

THE ECONOMICS OF UNCERTAINTY

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Econometric Research Program
Research Memorandum No. 54
20 March 1963

The research described in this paper was supported partially by The Carnegie Foundation of New York, and partially by The Rockefeller Foundation.

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1. Introduction

1.1 In this paper we shall discuss a few of the new and rather unexpected problems which we encounter when we try to introduce uncertainty in some of the classical economic models. In Section 2 we shall study some simple numerical examples which may help the reader to get an intuitive grasp of the problems involved. In Section 3 we shall discuss some practical implications indicated by some slight generalization of our numerical examples. In Section 4 we shall give a heuristic survey of the mathematical problems involved in a full generalization of the simple models developed in the preceding sections.

1.2 The key to the economics of uncertainty appears to be Bernoulli's utility principle, or the "expected utility hypothesis." This principle was first proposed by Daniel Bernoulli (3) in a paper published in 1738, and was applied occasionally and reluctantly by a few economists and statisticians during the following two centuries. One of these was Barrois (2) who published a fairly complete theory of insurance based on the Bernoulli principle as early as 1834. However, in general the principle was ignored until 1947, when it was made respectable and even fashionable by von Neumann and Morgenstern (12).

Von Neumann and Morgenstern proved that the Bernoulli principle can be derived as a theorem from a few simple assumptions as to how rational

people make their decisions under uncertainty. There has been a considerable amount of discussion, often confused, over the general validity of these assumptions. We shall not take up these questions in any detail in the present paper, but we shall discuss the subject briefly in Section 4.

2. Two Numerical Examples

2.1 We shall consider two persons, and assume that each of them owns a business.

Person 1 owns Business 1, which will give a profit of either 1 or zero with equal probability. We shall write $(\frac{1}{2}, 1)$ for this business.

Person 2 owns a business $(\frac{1}{4}, 2)$ which will give a profit of 2 with probability $\frac{1}{4}$, and nothing with probability $\frac{3}{4}$.

According to the Bernoulli principle, the preferences of a person can be represented by a function, which it is convenient to call "utility of money."

If the utility of money to Person 1 is $u_1(x) = 8x - x^2$ he will, according to the Bernoulli principle, assign the following utility to Business 1:

$$U_1(1,0) = \frac{1}{2}u_1(0) + \frac{1}{2}u_1(1) = 3.5$$

Assume further that the utility of money to Person 2 is $u_2(x) = 8x - \frac{1}{2}x^2$. The utility which he assigns to Business 2, i.e., to his own business, is then given by

$$U_2(0,1) = \frac{3}{4}u_2(0) + \frac{1}{4}u_2(2) = 3.5$$

2.2 We now assume that the two persons can exchange shares in their businesses. It is easy to see that they can both increase their utility by such transactions. For instance, if the outcome of an exchange is that Person 1 owns 75% of Business 1 and 50% of Business 2, the utilities of the two persons will be (assuming statistical independence):

$$U_1(0.75, 0.5) = \frac{3}{8}u_1(0) + \frac{3}{8}u_1(0.75) + \frac{1}{8}u_1(1) + \frac{1}{8}u_1(1.75) = 4.42$$

and

$$U_2(0.25, 0.5) = \frac{3}{8}u_2(0) + \frac{3}{8}u_2(0.25) + \frac{3}{8}u_2(1) + \frac{1}{8}u_2(1.25) = 2.84$$

This exchange implies that Person 1 has received a 50% interest in Business 2 in exchange for a 25% interest in Business 1. This obviously means that shares in the two businesses have been traded at a price ratio of 2:1.

The exchange in this example gives Person 1 a substantial utility increase, but it gives Person 2 a lower utility than he has in the initial situation. If Person 2 acts rationally, he will obviously refuse to take part in this exchange.

If the two persons agree to split even, i.e., that both should hold a 50% interest in each business, the utilities will be

$$U_1(0.5, 0.5) = 3.50 \quad \text{and} \quad U_2(0.5, 0.5) = 3.75$$

This means that only Person 2 benefits from the exchange, since the utility of Person 1 remains unchanged. If Person 1 acts rationally, he may be indifferent as to whether he should take part in this exchange or not, or he may suggest another exchange arrangement which will give him a part of the gain.

