A SPECTRAL ANALYSIS OF BUSINESS CYCLE INDICATORS: LEAD-LAG IN TERMS OF ALL TIME POINTS

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Sections IV, V, VI, and VII partially overlap with some sections of Chapter XII of C. W. J. Granger in association with M. Hatanaka, An Analysis of Economic Time Series; but the present paper incorporates some important revisions that have been made due to G. H. Moore's advice since the above book was written. There are also some additions and other changes.

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Summary

The author has tried to accomplish:

1) an elementary explanation of spectral analysis and cross-spectral analysis on the basis of the concepts and techniques that are presently a part of the economist's common knowledge, e.g., regression analysis, time series decomposition, and macro-dynamics, rather than on the basis of Cramer's representation of stationary stochastic process derived from advanced mathematics, or, electric engineer's "jargon" such as signal, impulse, and response (however, our explanation treats only the concepts used in the spectral and the cross-spectral analysis; it does not treat technicalities of the estimation procedures);

2) a comparison between the estimates of the lead-lag by the National Bureau method involving peaks and troughs and by the cross-spectral analysis involving the data for all time points (the comparison uses about 30 business cycle indicators);

3) a study of the lead-lag relationships involved in the cyclical adjustment of labor inputs to changes in outputs in manufacturing in order to obtain reliable leading indicators that promptly reveal changes in expectation (this study uses time series for the labor inputs in the set of business cycle indicators); and

4) the derivation of some crude, general statements as to the significance of the cyclical component from the spectral and cross-spectral analysis of important economic time series as to different sectors of the economy, which the business cycles indicators are.

Actually 3) and 4) have not been very successful, but, a suggestion for model-building is derived from 4). As for 2) the estimates of the lead-lag by two different methods have come very close in most cases. There are still several ambiguous points in the results, and a more extensive study is suggested.

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I. THE SETTING OF THE PROBLEM

In the National Bureau method for studying business cycle indicators, peaks and troughs are marked for each individual time series in which the cyclical component is visible. This is done in order to define the expansion and contraction phases of the specific cycles of individual time series and also to derive from them the peak and trough reference dates of business cycles. The lead or lag of business cycle indicators is measured as the time difference between their peaks and the peak reference dates, and between their troughs and the trough reference dates. Since many stabilization policies must be executed once (or, at most, a few times) at the right time during the expansion or contraction phase in an attempt to mitigate the economic stresses about the reference peaks and troughs, the time difference between the peak or trough of each indicator and the possible peak or trough reference date is an important piece of information for policy-planners.

Selection of the peaks and troughs for each individual time series involves a great many difficulties. Exact dating is meaningless because of random disturbances in the economic system, and difficult because of estimation errors. Further, some stabilization policies, the most important of which would probably be open-market operations, must be executed constantly, or their possible use must be considered constantly. For these policies the lead-lag relations in terms of all time points is more desirable than the lead-lag in terms of peaks and troughs only.[1] It is an open question whether or not the two kinds of lead-lag are equal.

In the present paper a study of the lead-

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1. Many economists have presented the hypothesis that the lead-lag relation would vary among the different phases of business cycles. Although this is an important hypothesis, the author does not know of any statistical techniques for testing this hypothesis appropriately. The cross-spectral analysis does not seem to be applicable to this problem.

lag of business cycle indicators in terms of all time points rather than of peaks and troughs alone is presented. A comparison between the two types of lead-lag is also undertaken.

The traditional method for the estimation of lead-lag in terms of all time points has been to estimate the auto-correlation coefficients. This method does not seem to be applicable to the present study of business cycle indicators because we would like to obtain the lead-lag in terms of the cyclical components and not in terms of any other components. Cross-spectral analysis was designed by Prof. John Tukey to handle aperiodic time series data and to estimate the lead-lag in any component defined as the weighted average of frequency intervals. In this paper, we present an application of the cross-spectral analysis to the study of business cycle indicators.

Spectral and cross-spectral analyses are not only new statistical techniques but also provide a way to represent various economic concepts used in the study of business fluctuations. Since the business cycle indicators form a sample of important time series covering different areas of our economy, we shall try to examine these concepts by using the results of spectral and cross-spectral analyses of many business cycle indicators.[2]

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2. The present study has another purpose which is not discussed here. Estimation of the spectrum and cross-spectrum involves some filtering techniques, and the selection of appropriate filtering techniques depends upon knowledge of the specific shape of the spectrum which we wish to estimate. Since the business cycle indicators are a sample of important time series in different sectors of our economy, it is hoped that we can develop filtering techniques that are appropriate for the study of a wide category of economic time series. The results of this study are in Chapter XII of An Analysis of Economic Time Series, by C. W. J. Granger in association of M. Hatanaka, 1953.

II. Elementary Explanation of Spectrum and Cross-spectrum

A. Introduction

Time series decompositions have been used in economics for a long period. According to a traditional decomposition method, a time series is decomposed into trend, cyclical component, seasonal variation, and irregular component. The ultimate purpose of this decomposition has been the study of the relationship among different economic time series with regard to the cyclical component. Spectral analysis performs a kind of time series decomposition indicating the relative importance of the different components. Cross-spectral analysis analyzes the relationship among different time series, component by component.

The concept of spectrum was developed before World War II; but it was through the efforts of M. S. Bartlett and J. W. Tukey, working independently of each other in 1949, that this mathematical concept became amenable to statistical estimation procedures. Since then Tukey has developed many techniques, such as cross-spectral analysis, for the application of spectral analyses to practical purposes. In order to understand the concept of the cross-spectrum, it is necessary to grasp the meaning of spectrum.

The explanation of spectral analysis in standard reference books[3] is written primarily for mathematicians. The explanation is based upon Cramér's representation of stationary stochastic processes, the understanding of which requires an exceedingly advanced knowledge

3. For example, Grenander, U. and Rosenblatt, M., Statistical Analysis of Stationary Time Series, and Hannan, E. J., Time Series Analysis are written definitely for highly-trained mathematicians. Granger, C. W. J. in association with M. Hatanaka, op.cit., is more elementary, but it is written primarily for statisticians who might work with economists, or for highly mathematically-minded economists.

of mathematics. The author has attempted elsewhere to base the explanation of spectral analysis <u>on the knowledge of statistics which many economists have</u>;[4] to be more concrete, 1) simple trigonometry and the theory of complex numbers as used, for example, in elementary economic dynamics, and 2) a basic knowledge of regression analysis. We shall present here such an exposition of spectral analysis emphasizing those aspects which are important in studying the lead-lag relation.

B. Sinusoid

Sinusoid is defined as

$$A \cos (2\pi f t + \theta) \tag{1}$$

which can also be expressed as

$$a \cos 2\pi f t + b \sin 2\pi f t \quad . \tag{1'}$$

The relation between (1) and (1') is [5]

$$A = \sqrt{a^2 + b^2} \qquad \theta = -\tan^{-1} \frac{b}{a} \tag{2a}$$

$$a = A \cos \theta \qquad b = -A \sin \theta . \tag{2b}$$

Time, t , is the argument of this function (1) or (1'), and , f, A and θ in (1) or, alternatively, f , a and b in (1') are the parameters. When a and b are given, A and θ can be obtained from them by (2a); and when A and θ are given, a and b can be

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4. Part I of Hatanaka, M., <u>Spectral Analyses of Business Fluctuations</u>, now being written. The present section is a concise excerpt from these chapters.

5. In obtaining $\tan^{-1} \frac{b}{a}$,
take the principal value of $\tan^{-1} \frac{b}{a}$ if $a \geq 0$, $b \geq 0$
take π minus the principal value of $\tan^{-1} \frac{b}{a}$ if $a \leq 0$, $b \geq 0$
take π plus the principal value of $\tan^{-1} \frac{b}{a}$ if $a \leq 0$, $b \leq 0$
take 2π minus the principal value of $\tan^{-1} \frac{b}{a}$ if $a \geq 0$, $b \leq 0$

obtained from them by (2b) . f is called <u>frequency</u> and is equal to the inverse of the period of the sinusoid. This is because when t moves from zero to $\frac{1}{f}$, $2\pi ft$ moves from zero to 2π and $\cos 2\pi ft$ completes its one cycle. A is <u>amplitude</u> and θ is <u>phase angle</u>. θ indicates the position of the sinusoid in relation to the origin of time by using angle. One can easily see from Fig. 1 that the diagram of $A \cos (2\pi ft + \theta)$ can be obtained by shifting the diagram of $A \cos 2\pi ft$ (i.e. $\theta = 0$) either to the left or to the right (depending on whether $\theta > 0$ or $\theta < 0$) by $-\frac{\theta}{2\pi f}$. Since $\frac{1}{f}$ is the period of the sinusoid, $-\frac{\theta}{2\pi}$ indicates what portion of the period this shift amounts to.

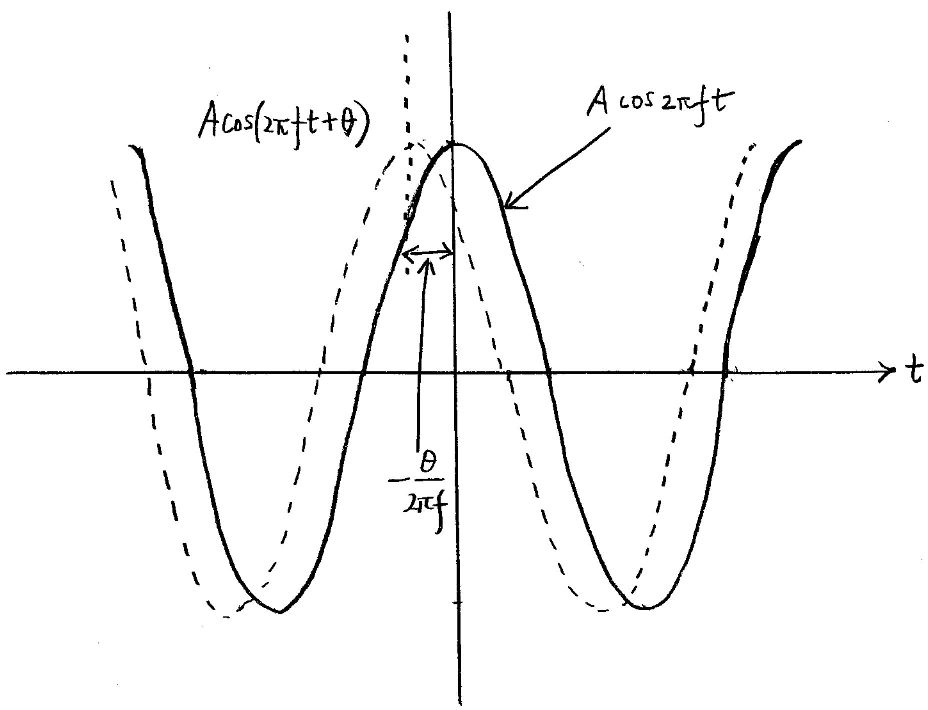

Fig. 1

Throughout the following presentation we assume that the data are discrete, i.e., they are defined in terms of some time unit such as a month. Then f is measured by that portion of one cycle movement which is contained in the given time unit. E.g., if the period is 12 months and a month is the time unit, $\frac{1}{12}$ cycle is contained in the time unit. Thus f is $\frac{1}{12}$ cycle per month.

C. Fourier Analysis

Since Fourier established what is now called Fourier's theorem for a periodic function of time many different versions of Fourier analysis have appeared. For our purpose, Fourier analysis can be regarded as a regression analysis of a given time series, $\{x_t\}$, $t = 1,2,\ldots,N$ upon the following set of sinusoids:

$$\begin{array}{lllll} a_1 \cos 2\pi \frac{1}{N} t + b_1 \sin 2\pi \frac{1}{N} t & \text{having the frequency} & \frac{1}{N} & & \\ a_2 \cos 2\pi \frac{2}{N} t + b_2 \sin 2\pi \frac{2}{N} t & \text{" \quad " \quad "} & \frac{2}{N} & & (3) \\ \cdots\cdots & \cdots & & & \\ a_{N/2} \cos 2\pi \frac{1}{2} t + b_{N/2} \sin 2\pi \frac{1}{2} t & \text{" \quad " \quad "} & \frac{1}{2} . & & \end{array}$$

(We ignore a minor point as to whether N is odd or even.) $\{x_t\}$ need <u>not</u> be periodic in the sense that $x_{t+t_0} = x_t$ for some t_0 and all t, i.e. $x_{1+t_0} = x_1$, $x_{2+t_0} = x_2$, ... The reason why we exclude from (3) sinusoids with frequencies higher than 1/2 is that, as long as we are treating discrete data, these higher frequencies are indistinguishable from the frequencies which are included in the above set of sinusoids. Since this may be considered a technicality we shall not elaborate on it.

It can be proved that the estimates of a's and b's in (3) by the least squares method are given by[6]

$$\hat{a}_j = \frac{2}{N} \sum_t x_t \cos 2\pi \frac{j}{N} t \qquad \hat{b}_j = \frac{2}{N} \sum_t x_t \sin 2\pi \frac{j}{N} t \qquad (j = 1, 2, \ldots, \frac{N}{2}) . \tag{4}$$

The variance of $\{x_t\}$,

$$\frac{1}{N} \sum_{t=1}^{N} (x_t - \bar{x})^2 \quad \text{where} \quad \bar{x} = \frac{1}{N} \sum_{t=1}^{N} x_t ,$$

can be decomposed into $N/2$ parts associated with $N/2$ sinuosids in (3). That part of the variance of $\{x_t\}$ which can be accounted for by the j^{th} sinusoid with the frequency point, j/N , can be represented as [7]

$$\frac{\hat{a}_j^2}{2} + \frac{\hat{b}_j^2}{2} = \frac{2}{N^2} \left[\left(\sum_t x_t \cos 2\pi \frac{j}{N} t \right)^2 + \left(\sum_t x_t \sin 2\pi \frac{j}{N} t \right)^2 \right] . \tag{5}$$

It can be shown, if we are concerned with the number of data, that the number of parameters to be estimated in the above least squares method and the number of data are identical no matter what N may be. Anyone who has some experience with the least squares method can see how poor the

6. The proof is given in any standard textbook on Fourier analysis, e.g., E. T. Whittaker and G. Robinson, The Calculus of Observations, pp. 264-267.

