# REFORMULATION OF SOME PROBLEMS IN THE THEORY OF RISK

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# 1. Introduction

1.1 In classical actuarial theory we are concerned exclusively with expected values.

The net premium of an insurance contract is by definition equal to the expected value of the claim payments which will be made under the contract. Similarly the technical reserves of an insurance company are defined as the expected value of the payments to be made under all contracts in the company's portfolio.

If the insurance contracts are of long duration, interest is usually taken into account by discounting the value of all payments to some particular point of time. In the following we shall ignore interest, since it is fairly clear that this element can be brought into all formulae without any serious difficulty.

1.2 It is, however, evident that an insurance company must consider the possibility of deviations from the expected values. In practice this is done by adding a "safety loading" to the net premium, and by keeping "special reserves" in addition to the technical reserves.

There is a considerable literature about the measures which insurance companies can take to allow for such deviations from the expected values. It is convenient to refer to this heterogeneous body of literature as the "theory of risk", although "non-classical actuarial mathematics" would have been a more appropriate and more correct term.

1.3 In this paper we shall re-examine some of the basic ideas and objectives behind the studies which one--more or less appropriately--refers to as theory of risk. We shall try to show that recent developments outside the field of actuarial mathematics make it possible to formulate these objectives in a precise manner, and in some cases to find clear-cut solutions to problems which have been discussed by actuaries in a rather inconclusive manner for more than a generation.

## 2. The Basic Model

2.1 An insurance contract defines a probability distribution $F_i(x)$, where $F_i(x)$ is the probability that claim payments under the contract shall not exceed x. If we ignore interest, the net premium of the contract is:

$$P_i = \int_0^\infty x dF_i(x) .$$

The technical reserves of an insurance company holding n contracts are:

$$V = \sum_{i=1}^{n} P_i = \int_0^\infty x dF(x)$$

where F(x) is the convolution of the distributions $F_1(x) \dots F_n(x)$.

2.2 We shall now consider an insurance company which holds a portfolio of insurance contracts. We shall write F(x) for the probability that the total amount of claim payments under these contracts shall not exceed x. We shall assume that the premiums for all contracts have been paid to the company in advance. We shall further assume that the pre-paid premiums together with the company's initial capital amount to S.

Claims may become payable at any time within the contract period. If we assume that all contracts are of short duration, it will not matter at which particular time the claims occur. The <u>risk situation</u> of the company will then be completely determined by the two elements S and F(x).

2.3 When all contracts in the portfolio have expired, the amount of money left with the company will be

$$y = S - x \quad .$$

y is obviously a stochastic variable, and its distribution is easily found to be

$$G(y) = 1 - F(S - y) \quad .$$

$$-\infty \leq y \leq S$$

We shall refer to this probability distribution as the <u>profit distribution</u> associated with the risk situation $\{S, F(x)\}$.

2.4 If an insurance company has a well-defined policy, it must have some consistent rule which makes it possible to decide when one profit distribution $G_1(y)$ is preferable to another $G_2(y)$.

A perfectly consistent rule would be to go exclusively by expected profits, and prefer $G_1(y)$ to $G_2(y)$ if and only if

$$\int_{-\infty}^{+\infty} y dG_1(y) > \int_{-\infty}^{+\infty} y dG_2(y) \quad .$$

There is nothing wrong with this rule, except that it does not seem to be followed by any insurance company. The fact that reinsurance exists is a sufficient proof that the possibility of deviations from expected profits is taken into account when insurance companies make their decisions.

2.5 It may be useful to illustrate the point above with a simple example.

We consider an insurance company which has underwritten a contract which can lead to a claim of $1 million with probability 0.001. We assume that no other claims can occur, and we assume further that the company's funds, including the premium collected for the contract mentioned, amount to just $1 million. It is easy to see that the profit distribution in this case is:

$ 1 million with probability 0.999

$ 0 with probability 0.001

If the company pays $50,000 in order to reinsure one-half of this risk, the profit distribution will change to

$ 950,000 with probability 0.999

$ 450,000 with probability 0.001

If the actuary of the company is used to reason along classical lines, he may point out that the net premium for this reinsurance is $500, and that it is sheer madness to pay 100 times this amount for reinsurance cover. His directors may, however, still prefer to take the reinsurance, rather than carrying the whole risk alone. This means that they consider the second profit distribution as better than the first.