2.3 The two examples we have considered show that both persons can increase their utility by an exchange of shares. If the two persons act rationally, we would expect them to reach agreement on an exchange arrangement which will give them both a "fair" increase in utility.

Assume now that the two persons have agreed on an exchange (x, y) such that Person 1 holds $100 - x\%$ and $100 - y\%$ interests in the two businesses. If our two persons behave rationally, they will agree on the exchange (x, y) only if there exists no exchange (x_0, y_0) which gives both persons a higher utility. If we further assume that a rational person will not agree to an exchange which gives him a lower utility than he has in the initial situation, we arrive at the following result:

If our two persons behave rationally, they will agree on an exchange (x, y) which satisfies the conditions:

(i) There exists no exchange (x_0, y_0) such that:

$$U_1(x_0, y_0) > U_1(x, y)$$

$$U_2(1-x_0, 1-y_0) > U_2(1-x, 1-y)$$

(ii)

$$U_1(x, y) > U_1(1, 0)$$

$$U_2(1-x, 1-y) > U_2(0, 1)$$

2.4 Condition (i) is obviously satisfied if there exists an exchange (x, y) such that the total differentials

$$dU_1 = \frac{\partial U_1(x,y)}{\partial x} dx + \frac{\partial U_1(x,y)}{\partial y} dy$$

$$dU_2 = \frac{\partial U_2(1-x,1-y)}{\partial x} dx + \frac{\partial U_2(1-x,1-y)}{\partial y} dy$$

have opposite signs for any values of dx and dy . This can happen only if there exists a constant k such that

$$\frac{\partial U_1(x,y)}{\partial x} = k \frac{\partial U_2(1-x,1-y)}{\partial x}$$

$$\frac{\partial U_1(x,y)}{\partial y} = k \frac{\partial U_2(1-x,1-y)}{\partial y}$$

2.5 If we apply this result to our numerical example, we obtain the conditions

$$16 - 2(2x + y) = k (13 + (2x + y))$$

$$16 - 2(x + 4y) = k (11 + (x + 4y))$$

Eliminating k from these equations we obtain

$$65 y = 16 + 17 x$$

This equation together with condition (ii) in para. 2.3 determines the optimal exchanges (x, y) , i.e., the set of all exchanges which two rational persons can agree upon.

Table 2 below shows some of the optimal arrangements which can be reached by an exchange of shares.

2.6 We shall now assume that instead of exchanging shares, our two persons consider a more general arrangement of the following kind:

- (i) If Business 1 succeeds, and Business 2 fails, the profit shall be split so that Person 1 receives x_1 and Person 2 receives $1 - x_1$
- (ii) If Business 1 fails, and Business 2 succeeds, the profit of 2 shall be split as follows:
 - Person 1: x_2 , Person 2: $2 - x_2$
- (iii) If both businesses succeed, total profits , i.e., $1 + 2$ shall be split:

Person 1: x_3 , Person 2: $3 - x_3$

This means that our two persons realize that there are three "states of the world," and that they need an arrangement as to how profits should be divided for each of these states. It is easy to see that an arrangement of this kind is equivalent to an exchange of shares only if $x_1 + x_2 = x_3$.

2.7 Applying the same reasoning as in para. 2.3, we find that a general arrangement (x_1, x_2, x_3) is optimal only if there exists a constant h such that

$$\frac{\partial U_1}{\partial x_1} = h \frac{\partial U_2}{\partial x_1}$$

$$\frac{\partial U_1}{\partial x_2} = h \frac{\partial U_2}{\partial x_2}$$

$$\frac{\partial U_1}{\partial x_3} = h \frac{\partial U_2}{\partial x_3}$$

Applied to our numerical example these conditions give:

$$4 - x_1 = h(7 + x_1)$$

$$4 - x_2 = h(6 + x_2)$$

$$4 - x_3 = h(5 + x_3)$$

Solving with respect to x_1 , x_2 , and x_3 , we obtain

$$x_1 = 4 - \frac{11h}{1+h}$$

$$x_2 = 4 - \frac{10h}{1+h}$$

$$x_3 = 4 - \frac{9h}{1+h}$$

Table 1 below shows some of the optimal situations which can be reached by such general profit-split arrangements.

