MARKETS WITH A CONTINUUM OF TRADERS III

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Abstract

Conditions are given for the core of a market with a continuum of traders to coincide with the set of its equilibrium allocations, that are wider than previously known conditions.

1. Introduction

In two previous papers[1] we introduced the notion of markets with a continuum of traders, and demonstrated their significance as mathematical models for the intuitive concept of "perfect competition". In particular, it was shown that under wide conditions, the core of such a market equals the set of its equilibrium allocations.[2] The main purpose of this paper is to widen still further the conditions under which these results hold.

It is chiefly MCT I that concerns us. Two conditions were imposed in that paper that may be considered restrictive: saturation of desire was ruled out, and it was demanded that each trader come to market with a positive amount of each commodity. Here we show that the proof still remains valid if saturation of desire is permitted for bundles that are coordinate-wise larger than the initial bundles. Further, it will be shown that the requirement of positive initial bundles can be essentially eliminated if the "desirability of commodities" condition is slightly strengthened. More precisely: In MCT I, the desirability condition demands that if a bundle x is coordinate-wise larger than a bundle y, then x is preferred to y. (It is this assumption that implies non-saturation.) Here we show that if this is strengthened so that x is preferred to y even when x is coordinate-wise no smaller than y--as long as it is actually greater in at least one commodity--then the requirement of positive initial bundles can be essentially eliminated; it need only be required then that each commodity is actually present in the market, i.e. that it is held by <u>some</u>

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[1]"Markets with a Continuum of Traders I, II", Econometric Research Program, Research Memoranda No. 39 (5 July 1962) and No. 40 (16 July 1962). These papers will be denoted MCT I and MCT II respectively. Reference to them will be made thus: "Lemma II.4.1" means lemma 4.1 of MCT II; "Section I.5" means section 5 of MCT I.

[2]An "equilibrium allocation" is one which, when combined with an appropriate price structure, yields a competitive equilibrium. The notions of "competitive equilibrium" and of "core" are formally defined in section 2 below; an intuitive discussion may be found in section I.1.

traders. Moreover, in order to permit saturation, the strengthened desirability condition is assumed only when y is not coordinate-wise larger than the initial bundle.

In section 2 we present our model and state our two theorems. Section 3 contains the proofs. In section 4 we present some counter-examples to show that various assumptions in our model cannot be weakened.

2. Mathematical Model and Statement of Results

The set of commodity bundles is the non-negative orthant Ω of a fixed Euclidean space R^n. A member of R^n is called a vector, and its coordinates are denoted by superscripts. A price vector is an n-tuple of non-negative numbers, not all 0; though formally it is in Ω, it should not be thought of as a commodity bundle. The inner product $\Sigma_{i=1}^{n} p^i x^i$ is denoted $p \cdot x$. Relations between vectors and operations on vectors are to be understood coordinate-wise, unless otherwise specified. Thus $x \geq y$ means $x^i \geq y^i$ for all i, $x > y$ means $x^i > y^i$ for all i, and if X is a vector function, then

$$\int X = (\int X^1, \ldots, \int X^n) .$$

The set T of traders is the closed unit interval $[0,1]$. The words "measure", "measurable", "integral", and "integrable" are to be understood in the sense of Lebesgue. All integrals are with respect to the variable t, ranging over all of T or a subset thereof. In writing an integral, we will omit the symbol dt and the indication of dependence of the integrand on t. When the range of integration is not specified, all of T is meant; thus $\int X$ means $\int_T X(t)dt$. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored; in particular, the following conventions are adopted:

(i) When referring to traders, the word "all" means "almost all", i.e. "all except for a null set".

(ii) An assertion that there "is" a trader with a certain property means that there is a non-null set of traders with this property.

These conventions must be constantly kept in mind if the sequel is to be understood.

An <u>assignment</u> is an integrable function on T to Ω. There is a fixed <u>initial assignment</u> I; it is assumed to satisfy either the strong or the weak form of the following condition (depending on the theorem being proved):

(2.1) <u>Strong Form</u>: $I(t) > 0$ for all t.

<u>Weak Form</u>: $\int I > 0$.

The strong form is the same as I.2.1. The weak form is the same as I.2.1(i). Analogues of both appear elsewhere in the economic literature.[3] It asserts that no commodity is completely absent from the market. It is not merely a normalization; the theorems using this assumption are false without it,[4] as will be shown in section 4. On the other hand, one feels intuitively that commodities that are absent from the market are "irrelevant", and should not be considered in the analysis. Unlike the strong form or the subsequent assumptions, the weak form doesn't actually restrict the real-life markets to which our model is applicable; it only gives directions for building the model from a given real-life market, i.e. by considering as commodities in the model, precisely those commodities that are actually present in the real-life market.

