MARKETS WITH A CONTINUUM OF TRADERS IV

Robert J. Aumann

Econometric Research Program
Research Memorandum No. 57
October 23, 1962

The research described in this paper was supported by The Carnegie Corporation of New York.

Princeton University
Econometric Research Program
92-A Nassau Street
Princeton, N.J.

Abstract Conditions are given for the existence of competitive equilibria in markets with a continuum of traders.

1. Introduction

In three previous papers[1] we introduced the notion of markets with a continuum of traders, and demonstrated their significance as mathematical models for the intuitive concept of "perfect competition". In particular, it was shown that under very wide conditions, the core of such a market equals the set of its equilibrium allocations.[2] On the other hand, these theorems do not establish the existence of competitive equilibria; it may well happen that both the core and the set of equilibrium allocations are empty. It is the purpose of this paper to lend additional substance to the results of the previous papers, by giving conditions for the existence of competitive equilibria--and thus for the non-emptiness of the core--in markets with a continuum of traders.

The assumptions we make, though far stronger than those of MCT I and II, are similar to those used by other authors, in connection with finite markets.[3] Thus we here assume that preferences are transitive, complete, continuous and convex; these assumptions are perfectly standard in the literature, but of them, only continuity was used in MCT I and II. Two theorems are proved, using (in addition to the above assumptions), different variations of assumptions concerning the initial bundles, the desirability of commodities, and saturation of desire. In Theorem A we assume that each trader holds a positive amount of each commodity; that the goods are "desirable" in the sense that no matter what bundle is held by a trader, he wants more of at least one commodity, unless his

---

[1]Aumann 1962, I, II, III. These papers will be denoted MCT I, MCT II, and MCT III respectively.

[2]An equilibrium allocation is one which, when combined with an appropriate price structure, yields a competitive equilibrium. For definitions and an intuitive discussion of "core" and "competitive equilibrium", see section 1 of MCT I.

[3]See for example Arrow and Debreu (1954), Gale (1955), Nikaido (1956), McKenzie (1959), Debreu (1959), and Karlin (1959).

current holdings saturate his desire;[4] that saturation never takes place unless more is held of each commodity than in the initial bundle; and that each trader's desire is "commodity-wise saturated" by some bundle y, in the sense that under no circumstances does a trader want more of any commodity than is present in y. In Theorem B we assume only that each commodity in the model is actually present in the market, in the sense that a positive amount of it is held by some trader; to compensate for this weakening of the "initial resources" assumption, we strengthen the "desirability assumption" to say that no matter what bundle is held by a trader, he wants more of every commodity, unless his current holdings saturate his desire; and as in Theorem A, that saturation never takes place unless more is held of each commodity than in the initial bundle. There is no "commodity-wise saturation" assumption in Theorem B. Theorems A and B are approximately comparable to the "special" and "general" theorems of McKenzie (1959).

Theorems A and B are parallel to Theorems A and B of MCT III, but the latter use considerably weaker assumptions (for instance they do not use convexity or that the preferences are quasi-orders, nor the notion of "commodity-wise" saturation). Precisely: Theorem A of MCT III holds under the conditions of Theorem A of this paper, and similarly for Theorem B, but not conversely.

The proof of our Theorem A follows McKenzie's beautiful proof for the case of finitely many traders (1959); but there are several points at which the passage to a continuum of traders is not routine. Of particular interest is our use of Banach-Space methods, in Section 4 and again in Section 6.

---

[4]This sounds tautologous, but is not; it involves the assumption of "free disposal", i.e. a bundle with fewer commodities is not preferred.

We also establish the existence of competitive equilibria under conditions that specialize those of the main theorem of MCT II.

In Section 2 we describe our model and state Theorems A and B. Theorem A is proved in Sections 3 through 5. In Sections 6 and 7 we apply a limiting process to deduce Theorem B from Theorem A. Section 8 contains a comparison of our proof with McKenzie's and Section 9 mentions counter-examples if some assumptions are weakened. Finally Section 10 establishes our Theorem C, which deals with conditions that specialize those of MCT II.

We wish to emphasize that this is not merely an extension, for its own sake, of known theorems for finitely many traders to the case of a continuum of traders. If that were the case, it would hardly be worth doing. The purpose of this paper is to complement and lend substance to MCT I, II, and III; and the theorems proved in <u>those</u> papers are true <u>only</u> for a continuum of traders--they are false for finitely many traders.

