitation — while formally correct — is not a serious barrier to practical use of numerical functions in the lexicographic case.

Assume that there exists a real function which is order preserving on the points in Figure 18. Then such a function will associate two values with points x_3 and x_5, which contain the same value of the Y-coordinate. We know that V_3, the value associated with x_3 by the function, is $\neq$ to V_5, the value associated with x_5, since by our rule $V_3 > V_5$. If a_3 designates the X-coordinate of x_3 , b_3 its Y-coordinate, and a_5 and b_5 the same for x_5 , then, as b_3 and b_5 are identical, and with this coordinate $(b_3 = b_5)$ we can associate a disjoint interval $(N_3 - N_5)$. Assuming that the Y-coordinate extends indefinitely, i.e., comprises the whole set of real numbers, then with each value (b component) we can associate one or several such disjoint intervals; and conversely, the same is true for every value of a . Cantor's Theorem, however, states that the set of all real numbers is not countable, while the set of all disjoint intervals is. Therefore, the assumption that we can define a real-valued function over a set of points ordered by a non-Archimedean criterion inevitably leads to contradiction, and is thus false. Debreu, to whom the above proof is due, states that the following condition must hold for any set of ordered points in order that a real-valued function may be defined on it.

> If for every $x \in X$ the sets $\{x\{X \mid x \leqq x'\}$, $\{x\{X \mid x' \leqq x\}$ are closed (in X), there exists on X a continuous, real, order-preserving function.[59]

However, it is significant to note that the discontinuity, which the assumption of lexicographic orderings will inevitably introduce into any attempt to define a continuous, real-valued function across such points, can be reduced to any size we choose. Thus, if the test scores can run from 0 to 100, let us add to the test score of every veteran the value 101 + the number of days of his service. In the resulting

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[59] op. cit. n. 48, p. 160. It is easily seen that this condition does not hold in the case of lexicographic orders. Consider Figure 19 shown below. The assumption of lexicographic orderings is

order, all veterans will obviously rank above all non-veterans, all non-veterans are ranked according to their test scores, and all veterans with equal test scores are ranked by their length of service, and the function ranking each man is continuous in the real integers. The discontinuity, concealed in this function, can, however, be still further reduced. Consider the case in which the scores on the qualifying test can range between $\rightarrow 0$ and $\rightarrow 100$ — that is, approach 100 and 0 as limits, without these two values actually ever being awarded to any candidate. Then add 100 points plus the number of days of military

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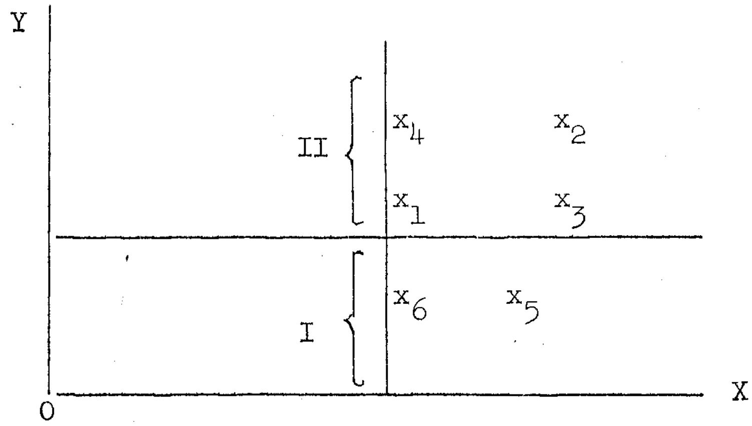
that $x_1 \; P \; x_2$, if its coordinates obey the relations $x_1 > x_2$, or $x_1 = x_2$, $y_1 > y_2$. Thus, points x_2, $x_3 \; P \; x_5$, $x_4 \; P \; x_1$. But all points like x_6 are ranked inferior to x_1. Thus, the set of all points preferred to x_1 is not closed, since it does not include all its own boundary (set of all points I, i.e., like x_6). The same is true for the set of all points inferior to point x_1, since it also does not include all its own boundary (i.e. set of points II, like x_4, which are preferred to x_1 by the lexicographic rule.