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[3] The strong form is the more popular, though many writers have stressed the desirability of replacing it. For an analogue of the weak form, see L. W. McKenzie, "On the existence of equilibrium for a competitive market", <u>Econometrica</u> 27 (1959), p. 58, assumption 5.

[4] The statements made in this connection in MCT II (bottom of p.3) are incorrect; the theorem of MCT II is false if assumption II.2.1(i) is omitted.

For each trader t there is defined a relation $\succ_t$ on Ω, which is called the preference relation of t and is assumed to satisfy the following conditions:

(2.2) Lower-semicontinuity (in the commodities): For each $y \in \Omega$, the set $\{x : x \succ_t y\}$ is open (relative to Ω).

(2.3) Measurability: For all assignments X and Y, the set $\{t : X(t) \succ_t Y(t)\}$ is Lebesgue measurable.

Moreover, the relation is assumed to satisfy either the strong or the weak form of the following condition (depending on the theorem being proved):

(2.4) Desirability-Strong Form: If $x \geq y$ and $x \neq y$, then $x \succ_t y$, unless $y > I(t)$.

Desirability-Weak Form: If $x > y$ then $x \succ_t y$, unless $y > I(t)$.

Condition (2.2) is slightly weaker than I.2.2, which was called "continuity" and included also the assumption that $\{x : y \succ_t x\}$ is open. The latter half was never used in MCT I, as was pointed out in section II.7. In case the given preference relation $\succ_t$ is derived from a transitive, reflexive and complete preference-or-indifference relation $\succsim_t$, and (2.2) holds, then it is possible to define a lower-semicontinuous utility function $u_t(x)$ for $\succ_t$; this fact inspired the name of the condition.

The measurability condition (2.3) is identical with those used in the previous papers, and was discussed there (see MCT I, p. 14--particularly footnote 8; also MCT II, pp. 17 and 18).

The only difference between the weak form of the desirability condition (2.4) and I.2.4 is the phrase "unless $y > I(t)$", which is absent from I.2.4. It is this phrase that permits saturation here. Both forms of

(2.4) forbid saturation except for bundles that contain more of each commodity than the initial bundle; for such bundles saturation is permitted but not required. For other bundles, the weak form permits saturation in one or more commodities, as long as total desire is not saturated, i.e. as long as <u>some</u> commodities are desired; whereas the strong form rules out saturation in <u>any</u> commodity. Geometrically, the indifference curve illustrated in Figure 1 is permitted by the weak form, but ruled out by the strong form.

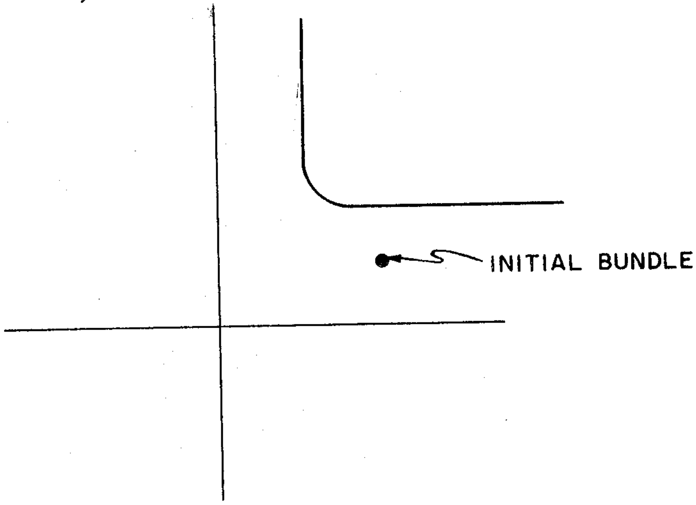

FIG. 1

A <u>coalition</u> is a measurable subset of T. An <u>allocation</u> is an assignment X such that $\int X = \int I$. An allocation Y <u>dominates</u> an allocation X <u>via</u> a coalition S if S is non-null, $Y(t) \succ_t X(t)$ for all $t \in S$, and S is <u>effective</u> for Y, i.e. $\int_S Y = \int_S I$. The <u>core</u> is the set of all allocations that are not dominated via any coalition.

A <u>competitive equilibrium</u> is a pair consisting of a price vector p and an allocation S, such that for all t, $X(t)$ is maximal w.r.t. $\succ_t$ in the <u>budget set</u> $\{x: p\cdot x \leq p\cdot I(t)\}$. An <u>equilibrium allocation</u> X is one for which there is a price vector p such that (p,X) is a competitive equilibrium.

