

A THEORY OF THE PSEUDOSPECTRUM
AND ITS APPLICATION TO NONSTATIONARY
DYNAMIC ECONOMETRIC MODELS

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Errata sheet for Research Memorandum No. 52

<u>Page</u>	<u>Line</u>	<u>From</u>	<u>To</u>
		+ from above - from below	
7	+ 10	$r(v) = E (x_t x_{t+v})$	$r(v) = E \{x_t \overline{x_{t+v}}\}$
7	+ 12	$\sum_{v=-2N}^{2N} (1 - \frac{ v }{2N+1}) r(v) e^{-i\omega v}$	$\sum_{v=-2N}^{2N} (1 - \frac{ v }{2N+1}) r(v) e^{-i\omega v}$
7	- 2	$\lim_{N \rightarrow \infty} F(\omega) \omega'; N$	$\lim_{N \rightarrow \infty} F(\omega - \omega'; N)$
7	- 1	$\frac{\sum_{v=-2N}^{2N} e^{-iv\omega} r(v)}{v=-2N}$	$\sum_{v=-2N}^{2N} e^{-iv\omega} r(v)$
9	Note 3 - 3	$r(v) = \sum_{t= v +1}^N E \{x_t \overline{x_{t+v}}\}$	$r(v) = \frac{1}{N} \sum_{t= v +1}^N E \{x_t \overline{x_{t+v}}\}$
10	- 1	a complex number	a complex function of time.
15	+ 5	$\int_{\omega-\pi}^{\omega+\pi} p_{x_j x_k}(\omega) \omega' p_{d_j d_k}(\omega') d\omega'$	$\int_{\omega-\pi}^{\omega+\pi} p_{x_j x_k}(\omega - \omega') p_{d_j d_k}(\omega') d\omega'$
17	Note 5 - 1	value of $\frac{v}{u}$	value of $\tan^{-1} \frac{v}{u}$
	- 2	value of $\frac{v}{u}$	value of $\tan^{-1} \frac{v}{u}$
	- 3	value of $\frac{v}{u}$	value of $\tan^{-1} \frac{v}{u}$
	- 4	value of $\frac{v}{u}$	value of $\tan^{-1} \frac{v}{u}$

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19	- 1	between $e^{i\phi_{jt}}$ and $e^{i\phi_{kt}}$ and $e^{-i\phi_{jt}}$ and $e^{-i\phi_{kt}}$	between $e^{i\phi_{jt}}$ and $e^{i\phi_{kt}}$ and between $e^{-i\phi_{jt}}$ and $e^{-i\phi_{kt}}$.
19	- 3	$+ \int_{\omega-\pi}^{\omega} p_{x_j x_k}(\omega) \omega' p_{\phi_j \phi_k}(\omega') d\omega'$	$+ \int_{\omega-\pi}^{\omega} p_{x_j x_k}(\omega-\omega') p_{\phi_j \phi_k}(\omega') d\omega'$
20	- 4	in $(\omega_0 - \epsilon, \omega_0 + \epsilon)$.	in $(\omega_0 - \epsilon, \omega_0 + \epsilon)$.
21	+ 1	$\int_{-\epsilon}^{\epsilon} p_{\phi_j \phi_k}(\omega') d\omega'$	$\int_{-\epsilon}^{\epsilon} p_{\phi_j \phi_k}(\omega') d\omega'$
29	+ 8	to a solvable form.	to a form that is solvable even when the parameters change over time.
37	Table 2	95% app. 3	95% approximately 3
8	+ 5	complex number.	complex time series.

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I. THE PURPOSE AND CONCLUSIONS OF THIS PAPER

The spectrum is a parameter of a stochastic process indicating its variance decomposition by frequencies. In mathematical analysis the spectrum has been defined for the stochastic process which has an infinitely long time domain and satisfies the stationarity conditions in the wide sense:

$$E(x_t) = \text{a constant}$$

$$E(x_t - E(x_t))^2 = \text{a constant}$$

$$E(x_t - E(x_t))(x_s - E(x_s)) \text{ depends only on the time distance } t-s.$$

The stationarity conditions mean, among other things, that the first two moments of the stochastic process do not depend upon the origin of time; i.e., the process is essentially "historyless" in the sense in which history is used in the social sciences.

On the other hand, in the practical applications of spectral analysis, samples of finite length have been used for the study of variance decomposition of the stochastic process which may not satisfy the stationarity conditions. As a first step toward filling the gap between the mathematical analysis and the practical applications, Blackman and Tukey⁽¹⁾ developed a method for making estimations of the spectrum of the stationary stochastic process by using samples of finite length. Thus, the practical applications of spectral analysis can be justified if the stochastic process is indeed stationary. In practice, however, spectral analysis has been used for the study of the stochastic process which is very unlikely to be stationary.

¹R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra. E. Parzen has also made an important contribution. E. Parzen, "Mathematical Considerations in the Estimation of Spectra," Technometrics, May, 1961.

Particularly in economic applications we cannot assume that the stationarity conditions hold. The institutional, behavioral, and technological backgrounds of economic time series are always changing and, therefore, there is no reason to believe that the first two moments of the economic stochastic process do not depend upon the origin of time.⁽²⁾

Under these conditions the finite length of the data has an entirely different implication than under the stationarity conditions. If the stationarity conditions hold, the estimate of the spectrum from samples of finite length can be considered as a kind of representation of the spectrum of the process having an infinitely long time domain. If the stationarity conditions do not hold, assuming that the "spectrum" of such a process would be meaningful, the estimate from samples of finite length cannot be considered a representation of the "spectrum", as long as the "spectrum" is a characteristic of the process over the entire time domain from $-\infty$ to $+\infty$. The only way to get around this difficulty is to recognize explicitly the historical nature of the economic stochastic process and to define the "spectrum" as a characteristic of the (nonstationary) process over a finite time domain. In other words, we must recognize that the "spectrum" would depend upon what finite portion of the time domain is taken, e.g., the "spectrum" during the second half of the 19th century would be different from the "spectrum" during the first half of the 20th century.

In the present paper we have defined the pseudospectrum and the cross-pseudospectrum for those processes which are not necessarily stationary and which are defined over a finite time domain. We shall show first that the basic characteristics of spectrum as defined in the usual mathematical analyses are maintained for our definition of pseudospectrum and cross-pseudospectrum. By this we mean the following: The pseudospectrum is the frequency decomposition of the mean of the

²Although we can apply any transformations to a given time series with the purpose of eliminating nonstationarity, it is difficult to ascertain whether or not the series comes from a stationary stochastic process after the transformation.

time-changing variance, just as the spectrum is the frequency decomposition of the (constant) variance. The mean of the time-changing autocovariance (or cross-covariance) and the pseudospectrum (or cross-pseudospectrum) is a Fourier transform pair, just as the (constant) autocovariance (or cross-covariance) and the spectrum (or cross-spectrum) is a Fourier transform pair. The coherence inequality holds for the cross-pseudospectrum and pseudospectra just as it does for the cross-spectrum and spectra. Further, if the stochastic process is defined over an infinitely long time domain and if the stationarity conditions do hold, then the pseudospectrum is identical to the spectrum.

It is important to note that pseudospectrum and cross-pseudospectrum are, apart from the complications due to the spectral window, the mathematical expectations of the estimates of spectra and cross-spectra which we would obtain if we had made this estimation without considering the problems of nonstationarity. (The electronic computers do not know whether the data come from a stationary stochastic process, and, regardless of the data given to them they "read out" some outputs.)

Although the exact specification of the types of nonstationarity is not possible at the present stage of economics, some observations and fairly reasonable hypotheses have been presented to characterize vaguely the nonstationarity of the economic stochastic process. In the present paper we study the nature of the pseudospectrum assuming that these vague observations and hypotheses are acceptable. The nature of the pseudospectrum then obtained can be used for the interpretation of the estimates of spectra and cross-spectra without considering the problems of nonstationarity. (The nature of the pseudospectrum derived from a too exact assumption as to the nonstationarity would not be useful because economics has not reached the stage in which an exact statement can be made as to the nonstationarity.)

First, we study the stochastic process in which the variance changes over time. The apparent variances of many economic time series for the United States show a secular change usually with a clear discontinuity about World War II. If d_t is a deterministic function of time and x_t is a stationary stochastic process so that $d_t x_t$ is a nonstationary stochastic process whose variance changes with time in

proportion to d_t^2 , then the pseudospectrum of $d_t x_t$ is the convolution of the pseudospectrum of d_t and the spectrum of x_t . (A similar formula also holds for the cross-pseudospectrum.) Thus, if d_t is either a smooth function of time or involves one or two jumps in addition to a smooth trend so that the pseudospectrum of d_t is concentrated in the very low frequencies, the pseudospectrum of $d_t x_t$ is roughly equal to the mean of the time-changing, instantaneous spectra, $d_t^2 P_x(\omega)$, where $P_x(\omega)$ is the spectrum of x_t .