Fig. 19

Non-Closure of Lexicographic Orderings



service to the test score of every veteran. In the final ranking, a non-veteran who does perfectly on the test will be assigned a rating approaching 100 as an upper limit, while the rating assigned to a veteran will approach 100 as a lower limit — i.e., the order assigned by the lexicographic ordering on the candidates will be preserved, while the numerical function assigned by our rule will be almost everywhere continuous.

From the above, we see that by selection of a proper conversion factor it is possible to "collapse" a multi-dimensional ranking into a single-dimensional one, even if the ranking is lexicographic. We would add, however, that multi-dimensional lexicographic measures may be used for certain decisions, without such conversion, thereby by-passing some very considerable difficulties. In this connection, two papers are relevant.[60] Hausner and Thrall generalize the von Neumann-Morgenstern index of measurable utility by showing that it is possible to dispense with the Archimedean Property. It will be recalled that this Property enters into the construction of the index in the proposition that, if C P B P A , then it is possible to construct a lottery ticket of the form [A(p) ; C(1-p)] for some probability p $(0 \geq p \leq 1)$ which will be equivalent to a ticket [B, p = 1] . Thrall offers the following illustration of a situation where this property

[60] Melvin Hausner, "Multidimensional Utilities," and Robert M. Thrall, "Application of Multidimensional Utility Theory," in R. M. Thrall, C. H. Coombs, and R. L. Davis, eds., Decision Processes (New York, Wiley, 1954); also L. S. Shapley, "Equilibrium Points in Games With Vector Payoffs," Naval Research Logistics Quarterly, Vol. 6, No. 1 (March, 1959), 57-62; also Oskar Morgenstern, "A Political Economist Views the Navy|Industry Look at the Future," Aerospace Engineering, Vol. 20, No. 12 (December, 1961), 48.

may not hold. The alternatives are: A, to be hanged at sundown; B, to be given one pin; C, to be given two pins. The order C P B P A still holds, but Thrall suggests that no rational person would ever pick the ticket [A(p) ; C(1-p)], with p other than absolutely zero. Thrall notes, however, that in certain situations a utility function which had such non-Archimidean properties can be profitably used:

> However, a two dimensional utility will fit this case if we let the dominant component describe the probability of being hanged and the subordinate component refer to the relative probability of one pin versus two pins. In more detail, we assign to A, B, C, respectively, the coordinates (0,0), (1,0), (1,1). Then the utility space can be considered as the triangle with vertices A, B, C. If D and E are two prospects, we have $D < E$ if E is closer to the side $\overline{BC}$, or if the directed line segment $\overline{DE}$ is parallel to and has the same sense as $\overline{BC}$; i.e., if $D = (a,b)$ and $E = (c,d)$ we have $D < E$ if $a < c$ or if $a = c$ and $b < d$.
>
> Another example arises in appraising various alternatives in a military situation. The commanding general must give top priority to not losing the war; subject to this priority he tries to conserve manpower; and, finally, other things being equal he attempts to conserve supplies. This could be described by a three dimensional utility....
>
> Suppose a commanding general is using two dimensional utility to evaluate a military situation. If he has n basic strategies $P_1, \ldots, P_n$ and his enemy has m basic strategies $Q_1, \ldots, Q_m$, he must first evaluate the outcome if he chooses P_i and the enemy chooses Q_j. This evaluation will be an ordered pair of numbers (a_{ij}, b_{ij}) where, say, the first component is the dominant one. If we assume that the general is using game theory, his action will be to choose some mixed strategy. If he chooses the mixed strategy which assigns probability p_i to P_i, i.e., if he chooses a probability vector $\pi = (p_1, \ldots, p_n)$ then his expected outcome will be at least as good as the vector
>
> $$f(\pi) = \min_j \Sigma p_\lambda (a_{\lambda j}, b_{\lambda j}) .$$
>
> Hence his task is to choose a probability vector π so as to maximize $f(\pi)$. Since his order relation is lexicographic, he first finds those vectors π for which the first component
>
> $$f_1(\pi) = \min \Sigma p_\lambda a_{\lambda j} \quad \text{of} \quad f(\pi)$$

is a maximum. According to the general theory of games these vectors form a non-empty convex subset S of the set of all probability vectors. If S is a one element set his problem is solved, but if S is larger than one element he now must search for vectors π in S which maximize the second component

$$f_2(\pi) = \min_j \Sigma p_\lambda b_{\lambda j} \quad \text{of} \quad f(\lambda) .$$

Again the theory of games guarantees a solution. If instead of two dimensions the general had three or more dimensions, he would merely iterate this process.