2.6 We shall now assume that the company's preferences over various profit distributions constitute a complete ordering over the set of all probability distributions $G(y)$. This is just the precise mathematical way of saying that the company has a well-defined policy, a term which we used rather loosely in paragraph 2.4.

A complete ordering can under very general conditions be represented by an index number or a functional $U(G)$ such that

$$U(G_1) > U(G_2)$$

if and only if $G_1(y)$ is preferred to $G_2(y)$, and

$$U(G_1) = U(G_2)$$

if the two distributions are considered as equivalent.

We shall refer to $U(G)$ as the utility attached to the profit distribution $G(y)$.

The ordering is assumed to include all probability distributions. Hence it must also include degenerate distributions of the type $\epsilon(y-a)$. For this distribution profits will be exactly $a$ with probability 1.

We shall write $U(G) = u(a)$ if $G(y) = \epsilon(y-a)$.

2.7 If the preference ordering is consistent, one can prove that

$$U(G) = \int_{-\infty}^{+\infty} u(y)dG(y) \quad .$$

This formula was first presented by Daniel Bernoulli (3) in 1738 as a reasonable hypothesis concerning rational decision-making under uncertainty. The principle was first applied to insurance problems by Barrois (1) in 1834. In 1947 von Neumann and Morgenstern (12) proved that Bernoulli's hypothesis could be derived as a theorem from a few simple, and intuitively very acceptable axioms. These axioms must be fulfilled if the preference ordering over the set of profit distributions shall be consistent in any acceptable sense, and it is almost self-evident that the axioms hold for a rationally managed insurance company.

2.8 The proof given by von Neumann and Morgenstern is elementary. However, the authors present their proof with apologies because it has become "lengthy and tiring" in order to be complete, and they express the hope that a shorter exposition may be found later.

Shorter ways to the theorem have indeed been found, but usually by sacrificing the elementary character of the proof. It has been demonstrated by Chipman (6), Debreu (8) and others that the theorem is an almost trivial consequence of the axioms when the problem is formulated in topological terms.

The more elementary discussion around the theorem has not tended to shorten the proof. Efforts seem to have concentrated on deriving the theorem from the simplest and most basic of axioms, and this obviously tends to lengthen the proof. Savage (13) may be less tiring than von Neumann and Morgenstern, but he is certainly not shorter. Some textbooks (9) give elementary proofs, which although not mathematically complete, give a good intuitive idea of the contents of the theorem.

2.9 From the formula in paragraph 2.6 we see that the function u(y) determines the preference ordering over the set of profit distributions. In the following we shall refer to u(y) as the company's <u>policy function</u>, since it determines the "attitude to risk", and hence the policy which the company will follow.

The function u(y) can evidently be interpreted as the utility attached to a profit y which will be received with certainty. For this reason the function is usually referred to as "the utility of money" in economic literature. We shall, however, avoid using this term, since it carries a number of undesirable connotations.

2.10 The policy function determines a unique preference ordering. However, the contrary is not true. It is easy to see that if a given preference ordering can be represented by a function u(y), it can also be represented by any function $Au(y) + B$, where A and B are constants, and $A > 0$. Hence a preference ordering determines the policy function only up to a positive linear transformation.

It is evident that any "reasonable" preference ordering is represented by functions u(y) which increases with y.

2.10 If an insurance company acts rationally, it will seek to manage its affairs so that it reaches the profit distribution which according to its particular policy is the "best" among the distributions which are attainable. This means that the objective of the company will be to maximize the utility index U(G) over the set of attainable profit distributions G(y).

This formulation of the problems which lie behind the theory of risk will in a number of cases make it possible to find definite solutions to familiar unsolved problems. We shall illustrate this with two simple examples.

## 3. Two Simple Examples

3.1 We shall consider a company which as a result of its direct underwriting has arrived at a risk situation determined by S and F(x).