Second, we proceed to the study of the stochastic process in which the amplitudes and the phases of different frequencies change with time. We know, for example, that the amplitudes of the seasonal variations of many economic time series have a downward trend and also that the phases of the seasonal variations change from year to year. Further, the phases of the cyclical components are very likely to be affected by external events such as wars. The convolution theorem that is similar to the one mentioned above holds in this case too. If the phase changes are uniform over all different frequencies, and if the phase changes are either smooth or involve one or two jumps in addition to a smooth trend, then the phase of the cross-pseudospectrum between two such processes $y_t^{(1)}$ and $y_t^{(2)}$ shows the average of the time-changing differences between the phases of the two processes. If the phase changes are uniform over only a narrow frequency interval, the phase of the cross-pseudospectrum is more complicated, but its meaning is straightforward. Let $\phi_\omega(t,v)$ represent the difference between the phase of $y_t^{(1)}$ at time t for frequency ω and the phase of $y_t^{(2)}$ at time $t+v$ for frequency ω . Then the phase of the cross-pseudospectrum at frequency ω is the double average of $\phi_\omega(t,v)$ over t and $|v|$, where the averaging over v is done with weights that are roughly in inverse proportion to v .

Third, we apply the concept of the pseudospectrum to the time series generated by the nonstationary dynamic econometric models. As for the pseudospectral matrix of the endogenous variable in an explosive dynamic econometric model with constant parameters, we can show that it is related to the spectral matrix of the random disturbance and the pseudospectral matrix of the exogenous variable, in the same way as

the spectral matrix of the endogenous variable in a stable dynamic econometric model is related to the spectral matrix of the random disturbance.

Finally, we study the stochastic process of the deviation from the equilibrium solution of the model, which deviation is used to represent business fluctuations. For the case in which the parameters of the model change over time, we can show that the pseudospectral matrix of the deviation is a convolution which involves the spectral matrix of the random disturbance and a transfer function of the parameters. However, even in the case in which the parameters are smooth functions of time and the spectrum of the random disturbance is smooth, the pseudospectral matrix has a more complex form than an average of the time-changing, instantaneous spectra. For a given time point t , let us consider the impacts of the random disturbances in the periods prior to t (and including t) upon the values of the deviation at time t . Let $C_{t,j}$ be the impact of the random disturbance in period $t-j$, and, let $\tilde{G}(t,\omega)$ be the transfer function of $C_{t,0}, C_{t,1}, \dots, C_{t,t-1}$ at frequency ω . If the parameters of the model are constant, $C_{t,0} = B^0, C_{t,1} = B^1, \dots, C_{t,j} = B^j, \dots$. The transfer function is independent of t and can be written as $G(\omega)$. The spectral matrix of the deviation is $G(\omega) P_U(\omega) G(\omega)^*$, where $P_U(\omega)$ is the spectral matrix of the random disturbance and $*$ means the Hermitian conjugate. If the parameters of the model are not constant, the pseudospectral matrix of the deviation is

$$\frac{1}{\pi N} \sum_t \left[\sum_v \tilde{G}(t,\omega) P_U(\omega) \tilde{G}(t+v,\omega)^* \frac{\sin \epsilon v}{v} \right]$$

under the same smoothness conditions, a double average of $G(\omega) P_U(\omega) G(\omega)^*$ over t and v .

So far we have mentioned the cases in which the pseudospectrum is directly amenable to a reasonable interpretation without the use of some specific a priori knowledge (e.g., the exact time function representing the changes in the parameters) about non-stationarity. There are many possible cases in which such an interpretation is not available. For example, as for the case of $d_t x_t$ mentioned above, if d_t is dominated by some irregular cycles, the pseudospectrum of $d_t x_t$ can be expressed as the result of smoothing the spectrum of x_t and then shifting it along the axis of ω .

This follows from the convolution theorem which holds for any movements of d_t and from the fact that the pseudospectrum of d_t is not concentrated in the very low frequencies around zero but rather spreads around a certain non-zero frequency. For the other models of nonstationarity treated above, basically the same convolution theorem holds for any types of changes of the parameters, although the relevant parameters vary from one model to another as described above. If the pseudospectra of the time series of the (changing) parameters are not concentrated in the very low frequencies, the interpretation of the spectral matrix is very difficult.

Thus the contribution of the present paper to economic applications of spectral analysis lies in the convolution theorem which holds for a broad class of non-stationary economic stochastic processes and which enables us to discern the cases in which a reasonable interpretation of the spectral matrix is available without the use of some specific a priori knowledge about nonstationarity from the cases in which such an interpretation is not possible.

II. PSEUDOSPECTRUM AND CROSS-PSEUDOSPECTRUM

One definition of the power spectrum (density) for a stationary stochastic process with zero mean is

$$f_x(\omega) = \lim_{N \rightarrow \infty} \frac{1}{2\pi(2N+1)} E \left\{ \left| \sum_{t=-N}^N x_t e^{-i\omega t} \right|^2 \right\}, \quad -\pi \leq \omega \leq \pi \quad (1)$$

where ω is the (angular) frequency, and x_t is the discrete time series data produced by the stochastic process. The power spectrum is the decomposition of the variance of the process in terms of frequency ω . (1) is equivalent to the more commonly used definition of the spectrum,

$$f_x(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} e^{-iv\omega} r(v) \quad (1')$$

where $r(v) = E(x_t x_{t+v})$. This is because

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{2\pi(2N+1)} E \left\{ \sum_{-N \leq t, s \leq N} x_t \bar{x}_s e^{-i\omega(t-s)} \right\} \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{v=-2N}^{2N} \left(1 - \frac{|v|}{2N+1}\right) r(v) e^{-i\omega v} \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega - \omega'; N) \sum_{v=-2N}^{2N} r(v) e^{-i\omega' v} d\omega' \end{aligned}$$

$$\left(\text{where } F(\omega - \omega'; N) = \sum_{v=-2N}^{2N} \left(1 - \frac{|v|}{2N+1}\right) e^{-i(\omega - \omega')v} \right)$$

and $\lim_{N \rightarrow \infty} F(\omega - \omega'; N)$ is a Delta function of $\omega - \omega'$,

$$= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{v=-2N}^{2N} e^{-iv\omega} r(v)$$

Let us consider a general (stationary or nonstationary) stochastic process $\{x_t\}$ and suppose that the data are available for a finite time period, $t=1, 2, \dots, N$. The pseudospectrum $p_x(\omega)$ of x_t is defined as

$$p_x(\omega) = \frac{1}{2\pi N} E \left\{ \left| \sum_{t=1}^N x_t e^{-i\omega t} \right|^2 \right\}, \quad -\pi \leq \omega \leq \pi \quad (2)$$

where x_t is considered, in general, as a complex number. For the purpose of our mathematical treatment, we find it convenient to work with the pseudospectrum for the variance about zero rather than the mean. This is why the mean is not subtracted in (2).

(a) The integral of $p_x(\omega)$ over the frequencies from $-\pi$ to π is the mean of the variance (about zero) over the given period.

$$\begin{aligned} \int_{-\pi}^{\pi} p_x(\omega) d\omega &= \frac{1}{2\pi N} \int_{-\pi}^{\pi} E \left\{ \left| \sum_{t=1}^N x_t e^{-i\omega t} \right|^2 \right\} d\omega \\ &= \frac{1}{2\pi N} \int_{-\pi}^{\pi} \sum_{t=1}^N \sum_{s=1}^N E \{ x_t \bar{x}_s \} e^{-i\omega(t-s)} d\omega \\ &= \frac{1}{N} \sum_{t=1}^N E \{ |x_t|^2 \} \end{aligned} \quad (3)$$

(b) Let us define the pseudoautocovariance $r(v)$ for lag v of a general process x_t as

$$r(v) = \frac{1}{N-v} \sum_{t=1}^{N-v} E \{ x_t \bar{x}_{t+v} \} \quad \text{for } v \geq 0$$

$$r(v) = \frac{1}{N-|v|} \sum_{t=|v|+1}^N E \{ x_t \bar{x}_{t+v} \} \quad \text{for } v < 0$$

Obviously $r(v) = \overline{r(-v)}$. Then the pseudospectrum $p_x(\omega)$ and the pseudoautocovariance $r(v)$ are a Fourier transform pair when $r(v)$ is weighted by $1 - \frac{|v|}{N}$, (3)

i.e.,

$$p_x(\omega) = \frac{1}{2\pi} \sum_{v=-(N-1)}^{N-1} \left(1 - \frac{|v|}{N}\right) r(v) e^{-i\omega v} \quad (4)$$

and

$$\left(1 - \frac{|v|}{N}\right) r(v) = \int_{-\pi}^{\pi} p_x(\omega) e^{i\omega v} d\omega \quad \text{for } v = 0, \pm 1, \dots, \pm (N-1) \quad (5)$$