2. Mathematical Model and Statement of Results

The set of commodity bundles is the non-negative orthant Ω of a fixed Euclidean space R^n. A member of R^n is called a <u>vector</u>, and its coordinates are denoted by superscripts. A <u>price vector</u> is an n-tuple of non-negative numbers, not all 0; though formally it is in Ω, it should not be thought of as a commodity bundle. The inner product $\Sigma_{i=1}^n p^i x^i$ is denoted $p \cdot x$. Relations between vectors and operations on vectors are to be taken coordinate-wise, unless otherwise specified. Thus $x \geq y$ means $x^i \geq y^i$ for all i, $x > y$ means $x^i > y^i$ for all i, and if X is a vector function, then

$$\int X = (\int X^1, \ldots, \int X^n) .$$

The set T of traders is the closed unit interval [0,1] with Legesgue measure μ. The words "measure", "measurable", "integral", and "integrable" are to be understood inthe sense of Lebesgue. All integrals are with respect to the variable t, and in most cases the range of integration is all of T. We will therefore always omit the symbol dt in an integral, will usually omit indication of dependence of the integrand on t, and will specifically indicate the range of integration only when it differs from all of T. Thus $\int X$ means $\int_T X(t)dt$. A null set is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. In view of this, we adopt the following conventions, which simplify the exposition considerably, but which must be constantly kept in mind by the reader.

Conventions

i) A statement asserted for "all" traders, or for "all" traders in a certain set, holds for all such traders except possibly for a null set.

ii) If it is asserted that there "is" a trader satisfying a certain property, or that the property is satisfied for "some" traders, this means that there is a non-null set of traders satisfying the property.

Convention ii), though perhaps less familiar than i), is its natural complement. It is particularly this convention that enables a quite considerable shortening of the exposition, with a corresponding gain in clarity.

An assignment is an integrable function on T to Ω. There is a fixed initial assignment I; it will be assumed to satisfy either the strong or the weak form of the following condition (depending on the theorem being proved):

(2.1) Strong Form: $I(t) > 0$ for all t

Weak Form: $\int I > 0$.

The strong form asserts that each trader comes to market with a positive amount of each commodity. The weak form asserts only that no commodity is totally absent from the market. Both forms occur also as (2.1) of MCT III, where a complete intuitive discussion is given. The weak form will be recognized as the analogue of assumption 5 of McKenzie (1959, p. 58).

For each trader t there is defined on Ω a relation $\succsim_t$ called preference-or-indifference. This relation is assumed to be a quasi-order, i.e. transitive, reflexive, and complete.[5] From $\succsim_t$ we define relations $\succ_t$ and $\sim_t$ called preference and indifference respectively, as follows:

$$x \succ_t y \text{ if } x \succsim_t y \text{ but not } y \succsim_t x \;;$$

$$x \sim_t y \text{ if } x \succsim_t y \text{ and } y \succsim_t x \;.$$

The following assumptions are made:

(2.2) Continuity (in the commodities): For each $y \in \Omega$, the sets $\{x : x \succ_t y\}$ and $\{x : y \succ_t x\}$ are open in the relative topology of Ω.

(2.3) Measurability: For all $x, y \in \Omega$, the set $\{t : x \succ_t y\}$ is measurable.

For a given trader t, a bundle y is said to saturate desire, or simply to saturate, if no bundle in Ω is preferred to y by t. The next

---

[5]A relation $\mathcal{R}$ is called transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$; reflexive if $x\mathcal{R}x$ for all x; and complete if for all x and y, either $x\mathcal{R}y$ or $y\mathcal{R}x$.

assumption, like assumption (2.1), is stated in two forms; which form is used depends on the theorem being proved.

(2.4) Desirability (of the commodities):

Strong Form: Unless y saturates, $x \geq y$ and $x \neq y$ imply $x \succ_t y$.

Weak Form: Unless y saturates, $x > y$ implies $x \succ_t y$.

For each t, let $\Upsilon(t) = \{y: \text{not } y > I(t)\}$.

(2.5) Saturation Restriction: No bundle in $\Upsilon(t)$ saturates.

The last assumption is

(2.6) Convexity: For each α such that $0 < \alpha < 1$, $x \succ_t y$ implies $\alpha x + (1-\alpha)y \succ_t y$.

An allocation is an assignment X such that $\int X = \int I$. A competitive equilibrium is a pair consisting of a price vector p and an allocation X, such that for all t, $X(t)$ is maximal w.r.t. $\succ_t$ in the "budget set" $\{x \in \Omega: p \cdot x \leq p \cdot I(t)\}$.