Theorem A: *Under the strong form of (2.1) and the weak form of (2.4), the core coincides with the set of equilibrium allocations*.

Theorem B: *Under the weak form of (2.1) and the strong form of (2.4), the core also coincides with the set of equilibrium allocations*.

Both theorems, of course, assume (2.2) and (2.3).

3. Proofs

The proofs are similar to those in MCT I; essentially the only additions here are lemma 3.2 and formula (3.5).

That every equilibrium allocation is in the core was established in section I.3 using none of the assumptions in any of their forms (cf. Remark I.3.1). For the converse, let X be in the core. We wish to prove that X is an equilibrium allocation.

Lemma 3.1 *There is a price vector* p, *such that* $x \succ_t X(t)$ *implies* $p \cdot x \geq p \cdot I(t)$.

This is an immediate consequence of lemma I.4.1 and the separating hyperplane theorem; in fact it is stated explicitly on p. 9 of MCT I. The proof uses neither (2.1) nor (2.4) in any of its forms; cf. Remark I.4.2.

Lemma 3.2 *For all* t, *not* $X(t) > I(t)$.

Proof Let $X = \{t : X(t) > I(t)\}$; assume S is non-null. Then $\int_S X > \int_S I$, so since X is an allocation, $\int_{T-S} X < \int_{T-S} I$. In particular, $T-S$ is not null.

Applying either form of the desirability condition (2.4), we obtain

$$X(t) + \int_{T-S}(I-X) \succ_t X(t)$$

for all $t \in T-S$. From this and

$$\int_{T-S}[X + \int_{T-S}(I-X)] = \int_{T-S} I$$

we deduce that $X + \int_{T-S}(I-X)$ dominates X via $T-S$; so X is not in the core, a contradiction. This proves lemma 3.2.

Let p be as in lemma 3.1.

Lemma 3.3 $p \cdot X(t) = p \cdot I(t)$ *for all* t.

Proof From lemma 3.2 and either form of desirability (2.4), we conclude that $X(t)$ is a limit point of points preferred to $X(t)$; hence by lemma 3.1, $p \cdot X(t) \geq p \cdot I(t)$ for all t. If there were a trader t such that $p \cdot X(t) > p \cdot I(t)$, then we would conclude that $\int p \cdot X > \int p \cdot I$, contrary to the fact that X is in the core and hence an allocation.

Lemma 3.4 *If* $p \cdot I(t) > 0$, *then* $x \succ_t X(t)$ *implies* $p \cdot x > p \cdot I(t)$.

Proof From lemma 3.1 and the hypotheses of our lemma, we obtain $p \cdot x \geq p \cdot I(t) > 0$. Hence there is a coordinate j such that $p^j > 0$ and $x^j > 0$; w.l.o.g. (without loss of generality) $j = 1$. Define $y_\delta = x - (\delta, 0, \ldots, 0)$; then for δ positive but sufficiently small, $y_\delta \in \Omega$, and by continuity (2.2), $y_\delta \succ_t X(T)$. Again applying lemma 3.1, we obtain

$$p \cdot I(t) \leq p \cdot y_\delta = p \cdot x - \delta p^1 ,$$

and hence $p \cdot x > p \cdot I(t)$, as was to be proved.

Under the hypotheses of Theorem A--i.e. the strong form of (2.1) and the weak form of (2.4)--we can now prove that (p,X) is a competitive equilibrium. Since X is in the core it is an allocation; and by lemma 3.3 $X(t)$ is in the appropriate budget set for all t. Finally, the strong form of (2.1) and the fact that $p \neq 0$ imply that $p \cdot I(t) > 0$, so we deduce from lemma 3.4 that

$$x \succ_t X(t) \text{ implies } p \cdot x > p \cdot I(t) ;$$

but this says precisely that $X(t)$ is maximal in its budget set, and completes the proof that (p,X) is a competitive equilibrium and X an equilibrium allocation. So Theorem A is proved.