Proof:

$$\begin{aligned} p_x(\omega) &= \frac{1}{2\pi N} E \left\{ \sum_t \sum_s x_t \overline{x_s} e^{-i\omega(t-s)} \right\} \\ &= \frac{1}{2\pi N} \left[\sum_{v=0}^{N-1} \sum_{t=1}^{N-v} E \left\{ x_t \overline{x_{t+v}} \right\} e^{-i\omega v} + \sum_{v=-(N-1)}^{-1} \sum_{t=|v|+1}^N E \left\{ x_t \overline{x_{t+v}} \right\} e^{-i\omega v} \right] \\ &= \frac{1}{2\pi} \sum_{v=-(N-1)}^{N-1} \left(1 - \frac{|v|}{N}\right) r(v) e^{-i\omega v} \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} p_x(\omega) e^{i\omega v} d\omega &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{v'=-N+1}^{N-1} \left(1 - \frac{|v'|}{N}\right) r(v') e^{i\omega(v'-v)} d\omega \\ &= \left(1 - \frac{|v|}{N}\right) r(v) \end{aligned}$$

Alternatively $r(v) = \frac{1}{N} \sum_{t=1}^{N-v} E \{ x_t \overline{x_{t+v}} \}$ for $v \geq 0$

$$r(v) = \frac{1}{N} \sum_{t=|v|+1}^N E \{ x_t \overline{x_{t+v}} \} \quad \text{for } v < 0$$

may be used. Then $p_x(\omega)$ and $r(v)$ are a Fourier transform pair without using the weight, $1 - \frac{|v|}{N}$.

(c) The pseudospectrum $p_x(\omega)$ is the mathematical expectation of sample estimates of spectra ⁽⁴⁾ obtained by using $(1 - \frac{|v|}{N})$ as the lag window for the estimation.

When we define the sample estimates of $r(v)$ and $p_x(\omega)$ respectively as

$$\left\{ \begin{array}{l} \hat{r}(v) = \frac{1}{N-v} \sum_{t=1}^{N-v} x_t \bar{x}_{t+v} \quad v \geq 0 \\ \hat{r}(v) = \frac{1}{N-|v|} \sum_{t=|v|+1}^N x_t \bar{x}_{t+v} \quad v < 0 \end{array} \right.$$

and

$$\hat{p}_x(\omega) = \sum_{v=-(N-1)}^{N-1} (1 - \frac{|v|}{N}) \hat{r}(v) e^{-i\omega v},$$

the mathematical expectation of $\hat{p}_x(\omega)$ is

$$E(\hat{p}_x(\omega)) = \sum_{v=-(N-1)}^{N-1} (1 - \frac{|v|}{N}) E(\hat{r}(v)) e^{-i\omega v} = p_x(\omega).$$

(d) If x_t is stationary with zero mean and the time period is infinite in length, the pseudospectrum converges to the spectrum, as can be seen from (1) and (2).

We can now define the pseudospectrum $p_d(\omega)$ of a deterministic process d_t in the given time period $t = 1, \dots, N$ as

$$p_d(\omega) = \frac{1}{2\pi N} \left| \sum_{t=1}^N d_t e^{-i\omega t} \right|^2$$

where d_t is considered, in general, a complex number. If a nonstationary process

⁴As Bartlett pointed out, the variance of $\hat{p}_x(\omega)$ is not reduced to zero when $N \rightarrow \infty$ even if x_t is stationary. This point, however, is totally irrelevant to the pseudospectrum of nonstationary process. The pseudospectrum is defined for and dependent upon a given finite length of time period covering a specific portion of our economic history; thus the consistency of the estimate $\hat{p}_x(\omega)$ is not a relevant problem.

$y_t = d_t + x_t$, where d_t is a deterministic process and x_t is a stationary stochastic process with zero mean, the pseudospectrum of y_t is obviously the sum of the pseudospectra of d_t and x_t . Especially when d_t is a trend, it is well known that the pseudospectrum of d_t is significant only at the low frequencies. This will be elaborated on later in Section VI.

For the multi-variate stochastic process we can define the cross-pseudospectrum and pseudocovariance. The cross-pseudospectrum between two stochastic processes, $x_t^{(1)}$ and $x_t^{(2)}$ is

$$p_{12}(\omega) = \frac{1}{2\pi N} E \left\{ \sum_{t=1}^N \sum_{s=1}^N x_t^{(1)} \bar{x}_s^{(2)} e^{-i\omega(t-s)} \right\}$$

and the cross-pseudocovariance is

$$r_{12}^{(v)} = \frac{1}{N-v} \sum_{t=1}^{N-v} E \left\{ x_t^{(1)} \bar{x}_{t+v}^{(2)} \right\} \quad v \geq 0$$

$$r_{12}^{(v)} = \frac{1}{N-|v|} \sum_{t=|v|+1}^N E \left\{ x_t^{(1)} \bar{x}_{t+v}^{(2)} \right\} \quad v < 0.$$

(e) The cross-pseudospectrum $p_{12}(\omega)$ and the cross-pseudocovariance $r_{12}(v)$ are a Fourier transform pair when $r_{12}(v)$ is weighted by $1 - \frac{|v|}{N}$.

The cross-pseudospectrum between two deterministic functions of time, λ_t and μ_t , is defined as

$$\frac{1}{2\pi N} \sum_t \sum_s \lambda_t \mu_s e^{-i\omega(t-s)}$$

In general, let $x_t = (x_t^{(1)}, x_t^{(2)}, \dots, x_t^{(m)})$ be a m-variate stochastic vector process. Then

$$R(v) = [r_{jk}(v)]$$

is the pseudocovariance matrix, and $R(-v) = \overline{R(v)}$.

The matrix of pseudospectral and cross-pseudospectral functions, i.e., the pseudospectral matrix, is defined as

$$P(\omega) = [p_{jk}(\omega)]$$

where

$$\begin{aligned} p_{jk}(\omega) &= \frac{1}{2\pi} \sum_{v=-(N-1)}^{N-1} \left(1 - \frac{|v|}{N}\right) r_{jk}(v) e^{-i\omega v} \\ &= \frac{1}{2\pi N} E \left\{ \sum_{t=1}^N \sum_{s=1}^N x_t^{(j)} x_s^{(k)*} e^{-i\omega(t-s)} \right\}. \end{aligned}$$

Another representation of the matrix $P(\omega)$ is

$$P(\omega) = \frac{1}{2\pi N} E \left\{ \sum_t \sum_s X_t X_s^* e^{-i\omega(t-s)} \right\}$$

where X_t and X_s^* are the vectors of m components, and where X_s^* is defined as the complex conjugate of the transpose of X_s , i.e., the Hermitian conjugate of X_s . This representation will be used extensively in Sections IV and V.

(f) The coherence inequality holds for the pseudospectra and cross-pseudospectrum.

Let $f_j(\omega)$ and $f_k(\omega)$ be the Fourier transform of $x_t^{(j)}$ and $x_t^{(k)}$, i.e.

$$f_j(\omega) = \sum_{t=1}^N x_t^{(j)} e^{-i\omega t}$$

$$f_k(\omega) = \sum_{t=1}^N x_t^{(k)} e^{-i\omega t}$$

Then from the Schwartz's inequality, the inequality

$$E \{ |f_j(\omega) f_k(\omega)|^2 \} \leq E \{ |f_j(\omega)|^2 \} E \{ |f_k(\omega)|^2 \}$$

holds. Then

$$\mathbb{E} \left\{ \left| \frac{1}{2\pi N} f_j(\omega) f_k(\omega) \right|^2 \right\} \leq \mathbb{E} \left\{ \frac{1}{2\pi N} |f_j(\omega)|^2 \right\} \mathbb{E} \left\{ \frac{1}{2\pi N} |f_k(\omega)|^2 \right\},$$

$$\text{i.e.,} \quad |p_{jk}(\omega)|^2 \leq p_{jj}(\omega) \cdot p_{kk}(\omega).$$

This is the coherence inequality.

III. THE PSEUDOSPECTRAL MATRIX OF THE STOCHASTIC PROCESS

WITH CHANGING AMPLITUDES AND CHANGING PHASES

The present section deals with the stochastic process in which either the amplitudes or the phases of different frequencies change over time. It also serves as the mathematical background for Sections IV and V.

(a) The pseudospectral matrix of the stochastic process in which the amplitudes change uniformly over different frequencies but the phases do not change.

The nonstationary stochastic process in which the amplitudes change uniformly over different frequencies but the phases do not change can be represented by the product of some function of time and some stationary stochastic process.