The utility attached to this situation is according to the formula in paragraph 2.6

$$U(0) = \int_0^\infty u(S-x)dF(x) \quad .$$

We now assume that the company can reinsure a quota k of its portfolio by paying a reinsurance premium of P(k). This arrangement will lead to a risk situation with utility

$$U(k) = \int_0^\infty u(S-P(k) - (1-k)x)df(x) \quad .$$

Since the company's objective is to maximize utility, its task will be to determine the value of k which maximizes U(k). This is a straightforward, although not always simple, mathematical problem.

There are obviously no difficulties involved in generalizing this model and considering other forms of reinsurance than quota share treaties.

3.2 The weakness of the model in the preceding paragraph is the assumption that a function $P(k)$ exists. We have no right to assume that reinsurance cover has so to speak a market price, and that a company can buy exactly the quantity it wants.

This problem has been investigated in some other papers (4) and (5). It appears that in a reinsurance market there will not be a unique market price determined by supply and demand for reinsurance cover. It seems that a complete analysis of reinsurance markets will have to be carried out in terms of the general theory of n-person games by von Neumann and Morgenstern (12).

3.3 As another example let us consider an insurance company which is about to market a new insurance contract. Let $F(x)$ be the claim distribution defined by this contract, let the premium be $P$, and the initial capital of the company $S$.

Assume now that the number $n$ of contracts which the company can sell depends on the amount $s$ spent on sales promotion, i.e. we assume that $n = n(s)$.

Hence an expenditure of $s$ for sales promotion will give the company a utility

$$U(s) = \int_0^\infty u(S + nP - s - x)dF^{(n)}(x)$$

where $F^{(n)}(x)$ is the n-fold convolution of $F(x)$ with itself.

The problem of the company is then to determine the value of $s$ which maximizes $U(s)$, given that $n = n(s)$.

3.4 The weakness of the model above is the assumption that there exists a function $n(s)$ which gives the market reaction to a certain expenditure on sales promotion. It is natural to assume that the reaction of the market will depend also on the sales efforts of all competing companies. If we want to analyze the problem in this more realistic manner, we will again have to resort to the general theory of n-person games.

This analysis will be more complicated than the analysis of a reinsurance market which we discussed in paragraph 3.2. In a reinsurance market we have to consider the policy functions of all participating companies. In the marketing problem discussed above we will also have to consider the market reaction to the sales effort made by the different companies.

## 4. Some Generalizations of the Model

4.1 In the preceding sections we have assumed that all insurance contracts were of short duration, and that premiums were paid in advance for the whole contract period. These simplifying assumptions made it possible to solve our problem by considering only the probability distribution of profits at the end of the contract period.

If we drop these assumptions, it may be of some importance whether claims occur early or late in the contract period. This may mean that we have to consider a stochastic process instead of a simple probability distribution, and this will clearly lead to considerable complications.

Under the more general assumptions it may be necessary to take interests into account. This will lead to some complications, which, however, seem to be of a fairly trivial nature, and which we will ignore in the following.

4.2 Let us now consider a portfolio of long-term insurance contracts, all of which will have expired by the time T. We can of course define a profit distribution $G(y,T)$ as the probability distribution of the amount of money $y$ which is left with the company when all contracts have expired. However, $G(y,T)$ will be of little interest if there is a possibility that the company may have to go into liquidation before the time T.

We shall illustrate this point by a simple example.

4.3 We shall consider a company with initial capital 1, and we shall assume that the company receives a premium of 2 by underwriting a portfolio which can lead to a claim of 4 with probability $p$.

The profit distribution $G(y,1)$ will then be

$$1 + 2 = 3 \qquad \text{with probability } q = 1 - p$$
$$1 + 2 - 4 = -1 \qquad \text{with probability } p$$

If the company underwrites a portfolio of this kind in two successive periods, it will get a profit distribution $G(y,2)$ given by the following table:

$$3 + 2 = 5 \qquad \text{with probability } q^2$$
$$3 + 2 - 4 = 1 \qquad \text{" " } pq$$
$$-1 + 2 = 1 \qquad \text{" " } pq$$
$$-1 + 2 - 4 = -3 \qquad \text{" " } p^2$$

The last two lines in this table concern the case where the company was insolvent after the first period, but still underwrote a portfolio for the second period. If this is illegal, so that the company had to liquidate after the first period, we obviously have to consider the modified profit distribution G'(y,2) given by

| | | |
|---|---|---|
| 5 | with probability | $q^2$ |
| 1 | " " | $pq$ |
| -1 | " " | $pq + p^2 = p$ |

4.4 In order to generalize these considerations, we shall formulate the problem in terms of the so-called <u>collective theory of risk</u> due to Lundberg (11). We shall use the notation of Cramér (7).