This discussion illustrates the fact that non-Archimedean utilities are perfectly satisfactory for game theory. The equivalence of game theory and linear programming guarantees that non-Archimedean utilities will be satisfactory also for linear programming problems.[61]

But while this type of measurement will indeed be satisfactory — if a lexicographic order conveys an accurate picture of the decision situation — in game theory or in linear programming problems where the question is one of selecting the preferred alternative, it will not serve in the case where we must compare the payoffs from different weapons systems or select an optimal set of alternatives, for example. Moreover, our previous strictures about the limited applicability of orders which are truly lexicographic still apply.[62]

In terms of our personnel selection problem, we note that it is possible to weaken somewhat the lexicographic relation, by allowing the dominant vector to take only the values of 1 or 0. Thus, we establish the following rule: All job applicants are divided into two

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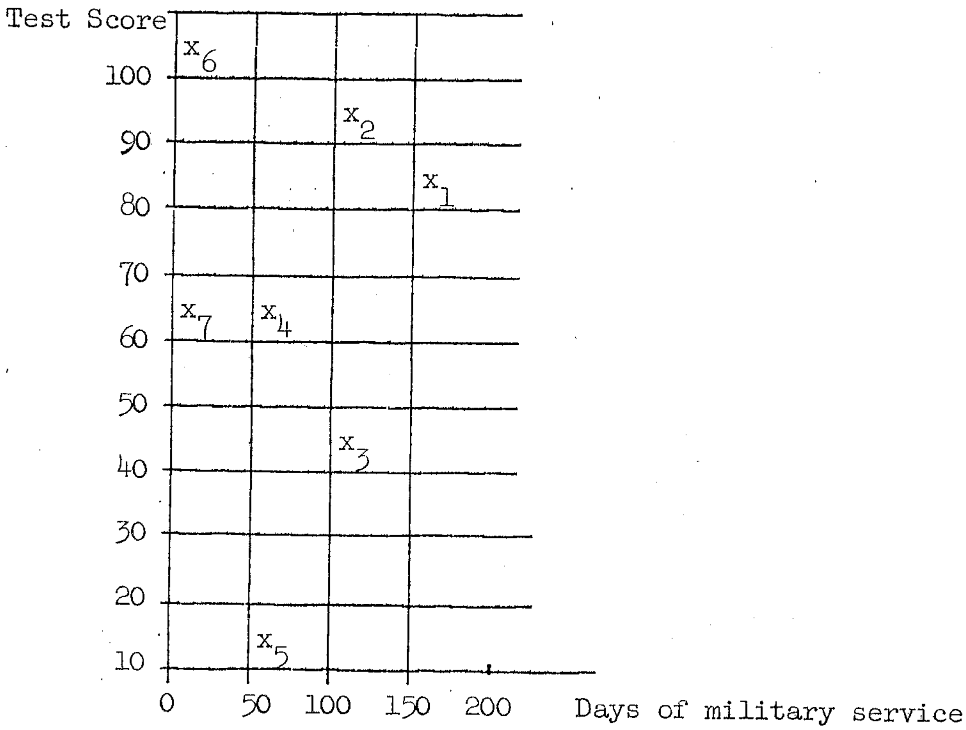
[61] Ibid., pp. 185-186.

[62] But see the remarks of Morgenstern, op. cit., n. 2. It is possible to conceive of orders which are "almost lexicographic" or "almost non-Archimidean" where the assumption of lexicographicity is so near true that it can be used as an approximation of the truth. To turn Thrall's example around, it is possible to conceive of a stack of pins so high, and therefore so valuable, that a person would run some incredibly minute chance of being hanged to attain it.

classes, veterans and non-veterans; all veterans are to be preferred to all non-veterans, but within each group ranking is to be on the basis of test scores. From Figure 20, we see that this results in a ranking of the candidates: x_2 P x_1 P x_4 P x_3 P x_5 P x_6 P x_7 , while the application of our previous rule would have resulted in a ranking x_1 P x_2 P x_3 P x_4 P x_5 P x_6 P x_7 . . . . A nearly continuous numerical scale is obtained by adding 100+ points to the test score of every veteran.