Theorem 1. Let $d_t^{(j)}$ ($j=1, \dots, M$) be deterministic processes and $x_t^{(j)}$ ($j=1, \dots, M$) stationary stochastic processes with $E(x_t^{(j)}) = 0$. Then a column vector of the stochastic process

$$y_t = \begin{bmatrix} y_t^{(1)} \\ \vdots \\ y_t^{(M)} \end{bmatrix} = \begin{bmatrix} d_t^{(1)} x_t^{(1)} \\ \vdots \\ d_t^{(M)} x_t^{(M)} \end{bmatrix}$$

, $t = 1, 2, \dots, N$, has the pseudospectral matrix $P_y(\omega) = [p_{y_j y_k}(\omega)]$

$$\text{where } p_{y_j y_k}(\omega) = \int_{\omega-\pi}^{\omega+\pi} p_{x_j x_k}(\omega-\omega') p_{d_j d_k}(\omega') d\omega' \quad (6)$$

$p_{x_j x_k}(\omega)$ and $p_{d_j d_k}(\omega)$ are respectively the cross-spectrum between $x_t^{(j)}$ and $x_t^{(k)}$, and the cross-pseudospectrum between $d_t^{(j)}$ and $d_t^{(k)}$.

Proof.

$$\begin{aligned}
 p_{y_j y_k}(\omega) &= \frac{1}{2\pi N} E \left\{ \sum_t \sum_s d_t^{(j)} x_t^{(j)} \bar{d}_s^{(k)} \bar{x}_s^{(k)} e^{-i\omega(t-s)} \right\} \\
 &= \frac{1}{2\pi N} \sum_t \sum_s d_t^{(j)} \bar{d}_s^{(k)} E \left\{ x_t^{(j)} \bar{x}_s^{(k)} \right\} e^{-i\omega(t-s)} \\
 &= \frac{1}{2\pi N} \int_{-\pi}^{\pi} \sum_t \sum_s d_t^{(j)} \bar{d}_s^{(k)} p_{x_j x_k}(\omega') e^{-i\omega'(t-s)} e^{-i\omega(t-s)} d\omega' \\
 &= \int_{-\pi}^{\pi} p_{x_j x_k}(\omega') p_{d_j d_k}(\omega - \omega') d\omega' \\
 &= \int_{\omega - \pi}^{\omega + \pi} p_{x_j x_k}(\omega) p_{d_j d_k}(\omega') d\omega'
 \end{aligned}$$

q.e.d.

The pseudospectrum of $y_t^{(j)} = d_t^{(j)} x_t^{(j)}$ is the convolution of the pseudospectrum of $d_t^{(j)}$ and the spectrum of $x_t^{(j)}$, i.e., a sort of weighted moving average of the spectrum of $x_t^{(j)}$ by using the pseudospectrum of $d_t^{(j)}$ as the weights.

(i) If $x_t^{(j)}$ is a white noise, ϵ_t , with its variance σ^2 , i.e., $y_t^{(j)} = d_t^{(j)} \epsilon_t$, then the pseudospectrum of $y_t^{(j)}$ is

$$p_{y_j}(\omega) = \frac{\sigma^2}{2\pi} \frac{\sum_t |d_t^{(j)}|^2}{N}$$

This means that $p_{y_j}(\omega)$ is the mean of the changing, instantaneous spectra of $d_t^{(j)} \epsilon_t$, i.e., $\frac{\sigma^2}{2\pi} |d_t^{(j)}|^2$ over time.

(ii) Suppose that the pseudospectrum (for the variance about zero) of $d_t^{(j)}$ is significant only in the frequency band $[-\epsilon, \epsilon]$, where $\epsilon > 0$ is a certain small number. (This is possible when $d_t^{(j)}$ is either a very smooth function of time or $d_t^{(j)}$ involves one or two jumps in addition to the trend. The fact that $p_{d_j}(\omega)$ is a pseudospectrum for the variance about zero is important in judging the

plausibility of the concentration of this spectrum in the low frequencies in economic studies.) Further assume that $p_{x_j}(\omega)$ is smooth so that

$$p_{x_j}(\omega) = p_{x_j}(\omega_0) \text{ for all } \omega\text{'s in } (\omega_0 - \epsilon, \omega_0 + \epsilon) .$$

Then

$$\begin{aligned} p_{y_j}(\omega_0) &\approx p_{x_j}(\omega_0) \int_{-\epsilon}^{+\epsilon} p_{d_j}(\omega') d\omega' \\ &\approx p_{x_j}(\omega_0) \int_{-\pi}^{\pi} p_{d_j}(\omega') d\omega' \\ &\approx p_{x_j}(\omega_0) \frac{\sum |d_t(j)|^2}{N} \end{aligned}$$

This is again the mean of the changing spectra.

(iii) If the movements of $d_t(j)$ are dominated by irregular cycles, the pseudospectrum of $y_t(j) = d_t(j)x_t(j)$ can be expressed as the result of smoothing the spectrum of x_t and then shifting it along the axis of ω . When only one sample of $y_t(j)$ is available and when no a priori information about d_t is given, there would be no easy way to interpret the pseudospectrum of $y_t(j)$.

(b) The pseudospectral matrix of the stochastic process where the phases change uniformly over different frequencies but the amplitudes do not change.

Any real, stationary stochastic process with continuous spectrum can be represented as

$$\begin{aligned} x_t(j) &= \int_{-\pi}^{\pi} e^{i\omega t} \{u_j(\omega) - iv_j(\omega)\} d\omega \quad (7) \\ &= 2 \int_0^{\pi} (\cos \omega t) u_j(\omega) d\omega + 2 \int_0^{\pi} (\sin \omega t) v_j(\omega) d\omega, \quad j=1, \dots, M \end{aligned}$$

where

$$u_j(\omega) = u_j(-\omega),$$

$$v_j(\omega) = -v_j(-\omega),$$

$$E(u_j(\omega)u_j(\omega')) = E(v_j(\omega)v_j(\omega')) = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ \frac{1}{2}p_{x_j}(\omega) & \text{if } \omega = \omega' \end{cases}$$

$$E(u_j(\omega)v_j(\omega')) = 0 \quad \text{for all } \omega, \omega'$$

$$E(u_j(\omega)u_k(\omega')) = E(v_j(\omega)v_k(\omega')) = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ \frac{1}{2}R_e [p_{x_j x_k}(\omega)] & \text{if } \omega = \omega' \end{cases}$$

$$E(u_j(\omega)v_k(\omega')) = -E(v_j(\omega)u_k(\omega')) = \begin{cases} 0 & \text{if } \omega \neq \omega' \\ \frac{1}{2}I_m [p_{x_j x_k}(\omega)] & \text{if } \omega = \omega' \end{cases}$$

($R_e []$ and $I_m []$ mean respectively the real part and the imaginary part of [].)

The phase of (7) is defined as ⁽⁵⁾

$$\tan^{-1} \frac{v_j(\omega)}{u_j(\omega)} .$$

⁵To be more precise, take the principal value of $\frac{v}{u}$ if $u \geq 0, v \geq 0$

π - the principal value of $\frac{v}{u}$ if $u \leq 0, v \geq 0$

$\pi +$ the principal value of $\frac{v}{u}$ if $u \leq 0, v \leq 0$

$2\pi -$ the principal value of $\frac{v}{u}$ if $u \geq 0, v \leq 0$

We are concerned with a real, stationary process of which the spectrum is identical to $p_{x_j}(\omega)$ and of which the phase of frequency ω is greater than that of $x_t^{(j)}$ by $\varphi_j(\omega)$. If $\varphi_j(\omega) = -\varphi_j(-\omega)$, then such a process is represented by

$$x_t^{(j)}\{\varphi_j(\omega)\} = \int_{-\pi}^{\pi} e^{i\omega t} e^{-i\varphi_j(\omega)} (u_j(\omega) - iv_j(\omega)) d\omega \quad (8)$$

$$= 2 \int_0^{\pi} \cos \omega t [\cos \varphi_j(\omega) u_j(\omega) - \sin \varphi_j(\omega) v_j(\omega)] d\omega \quad (8')$$

$$+ 2 \int_0^{\pi} \sin \omega t [\sin \varphi_j(\omega) u_j(\omega) + \cos \varphi_j(\omega) v_j(\omega)] d\omega .$$

Actually (8) is a real process, because $\cos \varphi_j(\omega) u_j(\omega) - \sin \varphi_j(\omega) v_j(\omega)$ is an even function of ω , and $\sin \varphi_j(\omega) u_j(\omega) + \cos \varphi_j(\omega) v_j(\omega)$ is an odd function of ω .