Under the hypotheses of Theorem B--the weak form of (2.1) and strong form of (2.4)--we cannot conclude so easily that $p \cdot I(t) > 0$, since we need not have $I(t) > 0$. Instead, we shall prove

$$p > 0. \tag{3.5}$$

If not, assume w.l.o.g. that $p^1 = 0$. There is a non-vanishing coordinate of p; w.l.o.g. let it be the second. By the weak form of (2.1), there is a t such that $I^2(t) > 0$. For this t, let $y = X(t) + (1,0,\ldots,0)$. Then by lemma 3.2 and the strong form of (2.4), we have

$$y \succ_t X(t) \quad ; \tag{3.6}$$

but from (3.3) and $p^1 = 0$ we deduce

$$p \cdot y = p \cdot X(t) = p \cdot I(t) \geq p^2 I^2(t) > 0. \tag{3.7}$$

Hence there must be an i such that $p^i > 0$ and $y^i > 0$. Now define $z_\delta = y - (0,\ldots,0,\delta,0,\ldots,0)$, where δ appears in the i^{th} coordinate;

then $z_\delta \in \Omega$ for δ sufficiently small. By (3.6) and continuity (2.2), we can find a δ so small that $z_\delta \succ_t X(t)$; but then by (3.7),

$$p \cdot z_\delta = p \cdot y - \delta p^i < p \cdot y = p \cdot I(t) \ .$$

Since $z_\delta \succ_t X(t)$, this contradicts lemma 3.1, and so establishes 3.5.

Now if $I(t) \neq 0$, then it follows from (3.5) that $p \cdot I(t) > 0$, and the remainder of the proof of Theorem B is as for Theorem A. If $I(t) = 0$, it follows from (3.5) that the origin is the only point in the budget set. Since $X(t)$ is in the budget set (by lemma 3.3), it is a fortiori maximal in it;[5] and thus Theorem B as well as Theorem A has been proved.

4. Counter-Examples

Our first example shows that (2.1) cannot be entirely eliminated; if the weak form of (2.1) fails, then even the strong form of (2.4) does not suffice to ensure that the core and the set of equilibrium allocations coincide. There are two commodities in the example. All traders have the same initial bundle, namely (1,0), and the same preference relation. The preference relation arises from a transitive, reflexive and complete preference-or-indifference order whose indifference curves are the south-western quarters of circles with center at (α,α) and radius α; α is a parameter ranging from 0 to ∞ (see Figure 2). More precisely: all points in Ω are on the southwestern quarter of one and only one such circle; $x \succ_t y$ if and only if the radius of the circle containing x is greater than the radius of y's circle. The determining feature of the example is that the indifference curves are tangent to the x-axis, yet do not run along it; we have $(x^1,0) \succ_t (y^1,0)$ if and only if $x^1 > y^1$.

[5] The possibility $0 \succ_t 0$ may be ruled out if Lemma I.4.1 is used.

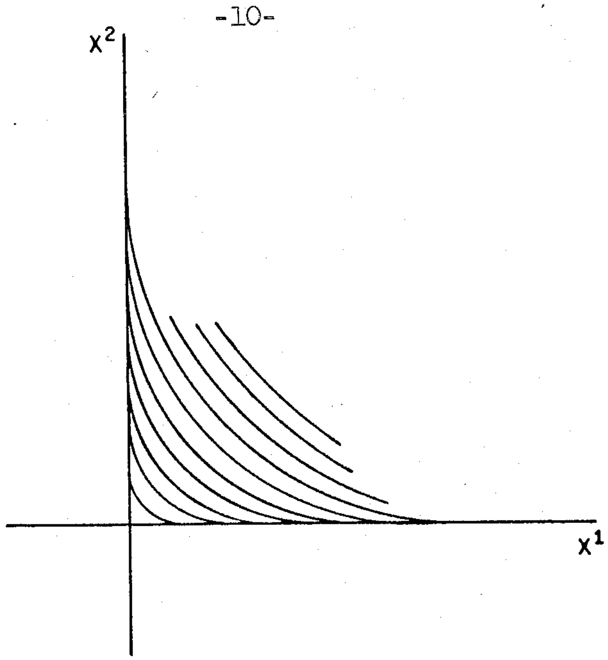

FIG. 2

It is easily verified that the core consists of the single allocation I, i.e. the allocation of the original bundles. So if there is a competitive equilibrium, it must be of the form (p, I). But then the line $p \cdot x = p \cdot I(t)$ must be tangent to the indifference curve passing through $I(t)$, for otheriwse $I(t)$ would not be maximal in the budget set (cf. Figure 3). Thus the line $p \cdot x = p \cdot I(t)$ must necessarily coincide with the