Fig. 20



Example 2. Non-Lexicographic Orders.

We turn to a consideration of those cases where the assumption of lexicographicity does not hold. In these instances, there are two basic methods of dealing with the problems presented by more than one dimension of measurement.

<u>Conversion</u>: The first of these is what may be termed "conversion." In this method, a multi-dimensional problem with numerical scales along each dimension of measurement is "converted" into a single-dimension by selecting a factor which converts measurement along one axis into measurement along another. Thus, in the example of veteran's preference given in the preceding section, we might add a fixed amount to every veteran's test score. Assuming this amount at, say, 20 points, then the candidates in Figure 20 above would be ranked x_2 P x_1 In x_6 P x_4 P x_3 In x_7 P x_5 . Again, we might add one point to the test scores for each 10 days of military service, which would rank the candidates x_2 In x_6 P x_1 P x_4 P x_7 P x_3 P x_5 .

A method of this sort may be used to resolve a number of multi-dimensional priority problems, especially when we are seeking a convenient rule-of-thumb. On the other hand, it does involve a very strong assumption which is probably inapplicable to most situations: that the contribution of the vector, which is converted into the other, to the total numerical score of each position, is constant — in our example, that one month's service is always equal to 1 point of test score, no matter what the score may be.

Conversion rule-of-thumb methods can be used to convert priority measurement in one dimension into priority measurement in another. In addition, priority — that is, subjective — measurement can sometimes be usefully converted into objective measurement in the context of certain decision problems. The usual meaning of the word "objective" implies something which can be expressed in dollars or pounds, while "subjective" usually implies something which cannot be expressed in such terms. But a middle ground may exist between the two types of indicators, in the sense that indicators which are subjective in nature may be usefully translated into objective terms. If we were attempting to estimate whether or not a proposed flood control was desirable, we would want to be able to compare costs and returns. Since one of the costs is danger to human life if the project is not established, an arbitrary value in dollars can be assigned to such loss, and the concept of expected costs used to combine these costs with expected dollar damage to property.[63] Such an approach can also be used to manipulate the results of programs in a desired direction. If one were designing a program for the optimal decision in the assignment of funds for alternative weapons systems where one of the evaluative factors is the fact that it is estimated that some systems will be ready for use before some others, we might include in the cost function for the program an arbitrary dollar penalty for not having the weapon available by a certain time. Not that we are trying to seriously estimate the dollar costs of military defeat. There is no meaningful way to estimate the money costs of a 20 percent choice of a weapon's unavailability resulting in a 2 per cent (estimated) choice of the nation losing a war. But a sufficiently high dollar penalty will reduce this risk to any desired estimated value. The choice between the degrees of risk we are willing to risk is, of course, a purely subjective one and not subject to computation.

Combination: The second method which may be used to resolve this problem is the method of "combination," in which the scores of an alternative along all axes are "combined" — that is, added and|or

[63]Cf. op. cit., n. 2.

multiplied together to give a total score for each alternative. This is, of course, adding apples and peaches since the measurements are not in comparable terms; but if we are careful to always remember the dubious numerical nature of the scales with which we are dealing, we will find the method as satisfactory as any of the others we have mentioned. The general choice in this instance is between addition and multiplication. If properly used, both of these methods will usually preserve the property of Pareto Optimality, and will indicate whether each alternative is $>$, is equal to, or $<$ than any other. In general, the choice of whether one uses addition or multiplication would depend on the particular problem at hand.[64] Thus, if your lowest ranking alternative was, in fact, completely valueless in one dimension (as in the earlier example in this discussion, if one of the outlying bases were of no positive value to the military effort), then it would make sense to assign this alternative the value of zero and to use the rule of multiplication. In our example this would be equivalent to stating that a request from an activity is worthless, no matter what is the degree of urgency. On the other hand, it would be obviously absurd to give the lowest degree of urgency a value of zero, since this would assert that the lowest ranking request from the highest valued base (in terms of importance) should be treated exactly the same as the lowest ranked request from the lowest ranked base. This difficulty can be easily avoided with the following procedure: let the numerical scale,