The spectrum of $x_t^{(j)}\{\varphi_j(\omega)\}$ is identical to the spectrum of $x_t^{(j)}$ because

$$E\{|e^{-i\varphi_j(\omega)}(u_j(\omega) - iv_j(\omega))\|^2\} = E\{u_j(\omega)^2\} + E\{v_j(\omega)^2\} .$$

The phase of $x_t^{(j)}\{\varphi_j(\omega)\}$ is greater than the phase of $x_t^{(j)}$ by $\varphi_j(\omega)$ at the frequency ω because

$$\tan^{-1} \frac{\sin \varphi_j(\omega) u_j(\omega) + \cos \varphi_j(\omega) v_j(\omega)}{\cos \varphi_j(\omega) u_j(\omega) - \sin \varphi_j(\omega) v_j(\omega)} = \tan^{-1} \frac{v_j(\omega)}{u_j(\omega)} + \varphi_j(\omega) .$$

The nonstationary, real stochastic processes, where the phases change with time, but the amplitudes do not change, can be represented as

$$y_t^{(j)} = \int_{-\pi}^{\pi} e^{i\omega t} e^{-i\varphi_{jt}(\omega)} (u_j(\omega) - iv_j(\omega)) d\omega \quad (9)$$

$$\begin{aligned}
 &= 2 \int_0^{\pi} \cos \omega t [\cos \varphi_{jt}(\omega) u_j(\omega) - \sin \varphi_{jt}(\omega) v_j(\omega)] d\omega \\
 &+ 2 \int_0^{\pi} \sin \omega t [\sin \varphi_{jt}(\omega) u_j(\omega) + \cos \varphi_{jt}(\omega) v_j(\omega)] d\omega .
 \end{aligned} \tag{9}$$

In the present section (b), we shall study the special case of $y_t^{(j)}$,

$$\varphi_{jt}(\omega) = \varphi_{jt} \quad \text{for} \quad 0 < \omega \leq \pi$$

$$\varphi_{jt}(\omega) = 0 \quad \omega = 0 \tag{10}$$

$$\varphi_{jt}(\omega) = -\varphi_{jt} \quad -\pi \leq \omega < 0$$

i.e., the case where the phases change uniformly over different frequencies.

Corollary 1. The nonstationary stochastic processes of $y_t^{(j)}$ and $y_t^{(k)}$ ($j=1, \dots, M, k=1, \dots, M$) defined by (9) and (10) have the cross-pseudospectrum $p_{y_j y_k}(\omega)$,

$$\begin{aligned}
 p_{y_j y_k}(\omega) &= \int_{\omega}^{\omega+\pi} p_{x_j x_k}(\omega-\omega') p_{\varphi_j \varphi_k}^-(\omega') d\omega' \\
 &+ \int_{\omega-\pi}^{\omega} p_{x_j x_k}(\omega)\omega') p_{\varphi_j \varphi_k}(\omega') d\omega'
 \end{aligned} \tag{11}$$

where $p_{\varphi_j \varphi_k}(\omega)$ and $p_{\varphi_j \varphi_k}^-(\omega)$ are the cross-pseudospectra, respectively, between $e^{i\varphi_j t}$ and $e^{i\varphi_k t}$ and $e^{-i\varphi_j t}$ and $e^{-i\varphi_k t}$.

Proof.

Since

$$y_t^{(j)} = e^{-i\varphi_{jt}} \int_{-\pi}^0 e^{i\omega't} \{u_j(\omega') - iv_j(\omega')\} d\omega' + e^{i\varphi_{jt}} \int_0^{\pi} e^{i\omega't} \{u_j(\omega') - iv_j(\omega')\} d\omega',$$

$$\begin{aligned} E(y_t^{(j)} \overline{y_s^{(k)}}) &= e^{-i(\varphi_{jt} - \varphi_{ks})} \int_{-\pi}^0 e^{i\omega'(t-s)} p_{x_j x_k}(\omega') d\omega' \\ &+ e^{i(\varphi_{jt} - \varphi_{ks})} \int_0^{\pi} e^{i\omega'(t-s)} p_{x_j x_k}(\omega') d\omega'. \end{aligned}$$

Therefore,

$$\begin{aligned} p_{y_j y_k}(\omega) &= \frac{1}{2\pi N} \sum_t \sum_s E(y_t^{(j)} \overline{y_s^{(k)}}) e^{-i\omega(t-s)} \\ &= \int_{-\pi}^0 p_{x_j x_k}(\omega') \overline{p_{\varphi_j \varphi_k}(\omega - \omega')} d\omega' + \int_0^{\pi} p_{x_j x_k}(\omega') p_{\varphi_j \varphi_k}(\omega - \omega') d\omega' \\ &= \int_{\omega}^{\omega+\pi} p_{x_j x_k}(\omega - \omega') \overline{p_{\varphi_j \varphi_k}(\omega')} d\omega' + \int_{\omega-\pi}^{\omega} p_{x_j x_k}(\omega - \omega') p_{\varphi_j \varphi_k}(\omega') d\omega'. \end{aligned}$$

q.e.d.

Suppose that the norms of $\overline{p_{\varphi_j \varphi_k}(\omega)}$ and $p_{\varphi_j \varphi_k}(\omega)$ are relatively significant only in the frequency band $[-\epsilon, \epsilon]$ where $\epsilon > 0$ is a certain small number. This is possible when the phase changes of $y_t^{(j)}$ and $y_t^{(k)}$ are smooth, or, otherwise involve one or two jumps. Further assume that $p_{x_j x_k}(\omega)$ is smooth so that

$$p_{x_j x_k}(\omega) = p_{x_j x_k}(\omega_0) \quad \text{for all } \omega\text{'s in } (\omega_0 - \epsilon, \omega_0 + \epsilon).$$

Then, if $\pi \geq \omega_0 > \epsilon$, then the first integral of (11) for $\omega = \omega_0$ is not significant, because the interval $[\omega_0, \omega_0 + \pi]$ does not include the frequency band $[-\epsilon, \epsilon]$. Therefore,

$$p_{y_j y_k}(\omega_0) \approx p_{x_j x_k}(\omega_0) \int_{-\epsilon}^{\epsilon} p_{\varphi_j \varphi_k}(\omega') d\omega' \quad (12)$$

$$\approx p_{x_j x_k}(\omega_0) \int_{-\pi}^{\pi} p_{\varphi_j \varphi_k}(\omega') d\omega' = p_{x_j x_k}(\omega_0) \cdot \frac{1}{N} \sum_t e^{i(\varphi_{jt} - \varphi_{kt})}$$

If $-\pi \leq \omega_0 < -\epsilon$, then the second integral of (11) for $\omega = \omega_0$ is not significant and, we obtain

$$p_{y_j y_k}(\omega_0) \approx p_{x_j x_k}(\omega_0) \int_{-\epsilon}^{\epsilon} p_{\overline{\varphi_j \overline{\varphi_k}}(\omega') d\omega' \quad (12')$$

$$\approx p_{x_j x_k}(\omega_0) \cdot \frac{1}{N} \sum_t e^{-i(\varphi_{jt} - \varphi_{kt})}$$

Thus, the phase of the cross-pseudospectrum between $y_t^{(j)}$ and $y_t^{(k)}$ differs from the phase of the cross-spectrum between $x_t^{(j)}$ and $x_t^{(k)}$ by

$$\tan^{-1} \frac{-\sum_t \sin(\varphi_{jt} - \varphi_{kt})}{\sum_t \cos(\varphi_{jt} - \varphi_{kt})} \quad (\omega_0 > \epsilon) \quad (13)$$

or

$$\tan^{-1} \frac{\sum_t \sin(\varphi_{jt} - \varphi_{kt})}{\sum_t \cos(\varphi_{jt} - \varphi_{kt})} \quad (\omega_0 < -\epsilon) \quad (13')$$

In order to understand the meaning of the phase of the cross-pseudospectrum between $y_t^{(j)}$ and $y_t^{(k)}$, let us consider a special case where $x_t^{(j)} \equiv x_t^{(k)}$ for all t 's. Then, the phase difference between $y_t^{(j)}$ and $y_t^{(k)}$ is solely due to the fact that $y_t^{(j)}$ in (9) involves $e^{-i\varphi_{jt}(\omega)}$ whereas $y_t^{(k)}$ involves $e^{-i\varphi_{kt}(\omega)}$. Indeed, $(\varphi_{jt} - \varphi_{kt})$ is the instantaneous phase difference at t

between $y_t^{(j)}$ and $y_t^{(k)}$. When $x_t^{(j)} \equiv x_t^{(k)}$, $p_{x_j x_k}(\omega_0)$ in (12) or (12') is real, and, the phase of $p_{y_j y_k}(\omega_0)$ is given by (13) and (13'). Therefore, the phase of the cross-pseudospectrum is a kind of average (over time) of the instantaneous phase difference. (13) or (13') shows that the average of any two angles θ_1 and θ_2 must be defined as

$$\tan^{-1} \frac{\sin\theta_1 + \sin\theta_2}{\cos\theta_1 + \cos\theta_2} .$$

(c) The pseudospectral matrix of the stochastic process where both the amplitudes and the phases change differently over different frequencies.

The real nonstationary stochastic process, $y_t^{(j)}$, where both the amplitudes and phases change differently over different frequencies, can be generated from a real stationary stochastic process

$$x_t^{(j)} = \int_{-\pi}^{\pi} e^{i\omega t} (u_j(\omega) - i v_j(\omega)) d\omega$$

by

$$y_t^{(j)} = \int_{-\pi}^{\pi} e^{i\omega t} d_t^{(j)}(\omega) (u_j(\omega) - i v_j(\omega)) d\omega \quad (14)$$

where

$$d_t^{(j)}(\omega) = a_{jt}(\omega) e^{-i\varphi_{jt}(\omega)}$$

$$\varphi_{jt}(\omega) = -\varphi_{jt}(-\omega)$$

$$a_{jt}(\omega) = a_{jt}(-\omega) .$$

$a_{jt}(\omega)$ represents the amplitude of the frequency ω at time t , and $\varphi_{jt}(\omega)$ the phase difference of the frequency ω between $x_t^{(j)}$ and $y_t^{(j)}$ at time t .