2.8 Comparison of Table 1 and Table 2 shows that an exchange of shares between the two persons inevitably will lead to a sub-optimal situation. This means that for any exchange (x, y) which appears optimal under the rules of para. 2.3, it will be possible to find a number of general arrangements (x_1, x_2, x_3) which will give both persons a higher utility.

Table 1

Optimal Arrangements

U_1	U_2	<u>Profits received by Person 1</u>		
		State 1	State 2	State 3
3.500	3.775	0.600	0.908	1.217
3.550	3.719	0.612	0.920	1.228
3.600	3.665	0.624	0.931	1.238
3.650	3.610	0.637	0.943	1.249
3.700	3.555	0.651	0.955	1.260
3.750	3.504	0.663	0.966	1.269

Table 2

Sub-optimal arrangements which can be reached
by an exchange of shares

U_1	U_2	<u>Shares held by Person 1</u>	
		Business 1	Business 2
3.500	3,764	0.588	0.400
3.550	3.708	0.600	0.403
3.600	3.650	0.613	0.406
3.650	3.593	0.626	0.410
3.700	3.536	0.638	0.413
3.750	3.479	0.651	0.416

2.9 The two tables do not look very different. The fact that the utility differences appear small has little significance, since the scale for measuring utility can be chosen arbitrarily. However, the arrangements which lead to a general optimum differ considerably from some exchange of shares. This means that a rearrangement of some importance is required in a move from a sub-optimal to an optimal situation.

The differences become even more important if the utility functions of the two persons are essentially different. To illustrate this point, let us consider a second example where

$$u_1(x) = \frac{1}{2} \quad \text{and} \quad u_2(x) = x^{\frac{3}{4}}$$

The conditions for an optimal general arrangement are

$$x_1^{-\frac{1}{2}} = h(1-x_1)^{-\frac{1}{4}}$$

$$x_2^{-\frac{1}{2}} = h(2-x_2)^{-\frac{1}{4}}$$

$$x_3^{-\frac{1}{2}} = h(3-x_3)^{-\frac{1}{4}}$$

Solving with respect to x_i we obtain

$$x_i = (h_0^2 + 2h_0 i)^{\frac{1}{2}} - h_0 \quad (i = 1, 2, 3)$$

where h_0 is an arbitrary constant. Some optimal arrangements for this example are given in Table 3.

Table 3

Optimal arrangements for the second example.

U_1	U_2	<u>Profits received by Person 1</u>		
		State 1	State 2	State 3
4.005	4.626	0.500	0.781	1.000
4.136	4.456	0.531	0.836	1.075
4.247	4.307	0.557	0.884	1.141
4.342	4.174	0.580	0.927	1.200
4.443	4.054	0.600	0.965	1.254
4.500	3.945	0.618	1.000	1.303
4.566	3.844	0.634	1.032	1.348
4.626	3.751	0.649	1.061	1.390
4.730	3.584	0.675	1.114	1.466
4.776	3.508	0.686	1.137	1.500
4.812	3.437	0.697	1.160	1.532
4.858	3.370	0.706	1.180	1.563

3. Economic Interpretation of the Model

3.1 The two examples which we have discussed are obviously too simple to have any economic significance. We shall discuss the various possibilities of generalizing the model in some detail in Section 4. For the time being we shall just note that there is no serious difficulty involved in extending

the model to n persons. If we consider n persons, each owning a business, we can study the various arrangements which they can make in order to reach an optimal distribution of potential profits. It is easy to see that this situation will be essentially the same as that of the two-person case.

In the n person case the number of "states of the world" will be

$$n + \binom{n}{2} + \dots + \binom{n}{n} = 2^n - 1$$

A general arrangement will then be determined by $n(2^n - 1)$ numbers:

$$x_i^{(j)} = \text{profits payable to person } i \text{ if state } j \text{ occurs.}$$

These numbers must obviously satisfy the equations

$$\sum_{i=1}^n x_i^{(j)} = x^{(j)} \quad \text{for all } j$$

where $x^{(j)}$ is the total amount of profits available for distribution in state j .

An exchange of shares is determined by n^2 numbers.

$$x_{ij} = \text{the share person } i \text{ holds in business } j.$$

These numbers must satisfy the conditions

$$\sum_{i=1}^n x_{ij} = 1 \quad \text{for all } j.$$

3.2 From the considerations in the preceding paragraph it follows that if our n persons seek a general arrangement, they have to agree on $n(2^n - 1)$ positive numbers, subject to $2^n - 1$ linear constraints. This means that their optimizing problem has $(n - 1)(2^n - 1)$ "degrees of freedom."