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[64] If we possess the final figures themselves (in the assignment problem, for example, the value of item X in position Z), then we can show that this figure can be broken up into the sum of two values which represent the contribution of each dimension to the total. Cf. Koopmans and Beckman, op. cit., n. 5.

which is assigned by the outlying bases to the relative urgency of a request, run from 1 to 2, and the relative importance of the base from which the request originates run from 0 to 1. All requests from the worthless base will be treated as worthless, all lowest value requests will be ordered according to the importance of the base from which they originate, and the principle of Pareto Optimality will be observed. If the lowest ranked base is, on the other hand, not worthless, but simply less important, we may have both scales run from 1 to 2.[65] To re-emphasize what we have said previously, the proper approach in designing numerical indicators to assist us in military computation is to decide first which criterion these methods of establishing the worth scales should meet and then to design them accordingly, since the degree of measurement involved in military-worth studies is necessarily limited.[66]

The simplest case in which the combinatorial approach can be used, and the case in which its use is least open to question, is where the numerical measure of value on one dimension is a subjective scale,

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[65] In simplest terms, the principle of additivity seems ill-adapted to this method because it will by definition give, for example, the value of a particular piece of equipment in a particular assignment, or the sum of the value of the equipment and of the place or operation to which it is assigned. Thus, if a very valuable piece of equipment is assigned to a worthless operation, the value of this assignment would be the value of the equipment itself if we were to add; whereas, in fact, the value of this assignment is patently zero. The process of multiplication, on the other hand, introduces (even though imperfectly) the concept that the value of the equipment in a particular operation is a unique alternative and depends on the value of the equipment and of its location in a fairly complex fashion.

[66] In a more general sense, it can be shown that an adoption of the procedures and problems introduced into economic literature by Professor Kenneth J. Arrow (cf. n. 10) can be used to show that any method of resolving the multidimensional priority problem, in which we have less than absolute-zero scales of measurement, will fail to meet a set of very reasonable criteria.

obtained by the questioning of expert personnel, but the numerical measure along the second dimension is in terms of some objective criterion of performance.

Example. The following method is used by the Department of Defense. Materials which are necessary for national defense are assigned index numbers, which are employed as follows:

1. The smaller the Alternative Material Index Number the more available is the metal. In choosing between several metals which meet military performance requirements, and that are required in the same quantity for a particular application, select the metal with the lowest index number.

2. If two or more metals which could be used in a particular application would be required in different quantities, multiply the Alternative Material Index Number of each by the respective quantity to determine their gross Index Number and adopt the metal with the lowest gross Index Number. If, for example, 100 tons of aluminum can be used in lieu of 150 tons of copper in a military application, the comparison would be thus:

   100 tons of aluminum times 0.60 = 60.0

   150 tons of copper times 0.77 = 115.5 .

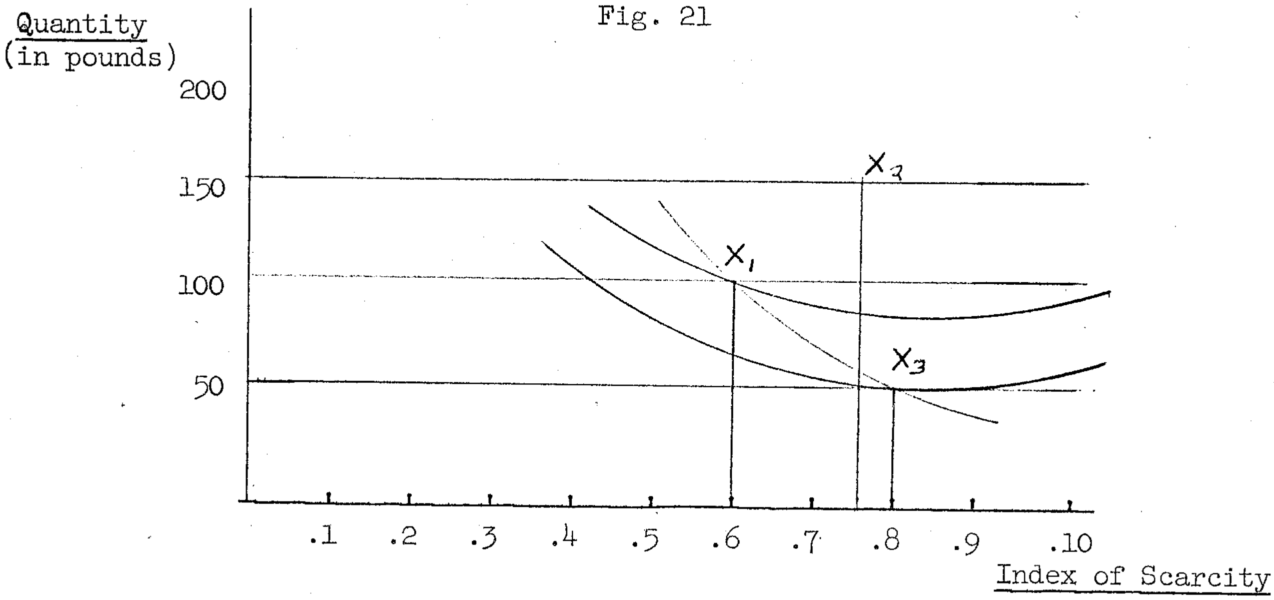
   In this instance preference would be given to aluminum.[67]