7. For a proof, see, e.g., H. T. Davis, The Analysis of Economic Time Series, pp. 65-66.

estimation would be under this circumstance. Indeed, this is so unless x_t is deterministic and furthermore a periodic function of time. (If x_t is deterministic and periodic, the mean of x_t as the constant term, plus, the sum of sinusoids in (3) using the estimates $\hat{a}_j$ and $\hat{b}_j$, fits x_t perfectly. This is a version of Fourier's theorem. As for the case of non-periodic x_t, even if we attempt to estimate any portion of the set of parameters, e.g., just $\hat{a}_j$ and $\hat{b}_j$ [for some number j], the result of this estimation is always identical to the result of the estimation which is calculated simultaneously with the rest of parameters using the least squares methods. This is due to the orthogonality condition.) In fact, this poor performance can be demonstrated in the following way. Suppose that $\{x_t\}$ $(t = 1,2,\ldots,N)$ is a stochastic process generated by a normal variate such that $E(x_t) = 0$, $E(x_t^2) = \sigma^2$, $E(x_t x_s) = 0$ if $t \neq s$. Then we can show that $\hat{a}_j$ is a normal variate with zero mean and $\frac{2\sigma^2}{N}$ as its variance, $\hat{b}_j$ is a normal variate with the same mean and variance as $\hat{a}_j$, and, $\hat{a}_j$ and $\hat{b}_j$ are uncorrelated. Then

$$E\left\{\frac{\hat{a}_j^2 + \hat{b}_j^2}{2}\right\} = \frac{2\sigma^2}{N}$$

Further, since $\frac{N}{\sigma^2}\left[\frac{\hat{a}_j^2 + \hat{b}_j^2}{2}\right]$ has χ^2 distribution with two degrees of freedom -- the variance of such χ^2 is known to be 4 --

$$\text{the variance of } \frac{\hat{a}_j^2 + \hat{b}_j^2}{2} = \frac{4\sigma^4}{N^2}.$$

Therefore, the ratio of the standard deviation of $\frac{\hat{a}_j^2 + \hat{b}_j^2}{2}$ to its mean is unity and is independent of N. In other words, the increase of N does not improve the estimate of $\frac{\hat{a}_j^2 + \hat{b}_j^2}{2}$. (This was first pointed out by M.Bartlett[8] in connection with periodogram. Periodogram is defined as $N \cdot \frac{\hat{a}_j^2 + \hat{b}_j^2}{2}$ where N is chosen to make one of j/N any desired rational values of frequency.)

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8. M. Bartlett, An Introduction to Stochastic Process, p. 278.

D. Spectral Analysis

Spectral analysis is based upon the concept of the stochastic process, which can be looked upon as a population having a time dimension. Thus, a given time series $\{x_t\}$ is a sample drawn from a stochastic process. The spectrum is a parameter of the stochastic process having an infinitely long time dimension from $-\infty$ to $+\infty$. It is a function of frequency f; when $E(x_t) = 0$ for all t, it is defined as[9]

$$S_x(f) = \lim_{n\to\infty} \frac{2}{2n+1} E\left\{\left(\sum_{t=-n}^{n} x_t \cos 2\pi ft\right)^2 + \left(\sum_{t=-n}^{n} x_t \sin 2\pi ft\right)^2\right\} \tag{6}$$

$$= \lim_{n\to\infty} \frac{2}{2n+1} E\left\{\sum_{-n\le t,s\le n} x_t x_s e^{-i2\pi f(t-s)}\right\} \tag{7}$$

for <u>any real number</u> of f between 0 and $\frac{1}{2}$. The existence of the limit is assumed. (When $E(x_t) = \mu \neq 0$, x_t in (6)-(7) must be replaced by $x_t - \mu$.)

When the spectrum is written in the form (6) and Fourier analysis is expressed by (5), the similarity between Fourier analysis and spectral analysis becomes obvious. Both are essentially the variance decomposition of $\{x_t\}$ using sinusoids. The differences between the two are more subtle. First, mathematical expectation is used in the definition of spectrum because it is a parameter of a stochastic process, whereas mathematical expectation is not used in Fourier analysis. Since, in the case of economic applications, we usually use just one sample drawn

9. To derive (7) from (6), transform (6) first into
$$\lim_{n\to\infty} \frac{2}{2n+1} E\left\{\left|\sum_{t=-n}^{n} x_t e^{-i2\pi ft}\right|^2\right\}.$$

from the stochastic process, we shall not emphasize this aspect of the difference between spectral analysis and Fourier analysis. The second difference is quite similar to the difference between discrete and continuous probability distributions. We first emphasize that (6)-(7) are defined for any real number of f between 0 and $\frac{1}{2}$. Just as $p(x)dx$ is the probability of x falling in the infinitesimal interval dx when $p(x)$ is a continuous (density) distribution defined for any real value of x within a given interval, $S_x(f)df$ is that part of total variance that can be accounted for by the infinitesimal frequency interval df. (Thus, $S_x(f)$ is called spectral density. At this point, we are using the word, variance, symbolically. The more exact meaning of total variance will be given later.) Using Fig. 2 we can contrast the Fourier and the spectral analyses. In the case of Fourier analysis the sum of the lengths of N lines erected at the frequency points $\frac{1}{N}$, $\frac{2}{N}$, ... must be equal to the total variance, whereas in spectral analysis the total area of the spectrum between 0 and $\frac{1}{2}$ must be equal to the total variance. This difference is reflected in the difference in the divisors of (5) and (7). (Notice that the length of the lines erected in the Fourier analysis is inversely related to N.)

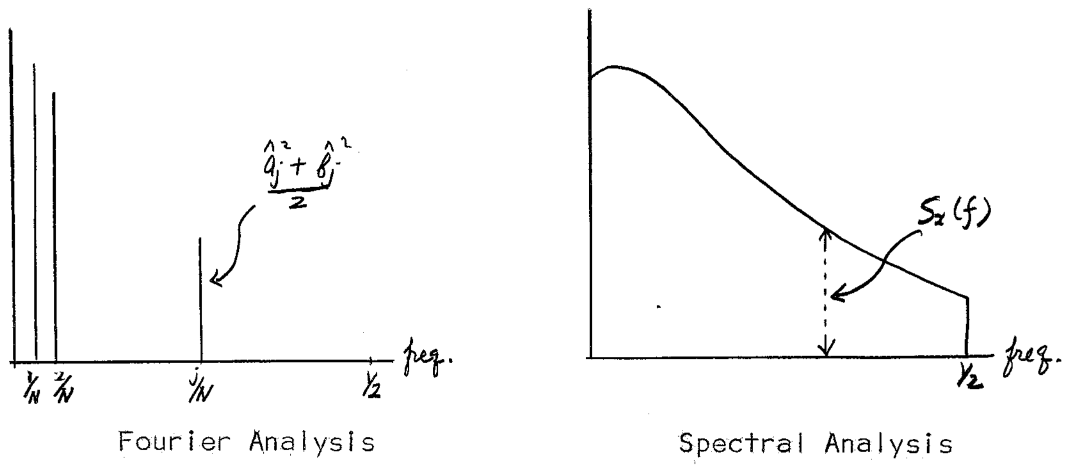

Fig. 2

It must be noted that in our definition of spectral density the variance contribution from a point of frequency is zero (just as the probability of x falling at a point is zero). This means that our definition is not applicable if the stochastic process contains a sinusoid, e.g. if $\{x_t\}$ can be represented by

$$\cos 2\pi f_0 t + \epsilon_t$$

where $\{\epsilon_t\}$ is a time series uncorrelated over time. This is because in such a process the point of frequency f_0 has a non-zero variance contribution. (It is possible to modify the definition of the spectrum so that it is applicable to stochastic processes containing sinusoids.) This restriction on the applicability of the definition of spectrum, however, does not seem to be a serious drawback as far as economic problems are concerned, because there is no reason to believe that the economic stochastic process contains a sinusoid. Even the seasonal variation does not have a constant structure. Its amplitude and phase do change over time and, hence, it cannot be represented as a sinusoid. Thus in the spectrum defined above, we are making a frequency decomposition of the variance of the stochastic processes which do not contain sinusoids. However perplexing this may be, this is an important point.

E. Spectra of Stationary Stochastic Processes

The spectrum has been traditionally defined only for stationary stochastic processes, which are defined by the following conditions:

$$\left.\begin{array}{l} E(x_t) = \text{constant, written as } \mu \\ E(x_t - \mu)^2 = \text{constant, written as } \sigma^2 \\ E(x_t - \mu)(x_s - \mu) \text{ depends only on } t-s\text{, written as } \gamma_h \; (h = t-s) \; . \end{array}\right\} \quad (8)$$

In other words, the first two moments of the stationary stochastic process do not depend on the origin of time. (Note that $\sigma^2 = \gamma_0$.) When the stochastic process is stationary, (7) is reduced to

$$2 \sum_{h=-\infty}^{\infty} \gamma_h e^{-i2\pi fh}$$

which is the definition of spectrum most commonly used.[10] $S_x(f)df$ is that portion of σ^2 which can be accounted for by the infinitesimal frequency interval df . In fact

$$\int_0^{\frac{1}{2}} S_x(f)df = \sigma^2$$

because $\gamma_h = \gamma_{-h}$. The spectrum of a random series which is uncorrelated over time -- such series are called white noise -- is horizontal as drawn in Fig. 3. Every infinitesimal frequency interval contributes to the total variance in an equal degree. (This is a way of defining "random".) The endogenous variable

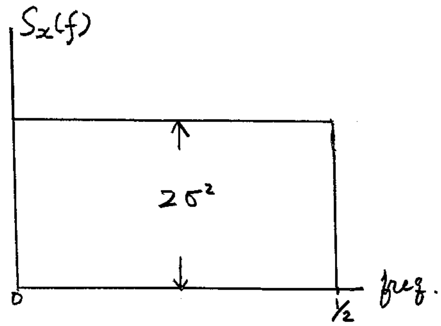

Fig. 3

in the linear dynamic econometric model, when the effect of exogenous variables is subtracted, can be represented as

10. From the standpoint of mathematical statisticians this is probably the most natural defintion of spectrum. See P. Whittle, "Some Recent Contributions to the Theory of Stationary Processes", Appendix 2 in Herman Wold, A Study in the Analysis of Stationary Time Series, p. 197, and R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra, p. 7.

$$a_0 x_t + a_1 x_{t-1} + \dots + a_k x_{t-k} = b_0 \epsilon_t + b_1 \epsilon_{t-1} + \dots + b_q \epsilon_{t-q} \qquad (9)$$

where ϵ_t is a white noise with variance σ^2. The stochastic process of $\{x_t\}$ which satisfies (9) is stationary if the stability condition is satisfied.[11] The spectrum of $\{x_t\}$ satisfying (9) is fairly smooth if k and q in (9) are fairly small. Figure 4 is a rough sketch of the spectrum of

$$x_t - 1.1\, x_{t-1} + 0.5\, x_{t-2} = \epsilon_t \quad .$$

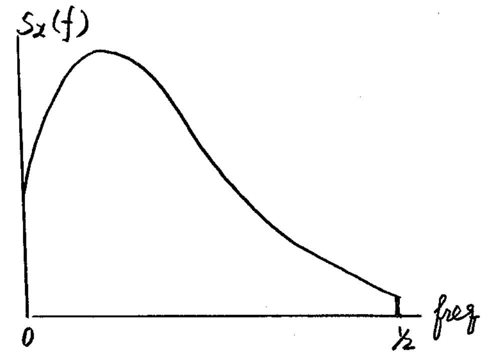

Fig. 4

Certain infinitesimal frequency intervals contribute more to the total variance than others, and this is why irregular cycles can be represented by the dynamic econometric model (9).

F. Pseudospectra of Non-stationary Processes

When the length of data is finite and when the stationarity conditions (8) are not satisfied by the stochastic process -- these are the conditions under which economists work -- we can define the pseudospectrum as

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11. I.e., if the absolute values of the roots $a_0 \lambda^{t-k} + a_1 \lambda^{t-k-1} + \dots + a_k = 0$ are less than 1, or if the solution of the difference equation $a_0 x_t + a_1 x_{t-1} + \dots + a_k x_{t-k} = 0$ converges to a limit when $t \to \infty$.

$$S_x(f) \equiv \frac{2}{N} E \left\{ \left(\sum_{t=1}^{N} x_t \cos 2\pi f t \right)^2 + \left(\sum_{t=1}^{N} x_t \sin 2\pi f t \right)^2 \right\}.$$

$S_x(f)df$ is that portion of the mean of the changing variance, $\frac{1}{N} \sum_{t=1}^{N} E(x_t^2)$, which can be accounted for by the infinitesimal interval df . It is proved elsewhere[12] that the pseudospectrum maintains the basic characteristics of the spectrum. Various types of non-stationarity can be considered. First, the pseudospectra of constant and trend (notice that the first condition of (8) is not satisfied here) have significant values only in the very low frequencies as illustrated by Fig. 5. The width of the frequency band over which the pseudospectrum is significant, relative to the mean of changing variance, is roughly in inverse proportion to the number of data , N . Second, let us consider the case where the variance is not constant, i.e., the case where the second condition of (8) does not hold.

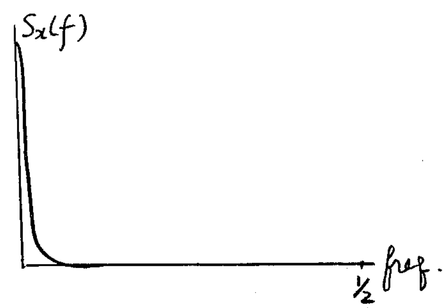

Fig. 5

12. M. Hatanaka and M. Suzuki, "A Theory of the Pseudospectrum and its Applications to Nonstationary Dynamic Econometric Models," Research Memorandum of Econometric Research Program, Princeton University. The present section F) is entirely drawn from this paper. The slight difference in the definition of pseudospectrum between the present paper and the paper cited above is due to the difference in the definition of frequency.

If the stochastic process of $\{x_t\}$ is stationary and d_t is a deterministic function of time (d_t can be normalized so that $\sum_{t=1}^{N} d_t^2 = N$), the pseudospectrum at frequency f_0 of $\{d_t x_t\}$ -- whose variance changes im proportion to d_t^2 -- is a weighted average of the spectrum of x_t about f_0, and the smoother the change in d_t the narrower the range of the weighted average. The smoothness of the changes involved in d_t is measured by the extent to which the pseudospectrum of d_t is concentrated in zero and the low frequencies. Thus, the d_t that involves a few jumps in addition to a trend is also considered as smooth.

A third type of non-stationarity is the case in which phase angles of various frequencies are not constant over time. (In studying the pseudospectrum this is a way to treat the case where the third condition of (8) does not hold.) This type of non-stationarity is important in relation to cross-spectrum, and, the explanation will be given later in the sections where the cross-spectrum is discussed.

Yet another type of non-stationarity is the dynamic econometric models of which the parameters change over time. We can say that if the cyclical components of the parameters dominate the others such as the constant part or trend in terms of their variance contribution, then the pseudospectrum becomes meaningless. If the constant or trend component dominates the cyclical component, the pseudospectrum (and, for that matter, the cross-pseudospectrum) is meaningful.

G. Spectral Window

When we come to the statistical estimation of the spectrum from time series data of finite length, we cannot estimate $S_x(f)$ at each distinct frequency point f. (This is analogous to the fact that we cannot estimate the value of a continuous probability density function at every point when only a finite number of data is given.) Therefore, we divide the frequency band between 0 and $\frac{1}{2}$ into an appropriate number of intervals and try to estimate the variance contribution from each frequency interval.[13] (This is analogous to a histogram.) In practice, what we try to estimate (for the spectrum for a frequency interval centered at f_0) is the weighted average of the true spectrum about f_0, i.e., a smoothed picture of the true spectrum. The weight is highest at f_0. When we use the Tukey estimation method, assuming that δ is the length of this interval, the weight is positive between $f_0 + 2\delta$ and $f_0 - 2\delta$, but it does not vanish completely beyond $f_0 + 2\delta$ and $f_0 - 2\delta$, though it is very small. The weighting function is called the <u>spectral window</u>.