Corollary 2. The cross-pseudospectrum between the real nonstationary stochastic processes $y_t^{(j)}$ and $y_t^{(k)}$ defined by (14) can be represented as

$$p_{y_j y_k}(\omega) = \int_{-\pi}^{\pi} p_{x_j x_k}(\omega') p_{f_j f_k}(\omega, \omega') d\omega' \quad (15)$$

where

$$p_{f_j f_k}(\omega, \omega') = \frac{1}{2\pi N} \sum_t \sum_s f_t^{(j)}(\omega') \overline{f_s^{(k)}(\omega')} e^{-i\omega(t-s)}$$

and

$$f_t^{(j)}(\omega') = d_t^{(j)}(\omega') e^{i\omega' t}$$

$$f_s^{(k)}(\omega') = d_s^{(k)}(\omega') e^{i\omega' s}$$

(The proof is omitted,)

$p_{f_j f_k}(\omega, \omega')$ is the cross-pseudospectrum at frequency ω between the deterministic processes $d_t^{(j)}(\omega') e^{i\omega' t}$ and $d_s^{(k)}(\omega') e^{i\omega' s}$.

Suppose (i) that $d_t^{(j)}(\omega')$ and $d_s^{(k)}(\omega')$ are smooth functions of time (for any ω') so that $|p_{f_j f_k}(\omega, \omega')|$ is significant only for the values of ω that are near or equal to ω' , which means

$$\int_{-\pi}^{\pi} p_{f_j f_k}(\omega_0, \omega') d\omega' \approx \int_{\omega_0 - \epsilon}^{\omega_0 + \epsilon} p_{f_j f_k}(\omega_0, \omega') d\omega'$$

and (ii) that $p_{x_j x_k}(\omega')$ is smooth so that

$$p_{x_j x_k}(\omega') \approx p_{x_j x_k}(\omega_0) \text{ for any } \omega' \text{ such that}$$

$$\omega_0 - \epsilon \leq \omega' \leq \omega_0 + \epsilon$$

Then from (15) we obtain

$$p_{y_j y_k}(\omega_0) \approx p_{x_j x_k}(\omega_0) \int_{\omega_0 - \epsilon}^{\omega_0 + \epsilon} p_{f_j f_k}(\omega_0, \omega') d\omega'$$

Further, if we can assume, in addition to (i) and (ii), that (iii)

$$\left. \begin{aligned} d_t^{(j)}(\omega') &= d_t^{(j)}(\omega_0) \\ d_s^{(k)}(\omega') &= d_s^{(k)}(\omega_0) \end{aligned} \right\} \text{ for any } \omega' \text{ in } (\omega_0 - \epsilon, \omega_0 + \epsilon)$$

then

$$p_{y_j y_k}(\omega_0) \approx p_{x_j x_k}(\omega_0) \cdot \frac{1}{\pi N} \sum_t \sum_s d_t^{(j)}(\omega_0) \overline{d_s^{(k)}(\omega_0)} \frac{\sin \epsilon(t-s)}{t-s}$$

The phase of $p_{y_j y_k}(\omega_0)$ differs from the phase of the $p_{x_j x_k}(\omega_0)$ by

$$\tan^{-1} \frac{\sum_t \sum_s a_{jt}(\omega_0) a_{ks}(\omega_0) \sin\{\varphi_{jt}(\omega_0) - \varphi_{ks}(\omega_0)\} \frac{\sin \epsilon(t-s)}{t-s}}{\sum_t \sum_s a_{jt}(\omega_0) a_{ks}(\omega_0) \cos\{\varphi_{jt}(\omega_0) - \varphi_{ks}(\omega_0)\} \frac{\sin \epsilon(t-s)}{t-s}} \quad (16)$$

(16) is a generalization of (13) and (13'). Notice that $\frac{\sin \epsilon(t-s)}{t-s}$ is inversely related to $|t-s|$ where ϵ is small.

In the special case in which the amplitude does not change at any frequency (16) becomes

$$\tan^{-1} \frac{\sum_t \sum_s \sin\{\varphi_{jt}(\omega_0) - \varphi_{ks}(\omega_0)\} \frac{\sin \epsilon(t-s)}{t-s}}{\sum_t \sum_s \cos\{\varphi_{jt}(\omega_0) - \varphi_{ks}(\omega_0)\} \frac{\sin \epsilon(t-s)}{t-s}} \quad (16')$$

Let $\nu = t-s$ and $\varphi_{jk, \omega_0}(t, \nu) = \varphi_{jt}(\omega_0) - \varphi_{ks}(\omega_0)$. Then (16') is a double

average of $\varphi_{jk, \omega_0}(t, v)$ over t and v , where the averaging over v is done with the weights $\frac{\sin v}{v}$.

IV. THE PSEUDOSPECTRAL MATRIX OF THE EXPLOSIVE DYNAMIC ECONOMETRIC MODEL

Most of dynamic econometric models are sets of linear difference equations such as

$$\begin{aligned}
 & B_0 \eta_t + B_1 \eta_{t-1} + \dots + B_p \eta_{t-p} \\
 & = \Gamma_0 \xi_t + \Gamma_1 \xi_{t-1} + \dots + \Gamma_q \xi_{t-q} + V_t
 \end{aligned}
 \tag{17}$$

where

$$\eta_t = \begin{bmatrix} \eta_t^{(1)} \\ \vdots \\ \eta_t^{(k)} \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \xi_t^{(1)} \\ \vdots \\ \xi_t^{(k)} \end{bmatrix}, \quad \text{and} \quad V_t = \begin{bmatrix} v_t^{(1)} \\ \vdots \\ v_t^{(k)} \end{bmatrix}$$

represent respectively the endogenous variables, exogenous variables and random disturbances. B_0, B_1, \dots, B_p and $\Gamma_0, \Gamma_1, \dots, \Gamma_q$ are $k \times k$ matrices of the parameters. $|B_0| \neq 0$ is assumed.

If the exogenous variables are removed from (17) and the stability condition holds, (17) represents a stationary stochastic process of the endogenous variable η_t . It is well known⁽⁶⁾ that the spectral matrix of η_t is related to $P_V(\omega)$ by

$$P_\eta(\omega) = (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p})^{-1} P_V(\omega) (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p})^{*-1} \tag{18}$$

⁶ Whittle, P., "The analysis of multiple stationary time series." Journal of the Royal Statistical Society (B) 15, (1953) pp. 125-139.

If the stability condition does not hold, η_t represents an explosive stochastic process. In this case, if we take a finite time period, we can prove that the pseudospectral matrix of η_t is related to the pseudospectral matrix of ξ_t and the spectral matrix of V_t just in the same way as $P_\eta(\omega)$ is related to $P_V(\omega)$ in (18).

Theorem 2. In the system (17), let us assume (i) that the end effects for p time lag of η_t and for q time lag of ξ_t are negligible for the period, $t=1, \dots, N$, (ii) that ξ_t is deterministic and V_t is stationary with $E(V_t) = 0$, and (iii) that

$$|B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p}| \neq 0, \quad -\pi \leq \omega \leq \pi.$$

Then

$$P_\eta(\omega) = (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p})^{-1} \{(\Gamma_0 + \Gamma_1 e^{-i\omega} + \dots + \Gamma_q e^{-i\omega q}) P_\xi(\omega) (\Gamma_0 + \Gamma_1 e^{-i\omega} + \dots + \Gamma_q e^{-i\omega q})^* + P_V(\omega)\} (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p})^{*-1}$$

where $P_\eta(\omega)$, $P_\xi(\omega)$ and $P_V(\omega)$ are respectively the pseudospectral matrices of η_t , ξ_t , and V_t .

Proof.