We see that the government is faced with the problem of how to purchase an optimal "bundle" of commodities, where each "bundle" has two components (instead of two types of commodities): the relative military value of the metal and its costs in terms of the amount used. In Figure 21, point x_1 is the aluminum combination, x_2 is the copper combination, and we have added x_3 — 50 pounds of manganese with a scarcity rating of .8 — for illustration. x_1 would have been preferred to x_2 in any case, since it is dominant in both directions of measurement, so that the whole purpose of the Defense Department method

[67] Memorandum, Department of Defense. The method by which this index (measure of priority) is established is classified information.

is to decide between combinations like x_1 and x_3 .

Fig. 21



If these were combinations of goods purchased by a consumer at different times, it would be problematical — depending on relative prices — as to whether x_3 could be compared to x_1 . But in the current situation we know that 100 pounds of aluminum is technologically equivalent to 50 pounds of manganese. Thus, we are forced only to choose the cheapest — where cost is defined as scarcity times amount — which can be read off by drawing iso-cost curves through x_3 and x_1 , and choosing the combination nearest to the origin. We thus see that we face no serious difficulties in multi-dimensional problems of this class — provided that we possess a satisfactory subjective index: therefore, the main interest must lie in those cases where more than one dimension of measurement is in subjective terms.

Summary.

Our discussion of numerical indicators in one dimension has been limited, for this topic has already been extensively considered elsewhere. It is, moreover, of considerably less interest than that of less-than-numerical indicators, since it is quite apparent what can be done with a numerical indicator of worth — once it has been derived — while the usefulness of less-than-numerical measures had not been previously dealt with in any systematic manner. What interest there is in the single-dimension numerical case lies, therefore, in the methods by which the priority indicators are derived.

In the case where we are dealing with measurement in more than one dimension, the problems encountered are of greater theoretical interest. As was shown, a number of more-or-less arbitrary methods can be used to induce comparability between the scales of measurement. As a check on the "validity" of these methods, rankings of certain of the points in the decision space could be solicited directly.

CHAPTER V

CONCLUSION

The Future Trend of Priority Research.

The present study is entitled "An Approach to a Theory of Priorities" in appreciation of the fact that extended future research will be necessary in order to apply its lines of reasoning to realistic decision problems. In particular, any significant advances will depend on the ability of those in authority — in the military or elsewhere — to separate problems where the solicitation of expert opinion approach is applicable from those where further technical study is implied — i.e., the ability to resolve the problem discussed under Assumption 1 in Chapter I. No less important, the question raised in the discussion of Assumption 2 — the development of logically satisfactory methods of amalgamation of expert opinions — must be dealt with. As the first is a problem to be handled by consultation with experienced military decision makers, so the second is a purely abstract problem which should be considered by those experienced in welfare economics and related disciplines.

A second, and less challenging, set of problems are related to the actual use of the method in decision situations. For realistically-sized problems where we use less-than-numerical scales of measurement, it will be necessary to develop programs which will quickly determine — from an initial set of rankings — whether or not these rankings have resolved the priority program, and what the probability is, if the program is not resolved, that movement to the next level of measurement will accomplish this resolution, e.g., the expected size of the non-dominated group of assignments in Example 1 in Chapters II and III.