We have commented earlier that the spectral analysis is based upon the methodology of modern statistics as to the relationship between the population and samples drawn from it. Thus, to the estimation of spectrum is attached the concept of degrees of freedom. In the case of the Tukey estimation method, it is given as $\frac{2N}{m}$, where N is the number of data and m is the number of intervals into which the entire range of frequencies from 0 to $\frac{1}{2}$ is divided. I.e., $\delta = \frac{1}{2m}$. If we want to maintain a given degree of freedom in the estimation of spectrum, the width of the interval of frequency δ is inversely proportional to the number of data; i.e., the longer the set of data, the finer the analysis of spectrum that can be made. The fact that <u>we can control the degree of freedom by choosing a</u>

13. R. B. Blackman and J. W. Tukey, <u>The Measurement of Power Spectra</u>.

desirable value of m (or δ) should be contrasted to the fact that the estimates of (5) in Fourier analysis have always only 2 degrees of freedom.

H. Periodicity of Oscillation

In mathematics a periodic function of time is defined as the $f(t)$ that satisfies $f(t) = f(t + t_0)$ for all t and a given t_0, i.e., $f(1) = f(1 + t_0)$, $f(2) = f(2 + t_0)$ ... . t_0 is called period. In economics we might call such functions periodic oscillations. Figure 6 shows an example of periodic oscillations. In this example, $t_0 = 6$.

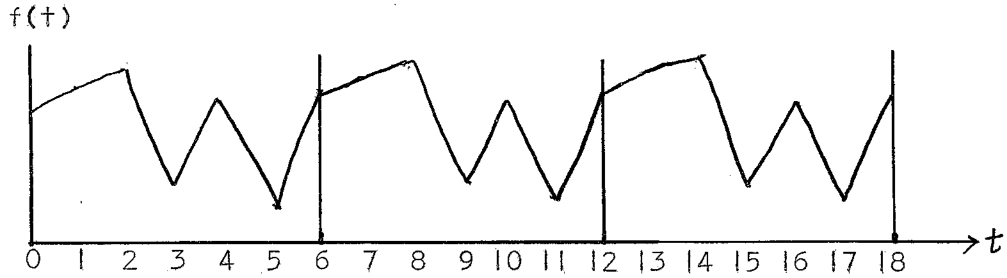

Fig. 6

and $f(0) = f(6)$, $f(1) = f(7)$, et cetera. There is a mathematical theorem, called Fourier's theorem, to the effect that any periodic oscillations with period t_0 can be approximated well by a set of sinusoids having only a finite number of the first portion of the sequence of frequencies $\frac{1}{t_0}$, $\frac{2}{t_0}$, ..., i.e., only those finite points of frequencies have significant variance components. We have commented earlier on the degrees of freedom for the spectral estimates. There is one (minor) difficulty in the presently available estimation methods of spectrum. The confidence band for the estimation of the spectrum is based upon the assumption that no single frequency dominates the adjacent frequencies in terms of their variance contribution. Thus, this assumption rules out the possibility of handling periodic oscillation. One very important difference between Fourier analysis and spectral analysis is the following: spectral analysis is not designed to handle periodic oscillations, whereas Fourier analysis is designed to handle periodic oscillations.

Thus, if we agree on two assumptions, 1) that the economic system contains a random element, and 2) but that it does not contain periodic oscillations, then spectral analysis is the method to be used.[14]

I. Cross-spectrum

When we have a multi-variate stochastic process, the cross-spectrum or cross-pseudospectrum can be defined. They are direct extensions of the spectrum or pseudospectrum. Thus, when $E(x_t) = E(y_t) = 0$ for all t, the cross-spectrum between $\{x_t\}$ and $\{y_t\}$ at frequency f (f is any real number between 0 and $\frac{1}{2}$) is

$$C_{xy}(f) = \lim_{n \to \infty} \frac{2}{2n+1} E\left\{ \sum_{-n \leqq t, s \leqq n} x_t y_s e^{-i2\pi f(t-s)} \right\} . \qquad (10)$$

Compare (10) with (7), which is an expression for spectrum. The cross-pseudospectrum is

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14. If the system contains peridoic oscillations and a random element, we can use Periodogram Analysis. If the system contains periodic oscillations but not a random element, then we can use Fourier Analysis. In fact, Fourier analysis produces a disagreeable result even when it is applied to a periodic function of time. Suppose that the analysis is applied to a sinusoid $A \cos(2\pi f_0 t + \theta)$ where $f_0 \neq j/N$ for any integer j but $(j_0/N > f_0 > (j_0-1)/N$. Then the variance contribution as estimated by Fourier analysis is spread around $(j_0/N$ and $(j_0-1)/N$ in spite of the fact that f_0 should be the only frequency which has a non-zero variance contribution.

$$C_{xy}(f) = \frac{2}{N} E \left\{ \sum_{1 \leqq t, s \leqq N} x_t y_s e^{-i2\pi f(t-s)} \right\}$$

Unlike the spectrum, the cross-spectrum is a complex function of f, having a real part $R_{xy}(f)$ and an imaginary part $I_{xy}(f)$. Although it is an analogy of the cross-products $\sum_t x_t y_t$ that appear in standard statistics, it is a far more complicated concept. In order to see the meaning of cross-spectrum intuitively we rewrite (10) as

$$C_{xy}(f) = \lim_{n\to\infty} \frac{2}{2n+1} E \left\{ \sum_t x_t e^{-2\pi ft} \sum_s y_s e^{i2\pi fs} \right\}$$

$$= \lim_{n\to\infty} \frac{2}{2n+1} E \left\{ (\sum_t x_t \cos 2\pi ft \sum_s y_s \cos 2\pi fs + \sum_t x_t \sin 2\pi ft \sum_s y_s \sin 2\pi fs) \right.$$

$$\left. + i(\sum_t x_t \cos 2\pi ft \sum_s y_s \sin 2\pi fs - \sum_t x_t \sin 2\pi ft \sum_s y_s \cos 2\pi fs) \right\} .$$

Let us put

$$a_x(f,n) \equiv \frac{2}{2n+1} \sum_{t=-n}^{n} x_t \cos 2\pi ft \quad , \quad a_y(f,n) \equiv \frac{2}{2n+1} \sum_s y_s \cos 2\pi fs$$

$$b_x(f,n) \equiv \frac{2}{2n+1} \sum_t x_t \sin 2\pi ft \quad , \quad b_y(f,n) \equiv \frac{2}{2n+1} \sum_s y_s \sin 2\pi fs \quad . \qquad (11)$$

Then

(i) Norm, or squared amplitude of $C_{xy}(f)$, i.e., $R_{xy}(f)^2+I_{xy}(f)^2$

$$= \lim_{n\to\infty} \frac{(2n+1)^2}{4}\left[\left\{E(a_x(f,n)a_y(f,n))\right\}^2 + \left\{E(b_x(f,n)b_y(f,n))\right\}^2 + \left\{E(a_x(f,n)b_y(f,n))\right\}^2 + \left\{E(b_x(f,n)a_y(f,n)\right\}^2\right]$$

because of

$$E(a_x(f,n)a_y(f,n))\cdot E(b_x(f,n)b_y(f,n)) = E(a_x(f,n)b_y(f,n))\cdot E(b_x(f,n)a_y(f,n))$$

which follows from (11) and the stationarity conditions (8).

(ii) phase of $C_{xy}(f) = \tan^{-1}\dfrac{I_{xy}(f)}{R_{xy}(f)}$ [15]

$$= \lim_{n\to\infty} \tan^{-1} \frac{E\{a_x(f,n)b_y(f,n)-b_x(f,n)a_y(f,n)\}}{E\{a_x(f,n)a_y(f,n)+b_x(f,n)b_y(f,n)\}}.$$

Now we can consider an analogy with Fourier analysis. Let us take the j^{th} sinusoid

$$a_{x,j} \cos 2\pi \frac{j}{N} t + b_{x,j} \sin 2\pi \frac{j}{N} t \tag{12}$$

used for the variance decomposition of $\{x_t\}$, and the j^{th} sinusoid

$$a_{y,j} \cos 2\pi \frac{j}{N} t + b_{y,j} \sin 2\pi \frac{j}{N} t \tag{13}$$

used for the variance decomposition of $\{y_t\}$, where the estimates of $a_{x,j}$, $b_{x,j}$, $a_{y,j}$, and $b_{y,j}$ are given by (4). We can define the

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15. For the definition of $\tan^{-1}$ use the same rule as specified in footnote 5.

complex cross-product between (12) and (13) by the following two concepts:

Product of squared amplitudes -- see the equation for A in (2a) -- of (12) and (13)

$$(a^2_{x,j} + b^2_{x,j}) \times (a^2_{y,j} + b^2_{y,j}) = (a^2_{x,j}a^2_{y,j} + b^2_{x,j}b^2_{y,j} + a^2_{x,j}b^2_{y,j} + a^2_{y,j}b^2_{x,j})$$

Phase difference -- see the equation for θ in (2a) -- between (12) and (13)[16]

$$\tan^{-1}\frac{b_{y,j}}{a_{y,j}} - \tan^{-1}\frac{b_{x,j}}{a_{x,j}} = \tan^{-1}\frac{a_{x,j}b_{y,j} - b_{x,j}a_{y,j}}{a_{x,j}a_{y,j} + b_{x,j}b_{y,j}} \quad . \qquad (14)$$

Then, we can see that the cross-spectrum $C_{xy}(f)$ is analogous to the complex cross-product thus defined. The squared amplitude of $C_{xy}(f)$ is an analogy of the product of the squared amplitudes of the two sinusoids of the same frequency used for the decomposition of x_t and y_t. The phase of $C_{xy}(f)$ is an analogy of the difference between the phase angles of two sinusoids of the same frequency used for the decomposition of x_t and y_t. The differences between cross-spectrum and complex cross-product in Fourier analysis are that: 1) since the cross-spectrum is defined for a (multi-variate) stochastic process, it is defined as a mathematical expectation as to all possible sample time series, and that 2) the cross-spectrum uses all real numbers of frequencies between 0 and $\frac{1}{2}$, as previously explained.

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16. Using the rule in footnote 5.

J. The Treatment of Lead-Lag Relations by Cross-spectrum

When (11) is substituted into $C_{xy}(f)$, it becomes

$$\lim_{n \to \infty} \frac{(2n+1)}{2} \cdot [\, E(a_x(f,n)a_y(f,n)) + E(b_x(f,n)b_y(f,n)) \,] +$$

$$\lim_{n \to \infty} \frac{(2n+1)}{2} \cdot i\, [\, E(a_x(f,n)b_y(f,n)) - E(b_x(f,n)a_y(f,n)) \,]$$

J. Tukey calls the real part of this expression <u>co-spectrum</u>, representing the "in-phase relation", and the imaginary part <u>quadrature spectrum</u>, representing the "90° out-of-phase relation". The cross-spectrum represents any relations (involving a lead or lag at each frequency) by combining properly these "in-phase" and "90° out-of-phase" relations. In order to understand the meaning of these terms, it is again instructive to consider the Fourier counterpart of the above expressions. The "in-phase relation" is $a_{x,j}a_{y,j} + b_{x,j}b_{y,j}$ and the "90° out-of-phase relation" is $a_{x,j}b_{y,j} - b_{x,j}a_{y,j}$. Since the a's are the cosine components and the b's are the sine components of the two time series, it should be clear why $a_{x,j}a_{y,j} + b_{x,j}b_{y,j}$ is the "in-phase" relation. To explain why $a_{x,j}b_{y,j} - b_{x,j}a_{y,j}$ is the "90° out-of-phase relation" let us consider two sinusoids such as (12) and (13). There is a one-to-one correspondence between the whole class of sinusoids of a given frequency, $\frac{j}{N}$, and the whole set of points in the two-dimensional space formed by the coordinates for a and b. E.g., the point x in Fig. 7 corresponds to the sinusoid, $a_{x,j}\cos 2\pi \frac{j}{N} t + b_{x,j}\sin 2\pi \frac{j}{N} t$. Let us normalize the sinusoids and consider only those having $a_{x,j}^2 + b_{x,j}^2 = 1$. (The normalization is made only to simplify the geometry involved here. The essence of the following statements follows even without this normalization.) Then the whole class of sinusoids under consideration corresponds to the unit circle in the (a,b) space. Now the phases of these sinusoids have a one-to-one correspondence with the angles represented on the (a,b) space.

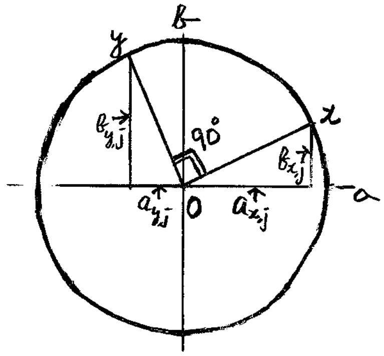

Fig. 7

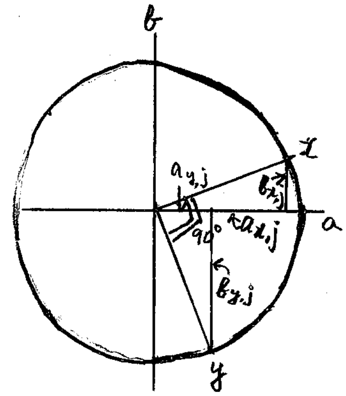

Fig. 8

E.g., the phase angle of the sinusoid that corresponds to x is essentially the angle x0a, i.e., $\tan^{-1}\frac{b_{x,j}}{a_{x,j}}$. (In fact, the minus sign must be attached in front. See (2a)). Suppose that the phase difference between the two sinusoids (12) and (13) is 90°. If the relative situation between the two phases is represented by Fig. 7, then $a_{y,j} = -b_{x,j}$, $b_{y,j} = a_{x,j}$. Therefore, $a_{x,j}a_{y,j} + b_{x,j}b_{y,j} = 0$, $a_{x,j}b_{y,j} - b_{x,j}a_{y,j} (= a_{x,j}^2 + b_{x,j}^2) = 1$. If the relative situation between the two phases is represented by Fig. 8, then $a_{y,j} = b_{x,j}$, $b_{y,j} = -a_{x,j}$. Thus, $a_{x,j}a_{y,j} + b_{x,j}b_{y,j} = 0$, $a_{x,j}b_{y,j} - b_{x,j}a_{y,j} = -(a_{x,j}^2 + b_{x,j}^2) = -1$. In other words, in either case, the "in-phase relation" is 0 and "90° out-of-phase relation" is 1 or -1.

It is important to note that equation (14) shows that the difference in the phase angles of the two sinusoids can be represented by the values of $a_{x,j}a_{y,j} + b_{x,j}b_{y,j}$ and the value of $a_{x,j}b_{y,j} - b_{x,j}a_{y,j}$.

As to the difference in the phase angles, we need to exercise caution in determining the direction of the lead or lag. For example, suppose that we observe a 12° phase difference at frequency 1/30 cycle per time unit. (see Fig. 9). In terms of time rather than of angles, this is $30 \times \frac{12}{360} = 1$, i.e., 1 time unit. The fact that the angle Yoa is greater than the angle Xoa by 12° on Fig. 9 means that the phase angle for the sinusoid Y is less than the phase angle for the sinusoid X by 12° (see (2a)). In other words, the sinusoid X leads the sinusoid Y by 12° or 1 time unit at frequency 1/30. This is illustrated in Fig. 10.