The Fourier transforms of the left and right hand sides of (17) are, respectively,

$$\begin{aligned} & \sum_{t=1}^N (B_0 \eta_t + B_1 \eta_{t-1} + \dots + B_p \eta_{t-p}) e^{-i\omega t} \\ & \approx (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p}) \sum_{t=1}^N \eta_t e^{-i\omega t} \end{aligned}$$

and

$$\begin{aligned} & \sum_{t=1}^N (\Gamma_0 \xi_t + \Gamma_1 \xi_{t-1} + \dots + \Gamma_q \xi_{t-q} + V_t) e^{-i\omega t} \\ & \approx (\Gamma_0 + \Gamma_1 e^{-i\omega} + \dots + \Gamma_q e^{-i\omega q}) \sum_{t=1}^N \xi_t e^{-i\omega t} + \sum_{t=1}^N V_t e^{-i\omega t} . \end{aligned}$$

This is because, for example, if the end effect for k time lag is negligible,

$$\begin{aligned} \sum_{t=1}^N B_k \eta_{t-k} e^{-i\omega t} &= \sum_{t=1}^N B_k e^{-i\omega k} \eta_{t-k} e^{-i\omega(t-k)} \\ &\approx B_k e^{-i\omega k} \sum_{t=1}^N \eta_t e^{-i\omega t} . \end{aligned}$$

The pseudospectral matrices of both sides of (17) are, respectively,

$$\begin{aligned} & \frac{1}{2\pi N} E \{ (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p}) \sum_{t=1}^N \eta_t e^{-i\omega t} \sum_{s=1}^N \eta_s e^{i\omega s} (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p})^* \} \\ &= (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p}) P_\eta(\omega) (B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p})^* . \end{aligned}$$

and

$$(\Gamma_0 + \Gamma_1 e^{-i\omega} + \dots + \Gamma_q e^{-i\omega q}) P_\xi(\omega) (\Gamma_0 + \Gamma_1 e^{-i\omega} + \dots + \Gamma_q e^{-i\omega q})^* + P_V(\omega) .$$

Since

$$|B_0 + B_1 e^{-i\omega} + \dots + B_p e^{-i\omega p}| \neq 0 \quad \text{for } \omega \text{ in } [-\pi, \pi] ,$$

we can get the theorem.

q.e.d.

V. PSEUDOSPECTRAL MATRIX OF THE DEVIATION FROM THE EQUILIBRIUM SOLUTION
IN THE DYNAMIC MODEL WITH CHANGING PARAMETERS

In econometric studies of business fluctuations the deviation of the solution of (17) from its equilibrium solution are frequently used to represent business fluctuations. We shall consider this deviation in the dynamic econometric model whose parameters change over time.

To treat the model with changing parameters, let us change the form of the system from (17) to a solvable form. Any linear difference equation of order p can be replaced by a system of p first order difference equations in p variables⁽⁷⁾, if we define the new variables as follows:

$$\begin{aligned}
 Y_t &\equiv \begin{bmatrix} \eta_t \\ \vdots \\ \eta_{t-p+1} \end{bmatrix} & U_t &\equiv \begin{bmatrix} B_0^{-1} V_t \\ \vdots \\ 0 \end{bmatrix} \\
 Z_t &\equiv \begin{bmatrix} z_{1t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} & &= \begin{bmatrix} B_0^{-1} \Gamma_0 \xi_t + B_0^{-1} \Gamma_1 \xi_{t-1} + \dots + B_0^{-1} \Gamma_q \xi_{t-q} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (18) \\
 B &\equiv \begin{bmatrix} -B_0^{-1} B_1 & , & -B_0^{-1} B_2 & , & \dots & -B_0^{-1} B_{p-1} & , & -B_0^{-1} B_p \\ I_k & & 0 & & & 0 & & 0 \\ 0 & & I_k & & & 0 & & 0 \\ \vdots & & \vdots & & & \vdots & & \vdots \\ \vdots & & \vdots & & & \vdots & & \vdots \\ 0 & & 0 & & & I_k & & 0 \end{bmatrix}
 \end{aligned}$$

where I_k is a $k \times k$ identity matrix and B is a $kp \times kp$ matrix.

⁷Paul A. Samuelson: Foundations of Economic Analysis Harvard University Press, (1947) Mathematical Appendix B.

The system (17) can now be represented by

$$Y_t - BY_{t-1} = Z_t + U_t \quad (19)$$

Let us consider the model with constant parameters by using the representation (19) in order to clarify the concepts that we use in this section.

The solution of (19) with Y_0 as the initial condition is

$$Y_t = B^t Y_0 + \sum_{m=0}^{t-1} B^m (Z_{t-m} + U_{t-m}) \quad (20)$$

Let X_t be the deviation from the equilibrium solution, i.e.,

$$X_t \equiv Y_t - (B^t Y_0 + \sum_{m=0}^{t-1} B^m Z_{t-m}) = \sum_{m=0}^{t-1} B^m U_{t-m}.$$

This is a moving average of random disturbances U_t . Then we can get the pseudospectral matrix of X_t ,

$$\begin{aligned} P_X(\omega) &= \frac{1}{2\pi N} E \sum_{t=1}^N \sum_{s=1}^N \sum_{m=0}^{t-1} \sum_{n=0}^{s-1} B^m (U_{t-m} U_{s-n}^*) B^{n*} e^{-i\omega(t-s)} \\ &= \frac{1}{2\pi N} \sum_t \sum_s \sum_m \sum_n B^m \left[\int_{-\pi}^{\pi} P_U(\omega') e^{i\omega'(t-m-s+n)} d\omega' \right] B^{n*} e^{-i\omega(t-s)}. \end{aligned}$$

Put

$$G(t, \omega') \equiv \sum_{m=0}^{t-1} B^m e^{-i\omega' m} \quad (21)$$

$$H(\omega - \omega', \omega') \equiv \sum_{t=1}^N G(t, \omega') e^{-i(\omega - \omega') t} \quad (22)$$

We shall call $H(\omega-\omega',\omega')$ the double transfer function of the parameters B^0, B^1, \dots, B^{t-1} . Then we obtain

$$P_X(\omega) = \frac{1}{2\pi N} \int_{-\pi}^{\pi} H(\omega-\omega',\omega') P_U(\omega') H(\omega-\omega',\omega')^* d\omega' \quad (23)$$

or

$$P_X(\omega) = \frac{1}{2\pi N} \int_{\omega-\pi}^{\omega+\pi} H(\omega',\omega-\omega') P_U(\omega-\omega') H(\omega',\omega-\omega')^* d\omega' .$$

In order to understand the meaning of (21), (22), and (23), let us assume that $\{B^m\}$ converges to zero when $m \rightarrow \infty$ and that, since X_t is now stationary, the time domain extends from $t = -\infty$ to $t = +\infty$, and accordingly the initial time point $t = 0$ is carried back to $t = -\infty$. Then

$$1) \quad G(t,\omega') = \sum_{m=0}^{\infty} B^m e^{-i\omega' m} = G(\omega') .$$

This is the transfer function of the coefficients of U 's .

$$2) \quad H(\omega-\omega',\omega') \text{ is a Delta function of } \omega-\omega' \text{ such that}$$

$$H(\omega-\omega',\omega') = 0 \text{ except } \omega-\omega' = 0 .$$

3) Further (23) becomes

$$P_X(\omega) = \left(\sum_{m=0}^{\infty} B^m e^{-i\omega m} \right) P_U(\omega) \left(\sum_{n=0}^{\infty} B^{n*} e^{i\omega n} \right) \quad (24)$$

$$= G(\omega) P_U(\omega) G(\omega)^* ,$$

which is a well known formula of the spectral matrix of a multi-variable moving average process.

Going back to the general case where the above assumptions need not hold, we can regard $G(t, \omega')$ as the instantaneous transfer function at t of the moving average with the weights $B^{t-1}, B^{t-2}, \dots, B^0$. We can also regard $H(\omega - \omega', \omega')$ for a given ω' as the transfer function of $G(t, \omega')$ evaluated at $\omega - \omega'$. If $G(t, \omega')$ is a smooth function of time t , $|H(\omega - \omega', \omega')|^2$ is significant only in the neighborhood around $\omega - \omega' = 0$.

Now let us consider the dynamic econometric model with the parameters changing over time. The system (19) should be replaced by

$$Y_t - B_{t-1} Y_{t-1} = Z_t + U_t \quad (25)$$

Let

$$\bar{Y}_t = B_{t-1} \bar{Y}_{t-1} + \dots + B_0 Y_0.$$

Then the deviation from the equilibrium solution with Y_0 as the initial condition is

$$\tilde{x}_t \equiv Y_t - (\bar{Y}_t + \sum_{m=0}^{t-1} C_{t,m} Z_{t-m}) = \sum_{m=0}^{t-1} C_{t,m} U_{t-m} \quad (26)$$

where

$$C_{t,m} = \begin{cases} I, & \text{at } m=0 \\ B_{t-1} \cdot B_{t-2} \cdot \dots \cdot B_{t-m}, & \text{at } m=1, \dots, t-1. \end{cases}$$

Then we can get the pseudospectral matrix of the deviation from the equilibrium solution as a convolution of the spectrum of the random disturbance U_t and the double transfer function of the parameters.