We consider as earlier a portfolio of insurance contracts which all will have expired at the time T, and we introduce the following symbols

$F(x,t)$ = the probability that claims occurred up to the time t shall not exceed x.

$Pt$ = the amount of premiums received up to the time t.

$S$ = the initial capital of the company.

The funds held by the company at time t, $Y_t = S + Pt - x$ is clearly a stochastic variable which can take both positive and negative values. If $Y_t < 0$ the company is insolvent or "ruined" at the time t, and may have to go into liquidation.

Let now

$$1 - \psi(T) = \Pr\{Y_t \geq 0 \text{ for all } t \leq T\} .$$

The complementary probability $\psi(T)$ is usually referred to as the ruin probability.

4.5 The profit distribution considered in paragraph 4.2 is obviously given by

$$G(y,T) = 1 - F(S + PT - y,T) .$$

However, as indicated by the example in paragraph 4.3 this distribution has to be modified if there are some rules which require the company to cease its operations if it becomes insolvent at some specified time. The nature of the modification will obviously depend on these rules.

The strongest possible rule is that the company shall go into liquidation immediately upon becoming insolvent. The probability that this shall happen is obviously $\psi(T)$. Hence we get a modified profit distribution $G'(y,T)$ of the form

$$\begin{array}{ll} \psi(T)G_1(y,T) & \text{for } y < 0 \\ (1-\psi(T))G_2(y,T) & \text{for } y \geq 0 . \end{array}$$

The two probability distributions $G_1(y,T)$ and $G_2(y,T)$ can be derived from the stochastic process $F(x,t)$. However, the computations involved are very heavy, and will not be carried through here.

4.6 The reinsurance problem of paragraph 3.1 will now consist of determining the attainable profit distribution which maximizes the expression

$$\int_{-\infty}^{+\infty} u(y) \, dG'(y,T) .$$

This formulation of the problem is unsatisfactory on at least the following two points:

(i) The value of T is fixed, and this seems unnecessarily rigid.

We shall deal with this problem in Section 5.

(ii) It is assumed that the reinsurance arrangements made at the time 0 remain fixed for the whole contract period.

The latter assumption can probably be relaxed by formulating the problem in terms of the <u>dynamic programming</u> of Bellman (2). Although his approach to this kind of problems appears very promising, we shall not explore its possibilities in the present paper.

4.7 An interesting aspect of the result in paragraph 4.5 is that it combines Lundberg's ruin probability with the von Neumann-Morgenstern theory based on the Bernoulli principle discussed in paragraph 2.7. We can see this result either as a generalization of Lundberg's theory, or as a special case obtained from the general decision theory by introducing restraints of particular relevance to insurance. The latter point of view is probably the more fertile.

4.8 Lundberg's theory has found virtually no application to practical insurance problems. The reason is clearly that it ignores the profit distribution, and hence deals only with one of the two elements which, according to paragraph 4.5, are essential to the problem. Most writers on collective risk theory focus their attention on the limiting case where T tends to infinity. In this case it may be logically justified to ignore the profit distribution. It seems, however, that by going to the limit these authors lose all contact with practical insurance problems, since they really assume that insurance companies are completely disinterested in profits.

## 5. A Different Approach

5.1 In this section we shall very briefly outline another approach which may give more satisfactory solutions to some of the problems studied under the general heading "theory of risk". This new approach will be discussed in more detail in a forthcoming paper.