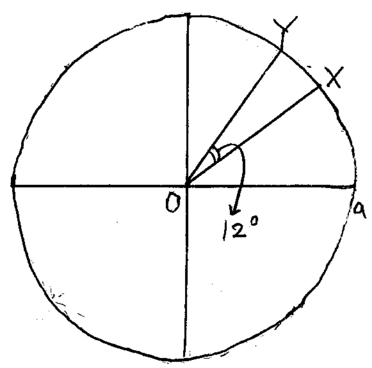

Fig. 9

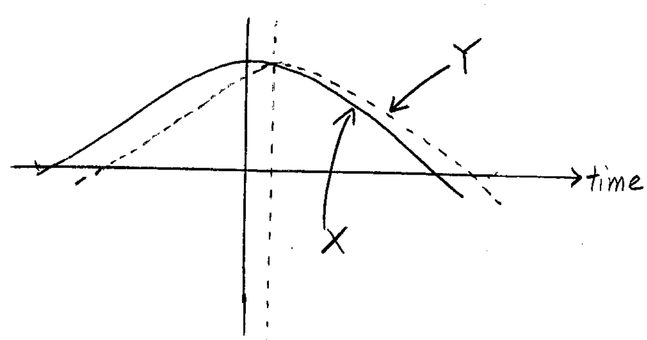

Fig. 10

Incidentally, all our mathematics used here are expressed modulus $2\pi = 360^{\circ}$, and the relative positions of X and Y in Fig. 9 can also represent the case where the angle Yoa is greater than the angle Xoa by 372° $(= 360^{\circ}+12^{\circ})$, i.e., X leads Y by 31 time units. Fig. 9 is also consistent with the possibility that the angle Yoa is greater than the angle Xoa by -348° $(= 12^{\circ}-360^{\circ})$, i.e., the angle of Y is less than the angle of X by 348° , i.e., X is lagging behind Y by 29 time units. The cross-spectrum cannot discriminate between these different possiblities.

K. Three Statistics Derived from Cross-spectrum; Black Box

In the applications to practical problems, we use three statistics derived from spectra and cross-spectra. (Each of the three can be defined for the spectra and cross-spectrum and also for the pseudospectra and cross-pseudospectrum.) First, coherence is defined as

$$\frac{|C_{xy}(f)|^2}{S_x(f)S_y(f)} \tag{14}$$

and indicates the coefficient of determination between x_t and y_t in regard to that part of these series which can be accounted for by the sinusoids of frequency f. The coherence always lies between 1 and 0. (This is also true for the coherence defined for the pseudospectra and cross-pseudospectrum.) In fact, the coherence is not the ordinary coefficient of determination, but this coefficient when the proper phase difference between x_t and y_t is adjusted. This is because $|C_{xy}(f)|^2$ is an analogy, not of the simple cross-product but of the product of the squared amplitudes.[17] Second, gain of y_t over x_t is defined as

17. Using an analogy with Fourier analysis, the expression corresponding to the coherence is $\dfrac{(a_{x,j}a_{y,j}+b_{x,j}b_{y,j})^2+(a_{x,j}b_{y,j}-b_{x,j}a_{y,j})^2}{(a_{x,j}^2+b_{x,j}^2)\,(a_{y,j}^2+b_{y,j}^2)}$

which is identically equal to 1, no matter what the values of a's and b's are, i.e., no matter what the phase difference between the two sinusoids is.

Although this analogy is useful to explain the relation between coherence and phase difference, it is quite misleading because it ignores the stochastic element involved in the definition of cross-spectrum. Indeed, the coherence for any two-variate stochastic processes is generally not unity. If the processes of $\{x_t\}$ and $\{y_t\}$ are totally uncorrelated, the coherence should be zero. We can visualize this by a two-variate stochastic process where the event of certain phase for $\{x_t\}$ occurring at a frequency is totally uncorrelated with the event of a certain phase for $\{y_t\}$ occurring at the same frequency. Since the cross-spectrum is defined using the mathematical expectation of all possible samples drawn from the stochastic process, the coherence is zero. When we are dealing with only one sample drawn from the stochastic process, as we do in most economic studies, the event of a certain phase difference occurring at a frequency f_o is totally uncorrelated with the occurrence of the same phase difference at a neighboring frequency $f_o+\varepsilon$. Therefore, over some frequency interval the coherence should be very small.

$$\frac{|C_{xy}(f)|}{S_x(f)} \tag{15}$$

and corresponds to the regression coefficient of y_t on x_t in regard to the frequency f. Third,

$$\tan^{-1} \frac{I_{xy}(f)}{R_{xy}(f)} \tag{16}$$

is the phase of the cross-spectrum $C_{xy}(f)$ and indicates the lead-lag between x_t and y_t in regard to frequency f.

When the length of data is finite, we must take a weighted average for all the functions of frequency f that appear in (14), (15), and (16), in just the same way as we had to take the weighted average of the true spectrum.

For a multi-variate stochastic process we can define the matrix of spectra and cross-spectra in the same way as we define the matrix of variance and covariance. From this matrix the partial cross-spectrum is derived in just the same way as the partial correlation is derived from the matrix of variance and covariance.[18]

Now we observe that the greatest advantage of cross-spectral analysis is that we can study certain relationships frequency-by-frequency. (In the next section, we shall explain that the time series components can be represented by frequencies.) Therefore, the cross-spectral analysis is useful when the relationship between $\{x_t\}$ and $\{y_t\}$ can be decomposed in terms of frequencies, i.e., the component of $\{y_t\}$ that is accounted for by frequency f is independent of the component of $\{x_t\}$ that is accounted for by any other frequencies f' and can

18. This work, done by T. Wonnacott, is described in Granger in association with Hatanaka, *op. cit.*, Chapter 5.

be dependent only on the component of $\{x_t\}$ that is accounted for by frequency f. Such relationships are called black box relationships. In the case where a portion of the variance of $\{y_t\}$ is accounted for by a deterministic function of $\{x_t\}$, $\{z_t\} \equiv F(\{x_t\})$, and the rest by a white noise, Tukey has proved that the relationship between $\{x_t\}$ and $\{y_t\}$ is a black box when

1) F is linear; i.e., if $\{z_{1t}\} = F(\{x_{1t}\})$ and $\{z_{2t}\} = F(\{x_{2t}\})$

   then $\lambda_1\{z_{1t}\} + \lambda_2\{z_{2t}\} = F(\lambda_1\{x_{1t}\} + \lambda_2\{x_{2t}\})$

and

2) F is time-invariant, i.e., if $\{z_t\} = F(\{x_t\})$ and if we define $\{z_t^*\} = \{z_{t+h}\}$, $\{x_t^*\} = \{x_{t+h}\}$, i.e., the new time series are obtained by shifting the old time series along the time axis, then $\{z_t^*\} = F(\{x_t^*\})$ no matter what h may be.

An example of a non time-invariant function is

$$z_t = ax_t + bt \quad .$$

The reason why this is not a time-invariant relation is

$$z_t^* = ax_t^* + bt + bh \neq ax_t^* + bt.$$

Thus, given the origin of time, when a time-invariant relationship is observed over a finite portion of the time, this observation can be just as good a sample (of this relationship) as an observation over any other portions of the time with equal length. In other words, the relationship is not "historical".

This condition certainly does not hold in economic time series. Particularly important in connection with the study of business cycle indicators is the time-changing lead-lag relationships.

L. Phase of Cross-pseudospectrum and Changing Lead-Lag

Let us consider a two-variate (non-stationary) stochastic process where the phase angles of the two series x_t and y_t can change over time. The difference between the phase angles of x_t and y_t, i.e., the lead or lag, can also change over time. Let us consider the case where the changes in the phase differences are smooth or involve only one or two jumps so that their pseudospectrum almost entirely concentrates in the low frequencies. If the changes in the phase angles in each of the two series are uniform over different frequencies, i.e, the way in which the phases change over time does not vary from frequency to frequency, then the phase of the cross-pseudospectrum at f is the average of the time-changing phase differences at f. If the changes in the phase angles are uniform over only a narrow frequency interval around f, the phase of the cross-pseudospectrum at frequency f is the double-average of $\varphi_f(t,v)$ over t and v, where $\varphi_f(t,v)$ represents the difference between the phase angle of x_t (at frequency f) at time t and the phase angle of y_t (at frequency f) at time $t+v$. The averaging over v uses the weights that are roughly in inverse proportion to $|v|$.

Therefore, we can conclude that <u>the phase of the cross-pseudospectrum represents the average of the time-changing lead or lag</u>.

M. Frequency Representation of Time Series Components, Especially the Cyclical Component

It is true that the representation of the time series components by frequencies may not be unique in the sense that two different components could be represented by practically the same weighted average of practically the same frequencies. This happens in the following two cases: (i) Different shapes of <u>trend</u> -- linear, exponential, quadratic, etc. -- are almost all similar in the frequency representation (see Fig. 5).

In all cases the pseudospectrum of the trend between 0 and $\frac{6}{N}$ frequencies contain more than 95% of the variance. (2) Fluctuations with changing amplitude appear on the spectrum almost exactly like the fluctuations with a constant amplitude, which is roughly the average of the changing amplitude, if this change is slow. For problems for which a fine distinction among slightly different time series components is necessary, the cross-spectral analysis would not be useful. The analysis would be very useful, however, for studying the relationship among several series in terms of any vaguely defined time series component.

Let us now try to represent the traditional time series components by frequencies. Since most of the cycle indicators are monthly data, let us use a month as the time unit to represent frequency. Then the frequency, $\frac{1}{12}$ cycle per month, is the frequency of a 12-month sinusoid. We shall abbreviate cycle per month as c/m . Let us consider an artificial seasonal component with a period of 12 months which is periodic but not a sinusoid. In order to represent this seasonal component we need the frequencies $\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}$ c/m . All but the first are called the harmonics of the first frequency. Since the real seasonal component is not an exact periodic function of time (because of amplitude changes and phase changes), some (usually very narrow) intervals of frequencies around $\frac{1}{12}, \frac{2}{12}, \ldots, \frac{6}{12}$ c/m have significant variance contributions in the spectrum of the seasonal component.

As for the cyclical component, it should be obvious that it is far more irregular than the seasonal and hence the representation of the cyclical component needs a wider frequency interval than the seasonal. The average duration of National Bureau business cycles is about fifty months. We can take $\frac{1}{48}$ c/m as the center of the frequency interval corresponding to the

National Bureau business cycles. ($\frac{1}{48}$ c/m is convenient because it is a quarter of $\frac{1}{12}$ c/m .) Recently, Mack proposed the consideration of sub-cycles, which are substantially shorter than the National Bureau business cycles,[19] and we would like to include subcycles in our definition of the cyclical component. Thus the cyclical component would be a weighted average of frequencies which include both $\frac{1}{48}$ c/m and $\frac{1}{24}$ c/m .

Now we must define the upper and lower bounds of the frequency interval corresponding to the cyclical component. We could use the well-known boundaries of business cycles set by Burns and Mitchell, i.e., two years for the lower bound and ten or twelve years for the upper bound[20] or in terms of frequencies, $\frac{1}{144}$ c/m as the lower bound and $\frac{1}{24}$ c/m as the upper bound.

We do not choose these upper and lower bounds, however. There are two reasons why the lower bound must be raised beyond $\frac{1}{144}$ c/m . The first reason is that there has never been a business cycle with a duration as long as 144 months in the United States. (The longest one for the United States was 101 months. G. H. Moore has informed the author that Burns and Mitchell considered European business cycles as well as those of the United States in setting the upper bound.) The second reason, which is more important than the first, is that cross-spectral analysis cannot analyze properly frequencies lower than $\frac{6}{N}$, where N

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19. R. P. Mack, "Notes on Subcycles in Theory and Practice", American Economic Review, May 1957, pp. 161-174.

20. A. Burns and W. C. Mitchell, Measuring Business Cycles, p. 3 and p. 57.

is the number of data.[21] The majority of business cycle indicators are 200-500 months long, and, for this majority $\frac{1}{144}$ c/m and even $\frac{1}{101}$ c/m are too low to be analyzed properly by spectral analysis. Since we have discarded all the indicators less than 200 months in length, the lower bound for the frequency band of cyclical component is at most $\frac{1}{33}$ c/m .

As to the upper bound of the frequency interval of cyclical component we choose $\frac{1}{12}$ c/m . Thus the interval contains Mack's sub-cycles, the average duration of which is 24 months. There are two reasons why the upper bound is raised from $\frac{1}{24}$ c/m to $\frac{1}{12}$ c/m . First, since 24 months is the average duration of a subcycle, individual durations can be shorter. The second reason, which is more important, is related to the observation that when we put the center of frequencies for the National Bureau business cycle at $\frac{1}{48}$ c/m , the center of the frequencies for the subcycles, $\frac{1}{24}$ c/m , is the first harmonic of the National Bureau business cycle. Once this is realized, there is no reason to stop at the first harmonic. Therefore, we have raised the upper bound as far as we can, stopping just before possible interference with the seasonal frequencies.[22]

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21. The phase of cross-spectrum at zero frequency is necessarily zero (because $b_x(f,n) = b_y(f,n) = 0$ at $f = 0$) and, in a sense, it is a meaningless concept because a sinusoid with zero frequency, i.e., infinite period, is a constant. The spectral window carries this meaningless cross-spectrum at zero frequency to the higher frequencies. For example, if we want to maintain 12 degrees of freedom for the estimates of cross-spectrum, the cross-spectrum at zero frequency is carried as far as the frequency 6/N . The distinction between the frequencies that can and cannot be analyzed properly by the cross-spectral analysis should <u>not</u> be taken rigidly. This is because this distinction depends upon the degree of freedom that one wishes to maintain and also because the spectral window smoothly approaches zero so that the upper bound to the region of frequencies that are affected by the meaningless zero frequency cannot be represented by a single number in practice.

22. We have commented on the spectral window. Since we would like to have the weights for 1/12 c/m to be zero, the center of the highest frequency interval to be included in the cyclical component is $(1/12 - 2\delta)$ c/m where δ is the width of the unit frequency interval.

In order to represent the cyclical component as a weighted average of frequencies, the appropriate distribution of weights within the upper and lower bounds should probably be the spectrum in this bound. In practice, we have taken a rather free attitude in this regard. For the average of phase and coherence over the frequencies for the cyclical component we have taken visually the average on the charts of phase and coherence. Since the estimates of these statistics are subjected to various kinds of errors (see the Appendix I to this paper), an elaborate weighting scheme would probably not be worthwhile.

The definition in the literature of irregular components has been vague. Economic statisticians tend to think that it is what is eliminated from a series by a moving average. Economists tend to associate the irregular components with exogenous factors. An unpublished study by J. Cunningham[23] shows that when we define the irregular component as whatever is obtained from the application of Shiskin's method, the spectra of irregular components are insignficiant in the frequencies less than $\frac{1}{12}$ c/m.

N. A Brief Comment on Filtering

In spectral analysis, linear transformations of a time series are called filtering. Thus, filtering can be represented as the operation to obtain from x_t

$$y_t = a_0 x_{t+h} + a_1 x_{t+h-1} + \cdots + a_k x_{t+h-k} \tag{17}$$

(h = some integer)

As to the cross-spectrum between the two time series $\{x_t\}$ and $\{y_t\}$ we have the following conditions:

23. J. Cunningham, "Some Observations on the Spectral Analysis of Economic Time Series", presented to the 1962 Meetings of the Econometric Society.