Theorem 3. The pseudospectral matrix $P_{\tilde{X}}(\omega)$ of \tilde{X}_t in (26) is,

$$P_{\tilde{X}}(\omega) = \frac{1}{2\pi N} \int_{\omega-\pi}^{\omega+\pi} \tilde{H}(\omega-\omega', \omega') P_U(\omega-\omega') \tilde{H}(\omega-\omega', \omega')^* d\omega' \quad (27)$$

where

$$\tilde{H}(\omega', \omega'-\omega) \equiv \sum_{t=1}^N \tilde{G}(t, \omega-\omega') e^{-i\omega't}$$

$$\tilde{G}(t, \omega-\omega') \equiv \sum_{m=0}^{t-1} C_{t,m} e^{-i(\omega-\omega')m}$$

Proof.

$$\begin{aligned} P_{\tilde{X}}(\omega) &= \frac{1}{2\pi N} E \sum_{t=1}^N \sum_{s=1}^N \sum_{m=0}^{t-1} \sum_{n=0}^{s-1} C_{t,m} U_{t-m} U_{s-n}^* C_{s,n}^* e^{-i\omega(t-s)} \\ &= \frac{1}{2\pi N} \sum_t \sum_s \sum_m \sum_n C_{t,m} \int_{-\pi}^{\pi} P_U(\omega') e^{i\omega'(t-m-s+n)} d\omega' \cdot C_{s,n}^* e^{-i\omega(t-s)} \\ &= \frac{1}{2\pi N} \int_{-\pi}^{\pi} \sum_t \tilde{G}(t, \omega') P_U(\omega') \sum_s \tilde{G}(s, \omega')^* e^{-i(\omega-\omega')(t-s)} d\omega' \\ &= \frac{1}{2\pi N} \int_{-\pi}^{\pi} \tilde{H}(\omega-\omega', \omega') P_U(\omega') \tilde{H}(\omega-\omega', \omega')^* d\omega' \\ &= \frac{1}{2\pi N} \int_{\omega-\pi}^{\omega+\pi} \tilde{H}(\omega-\omega', \omega') P_U(\omega-\omega') \tilde{H}(\omega-\omega', \omega')^* d\omega' \end{aligned}$$

q.e.d.

The pseudospectral matrix of the deviation of η_t in (17) from its equilibrium solution can be obtained as a portion of $P_{\tilde{X}}(\omega)$.

The meaning of \tilde{G} , \tilde{H} and (27) are now obvious from the explanation given for G , H , and (23). Furthermore, we can derive the pseudospectral matrices of the following case.

If (i) $P_U(\omega')$ is smooth so that for some small number $\epsilon > 0$

$$P_U(\omega') = P_U(\omega_0)$$

where $|\omega_0 - \omega'| < \epsilon$, (ii) $\tilde{G}(t, \omega)$ is a slowly changing function of time so that

$$\tilde{H}(\omega - \omega'; \omega') \cdot \tilde{H}(\omega - \omega'; \omega')^*$$

is negligible when $|\omega_0 - \omega'| > \epsilon$, and (iii) $\tilde{G}(t, \omega) = \tilde{G}(t, \omega')$ for any t and ω such that $|\omega_0 - \omega'| < \epsilon$, then we obtain

$$P_{\tilde{X}}(\omega) \approx \frac{1}{\pi N} \sum_t \sum_s \tilde{G}(t, \omega) P_U(\omega) \tilde{G}(s, \omega)^* \frac{\text{sinc}(t-s)}{t-s} \quad (28)$$

The above procedure can be summarized as follows. For a given time point t , let us consider the impacts of the random disturbances in the periods prior to t (including t) upon the values of \tilde{X} at time t . These impacts can be represented by $C_{t,0}, C_{t,1}, C_{t,2}, \dots, C_{t,t-1}$ ($C_{t,j}$ is the impact of the random disturbance in period $t-j$). The transfer function of $C_{t,0}, C_{t,1}, \dots, C_{t,t-1}$ at frequency ω is $\tilde{G}(t, \omega)$. — If the parameters of the model are constant, $C_{t,0} = B^0, C_{t,1} = B^1, C_{t,2} = B^2, \dots$. The transfer function is $G(\omega)$ and the spectral matrix is $G(\omega)P_U(\omega)G(\omega)^*$. — If the parameters of the model are not constant, the pseudospectral matrix is

$$\frac{1}{\pi N} \sum_t \left[\sum_{\nu} \tilde{G}(t, \omega) P_U(\omega) \tilde{G}(t+\nu, \omega)^* \frac{\text{sinc}\nu}{\nu} \right]$$

It should be noted that the pseudospectral matrix of the deviation can not be considered the mean of the time changing instantaneous spectra, i.e.,

$$\frac{1}{N} \sum_t \left(\sum_{m=0}^{t-1} B_t^m e^{-i\omega m} \right) P_U(\omega) \left(\sum_{n=0}^{t-1} B_t^n e^{-i\omega n} \right)^*$$

This is because $C_{t,m}$ can be very different from B_t^m , particularly when B_t has a trend, and also because (28) has a double summation over t and s .

If the cyclical changes rather than the constants or trends dominate the changes of the parameters of the dynamic econometric model, then the interpretation of the pseudospectrum is difficult, just as in the case of the cyclically changing variance discussed in Section III.

VI. PSEUDOSPECTRUM OF A CONSTANT AND A TREND

In the previous sections we have mentioned that the pseudospectra of a constant and a trend are significant only in the narrow frequency band around zero. In the present section we investigate how narrow the band really is. Since the pseudospectrum of a trend depends upon where the origin of time is, it is difficult to present results that have significant generality. Furthermore, the pseudospectrum of $a + bt$ is not generally equal to the sum of the pseudospectra of a and bt unless the origin of time is centered over the time span. This diminishes the significance of the following presentation to some extent. Thus we merely try to present a clue which might be useful for forming some idea as to the pseudospectra of constant and trend.

Let us define $p_{c,n}(\omega)$ and $p_{t,n}(\omega)$ respectively, as

$$p_{c,n}(\omega) = \frac{1}{2\pi N} \left| \sum_{t=0}^{N-1} e^{-i\omega t} \right|^2$$

$$p_{t,n}(\omega) = \frac{1}{2\pi N} \left| \sum_{t=0}^{N-1} t e^{-i\omega t} \right|^2 .$$

Let us define ω_0 for each preassigned value of α in such a way that

$$\int_{-\omega_0}^{\omega_0} p_{c,n}(\omega) d\omega = \alpha \int_{-\pi}^{\pi} p_{c,n}(\omega) d\omega = \alpha$$

$$\int_{-\omega_0}^{\omega_0} p_{t,n}(\omega) d\omega = \alpha \int_{-\pi}^{\pi} p_{t,n}(\omega) d\omega = \alpha \frac{(N-1)(2N-1)}{6}$$

We have studied the range of α between 90% and 99% and the range of N

between 200 and 1,000. When α is fixed, ω_0 depends upon N . This relation between ω_0 and N for a given α is almost an inverse proportion within the regions of α and N which we studied. Therefore, we can use $\frac{2\pi}{N}$ as the unit of frequency. As for $p_{c,n}(\omega)$, the relation between ω_0 and N for any given α is so nearly an inverse proportion that we have presented fairly exact figures in Table 1.⁽⁸⁾ For $p_{t,n}(\omega)$, the relation is slightly more complicated, and Table 2 for $p_{t,n}(\omega)$ does not have the same degree of exactness as Table 1.

$p_{c,n}(\omega)$

α	90%	95%	99%
$\frac{\omega_0}{2\pi N}$.9	2.1	10.2

Table 1.

$p_{t,n}(\omega)$

α	90%	95%	99%
$\frac{\omega_0}{2\pi N}$	1.6	app.3	12-15

Table 2.

It might be interesting to note that $p_{t,m}(\omega)$ near zero frequency has a violent movement, and we had to estimate the integral

$$\int_{-\omega_0}^{\omega_0} p_{t,n}(\omega)d\omega \text{ by subtracting } \int_{-\pi}^{-\omega_0} p_{t,n}(\omega)d\omega + \int_{\omega_0}^{\pi} p_{t,n}(\omega)d\omega \text{ from } \int_{-\pi}^{\pi} p_{t,n}(\omega)d\omega .$$

In view of equations (6), (11), (15) and (27) for the pseudospectra of non-stationary processes, and in view of the fact that the pseudospectrum is the average of all sample estimates of the spectrum (apart from the complications due

⁸ If we were treating continuous time, this relation should be exactly an inverse proportion. However, time element is discrete in the pseudospectrum.

to the spectral window) obtained with no regard to the problem of stationarity, an interesting question is how wide the unit of the frequency interval should be in order to make this average of sample estimates independent of the slow changes in the parameters of the stochastic process. Obviously the small numerical study summarized in the above two tables is not sufficient to give an answer to this question for the general class of slow movements of parameters. Let us suppose, however, that Table 2 represents the nature of the pseudospectra of the linear trend in general. Then we can answer the above question. When m represents the number of lags used in the estimation of the spectrum, the unit of frequency interval is $2\pi \times \frac{1}{2m}$. If d_t in Sec. III, and $G(t, \omega')$ in Sec. V are linear functions of time, then 95% of the variance of d_t or $\tilde{G}(t, \omega')$ is contained in the frequency interval having the width $6 \times \frac{2\pi}{N}$, where N is the number of data. If $2\pi \times \frac{1}{2m}$ is equal to $6 \times \frac{2\pi}{N}$, i.e., $N = 12m$, then the pseudospectra represented in the equations (6), (11) (15) and (27) become practically independent of the parameter changes.