Finally, there is a great need to incorporate stochastic and dynamic elements into the analysis. For example, the assertion has been often made that priorities, meaning classification schemes of the type discussed above, are completely ineffective when the set of alternatives to be reached is not constant, nor known in advance, and where there is a limitation on resources so that the organization cannot (eventually) satisfy all the alternatives. Thus, when the problem was one of allocating raw materials in wartime, priority systems eventually broke down when a number of new and extremely important projects, which rated high priority, came into existence after the priority system was established. When high priorities were assigned to these projects, the total number of A-1 priorities became so great that the rating became not a guarantee of obtaining its material, but a mere "hunting license." The authorities, it is true, attempted to restore the usefulness of the priority system by adding new, super-priority ratings, but this led to inflation of the priority currency, and then the problem had to be re-faced when a still more important project had to be considered. In the end, the priority systems were abandoned in favor of systems of direct allocation. Nearly all the material concerned with the subject of priorities is about these wartime experiences with priority (classification ranking) plans and why they failed. There is thus what appears to be general agreement among military supply experts at least that only direct allocation will be satisfactory in the limited supply-variable alternative.

From what we have said previously, however, it should be clear that direct allocation and priorities are not opposites or competitive, but rather mirror images of one another, since in any assignment problem, the values of the assignment matter dictate where each item shall be placed for an optimal solution of the assignment problem; and some system of priorities is reflected in even the most improvised system of direct allocation.

This is not to say that this particular case does not pose some special problems in the design of priority systems. In addition to the always present problem of whether or not priorities should be overriding, which can usually be solved by numerical utilities and some built-in systems of diminishing marginal worth, there are the special problems introduced by the fact that the number of alternatives to be ranked is variable. It would appear, however, that the problem has been somewhat misconstrued by those who maintain that only direct allocation can be effective in such cases. These critics point to the inflation of priorities as an indication of the breakdown in the priority system. But this type of inflation, in a situation where there is no meaning to "social justice," is not comparable with price inflation in the economy proper; but it is an essential part of a properly functioning system in non-price situations. As new and more urgent requests become evident, it is obviously correct that they be assigned higher and higher priorities. It is not this "inflation," but the fact that the priority systems were improperly designed to deal with a future stream of alternatives, that led to difficulties during the Second World War.

This, then, is one type of stochastic feature which should be incorporated into the analysis of priority problems — that new alternatives are constantly likely to appear, and that optimal allocation patterns may involve setting aside a portion of resources to accommodate these alternatives when they do appear. Provided that we can devise a rate of time discount or priority to compare meeting present goals with future goals, and provided that it is possible to specify in a meaningful way the values of these future alternatives combined with the chances that they will appear, our deterministic, one-period examples can be expanded into a much more useful form.

A second type of stochastic problem is where a fixed set of alternatives — to which priority indicators are attached — are subject to random demand. A typical example would be where we have a number of types of equipment which are needed for the efficient operation of a fleet, a limited appropriation with which to purchase equipment and stock the fleet, dollar costs for each type of equipment, and subjective measures of priority for each type of equipment which indicate the "military dis-utility" of "run-outs." If these priority indicators are in numerical form, then the problem is a typical inventory situation, if we can only define the probable demands for each type of equipment. The complications arise if the priority indicators attached to each type of equipment are of one of the less-than-numerical scales discussed in Chapters II and III — i.e., that we have information of the form that the difference in the costs of running out of type A equipment as opposed to type B equipment is greater than the difference in the costs of running out of type C equipment, as opposed to type D. Following the line of reasoning of our earlier chapters, we know that such information establishes a range of values for each alternative. Solving the inventory problem for each admissible value of each type of equipment run-out, we would establish a set of non-dominated inventory policies. The size of this set will depend on the degree of measurement on the costs of run-out, and how this measurement interacts with the other parameters of the problem, but, unlike the selection of an optimal set of projects example used in Chapters II and III, since the inventory plans are not composed of a set of "lumpy" objects, but are fairly continuously related to one another, it is not probable that this set will consist of a single plan. Selective solicitation of additional order information can be used to reduce the size of the set up to a point. And, as we have noted previously, the information that an optimal inventory program consists of stocking between X and $X + \Delta X$ units of A and Y and $Y + \Delta Y$ units of B, can prove a major advance over _ad hoc_ solutions. In most military supply problems we are seeking improvement rather than optimization, the concept which has come to be known in military logistics as "acceptability."