(i) the coherence is equal to 1

(ii) the gain of $\{y_t\}$ over $\{x_t\}$ is equal to $\frac{S_y(f)}{S_x(f)}$ and can be represented by using the parameters a's in (17) as

$$|a_0+a_1e^{-i2\pi f}+ a_2e^{-2i2\pi f} + --- + a_ke^{-ki2\pi f}|^2 \qquad (18)$$

(iii) the phase is

$$\tan^{-1} - \frac{a_1\sin 2\pi f + a_2\sin 2(2\pi f) + --- + a_k\sin k(2\pi f)}{a_0+a_1\cos 2\pi f+a_2\cos 2(2\pi f)+ --- + a_k\cos k(2\pi f)}. \qquad (19)$$

Since the relation between $\{x_t\}$ and $\{y_t\}$ is deterministic, the coherence is obviously 1. When the coherence is 1, the gain is the ratio of two spectra. Formulas (18) and (19) can be derived from (17), (7) and (10).

Apart from the general use of (17) as a transformation of variables, filtering is essential for the estimation of spectrum and cross-spectrum. Suppose that the true spectrum has a very sharp and high peak at a frequency f_0. Then, the "spectrum" which we estimate, i.e., the "spectrum" obtained by smoothing the true spectrum by a spectral window (see Section II, G) at a frequency f_1, which is located near f_0, is mostly the "leakage" of the true spectrum at f_0 rather than the true spectrum at f_1. In the case of economic time series, such sharp and high peaks in the (pseudo) spectrum appear at zero and near-zero frequencies (due to the trend and the long waves, see Fig. 5) and the seasonal frequencies, $\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{12}, \frac{1}{2}$ c/m. (This explanation anticipates the results described in Section V to some extent.) When we are concerned with frequencies other than these, e.g., the frequency of the cyclical component, the "spectrum" obtained by smoothing the true spectrum is frequently meaningless.

Filtering enables us to eliminate this difficulty. We can choose some appropriate values for the parameters a'_s in (17) so that the true spectrum of $\{y_t\}$ could not possibly have such extreme peaks which would obscure the frequencies of the cyclical component. Then we can first estimate the "spectrum" of $\{y_t\}$; by using (18), we can estimate the "spectrum" of $\{x_t\}$. The estimate of the "spectrum" of $\{x_t\}$ is far nearer the true spectrum of $\{x_t\}$ than is the direct estimate of the "spectrum" of $\{x_t\}$.

The elimination of the peak at and near zero frequency is similar to a trend adjustment, and, the elimination of the peaks at the seasonal frequencies is similar to a seasonal adjustment. However, the sole purpose of filtering is to reduce the peaks of the spectrum. There is usually great freedom as to the extent to which the reduction of peaks should be made. The effect of each filtering method upon the spectrum is mathematically known. Therefore, the problems of filtering are simpler than the problems involved in trend and seasonal adjustments, where the time series obtained by these adjustments are required to have certain rigorous properties. (See the comment on the non-uniqueness of frequency representation at the beginning of Section II, M.)

The author's experience with different types of filtering in relation to the study of cyclical component is described in Chapter XII of C. W. J. Granger in association with M. Hatanaka, An Analysis of Economic Time Series .

O. Estimates of Spectra and Cross-spectra in Relation to Pseudospectra and Cross-pseudospectra

In several previous sections, we have explained the conceptual meaning of pseudospectrum and cross-pseudospectrum. These concepts also have a great, practical significance. Suppose that we estimate the "spectrum" and "cross-spectrum" of some time series data by using a standard estimation method, without paying any attention to the possible non-stationarity of the mechanism behind the time series data. (The computer does not know whether or not the input data come from a stationary stochastic process, and it prints out something as the estimates of "spectrum" and "cross-spectrum".) The pseudospectrum and cross-pseudospectrum are the mathematical expectations of such estimates of "spectrum" and "cross-spectrum" obtained by using a special, spectral window. This window is not desirable from a certain standpoint of statistical estimation, and, hence, noone actually uses it. The major difference between this window and the commonly used windows, however, is only that the latter smoothes the true spectrum more than the special window for the pseudospectrum does. Therefore, unless we try to make a statement about a very narrow frequency interval, we can say that we actually estimate the pseudospectrum and cross-pseudospectrum when we estimate the spectrum and cross-spectrum without paying any attention to the possible non-stationarity.

III. STRUCTURE OF LEAD-LAG[23]

In past studies of economic relationships, generally two kinds of definitions of lead-lag relationships have emerged. In the field of business cycles many people have tried to define the lead-lag in terms of the cyclical component, which must be distinguished from the lead-lag in terms of the seasonal, or trend component. This is very clear in the methodology of the National Bureau.[24] On the other hand, in econometrics, the relationship

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23. The author is aware of the fact that the problems treated in the present section need a far more careful study.

24. E. g., G. H. Moore, "Introduction", pp. xxx-xxxiii, in Business Cycle Indicators, ed. by G. H. Moore.

between any two variables x_t and y_t is represented in the form of a distributed lag with a random disturbance ϵ_t :

$$y_t = a_0 x_t + a_1 x_{t-1} + \dots + \epsilon_t \tag{20}$$

without distinguishing the different time series components. For some indicators the economic relations underlying them are so vague that the consideration of (20) is irrelevant, whereas, for other indicators we even have some theories as to the relationships underlying them. It would be important to investigate the implications of these ideas from the standpoint of the cross-spectrum. In the following explanation we assume that $\{x_t\}$ and $\{\epsilon_t\}$ are independent so that there is no feed-back effect of $\{x_t\}$ upon $\{y_t\}$. However, $\{\epsilon_t\}$ is not assumed to be a white noise.

When we define any time series component by a weighted average of frequencies in given intervals of frequencies, a first hypothesis which one would naturally think of is that the magnitude of lead-lag between $\{x_t\}$ and $\{y_t\}$ is constant over an interval of frequency in terms of phase angle θ (or $\frac{\theta}{2\pi}$) (explained in Section II B) . This constancy means that the magnitude of lead-lag in terms of calendar time is in proportion to the period, i.e., in inverse proportion to frequency. Thus, between $\frac{1}{48}$ c/m and $\frac{1}{24}$ c/m , this hypothesis means that if there is a 2 months lag at $\frac{1}{48}$ c/m then there is a 1 month lag at $\frac{1}{24}$ c/m. Tukey calls this fixed angle lag. This hypothesis, however, is not generally consistent with the econometric relationship presented above. If there is only one term, apart from ϵ_t , involved on the righthand side of (20) , e.g.

$$y_t = a_h x_{t-h} + \epsilon_t \quad , \tag{21}$$

then the phase of cross-spectrum between $\{x_t\}$ and $\{y_t\}$ (when looked upon as a function of f) is in proportion to f . This should be obvious intuitively because (21) implies h months of lag of y_t behind x_t in every frequency. The constant magnitude of lag in terms of calendar time means that the magnitude of lag in terms of phase angle is in inverse proportion to the period, i.e., in proportion to the frequency. Tukey calls such a lag-relation fixed time lag.[25]

If the sequence of weights $a_o, a_1 \ldots$ in (20) is symmetric about h so that

$$a_{h-1} = a_{h+1}, \quad a_{h-2} = a_{h+2}, \ldots \quad a_o = a_{2h}, \; a_{2h+j} = 0 \quad (j>0)$$

then the phase of cross-spectrum between $\{x_t\}$ and $\{y_t\}$ satisfying (20) is just the same as the phase of cross-spectrum between $\{x_t\}$ and $\{y_t\}$ satisfying (21). In other words, what appears to be h months fixed time lag on the cross-spectrum could very well be a symmetric distributed lag with an average lag of h months. This point must be kept in mind in the interpretation of cross-spectrum.

In the general case of (20) where the weights $a_o, a_1, \ldots$ need not be symmetric, the phase of the cross-spectrum is generally neither fixed angle lag nor fixed time lag. The author has studied only the case where a_j's follow the law of the decaying exponential,

$$a_j = e^{-cj}$$

where c is some positive constant. Within the range of values of c which would be reasonable in economics, this phase declines very rapidly as the frequency moves from zero, and declines more slowly as the frequency comes to the region that corresponds to the cyclical component. The phase of the cross-spectrum between $\{x_t\}$ and $\{y_t\}$ related by (20) is

$$\tan^{-1} \frac{e^{-c} \sin 2\pi f}{1 - e^{-c} \cos 2\pi f} .$$

25. G. H. Moore has pointed out to the author that the National Bureau method has an analogy to both the fixed time lag and the fixed angle lag. The fixed time lag "occurs when we compare specific cycle peaks with reference peaks" (quoted from a private communication). Although the lag here is defined in terms of peaks and troughs alone, the lag is measured by using calendar time. The fixed angle lag "occurs when we determine the reference stages over which a series typically expands or contracts. The stages cover thirds of a reference phase, not a fixed number of months", (quoted from a private communication).

IV. SELECTION OF CYCLE INDICATORS AND REFERENCE SERIES

In Business Cycle Indicators, Vol. II, G. H. Moore lists about seventy cycle indicators (excluding the diffusion indexes). With his advice, we selected twenty-nine of them as the objects of our study. This selection was based upon the following three principles: 1) series that cover no more than 15-20 years are eliminated, because they are too short for the purpose of studying cyclical components by spectral analysis; 2) series which have lost their significance due to the appearance of better series are included if they are sufficiently long, because the purpose of our study is not the prediction of movements in the economy for a few months ahead of the time of this writing; 3) series for which only seasonally adjusted data are available are eliminated. Some of the cycle indicators are seasonally adjusted at the source, and not only are the unadjusted data unavailable but the adjustment methods are not known in detail or are extremely complicated so that its effect upon the spectrum cannot easily be ascertained. Some doubt has been expressed by econometricians as to the effects of the seasonal adjustment methods on each frequency. Therefore the safest approach is to avoid the use of data which have been seasonally adjusted until a large-scale study of the methods of seasonal adjustment is made.[26]

Of the twenty-nine cycle indicators, twenty-six are monthly series. Twenty-three of them start from 1919 or can be so treated without discarding a significant part of the available data. These twenty-three indicators are grouped as Class I. Three monthly cycle indicators start before 1919, and they are included in Class II. Three cycle indicators are quarterly and they are included in Class III.

Within each class of cycle indicator, one indicator is chosen as the reference time series with which the other indicators in that class are to

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26. The author is informed that . M. Nerlove is engaged in a study of this kind.

be compared in estimating the lead-lag. For Class I the reference time series is the Federal Reserve Industrial Production Index; this choice is made on the advice of G. H. Moore. His suggestion is that this is one of the best reference series because of the sensitivity of manufacturing and mining production to business cycle movements. For Class II the reference time series is bank clearings. For Class III the reference time series is GNP in current prices.[27] The cross-spectrum between the Federal Reserve Industrial Production Index and bank clearings indicate the absence of lead or lag between the two series in regard to the frequencies of the cyclical component.

V. STRENGTH OF THE CYCLICAL COMPONENT

In this section we shall digress from our main outline in order to comment on the importance of the cyclical components, as measured by their variance contribution, relative to the other time series' components. As judged from the estimates of spectra of business cycle indicators, frequencies lower than those for the cyclical components contribute far more to the total variance than frequencies for the cyclical components. In other words, the trend and long waves are far more important components than is the cyclical component. On the other hand, frequencies higher than those for the cyclical components contribute less to the total variance than the frequencies for the cyclical components for most of the business cycle indicators. (An exception is, e.g., Department Store Sales for which the seasonals are very strong.) In other words, the cyclical components are more important than the seasonals and other very short-run fluctuations for most of the

27. For many cycle indicators, as for most economic time series data, there are "breaks" in the series. We have used the longest possible piece of data without doing the work of "piecing together" a series; the one exception to this rule is the Retail Sales series.

Since recent data are frequently subject to revisions, we have used data only up to June 1961. Some of the cycle indicator series have been revised since the publication of Business Cycle Indicators and we have used the revised data supplied by Dr. Moore.

business cycle indicators.

Another interesting problem is whether or not the frequency interval for the cyclical component is well defined by the shape of the spectrum. If there is a gap in the spectrum between the frequencies for

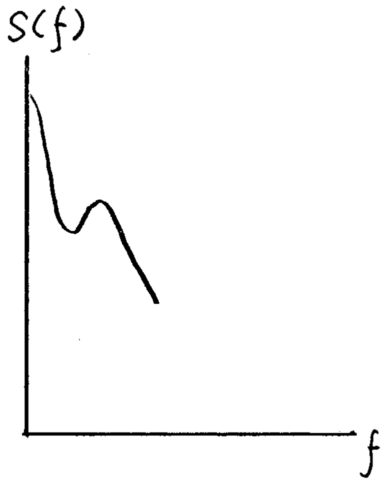

Fig. 11

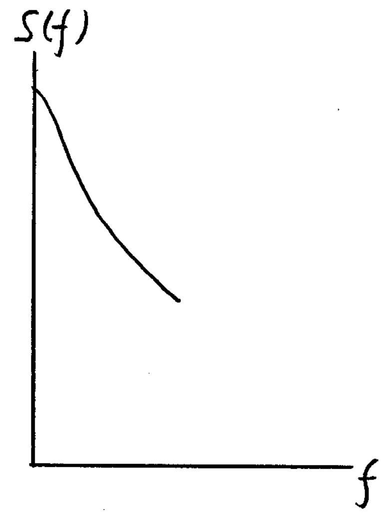

Fig. 12

the cyclical component and the lower frequencies, as illustrated in Figure 11, then the cyclical component can be considered as meaningful on this basis. If there is no such gap, as illustrated in Figure 12, then the cyclical component is not meaningful on the basis of spectrum alone. The gap which is being sought is located, if it exists, in a very low frequency, which is usually less than $\frac{6}{N}$ for most of the business cycle indicators. Therefore, this problem cannot be properly solved for most of the business cycle indicators. The general contours of the spectra of most of the cycle indicators covering the period since 1919, however, are found to be very high at the very low frequencies near zero, and steadily declining as we move toward the higher frequencies, but the decline stops or becomes less rapid or even reveals a slight rise as we reach the frequencies corresponding to the cyclical components.

In the case of two cycle indicators, the Standard and Poor Common Stock Price Index and bank clearings, the available data cover the period from the 1870's to the 1950's and are long enough for study of the possible gap in the spectrum mentioned above. The spectra of these two series indicate a peak about $\frac{1}{40}$ c/m and a gap about $\frac{1}{48}$ c/m. (The peaks would probably be statistically insignificant although the significance test is difficult because of non-stationarity.) However, these peaks and gaps occur over a very narrow frequency interval (about $\frac{1}{60}$ c/m from bottom to bottom or from peak to peak), and are observed not on the spectra of these series themselves but only on the spectra of the logarithm of these series. The reason why the peaks and gaps do not appear in the spectra of the original series could possibly be that the variances of the original series have increased over time, whereas the variances of the logarithm of the series appear to be roughly constant. (See the earlier comment on the pseudo-spectrum of $\{d_t x_t\}$.)[28] The author does not know whether or not any economic significance can be attached to these extremely narrow peaks and gaps.

Apart from this difficulty of interpretation, the general indications are that the cyclical component is not a meaningful concept on the basis of spectrum alone.