If our persons only consider arrangements which can be reached by exchanging shares, they will have to agree on n^2 positive numbers subject to

n linear constraints. It is clear that this in general will lead to a sub-optimal situation, since the optimizing is done with only $n(n - 1)$ degrees of freedom.

3.3 Let us now assume that these n persons create a stock exchange where they can buy and sell shares in their businesses. If all persons behave in accordance with the usual assumptions of classical economic theory, each share will find its equilibrium price, and the market as a whole will reach a competitive equilibrium. This competitive equilibrium will be stable in the sense that no further exchange of shares is possible without reducing the utility of at least one of the persons who take part in the transaction. This means that no further transactions will take place, if we make the usual assumptions of rational behaviour.

However, we found above that an exchange of shares could, in general, only lead to a sub-optimal situation. This means that if the market has reached a competitive equilibrium by trading of shares, it may still be possible to make more general arrangements which will increase the utility of all participants.

3.4 It is evident from the considerations in the preceding paragraph that there will be an inherent element of instability in the stock exchange created by our n persons. It is always possible that some person will discover that the competitive equilibrium is a sub-optimal situation, and that the market can be brought closer to the general optimum by the creation of new securities, such as preferred shares, premium bonds and investment trusts with different leverage.

If the general optimum is determined by some complicated expression as the one we found in para. 2.9, it is clear that the optimum cannot be

reached by creating a finite number of new securities. However, there is no limit to the number of different securities which can be created by ingenious brokers. This means that if the managers of our stock exchange have sufficient ingenuity, there will be a steady flow of new securities, and by trading in these new securities, the market will gradually approach, but never reach, a general optimum.

3.5 It is evident, even to the most casual observer, that stock markets in the real world are never in a state of stable equilibrium. Such markets usually seem to be moving all the time, in one direction or another. One can explain this by introducing dynamic consideration, and by assuming that the market never has time to reach an equilibrium before external conditions or personal probabilities change. One may also explain the fluctuations in the market as the "tatonnements" of the static Walras model of general equilibrium.

Our model offers another explanation, which may or may not be considered as simpler and more plausible. Heuristically we can formulate our explanation as follows: A competitive equilibrium in our model of a stock market will in general be sub-optimal as long as the number of different securities is finite.

3.6 The considerations in the preceding paragraphs suggest that the natural elements or units in our model may be not the businesses, but the "states of the world." This approach has been explored by Arrow (1) who suggested that the price should be attached to the "state of the world." He introduces a price concept defined as

q_i = the amount one has to pay in order to be assured of receiving one unit of mone if state i occurs.

Arrow shows that with this price concept, and the classical behaviour assumptions, there will exist a competitive equilibrium, which also is Pareto optimal.

With this ingenious device, Arrow is able to save the classical equilibrium model and to make it work also under uncertainty. However, it is not so easy to accept his underlying behaviour assumption that people in the market take a system of Arrow-prices as given, and buy and sell until their utility is maximized. There are certainly a number of people who believe that common shares have a market price, and who act accordingly.

3.7 Stock exchanges offer the most obvious real life example of markets where uncertainty is an essential element. A less familiar example is the reinsurance market. If we reverse the signs in our numerical examples, and assume that the persons seek an arrangement which will lead to an optimal distribution of losses, the model can be interpreted as a reinsurance market. Instead of persons owning businesses we will then have to consider insurance companies holding portfolios of insurance contracts. This interpretation of the model has been discussed in some detail in another paper (6), so we shall not pursue the subject any further.

It is, however, worth noting the interesting growth of the so-called "non-proportional" reinsurance during the last decade. The essence of such reinsurance arrangements is that the reinsurer is called to pay specified amounts only if certain combinations of events (or states of the world) occur. The older forms of "proportional" reinsurance correspond to the arrangements which we referred to as "exchange of shares,"

and as such they will lead to sub-optimal situations. It is possible that practical insurance men have discovered this, and that they found that the market could be brought closer to a general optimum through the introduction of non-proportional reinsurance treaties. If this is so, we have a very instructive example of how a businessman's hunch can be well ahead of economic theory.