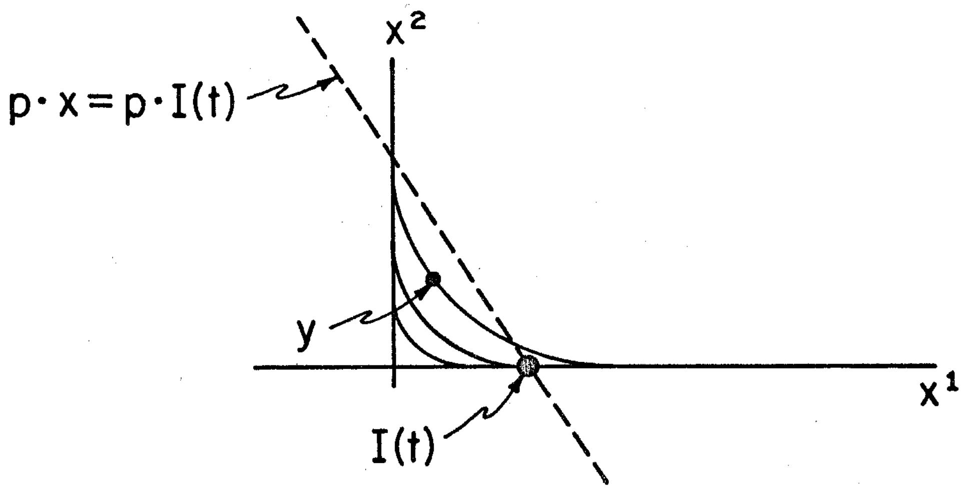

FIG. 3

y is preferred to $I(t)$ and is in the budget set.

x^1-axis. But then any point on the x^1-axis to the right of $I(t)$ is in the budget set and preferred to $I(t)$; so (p,I) is not a competitive equilibrium, and in fact no such equilibrium exists. Hence the core does not coincide with the set of competitive equilibria. Note that the example satisfies the strong form of (2.4).

This example has the following rather startling feature. The market contains, to start with, no amount whatsoever of commodity 2. Under these circumstances it might be thought that commodity 2 can be ignored in computing competitive equilibria. But this is not so; for when we consider the market as one in commodity 1 only, then obviously there _is_ a competitive equilibrium, namely the initial allocation, together with an arbitrary positive price. What we have done, therefore, is to take a market which does have a competitive equilibrium, and by merely _thinking_ of an additional commodity which is not at all available for trading, we have destroyed the equilibrium. This might have fascinating applications, such as the advertising of non-existing products, with the idea of creating preference patterns that would affect market behavior in the products that do exist.

The core does not behave in this curious way; as is easily verified, it is determined solely by the preferences among those bundles containing actually available commodities only. It is therefore unaffected by the consideration of additional "dummy" commodities.

The same example works for a finite number of traders or even when there is only one trader. But of course what we wanted to show is that the core need not coincide with the set of equilibrium allocations even when there is a continuum of traders, unless the appropriate assumptions are satisfied.

A modification of the previous example can be used to show that the strong form is needed either in (2.1) or in (2.4)--the weak form in both does not suffice for our theorem. In this example, some of the traders have the initial bundles and the preference relations of the previous example; the rest have the initial bundle (0,1), and the indifference curves are straight lines parallel to the x^1-axis--that is, $x \succ_t y$ if and only if $x^2 > y^2$. Intuitively, traders of the second kind are interested only in commodity 2. Here again it is easily verified that the core consists precisely of the initial allocation I; but no (p,I) can be a competitive equilibrium, for the same reason as in the previous example.

The last example is concerned with the question of saturation. Both forms of the desirability-of-commodities condition (2.4) permit (but do not require) saturation at bundles that are $> I(t)$, but forbid saturation at all other bundles. Roughly, this means that what a trader holds initially of each commodity will not satisfy him, under any circumstances, no matter how much he obtains of the other commodities. Of course this is far better than excluding saturation altogether (as in I.2.4), which can only be considered an idealization of reality; even so, it (the assumption of this paper) constitutes a considerable restriction. But some restriction on the bundles that saturate desire is necessary. Indeed, consider a two-commodity market in which all the initial bundles are (2,2). Define an allocation X by

$$X(t) = \begin{cases} (0,2) & \text{for } 0 \leq t < \frac{1}{4} \\ (2,0) & \text{for } \frac{1}{4} \leq t < \frac{1}{2} \\ (3,3) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases},$$

and suppose that $X(t)$ saturates t's desire for all t. Then clearly X is in the core; but (p,X) can never be a competitive equilibrium, because for $\frac{1}{2} \leq t \leq 1$, $X(t)$ is in none of the budget sets that are possible for different p.

Other saturation conditions can be obtained if one assumes "λ-desirability", i.e. a bounded marginal rate of substitution between any two commodities (see MCT II). There a saturating bundle must have at least some extremely large coordinates, but other coordinates can be smaller than the corresponding coordinates of the initial bundle. But λ-desirability is a very strong desirability-of-commodities type assumption, and in itself constitutes a considerable restriction.