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28. As far as the bank clearing data are concerned, the peak at $\frac{1}{40}$ c/m is revealed more clearly during the period before 1914 than during the period after 1919. We have also analyzed the spectrum of logarithm of pig iron production, which is a very long time series, although this series is not treated by the National Bureau as an indicator. Again a peak is revealed about $\frac{1}{40}$ c/m and its harmonic, $\frac{1}{20}$ c/m, for the periods both before 1914 and after 1919.

VI. RESULTS OF THE CROSS-SPECTRAL ANALYSIS

We have estimated the cross-spectra between the indicators and the reference time series of the classes to which they belong. Our main interest lies in the coherence and phase derived from the cross-spectra, as the coherence indicates how closely each cycle indicator is related to the reference time series and the phase measures the lead-lag between the indicator and the reference time series.

The results of the estimation are summarized in Table I. Let us study the lead-lag relations in Table I. We first observe that there are many cases where this relation is very complicated. In fact, no models of the relationships which the author knows fit the results of cross-spectral analysis in these cases. Second, some indicators such as lay-off rate and business failures have an inverse relationship with the general activity of the economy. This is revealed in the cross-spectrum as a nearly 180° phase difference. When the phase difference is $(180^{\circ} + x^{\circ})$ or $(180^{\circ} - x^{\circ})$, we show it in Table I as "inverted series leading (or lagging) by x°". Third, we observe some examples of fixed time lead or lag in Table I. The symmetric distributed lag is not necessarily an unreasonable model when a chain of relations is involved, e.g., $\{z_t\}$ depending upon $\{y_t\}$, and $\{y_t\}$ depending upon $\{x_t\}$. This is because even if the distribution of weights in the lag between $\{x_t\}$ and $\{y_t\}$ and this distribution between $\{y_t\}$ and $\{z_t\}$ are very skewed, this distribution between $\{z_t\}$ and $\{X_t\}$ is more symmetric. Further, symmetric distributed lag is not necessarily an unreasonable model when aggregate data are involved. This is because even if the distribution of weights in regard to a single individual economic unit is skewed, the different magnitudes of reaction time (the time for which the weights are zero) tend to make the distribution of weights for the aggregate more symmetric particularly when a short time unit, such as a month, is used.

In regard to the coherence in Table I, we can state that the cyclical component is the frequency band in which the coherence tends to be higher than in any other frequency band, except for the seasonal frequency points. It is true that the coherence in terms of the cyclical component is low for many cycle indicators, but these indicators generally have low coherence in all the frequency bands. Thus we can say that the cyclical component is a meaningful concept in terms of the cross-spectrum.

VII. JUDGMENT OF THE CYCLE INDICATORS: COMPARISON WITH THE NATIONAL BUREAU RESULTS

It must be emphasized that only about one-half of the cycle indicators have reasonably high values of coherence (with the reference time series) in regard to the frequency band corresponding to the cyclical component. The indicators having low coherence can hardly be treated as good indicators, whether leading or lagging.

A meaningful significance test for the estimates of coherence is extremely difficult because of the non-stationarity of the underlying processes. We might take 0.5 as a point to distinguish between high and low coherence. This value, 0.5, is the estimate of the coherence between the Federal Reserve index of total industrial production and the National Bureau zigzag business cycles obtained by connecting linearly the consecutive reference dates since 1919. This zigzag cycle $x(t)$ can be defined as

$x(t) = +1$, if t = a peak reference month

$x(t) = -1$, if t = a trough reference month

$x(t) = -1 + \frac{2(t-T_1)}{T_2-T_1}$, if $T_2 \geqq t \geqq T_1$, and if T_1 is a trough reference month and T_2 is the next peak reference month

$x(t) = 1 - \frac{2(t-T_1)}{T_2-T_1}$, if $T_2 \geqq t \geqq T_1$, and if T_1 is a peak reference month and T_2 is the next trough reference month

Among the monthly series that we have studied the following have a high coherence with the reference series: Average Work Week, Lay-off Rate*, New Orders of Durable Goods, Housing Starts, Residential Building Contracts, Raw Materials Spot Price, Employment*, Freight Car Loadings*, Bank Debits outside N.Y., Retail Sales, Department Store Sales*, and Manufacturer's Inventory. (Asterisk indicates series having extremely high coherence.) Between the two quarterly series we have studied, the corporate profits after tax has high coherence.

The direction of lead-lag as measured by the cross-spectral analysis is almost invariably the same as the direction as obtained by the National Bureau method. Whereas the National Bureau method measures the lead-lag only in terms of peaks and troughs, the cross-spectral analysis measures it in terms of all time points. Therefore, the coincidence of the two results is remarkable. There are three cases, however, where the results of the two methods do not coincide. One is the Industrial Raw Materials Spot Price index, which is treated as a leading indicator by the National Bureau, whereas in our results the phase is zero on the average over the frequency interval of the cyclical component. However, the phase of the cross-spectrum is very complicated in this particular case, and it would be worthwhile to describe its details.

About $\frac{1}{48}$ c/m , the Industrial Materials Spot Price leads the reference series (industrial production index) by about 35°, i.e., 48 months × $\frac{35}{360} \approx 5$ months . About $\frac{1}{24}$ c/m , where the coherence is higher than in the previous frequency, the industrial materials spot price lags by about 50°, i.e., 24 months × $\frac{50}{360} \approx 3$ months. However, the appropriate significance test for the phase angle is very difficult because of the possible non-stationarity, and the author does not know how much significance we can attach to these figures.

Another discrepancy is Retail Sales. This is treated as a coincident indicator by the National Bureau, but according to our results, it is slightly leading in all frequencies corresponding to the cyclical component. Yet another discrepancy is Department Stores sales, which is treated as a coincident indicator by the National Bureau. According to the cross-spectral analysis this series is roughly coincident about $\frac{1}{48}$ c/m and leading about $\frac{1}{24}$ c/m .

In order to make a proper comparison between the estimates of lead-lag by the National Bureau method and the results of cross-spectral analysis, the figures in Column (E) of Table I are obtained by using the reference series adopted above for each class rather than the National Bureau reference dates. (This procedure was proposed to the author by G. H. Moore, and he has kindly supplied the results of his calculations.) Figure 11 is the cross-tabulation of the lead-lag estimated by the National Bureau method and the cross-spectral analysis for Classes I and II. (Class III are quarterly series.) Figure 13 indicates that the two estimates are quite similar. It must be pointed out, however, that the discrepancy between the two estimates can be large in individual cases. (See, e.g., Class III, (2) of Table I.)

<table>
<tr><th colspan="7">NBER; column (E) of Table I</th></tr>
<tr><th>cross-spectrum; column (D3) of Table I*</th><th>I
less than or equal to -9</th><th>II
-5.5 to -8.5</th><th>III
-2.0 to -5.0</th><th>IV
+1.5 to -1.5</th><th>V
+5.0 to +2.0</th><th>VI
+8.5 to +5.5</th></tr>
<tr><td>I less than or equal to -9</td><td>||</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>II -5.5 to -8.5</td><td></td><td>|</td><td></td><td></td><td></td><td></td></tr>
<tr><td>III -2.0 to -5.0</td><td></td><td>|||</td><td>||</td><td>|</td><td>|</td><td></td></tr>
<tr><td>IV +1.5 to -1.5</td><td></td><td></td><td>|</td><td>||||||</td><td></td><td></td></tr>
<tr><td>V +5.0 to 2.0</td><td></td><td></td><td></td><td></td><td></td><td>|</td></tr>
<tr><td>VI +8.5 to +5.5</td><td></td><td></td><td></td><td></td><td></td><td>|||</td></tr>
</table>

* The lead-lag in terms of phase angles is converted into months by using 48 months as the standard duration of business cycles. Thus, the figures used for the construction of Fig. 13 are different from those in Table I for such cases as Class I (20).

Fig. 13

VIII. INVENTORY CHANGE AS A LEADING INDICATOR

Economic activity in general is based upon expectations as to the future movement of economic or non-economic "variables". A good strategy for finding leading indicators would therefore be to search among the series which are <u>directly and immediately</u> influenced by changes in expectations. One such series could be found from those which are related to the adjustment of labor input. Another series could be found from those which reflect the desired amount of inventory (especially of raw materials), unfilled orders from the standpoint of producers and outstanding orders from the standpoint of purchasing departments. Although we do not have the data for desired inventory, we can study the inventory series. In the rest of this paper we shall study the series related to labor inputs and the inventory series.

In Table I we have two series concerned with inventory. Change in Business Inventory is a quarterly series. The coherence is not too high, and the lead is slightly over one quarter, which makes the series a poor indicator. (Incidentally, the National Bureau method gives a far longer lead.) The other series is Manufacturers' Inventory, for which the coherence is a little higher even though this series itself lags. There is a possibility that the change in the series might lead. The reason is as follows. Let x_t be the inventory level. Then the inventory change y_t is

$$y_t = x_t - x_{t-1} \quad ,$$

a special case of (17). Thus

$$\begin{aligned} S_y &= |1-e^{i2\pi f}|^2 S_x \\ &= 2(1-\cos 2\pi f)\, S_x \end{aligned} \tag{22}$$

phase difference between y_t and x_t is

$$\tan^{-1} \frac{-\sin 2\pi f}{1-\cos 2\pi f} \tag{23}$$

The numerical evaluation of (22) and (23) for frequencies for the cyclical component and any lower frequencies shows (1) that $2(1-\cos 2\pi f)$ is very small there and (2) that the phase difference is slightly less than 90°, y_t leading x_t.

Using the result shown in Table I we have estimated the lead-lag between the change in Manufacturer's Inventory and the reference series of the class (the Federal Reserve Board production index) to which this series belongs. The result is that the change in Manufacturer's Inventory leads by about 8 months at $\frac{1}{48}$ c/m and by about 2 months at $\frac{1}{24}$ c/m. However, the coherence is low, about 0.4 at $\frac{1}{48}$ c/m and 0.5 at $\frac{1}{24}$ c/m. This makes the change in the manufacturer's inventory a poor indicator. It might be interesting to note, however, that the spectrum of the changes in the manufacturer's inventory shows a sharp peak at $\frac{1}{48}$ c/m in spite of that $2(1 - \cos 2\pi f)$ is very small for the frequencies lower than or equal to $\frac{1}{48}$ c/m.

The series, New Orders of Durable Goods, is also directly influenced by expectations. Unfortunately this series does not have a high coherence. Probably some other forms of New Orders might have a high coherence. A very careful study of New Orders as to its relationship to expectations and production scheduling is certainly called for in future studies of business fluctuations.

IX. LABOR INPUT SERIES AS LEADING INDICATORS

The Lay-off Rate is one of few leading indicators having a very high coherence with its reference series. Further, the Lay-off Rate has a number of other clearly connected time series, e.g., average work-week, accession rate, etc., all of which form the time series representing labor inputs in the manufacturing area. Thus, not only the one time series, but also the relationships among this whole family of indicators could be useful for indicating cyclical movements in the economy.

Labor input has three dimensions: (1) number of workers, (2) number of working hours per unit of time (usually a week), and (3) quality of the services rendered by workers. Fortunately the third dimension, which is the most difficult to treat, seems to be unimportant for the study of the cyclical component.

For the aggregate of workers employed in the manufacturing sector and also changes in the series over time, we have monthly time series data for (1) the index of total number of (production) workers, as of the middle of the month, (2) the rate of lay-offs per 100 workers, as of the middle of the month[29], (3) the rate of gross accession per 100 workers, and (4) the rate of quits and the rate of discharge per 100 workers.[30] In the present study, we have used the first two series without any transformation, but we have transformed the last two series into (5) the rate of gross accession minus quits minus discharge, which we call the rate of compensated accession, and (6) the rate of gross accession minus quits minus discharge minus lay-offs, which we call the rate of net accession.

The reason why these transformations are made is to get the variables which appear to be relevant to the cyclical adjustment by employ<u>ers</u> of labor input to changes in outputs. Quits are motivated (at least in the immediate sense) by employ<u>ees</u> rather than by employ<u>ers</u>, and, we might assume for the aggregate of manufacturing sector that any changes in the labor input caused by the initiative of the employ<u>ees</u>, other things -- especially output -- being equal, are offset by the actions of the employ<u>ers</u>. Thus quits are subtracted from gross accession. Discharge is quantitatively insignificant, and although it is initiated by employers, the motivation has nothing to do with the adjustment of labor input to the cyclical changes in output, and we might assume that the loss of labor input due to discharge, other things being equal, is offset by accession. Thus, discharge is also subtracted from gross accession.

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29. As of the end of the month before 1945.

30. Accession, lay-offs, quits, and discharge are for production workers only before 1943 and for production and non-production workers thereafter. This is a kind of non-stationarity which does not seriously affect the interpretation of the cross-spectra. See Section II, F, 0.

The reason why the rate of net accession is formed is the mathematical relation mentioned in (23), i.e., that for any time series $\{x_t\}$, the time series of the change in x_t, i.e., $\{x_t - x_{t-1}\}$ leads $\{x_t\}$ by about 90° at the frequencies of the cyclical component.

The purpose of our study is to find a consistent set of lead-lag relations between the output and the labor input series (1), (2), (5) and (6) mentioned above. We have used monthly data for the period from 1932 to 1958.[31] Table 2 summarizes the result. Naturally, the coherence between output and employment is very high. Employment (the number of workers) lags slightly behind output.[32] Later we shall find that the average work week multiplied by employment, i.e., the time series of manhours, virtually coincides with output. (There is some evidence, however, that the man-hours series leads the output series by about 4°, or about .5 months at $\frac{1}{48}$ c/m.) This is a somewhat puzzling result since the time necessary for a raw material to go through the production processes of a plant is about 3 months, on the average for all manufacturing plants. Even after we account for the fact that a large part of the output data is estimated by using man-hours data, we should expect the time series of man-hours to lead output by about a month.

The coherence between output and the lay-off rate is higher than the coherence between output and the compensated accession rate in the frequency band of the cyclical component. Because of this, the coherence between the output and the net accession rate lies about half way between the coherence between output and the lay-off rate and the coherence between the output and the compensated accession rate.

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31. Actually the data for the first 12 months and the last 12 months are lost because of the filtering process necessary for estimating the spectra and cross-spectra. Thus, in effect, the time period studied here is from 1933 to 1957.

32. Gordon says that the estimate of lead-lag by the National Bureau method also shows a slight lag of employment behind output. R. A. Gordon, Business Fluctuations, p. 289.

The high coherence between the output and the lay-off rate is consistent with our knowledge that under the present industrial man-hour policies cyclical adjustments of labor input to output are made mostly through lay-offs and rehiring of laid-off workers. Further, the low coherence between the output and the compensated accession rate implies either that the compensated accession differs widely from the rehiring of the laid-off workers or that the relation between output and lay-offs is not a reversal of that between output and rehiring. Lack of adequate data for re-hiring of laid-off workers prevents us from pursuing these questions. We can conclude that among the time series concerned with the number of workers (rather than the number of working hours), the lay-off rate seems to be the best indicator of business cycles.