3.8 As long as our n persons consider the exchange of shares and other familiar securities (such as cash) as the only way to reach an optimal situation, it seems natural to analyze their problem within the framework of classical economic theory. Concepts like "price," "supply" and "demand" are easily defined, and the usual equilibrium analysis can be carried through without any particular difficulty. The difficulty comes only at the end, when we realize that this approach leads to a sub-optimal situation.

In order to reach a general optimum our persons have to reach a profit-split agreement covering every state of the world. It seems natural to assume that an agreement of this kind is reached through a bargaining process rather than by the help of some semi-automatic price-mechanism. This again should indicate that the theory of n -person games is the appropriate tool for analyzing the problem. It has been shown in some other papers (4) and (5) that this theory can give a fairly realistic explanation of the transactions which take place in the international reinsurance markets.

Brokers seem to play an important part in markets where uncertainty is an essential element. This can be taken as evidence that the classical price mechanism does not work in such markets, since if it did, brokers would be unnecessary middle-men. If, however, these markets are analyzed

in terms of game theory, it becomes clear that brokers fulfill an essential function by helping the people in the market to negotiate their way towards a general optimum.

4. Generalization of the Model.

4.1 In Section 3 we have drawn a number of rather far-reaching conclusions from a mathematical model which is so simple that it must be considered as just a little more than a child's toy. It is hardly possible, without a considerable amount of goodwill, to see that this simple model contains any hints as to how important economic phenomena can be explained. In the following we shall indicate in a heuristic way how the model can be generalized to the extent that may make it useful in serious economic analysis.

We have already mentioned in para. 3.1 that generalization to an arbitrary number of persons does not present any difficulty, and we shall not discuss this question any further.

4.2 In order to obtain a more general concept of a "business" than the one introduced in para. 2.1, we can assume that the profit is X , where X is a stochastic variable determined by a probability distribution $F(x)$. This means that a business is completely described by a probability distribution. If persons owning such businesses shall be able to make rational decisions, we must assume that they have a consistent preference ordering over the set of all probability distributions. This preference ordering can be represented by a functional which assigns the utility $U(F)$ to the distribution $F(x)$. The Bernoulli principle states that there exists a function $u(x)$ such that

$$U(F) = \int_{-\infty}^{+\infty} u(x) dF(x)$$

This representation must hold also in the degenerate case when $F(x) = \epsilon(x-a)$, i.e., when profit is equal to a with probability 1. From this it follows that $u(x)$ can be interpreted as the utility assigned to profits which are certain, or in classical terminology, as the "utility of money."

4.3 It is obviously not very satisfactory to consider a business as defined just by a simple probability distribution. A business is usually a continuing affair so if we want a realistic model, we must also bring in a time element. The obvious way of doing this is to assume that a business can be completely described by a stochastic process $X_1, X_2, \dots, X_t, \dots$, where the stochastic variable X_t is the profit which the business will give in period t .

If the stochastic process is finite, it is completely determined by a joint probability distribution $F(x_1, \dots, x_n)$. We can then follow the argument used by von Neumann and Morgenstern and assume that a rational person has a consistent preference ordering over the set of all joint probability distributions with a finite number of variables.

It then follows that there exists a function $u(x_1 \dots x_n)$ so that

$$U(F) = \int_{-\infty}^{+\infty} u(x_1 \dots x_n) dF(x_1 \dots x_n)$$

There are no mathematical difficulties involved on this point. If the preference ordering is in some sense continuous, such an integral representation will exist.

If we want to remove the finite horizon, and let n go to infinity,

we run into mathematical problems of some complexity, which we shall return to in para. 4.5.

4.4 The immediate difficulty is to determine the shape of the function $u(x_1 \dots x_n)$ for a rational person. This function obviously expresses the "timing preference," i.e., a preference ordering over sequences of profit payments which all are considered as certain. Most economists who have worked with this concept have assumed that there is a preference for earlier payments, i.e., that a sequence such as

(3, 2, 3, 1, 1) is preferred to (2, 2, 2, 2, 2)

In many economic situations it seems natural to assume that such an "impatience element" (Böhm-Bawerk's Minderschätzung) exists. If, however, a steady or steadily increasing payment sequence is preferred to a fluctuating one, the preferences above must be reversed. This will often be a reasonable assumption, and will not present any serious mathematical difficulties as long as n is finite.