In assumption (2.2) we may avoid referring to the relative topology by replacing $\succ_t$ by $\succsim_t$, and "open" by "closed". Unlike MCT I, II, and III, this paper actually uses the second half of the continuity condition. Together with the assumption that $\succsim_t$ is a quasi-order, assumption (2.2) yields the existence of a continuous utility function $\upsilon_t(x)$ on Ω for each fixed trader t. Then assumption (2.3) says the υ_t can be chosen so that $\upsilon_t(x)$ is measurable in t for each fixed x. This version of the measurability assumption looks slightly weaker than the version used in MCT I, II, and III; but in fact, under the stronger conditions of this paper, the versions are equivalent (see lemma 7.1).

The strong form of assumption (2.4) says that until saturation is reached, each trader wants more of each commodity; the weak form only says that he wants more of at least one commodity. Of course it is not implied that saturation is ever reached. Either form implies[6]

$$x \geq y \quad \text{implies} \quad x \succsim_t y$$

for all y; the weak form is equivalent[7] to it. The two forms are further discussed in MCT III.[8]

The set $\Upsilon(t)$ is illustrated in Figure 1. It is the union of all budget sets of t for all possible price vectors. Most of the analysis will take place within this set. Thus the saturation restriction (2.5) says in effect that saturation does not occur in the area which interests us. Outside this area it may or may not occur.

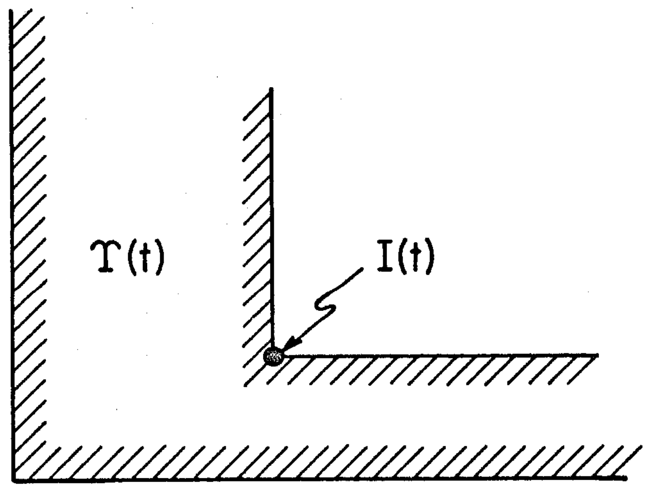

FIG. 1

[6]If y does not saturate this follows from continuity. If y saturates, suppose $x \geq y$ and $y \succ_t x$. On the straight line joining y to x, let z be the last point that saturates. Then for w between z and x on the line and close to z, we have $w \succ_t x$, $x \geq w$ and w does not saturate. But this contradicts the first conclusion.

[7]This uses (2.6).

[8](2.4) and (2.5) together are essentially the same as (2.4) of MCT III. The assumptions here are slightly stronger, but the difference is unimportant.

The convexity assumption is the usual one in economics; it is often called the "law of diminishing returns". Geometrically, the condition asserts that the indifference levels are convex surfaces[9] having no "thickness"--except at saturation, where the indifference level is a convex set with thickness.

There is another condition on the preferences, which however is used in only one of our theorems. Let N be an assignment. We say that t's desire is commodity-wise saturated at $N(t)$ if for all bundles x and commodities i such that $x^i \geq N^i(t)$, we have

$$x \sim_t (x^2, \ldots, x^{i-1}, N^i(t), x^{i+1}, \ldots, x^n) .$$

In other words, changing the value of the i^{th} coordinate above $N^i(t)$ does not change the indifference level. Intuitively, this means that desire for the i^{th} commodity is saturated when the quantity of that commodity is $N^i(t)$, though trader t may still want more of other commodities j of which he holds less than $N^j(t)$. To rephrase the condition, let $\Delta(t) = \{x \in \Omega : x \leq N(t)\}$ be the "hyper-rectangle" of bundles that are $\leq N(t)$, and define a mapping u_t from Ω into $\Delta(t)$ as follows: $u_t(x)$ is the bundle formed from x by replacing by $N^i(t)$ all coordinates x^i of x that exceed $N^i(t)$. Then commodity-wise saturation at $N(t)$ asserts that $u_t(x) \sim_t x$. It follows that the entire preference order is determined by its behavior in the hypercube $\Delta(t)$, since $x \succsim_t y$ if and only if $u_t(x) \succsim_t u_t(y)$. A preference order with commodity-wise saturation is illustrated in Figure 2.