Unfortunately, the magnitude of lead of the lay-off rate is not so large as one might hope for. Further, the structure of lead-lag, as described by the phase of cross-spectrum (see Section III), is a little complicated in this particular case. In the frequency band of the cyclical component, the phase somewhat resembles a fixed time lead of approximately 3 months. It is probably also consistent with some sort of distributed lag. Over the frequency between 0 and $\frac{1}{4}$ c/m, however, the phase of the cross-spectrum reveals a remarkable fixed angle lag, if we ignore the small oscillations in the phase diagram probably caused by sampling fluctuations, and if we exclude the harmonics of the seasonal frequencies.

Now, let us turn to another dimension of labor input, the number of working hours. G. Bry has made a very imaginative and instructive study of the lead of the average work week in manufacturing using the National Bureau method.[33] In terms of peaks and troughs, the average work week was revealed to be an excellent leading indicator. Table I

33. Gerhard Bry, "The Average Workweek as an Economic Indicator", Occasional Paper 69, the National Bureau of Economic Research. His study includes also the lay-off rate and other series.

indicates that according to the cross-spectral analysis, the average work week leads output only slightly and the coherence is not impressive. Thus, in terms of all time points, the average work week is not a very good indicator. An obvious explanation is that one of the main reasons why the average work week is an excellent leading indicator in terms of peaks and troughs, i.e., the removal of overtime when reduction of output is anticipated, is a fact characterized only by peaks; when all time points are included in the study, this important fact is just washed away.

Since the average work week times employment is labor input -- apart from quality change -- let us investigate this product. First, the cross-spectral analysis reveals that there is no lead or lag between output and labor input. The change in the product of the average work week and employment, which will be written as $\Delta(WE)$, can be represented, in its first approximation, as the sum of

(the change in the average work week) × employment
which will be written as $(\Delta W)E$,

and

(the change in employment) × average work week
which will be written as $(\Delta E)W$.

Since it is mathematically known (see Section IX) that $\Delta(WE)$ leads WE by about 90° in the frequency band of the cyclical component, it would be interesting to see if either $(\Delta W)E$ or $(\Delta E)W$ leads WE by more than 90°. To represent ΔE, we have used the data defined as the net accession rate previously. Our study indicates that $(\Delta W)E$ indeed leads output by more than 90°. The lead of $(\Delta W)E$ over the output is greater than the lead of $(\Delta E)W$ over the output; this is further confirmed by the phase of the cross-spectrum between $(\Delta W)E$ and $(\Delta E)W$. The coherence between output and $(\Delta W)E$ is about 0.7 in the frequency band of the cyclical component. Thus $(\Delta W)E$ may be a very good leading indicator.

Finally, one would wonder about the choice between the lay-off rate and $(\Delta W)E$. Our study indicates that $(\Delta W)E$ leads the lay-off rate although the coherence between $(\Delta W)E$ and the output is a little lower than the coherence between the lay-off rate and output.

X. SUGGESTIONS FOR FUTURE STUDIES

Economic relationships are presumably changing over time, and the magnitude of leads or lags is also changing. The author's preliminary study of the changes in the phase angles of economic time series indicates the drastic changes in the phase angles when the economy engages in war or just after a war. The estimate of leads or lags obtained from cross-spectra are at best some average of the changing leads or lags over the time period covered by the data which are used in the estimation (see Section II, 0). In order to use business cycle indicators for prediction, we must know how the magnitude of lead or lag has been changing in the past so that we can say how it will move in the future. There are many indications that when the lead-lag is defined in terms of all time points, there have been great changes in the magnitude of the lead-lag. Therefore, we must somehow estimate the changes in the lead-lag over time.[34]

Yet another problem for future work concerns the complicated lags which we observed for many cycle indicators as indicated in Table I. The relation of the phase of the cross-spectrum to the frequency is not linear and therefore no simple lead-lag relationship holds over all the frequencies of the cyclical component. This difficulty might be resolved by distinguishing the frequencies about $\frac{1}{48}$ c/m and the frequencies about $\frac{1}{24}$ c/m. In other words the possibility that the behavior of our economy

34. D. Brillinger and J. Tukey have suggested a method for making such estimates to the author, and a study is now being made by D. Brillinger and the author.

about $\frac{1}{48}$ c/m and the behavior about $\frac{1}{24}$ c/m are fundamentally different must be investigated in future studies of various economic relationships.

Finally, as a possible suggestion to a large research institution one might consider an extension of the present study to analyze a far greater number of business cycle indicators. The author feels that the estimates of lead or lag by the cross-spectral analysis might differ greatly from the National Bureau estimates in cases of choppy time series, and also in cases where the slope of the trend in a given series is greatly different from that of the reference series (Class III, (2) is such a case). However, any general statement as to the possible causes of the systematic discrepancies between the estimates of lead-lag by the two methods cannot be confirmed without the study of a large number of time series. This is because the estimates of cross-spectrum are subjected to many kinds of errors and only the average of the estimates of cross-spectra for many series is reliable.

TABLE 1

For the Cyclical Component

$\frac{6}{N}$ c/m - $\frac{1}{12}$ c/m

(A) Series title Abbreviated	(B) NBER Classification, their code number	(C) Period Covered	(D1) Frequency band in which coherence is fairly high	(D2) Average coherence in that band	(D3) Phase	(E) NBER estimate of lead-lag (median of -lead +lag) using the reference series of each class
CLASS I						
(1) Average Work Week	Leading 1.0	Jan. 1932-June 1961	$(\frac{1}{60}-\epsilon)^{*}-(\frac{1}{16}+\epsilon)$ c/m**	0.55	Fixed time lead 1 mo.	-3 mos.
(2) Gross Accession Rate	Leading 2.0	Jan. 1930-June 1961	$[(\frac{1}{60}-\epsilon)-(\frac{1}{24}+\epsilon)$ c/m***	0.35]	Complicated; avg. angle lead in this freq. band is $\sim 70^{\circ}$ e.g. 48 mos. x $\frac{70}{360}$ = 9 mos.	-8 mos.
(3) Lay-off Rate	Leading after inversion 3.0	Jan. 1930-June 1961	$(\frac{1}{80}-\epsilon)-(\frac{1}{22}+\epsilon)$ c/m - - - - - - - - - - - - - - - $(\frac{1}{22-\epsilon})-(\frac{1}{12}+\epsilon)$ c/m	0.85 .6	Somewhat like fixed time lead 3 mos. after inversion	-6 mos.
(4) New Orders	Leading 4.0	Jan. 1939-June 1961	$(\frac{1}{48}-\epsilon)-(\frac{1}{24}+\epsilon)$ c/m	0.55	Complicated; angle lead, $\sim 25^{\circ}$; e.g. 48 mos. x $\frac{25}{360}$ = 3 mos.	-3 mos.
(5) Housing Starts	Leading 5.0	Jan. 1939-Dec. 1959	$(\frac{1}{48}-\epsilon)-(\frac{1}{36}+\epsilon)$ c/m - - - - - - - - - - - - - - - $[(\frac{1}{30}-\epsilon)-(\frac{1}{12}+\epsilon)$ c/m	0.7 0.45]	Complicated; avg. angle lead in these two bands is 85°; e.g. 48 mos. x $\frac{85}{360}$ = 11 mos.	-9 mos.

TABLE 1 (cont'd)

(A)	(B)	(C)	(D1)	(D2)	(D3)	(E)
(6) Residential Building Contracts	Leading 5.1	Jan. 1925-Dec. 1956	$(\frac{1}{48}-\epsilon)-(\frac{1}{19}+\epsilon)$ c/m	0.7	Complicated; avg. angle lead is 90° 48 mos. x $\frac{90}{360}$ = 12 mos.	-10.5 mos.
(7) Com'l and Ind'l Bldg. Contracts	Leading	Jan. 1925-Dec. 1956	$\left[(\frac{1}{64}-\epsilon)-(\frac{1}{16}+\epsilon)\right.$ c/m	$\left.0.35\right]$	(Fixed time lead 30 mos. ??)	-2 mos.
(8) Business Failures since 1939	Leading after inversion 8.0	Jan. 1939-June 1961	$\left[(\frac{1}{48}-\epsilon)-(\frac{1}{24}+\epsilon)\right.$ c/m	$\left.0.25\right]$	Complicated; inverted series lead by $\sim 40^{\circ}$ in this freq. band; e.g. 48 mos. x $\frac{40}{360}$ = 5 mos.	-6 mos.
(9) Dow-Jones Ind'l Stock Price	Leading 10.1	Jan. 1919 June 1961	$\left[(\frac{1}{48}-\epsilon)-(\frac{1}{24}+\epsilon)\right.$ c/m	$\left.0.40\right]$	Angle lead $\sim 25^{\circ}$ in this freq. band; e.g. 48 mos. x $\frac{25}{360}$ = 4 mos.	-3.5 mos.
(10) Raw Materials Price	Leading 12.1 13.0	Jan. 1919-Dec. 1957	$(\frac{1}{48}-\epsilon)-(\frac{1}{24}+\epsilon)$ c/m	0.55	Complicated; avg. angle in this freq. band is zero.	-1 mo.
(11) Employment in Non-Agriculture	Coincident 13.0	Jan. 1939-June 1961	$(\frac{1}{48}-\epsilon)-(\frac{1}{14}+\epsilon)$ c/m	.8	Angle is zero	0 mos.
(12) Freight Car Loadings	Coincident 15.2	Jan. 1919-June 1961	$(\frac{1}{40}-\epsilon)-(\frac{1}{20}+\epsilon)$ c/m	0.8	Angle is zero	-1 mo.
(13) Factory Employment	Coincident 13.1	Jan. 1919 Dec. 1958	$(\frac{1}{80}-\epsilon)-(\frac{1}{13}+\epsilon)$ c/m	0.85	Angle is zero	0 mos.
(14) Bank Debits outside NY before 1943	Coincident 18.0	Jan. 1919-Dec. 1942	$(\frac{1}{60}-\epsilon)-(\frac{1}{18}+\epsilon)$ c/m	0.75	Angle is zero	+.5 mos.
(15) Bank Debits outside NY after 1943	Coincident 18.0	Jan. 1943-June 1961	$(\frac{1}{36}-\epsilon)-(\frac{1}{18}+\epsilon)$ c/m	0.65	Angle is zero	-.5 mos.

TABLE 1 (cont'd)

(A)	(B)	(C)	(D1)	(D2)	(D3)	(E)
(16) Wholesale Prices	Coincident 21.0	Jan. 1919- June 1961	$\left[(\frac{1}{60}-\epsilon)-(\frac{1}{30}+\epsilon)c/m\right.$	$\left.0.3\right]$	Complicated; avg. angle in this freq. band is zero	+1.5 mos.
(17) Retail Sales	Coincident 20.0	Jan. 1935- June 1961	$(\frac{1}{48}-\epsilon)-(\frac{1}{24}+\epsilon)c/m$	0.6	Complicated; angle lead is ~ 180° in this band; e.g. 48 mos. x $\frac{18}{360}$ = 2 mos.	0 mos.
(18) Dept. Store Sales	Coincident 20.1	Jan. 1919- June 1961	$(\frac{1}{40}-\epsilon)-(\frac{1}{21}+\epsilon)c/m$	0.7	Complicated; avg. angle lead is 17° 40 mos. x $\frac{17}{360}$ = 2 mos.	+4 mos.
(19) Wage cost per output	Lagging 23.1	Jan. 1919- Dec. 1958	$\left[(\frac{1}{40}-\epsilon)-(\frac{1}{24}+\epsilon)c/m\right.$	$\left.0.3\right]$	Complicated; avg. angle lag is 80° in this band, e.g. 40 mos. x $\frac{80}{360}$ = 7 mos.	+7 mos.
(20) Manufacturers' Inventory	Lagging 24.0	Jan. 1939- June 1961	$(\frac{1}{36}-\epsilon)-(\frac{1}{18}+\epsilon)c/m$	0.5	Complicated; avg. angle lag is 70° e.g. 24 mos. x $\frac{70}{360}$ = 5 mos.	+5.5 mos.
(21) Bank Interest	Lagging basically 26.0	July 1939- June 1961	[about $\frac{1}{36}$ c/m	0.45]	Fixed time lag 7 mos.	+6 mos.
(22) Consumer Installment Debt	Lagging 25.0	Jan. 1929- June 1961	$\left[(\frac{1}{48}-\epsilon)-(\frac{1}{24}+\epsilon)c/m\right.$	$\left.0.45\right]$	Complicated; avg. angle lag is 45°; e.g. 48 mos. x $\frac{45}{360}$ = 6 mos.	+5.5 mos.

TABLE 1 (cont'd)

(A)	(B)	(C)	(D1)	(D2)	(D3)	(E)
CLASS II						
(1) Business Failures before 1932	Leading after inversion 8.0	Jan. 1894-Dec. 1932	$0 - (\frac{1}{24}+\epsilon)$ c/m	0.8	Inverted series leads, fixed time lag of 2.2 mos.	-5.5 mos.
(2) Standard & Poor Stock Prices (1875-1956)	Leading 10.0	Jan. 1871-Dec. 1956	$(\frac{1}{240}-\epsilon)-(\frac{1}{80}+\epsilon)$ c/m	0.6	Nothing definite can be said about the phase in this band	-3 mos.

For the Cyclical Component

$\frac{6}{N}$ c/q**** - $\frac{1}{4}$ c/q

(A)	(B)	(C)	(D1)	(D2)	(D3)	(E)
CLASS III						
(1) Corporate Profits after Tax	Leading 9.0	1st qtr 1939-2nd qtr 1961	$(\frac{1}{24}-\epsilon)-(\frac{1}{8}+\epsilon)$ c/q	0.85	Avg. lead is ~ 40°; 12 qtr x 40/360 ≈ 1.3 qtr	-3 qtr.
			$(\frac{1}{8}-\epsilon)-(\frac{1}{4.8}+\epsilon)$ c/q	.70	Avg. angle is about zero	
(2) Change in Business Inventory	Leading 11.0	1st qtr 1939-2nd qtr 1961	$(\frac{1}{24}-\epsilon)-(\frac{1}{4.8}+\epsilon)$ c/q	0.4	Avg. lead is ~ 36°; e.g. 12 qtr x 36/360 ≈ 1.2 qtr	-9 qtr.

*
The reading of frequency 1/60 c/m is the center of the last frequency unit to which the description applies according to the estimates of cross-spectra. Because of the spectral window, the cross-spectrum at this center frequency actually contains a part of the information as to the frequencies lower than 1/60 c/m, and this fact is indicated by $-\epsilon$.

**
The estimates of cross-spectrum at 1/16 c/m contain some information as to the frequencies above this frequency.

***
The frequency bands and the coherence therein are placed within parenthesis in cases where the coherence is low.

****
Abbreviation for quarters per cycle.

TABLE 2

	for the frequency band of the cyclical component		for the frequency band $\frac{1}{12}-\frac{1}{4}$ c/m excluding the seasonal frequencies	
	coherence	phase	coherence	phase
output ⟷ factory employment	very high app. 0.9	0^{o}	invariably very high 0.8	employment lags behind output by 0.5 months
output ⟷ lay-off rate	very high app. 0.8	after inversion of the lay-off rate, it leads by app. 50^{o}	0.7 between $\frac{1}{12}-\frac{1}{6}$ c/m 0.5 between $\frac{1}{6}-\frac{1}{4}$ c/m	after inversion of the lay-off rate, it leads by app. 50^{o}
output ⟷ net accession rate	not so high app. 0.5	net accession rate leads by app. 80^{o}	low	
output ⟷ the rate of change of employment	high app. 0.6	the rate of change of employment leads by app. $80-90^{o}$	0.7 between $\frac{1}{12}-\frac{1}{6}$ c/m 0.5 between $\frac{1}{6}-\frac{1}{4}$ c/m	the rate of change of employment leads by about 50^{o}

APPENDIX I

Errors in the Data and Errors in the Spectral Estimates as Applied to Economic Studies

We shall ascertain the effect of the errors in the data upon the spectrum and cross-spectrum under otherwise ideal conditions, i.e., stationary stochastic processes defined from $t = -\infty$ and $t = +\infty$. Since the logic for this procedure is the same for both the spectrum and the cross-spectrum, we shall discuss the spectrum only.