The whole subject of timing preference under uncertainty is still relatively unexplored, and it is by no means certain that functions $u(x_1, \dots, x_n)$ which seem acceptable on intuitive reasons will correspond to "reasonable" preference orderings over the set of distributions $F(x_1 \dots x_n)$ — and vice versa.

4.5 It is obviously desirable to remove the finite horizon in our model. This problem has recently been studied by Koopmans (8).

Koopmans defines a utility function $u(x_1, x_2, \dots, x_t, \dots)$ over an infinite sequence of payments. From some innocent looking assumptions about this function he proves that we must have

$$u(x_1, \dots, c_t + a, \dots, x_{t+i}, \dots) > u(x_1, \dots, x_t, \dots, x_{t+i} + a, \dots)$$

for any positive i and a . This means that the existence of an impatience element can be derived as a mathematical consequence of some more basic assumptions about timing preferences.

The most critical of Koopmans' assumptions appears to be his Postulate 3, which in our terms can be stated as follows:

For any positive a and for any payment sequence $x_1, x_2, \dots, \dots, x_t, \dots$, the following inequality must hold

$$u(x_1 + a, x_2, \dots, x_t, \dots) > u(x_1, x_2, \dots, x_t, \dots)$$

The postulate is essentially one of inter-period independence, which has mathematical convenience as its main justification.

4.6 Koopmans (8) has pointed out that it follows from an earlier result of Debreu (7) that a slight strengthening of the independence assumption implies that the utility function must be of the form

$$u(x_1, \dots, x_t, \dots) = \sum_{t=1}^{\infty} r^{t-1} u(x_t)$$

where r is a positive constant.

If we write z^x for the infinite vector $\{x_3, x_4, \dots\}$, the stronger independence assumption can be formulated as follows:

$$u(x_1, x_2, z^x) > u(y_1, y_2, z^x) \text{ implies } u(x_1, x_2, z^y) > u(y_1, y_2, z^y)$$

and

$$u(x_1, x_2, z^x) > u(y_1, x_2, z^y) \text{ implies } u(x_1, y_2, z^x) > u(y_1, y_2, z^y)$$

hold for all x_1, x_2, z^x, y_1, y_2 and z^y .

4.7 It is tempting also to assume full independence in the stochastic process, i.e., that X_1, \dots, X_t, \dots are stochastically independent. This means if $F(x,t)$ is the probability distribution of X_t , the utility assigned to the infinite stochastic process $X_1 \dots X_t \dots$ will be given by

$$U(X_t) = \sum_{t=1}^{\infty} r^{t-1} \int_{-\infty}^{+\infty} u(x) d_x F(x,t)$$

If we introduce $r = e^a$, and write the sum as a Stieltjes integral, the expression becomes

$$U(X_t) = \int u(x) d\{e^{at} F(x,t)\}$$

where integration is over the whole $x t$ space.

This is in many ways a suggestive and interesting formula. It seems to indicate that uncertainty and the time element occur in an almost symmetrical manner in a general economic model. This has been suggested, although necessarily in vague terms by some economists. The most explicit seems to be Morgenstern (10) and (11).

4.8 We reached the result in the preceding paragraph by making strong independence assumptions. In order to construct a realistic model, we must clearly allow for inter-period dependence both in the utility function and in the stochastic process which determines business profits. This will, as far as we can see at present, lead to mathematical problems of a really formidable nature.

If we accept the independence assumptions, it should be possible to determine the optimal states of an n -person market. The optimality conditions of para. 2.3 can be generalized without any difficulties if we use the Kuhn-Tucker Theorem (9) instead of the primitive method of

solution which was adequate for our numerical examples.

4.9 Some readers may have doubts as to the usefulness of pursuing general equilibrium analysis to such generality as that of the preceding paragraphs. In order to illustrate the usefulness of a very general theoretical framework, we shall briefly consider the simple and "practical" problem of formulating the objectives of a firm.

Apparently no modern economist dares to assume that a firm seeks to maximize simple short-run profits. The literature is full of ingenious suggestions as to what firms want to maximize, such as market share, net worth, probability that profits shall stay above a certain threshold, etc., etc.