5.2 We consider first an insurance company with initial capital $S_o$, and we assume that this company receives a premium P for underwriting a portfolio with claim distribution F(x). This transaction will give the company a utility

$$U(S_o) = \int_0^\infty u(S_o + P - x)df(x) \quad .$$

We shall ignore that it may be possible for the company to increase this utility by suitable reinsurance arrangements. We next assume that things go well, so that when the contracts in this portfolio have expired, the company is left with a capital $S_1 > S_o$.

The company then decides to distribute an amount $s_1$ as dividend, so that it will enter the next underwriting period with capital $S_1 - s_1$. If in this second period the company underwrites a portfolio identical with the one in the first period, the utility will be

$$U(S_1 - s_1) = \int_0^\infty u(S_1 - s_1 + P - x)df(x) \quad .$$

5.3 It is clear that $U(S_1-s_1)$ will decrease with increasing $s_1$, so that high dividend payment appears as a disadvantage to the company. If, however, a high divident is considered desirable in itself, the company will have to balance the two elements.

If the company shall be able to make rational decisions in such cases, it must have a complete preference ordering over a set of pairs $\{s_1, U(S_1-s_1)\}$. This ordering can be represented by a utility function

$$V(s_1, U(S_1-s_1)) \quad .$$

The problem of the company is then reduced to determining the value of $s_1$ which maximizes this function.

It may be possible to extend these considerations to operations over several periods, so that our problem will be to determine values $s_1 \ldots s_T$ which maximize a function of the form

$$V(s_1, \ldots s_T, U(S_T - \Sigma s_t)) \quad .$$

5.4 With the considerations in the preceding paragraph we stepped on virgin soil, which appears very fertile. At present we can, however, only conjecture how this soil must be tilled if it shall yield solutions to the problems we have discussed.

It seems that a rational solution of our problem will require a preference ordering over dividend sequences of the type $\{s_1 \ldots s_t \ldots s_T\}$. This should present no mathematical difficulty, since such preference orderings over "commodity bundles" have been defined and used with considerable success in economic theory.

If T is finite, it is possible to make use of the Bernoulli principle to extend the definition to sequences where the elements are stochastic variables. It is, however, desirable to remove the "finite horizon" restriction on T, and this seems to involve considerable mathematical difficulties.

5.5 The first broad and systematic study of this problem seems to be one published by Koopmans (10) less than three years ago. The application Koopmans has in mind in choice of consumption levels at different points of time. For this application it is natural to postulate "impatience", i.e. that a sequence such as {3,2,2,1} is always preferred to sequences of the type {2,2,2,2} and {1,2,2,3}. Koopmans shows that with this postulate an acceptable preference ordering can exist over a set of infinite non-stochastic sequences.

The impatience element does not seem to be particularly relevant in insurance. On the contrary most statements from insurance companies seem to indicate that preferences are just the reverse of that indicated in the example above, i.e. the aim is a steady, or a steadily increasing dividend rate.

It is an open question whether such preferences can be formalized, and if they are consistent with a complete ordering over a set of infinite dividend sequences.

## 6. Conclusions

6.1 In this paper we have indicated that fairly advanced, and partly new mathematical methods may be required to solve some of the problems which intrigue actuaries today. It may, however, be useful to pause for a moment and ask if we are not engaging in a wild goose chase by developing such methods and putting them to application in our work.

6.2 The simplest solution to the problem in paragraph 5.2 would be to distribute a dividend $s_1 = S_1 - S_o$ whenever $S_1 > S_o$, and to distribute nothing if $S_1 < S_o$.

The traditional objection to this apparently sensible dividend policy is that it will lead to violent fluctuations in the dividend rate. However, why can we not accept such fluctuations as a fact of life?

Experts on collective risk theory may be horrified at this dividend policy, and point out that it will give a ruin probability equal to one. However, is it really a catastrophy that an insurance company is virtually certain to become insolvent at some time in the infinite future?

6.3 The main purpose of this paper has been to point out that the mathematical tools which are necessary to solve some of our problems seem to be available in non-actuarial literature.

There can be no excuse for continuing to attack our problems with inadequate tools. The purpose of these concluding remarks is merely to call for some reflection before we pick up the proper tools and set to work. It may be that the problems we have discussed should be formulated in another way, and solved with entirely different methods.

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