Let us first note that since the mean is irrelevant to the spectrum, the mean of the errors in the data does not affect the spectrum. This should be a great consolation for the users of spectral analysis in economics. It appears that a large part of the error in economic data is persistent over time. This persistent part can be treated as the mean of the errors. Thus, the non-zero mean seems to be a major part of the errors. The mean of the errors might have a significant trend component which, however, affects only the very low frequencies.

Let us now consider the variance of the errors in the data. Let x_t be the true value, ϵ_t be the error, so that $x_t + \epsilon_t$ will be the observed values. Let us assume that $E(x_t) = 0$, and $E(\epsilon_t) = 0$. Let S_x be the spectrum of x_t, S_ϵ be the spectrum of ϵ_t, $R_{x\epsilon}$ be the co-spectrum between x_t and ϵ_t, and $I_{x\epsilon}$ be the quadrature spectrum between x_t and ϵ_t. Then

$$\frac{R^2_{x\epsilon}}{S_x S_\epsilon} = \frac{R^2_{x\epsilon}}{R^2_{x\epsilon} + I^2_{x\epsilon}} \times C_{x\epsilon} ,$$

where $C_{x\epsilon}$ is the coherence between x_t and ϵ_t. (All the variables that appear here are functions of frequency, f, but to simplify the notations this is not explicitly indicated.) Let

$$\alpha_{x\epsilon} = \frac{R_{x\epsilon}^2}{R_{x\epsilon}^2 + I_{x\epsilon}^2} .$$

Then the spectrum of $(x_t + \epsilon_t) = S_x + S_\epsilon + 2R_{x\epsilon} = S_x + S_\epsilon + 2\sqrt{S_x S_\epsilon C_{x\epsilon} \alpha_{x\epsilon}}$.

Let us make an "educated" guess of the possible errors in the economic data. Suppose that the relative magnitude between x_t and ϵ_t is 10:1 (i.e., 10% estimation error in terms of the standard deviation <u>excluding the mean</u>). If the coherence $C_{x\epsilon}$ is zero, the spectrum of $x_t + \epsilon_t$ is 1% larger than S_x , i.e., the error in the spectrum of $x_t + \epsilon_t$ is negligible. It seems, however, that apart from their persistent means the errors in economic data are highly correlated with the true values. Thus, if $C_{x\epsilon} = 0.8$ (i.e., the correlation coefficient is about 0.9), and $\alpha_{x\epsilon} = 0.8$ (i.e., the phases of x_t and ϵ_t are almost identical), then the spectrum of $(x_t + \epsilon_t)$ is 17% larger than S_x . Under the same values of $C_{x\epsilon}$ and $\alpha_{x\epsilon}$, if the relative magnitude between x_t and ϵ_t is 10:2 (i.e., 20% estimation error), the spectrum of $(x_t + \epsilon_t)$ is 36% larger than S_x . The only consolation in this case is that, if x_t and ϵ_t are so much alike (as $C_{x\epsilon} = 0.8$ and $\alpha_{x\epsilon} = 0.8$), and if the same holds for y_t and its error η_t , then the coherence and angle between $x_t + \epsilon_t$ and $y_t + \eta_t$ must be practically identical to those between x_t and y_t .

Thus, although the effect of the variance of the errors in the data depends upon how the errors are correlated with the true values, this effect can be uncomfortably large. The author's feeling, however, is that the major portion of these errors would be due mainly to the mean and thus be irrelevant to spectral analysis.

As for the errors in the estimates of spectrum, assuming no errors in the data, we must first point out that we never obtain the unbiased estimate of spectrum from a series of finite length. This is because we are estimating a weighted average (smoothed form) of the true spectrum. (See the explanation about the spectral window in Section II, G.) All the currently available estimation methods for spectra and cross-spectra make the assumption of stationarity for the mathematical derivation of the variance of the estimates. Since we are actually dealing with the pseudospectrum in economic studies (see Section II, O), we must say that we know practically nothing about the variance of the estimates in connection with these studies. This is one of the important, unsolved problems in this area. Furthermore, although normality is not assumed in its strict form in the derivation of the variance of the estimates, it remains to be seen whether or not the non-normality which we actually observe in our economy is well taken care of by the currently available statistical methods. This is another important, unsolved problem.

APPENDIX II

Examples of Estimation of a Spectrum and a Cross-Spectrum

The estimations of spectra and cross-spectra involve a great deal of technicalities, which are far beyond the scope of the present paper. We shall illustrate some estimate of one spectrum and one cross-spectrum so that the readers can compare the results of spectral analysis with the information which they obtain from other sources.

Fig. 14 is an estimate of the spectrum of the National Bureau business cycles, on a monthly basis, from 1919 to 1958 obtained by connecting linearly the reference dates (see the definition of $x(t)$ in Section VII). Actually the logarithm of the spectrum is plotted on the Y axis. The width of the unit frequency interval in the X axis is $\frac{1}{240}$ c/m . The spectra for the frequencies higher than $\frac{20}{240}$ c/m are not shown because they are insignificant. The peak of the estimate of spectrum is located at about $\frac{1}{48}$ c/m and this corresponds to the average duration of business cycles, about 50 months, as estimated by the National Bureau. The harmonics of the business cycles are also observed at $\frac{1}{24}$ $(= \frac{2}{48})$ c/m and $\frac{1}{16}$ $(= \frac{3}{48})$ c/m .

Fig. 15 is an estimate of the cross-spectrum between

$$y_{\alpha,t} = x_t + x_{t-1} + x_{t-2} + x_{t-3} + x_{t-4} + \alpha\varepsilon$$

and x_t , covering the time points $t = 5,6 \ldots , 300$. $\{x_t\}$ is a sample of a white noise with normal distribution, and, $\{\varepsilon_t\}$ is another sample of a white noise with normal distribution having the same variance as $\{x_t\}$.

Of course, $\{x_t\}$ and $\{\epsilon_t\}$ are independent. When we put

$$z_t = x_t + x_{t-1} + .. + x_{t-4},$$

$S_z(f)$, the spectrum of $\{z_t\}$, and, $S_x(f)$, the spectrum of $\{x_t\}$, are related by

$$S_z(f) = (1 + 2\cos 2\pi f + 2\cos 4\pi f)^2 S_x(f) .$$

Therefore, the theoretical values of the coherence between $\{y_t\}$ and $\{x_t\}$, i.e. the coefficient of determination defined frequency by frequency, should be

$$C_\alpha(f) = \frac{S_z(f)}{S_z(f)+\alpha^2 S_\epsilon(f)} = \frac{(1+2\cos 2\pi f+2\cos 4\pi f)^2}{(1+2\cos 2\pi f+2\cos 4\pi f)^2+\alpha^2}$$

because $S_x(f) \equiv S_\epsilon(f)$. In the lower half of Fig. 15 are shown the values of $C_\alpha(f)$ computed by the above formulae and the estimates of the coherence of the cross-spectrum between $\{x_t\}$ and $\{y_{\alpha,t}\}$ as obtained from their sample values. The coherence for the frequencies above $\frac{1}{5}$ is not shown because the spectrum of $\{y_{\alpha,t}\}$ is not significant for these frequencies. In the upper half of Fig. 15 are shown the theoretical value of the phase, which is $2\pi - 2\pi \cdot 2f$ because $y_{\alpha,t}$ lags behind x_t by 2 time units and the lag relation is a fixed time-lag, and the actual estimates of the phase of the cross-spectrum between $\{x_t\}$ and $\{y_{\alpha,t}\}$. The estimates for $\alpha = 1$ and $\alpha = 2$ are so close to the theoretical values that they are visibly indistinguishable.

Fig. 14

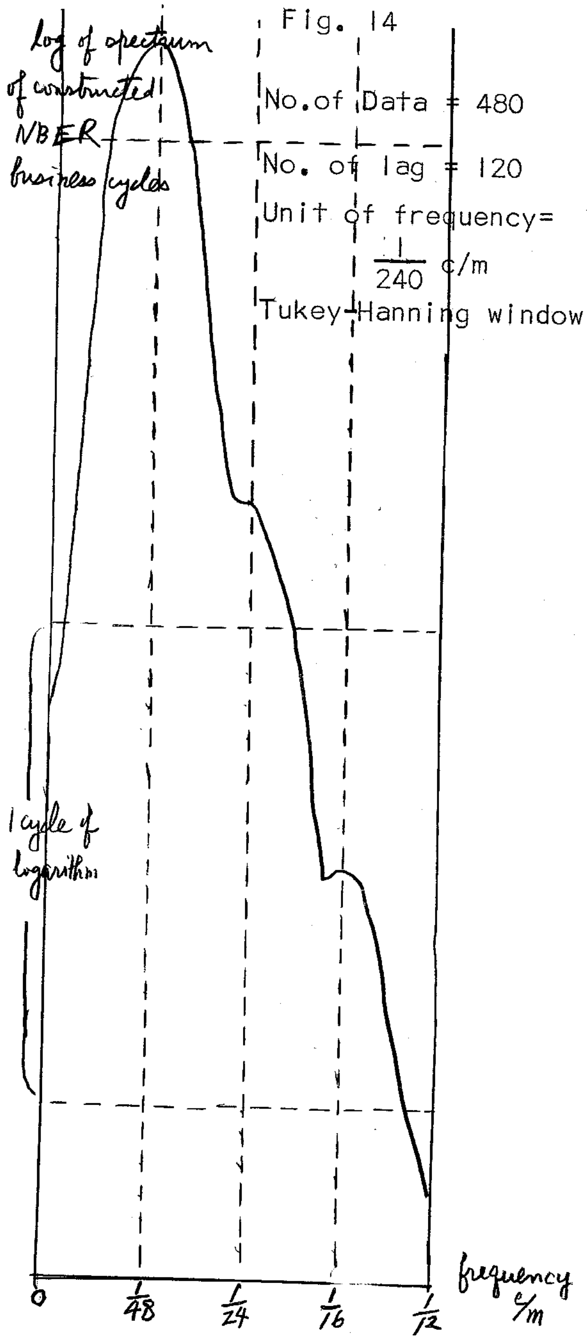
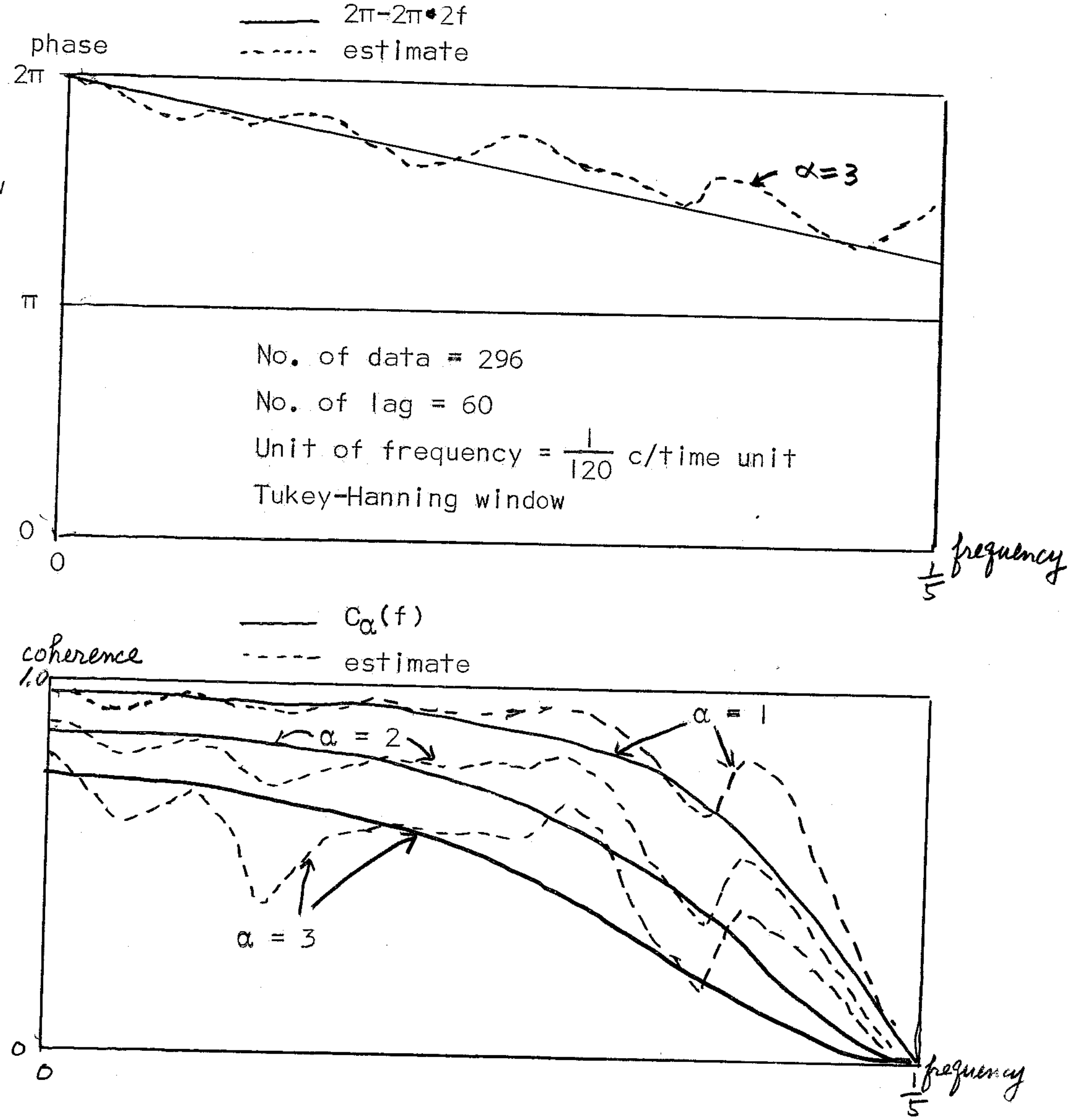

log of spectrum of constructed NBER business cycles
No.of Data = 480
No. of lag = 120
Unit of frequency= $\frac{1}{240}$ c/m
Tukey-Hanning window
1 cycle of logarithm
0
$\frac{1}{48}$
$\frac{1}{24}$
$\frac{1}{16}$
$\frac{1}{12}$
frequency c/m

Fig. 15


phase
2π-2π•2f
estimate
2π
π
0
α=3
No. of data = 296
No. of lag = 60
Unit of frequency = $\frac{1}{120}$ c/time unit
Tukey-Hanning window
$\frac{1}{5}$ frequency
$C_\alpha(f)$
estimate
coherence
1.0
α = 1
α = 2
α = 3
0
$\frac{1}{5}$ frequency