Instead of making arbitrary, and possibly contradictory assumptions of this kind, we can assume that the manager of the firm realizes that future profits can only be described by a stochastic process. If the manager by his decisions can alter this stochastic process, his job will obviously be to make the decisions which will give the firm the "best" attainable process. This means that if the manager shall be able to make intelligent decisions, he must have a preference ordering over the set of all attainable processes.

If the firm has a finite time horizon, the stochastic processes can be represented by joint probability distributions of the type $F(x_1 \dots x_n)$. The preference ordering can then by the Bernoulli principle be represented by a function $u(x_1 \dots x_n)$.

This means that the general objective of the firm will be to maximize an expression of the form

$$U(F) = \int_{-\infty}^{+\infty} u(x_1, \dots, x_n) dF(x_1, \dots, x_n)$$

This may be a difficult problem, and a practical businessman may seek some "rule of thumb" which gives an approximate solution. For instance, a high market share this year may not give a high profit this same year, but it may increase the probability of high profits in following years. Hence to maximize the market share in the short run may be a good working rule for solving the general maximizing problem.

The assumptions which we have made mean essentially that people are in business for the sake of profits, and that businessmen are more sophisticated about profits than most economists seem to be. These assumptions do not appear entirely unreasonable.

5. Conclusion.

5.1 In this paper we have used some mathematics which may be considered as "advanced," and we have indicated that mathematics of a far more advanced nature may be required in order to develop a complete theory for the economics of uncertainty. It may be useful to conclude with an attempt to explain why such mathematics is essential, owing to the very nature of the problem we set out to study.

5.2 In classical commodity markets people trade "commodity bundles" which can be thought of as vectors with a finite number of elements, or as points in an n -dimensional space. In the markets which we have considered the commodities traded are probability distributions or stochastic processes. A probability distribution can also be thought of as a vector, but this vector will in general have an infinite number of elements, and it can be represented as a point only in a Hilbert space.

The introduction of uncertainty in the classical economic theory means essentially that we go from the finite to the infinite. We cannot

expect this to be an easy step, and we should not be surprised that mathematics of a quite different level is required.

5.3 In classical models it is to a large extent a matter of taste whether we take the underlying market structure to be given in the form of indifference curves, demand and supply functions, substitution rates or utility functions. However, of these practically equivalent concepts utility is only one which can easily be generalized to the infinite case. It is for this reason that the Bernoulli utility concept plays such an important part in the theory which we have tried to outline in this paper.

References

- (1) ARROW, K. J.: Le role de valeur boursières pour la repartition la meilleure des risques. Colloques Internationaux du CNRS XL, pp. 41-48, Paris 1953.
- (2) BARROIS, T.: Essai sur l'application du calcul des probabilités aux assurances contre l'incendie, Lille 1834.
- (3) BERNOULLI, D.: Exposition of a new theory on the measurement of risk. Econometrica 1954, pp. 23-46. (Translation of: Specimen Theoriae Novae de Mensura Sortis, St. Petersburg 1738.)
- (4) BORCH, K.: Reciprocal Reinsurance Treaties seen as a Two-Person Cooperative Game. Skandinavisk Aktuarietidskrift 1960, pp. 29-58.
- (5) BORCH, K.: The Safety Loading of Reinsurance Premiums. Skandinavisk Aktuarietidskrift 1960, pp. 163-184.
- (6) BORCH, K.: Equilibrium in a Reinsurance Market. Econometrica 1962, pp. 424-444.
- (7) DEBREU, G.: Topological Methods in Cardinal Utility Theory. Mathematical Methods in the Social Sciences, pp. 16-26, Stanford 1960.
- (8) KOOPMANS, T.C.: Stationary Ordinal Utility and Impatience, Econometrica 1960, pp. 287-309.
- (9) KUHN, H.W. and TUCKER, A.W.: Nonlinear Programming. Proceedings of the Second Berkeley Symposium, pp. 481-492, Berkeley 1951.
- (10) MORGENSTERN, O.: "Das Zeitmoment in der Wertlehre," Zeitschrift für Nationalökonomie, 1934, pp. 433-458.
- (11) MORGENSTERN, O.: "Vollkommene Voraussicht und wirtschaftliches Gleichgewicht," Zeitschrift für Nationalökonomie, 1935, pp. 337-357.
- (12) NEUMANN, J. von and O. MORGENSTERN,: Theory of Games and Economic Behavior, 2nd edition, Princeton 1947.