Applications to Economic Decision Problems.

The discussion in this report has been confined almost entirely to problems of military decision. We would disagree with those commentators who might maintain that these are not _economic_ problems in the proper definition of the term. Charles Hitch comments:

> . . . men in their wealth producing and acquiring activities are especially inclined to _economize_; i.e., to minimize costs, or to maximize the achievement of objectives with resources, in a careful and explicit way. But people in other activities, outside the institutions of the economy as these are usually understood, also economize in this manner, or want to, or in some important sense ought to. These include people concerned with national security, and in consequence a lot of

> economic theory — or the analysis of economizing — can be applied, if appropriately adapted, to the solution of their problems. The design of a system for ordering and stocking spare parts for aircraft is an economic problem whether the aircraft are owned by a commercial airline or by the U. S. Air Force, although there are important differences stemming from the fact that the services of the former are sold on the market while the services of the latter are not. So is the choice between Bomarc and Nike-Hercules: which missile (or combination of the two) will provide the most economical (i.e., efficient) air defense of the U. S.? So, too, is the choice between military aid, on the one hand, and a larger budget for our own military forces on the other.[68]

We would add, however, that the possible applications of subjective indicators of priority of varying degrees of measurement have applications, as we have indicated in the introduction, which range far beyond the purely military. Subjective indicators of worth have already been utilized, for example, by Markowitz, in his method for selection of optimal investment portfolios, and has even been applied to macro-economic decision problems for the economy as a whole.[69] The use of subjective measures of worth even in discussions of the problems of military decision (in any rigorous manner) is only some ten to fifteen years old, and most non-military applications date from an even later period. It seems safe to suggest, therefore, that subjective indicators of worth will come to play an increasing role in economic decision making in the future, if certain prejudices against their use can be overcome.

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[68] Op. cit., n. 1, p. 4.

[69] In particular, the path-breaking (and almost unnoticed) work of Ragnar Frisch in translating preferences of political authorities into numerical terms, and the imbedding of these preferences as the objective function in a national macroeconomic decision problem. Cf. Professor Frisch's monograph, "Numerical Determination of a Quadratic Preference Function in Use in Macro-Economic Programming," Memorandum fra Sosial-Økonomisk Institutt (University of Oslo, February 14, 1957).

In a number of readings done for this study, the author has encountered the position that while theoretical analysis may be "all right in its place," it is no substitute for experience and judgment. But, it is the purpose of all attempts to use subjective indicators of priority in decision problems to make maximum use of experience and expert opinion. It is, therefore, quite false to suggest that there is a necessary dichotomy between analysis and the use of expert judgment. Analysis without judgment faces the danger of losing sight of its objectives and the intangibles of a problem through an over concern with internal rigor: judgment without analysis can become so involved in the complexities of a problem that the final decision may not reflect the best judgment of the decision maker. A proper use of data based on judgment in analytical models is what is called for — a form of "subjective programming."

Concluding Note.

To the extent that it may prove necessary, we conclude with a note of caution. These methods of constructing numerical or quasi-numerical utilities are best suited for the resolution of decision problems, where the number of alternatives is determinate and limited, and there are a limited number of dimensions of measurement. In these "sub-optimizing" situations, careful application of the methods described in this study can lead to answers which, if not optimal, are at least improvements over presently employed *ad hoc* procedures. As Charles Hitch writes:

> Economists will be successful in tackling these problems only if they accept modest objectives. Optima are unattainable in a world of incommensurables and uncertainties. But economizing does not necessarily require optimizing, in the military or in

the economy. It simply requires finding better solutions than those which would otherwise be adopted. And this, in very many cases, is both a feasible and the practically important objective.[70]

[70] Ibid., p. 12. This does not mean that one cannot sometimes make useful judgments on very broad problems. Cf. Giles Mellon, "Less Bang for the Buck?" Challenge, Vol. 8, No. 9 (June, 1960), 19-23.

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