The existence of an $N(t)$ which commodity-wise saturates desire is intuitively very acceptable; it simply means that there is an upper bound on the

[9] A "convex surface" is the boundary of a convex set of full dimension. It is not in general convex as a set. We will not use the term "convex surface" in the sequel.

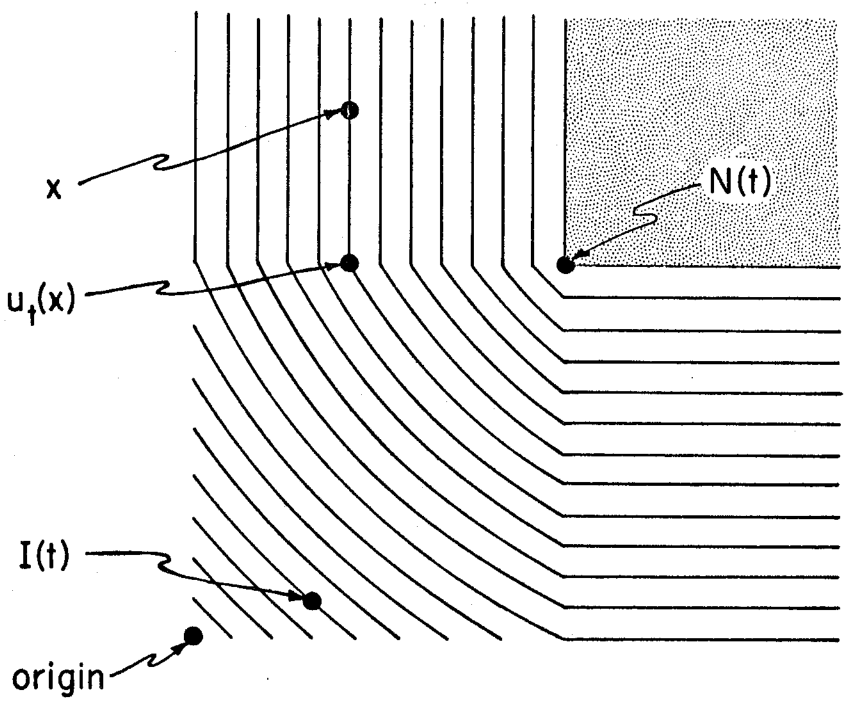

FIG. 2

(The light lines are indifference curves. The lightly shaded area to the northeast of $N(t)$ constitutes a single indifference "curve".)

amount of a commodity which can be profitably used by an individual, no matter what other commodities are or are not available. The demand that N be an assignment, i.e. integrable, means that "the market as a whole can be commodity-wise saturated"; more precisely, it means that there is a bundle (namely $\int N$) that can be distributed among the traders in such a way as to commodity-wise saturate each trader's desire. This too is intuitively very acceptable. If there is an assignment N such that each trader's desire is commodity-wise saturated at $N(t)$, then we shall say for short that <u>each trader's desire can be commodity-wise saturated</u>.[10]

[10] Use of this phrase will entail integrability of the saturating function.

Theorem A Assume the strong form of condition (2.1), and the weak form of condition (2.4). Assume further that each trader's desire can be commodity-wise saturated. Then there is a competitive equilibrium.

Theorem B Assume the weak form of (2.1) and the strong form of (2.4). Then there is a competitive equilibrium.

3. Outline of the Proof of Theorem A

The starting point of the proof is the preferred set $\Gamma_p(t)$, defined for each trader t and each price vector p to be the set of elements preferred or indifferent to all elements of the budget set; formally, denoting the budget set $\{x \in \Omega : p \cdot x \leq p \cdot I(t)\}$ by $B_p(t)$, we define

$$\Gamma_p(t) = \{y \in \Omega : \text{for all } x \in B_p(t),\ y \succsim_t x\} ,$$

(see Figure 3). Next, define

$$\int\Gamma_p = \{\int X : X \text{ is an assignment such that } X(t) \in \Gamma_p(t) \text{ for all } t\} ;$$

this is called the aggregate preferred set. $\int\Gamma_p$ is the set of all aggregate bundles that can be distributed among the traders in such a way that each trader is at least as satisfied as he is when he sells his initial bundle and buys the best (by his standards) that he can with the proceeds, at prices p. Let $c(p)$ be the unique[11] point in $\int\Gamma_p$ that is nearest to $\int I$, and let

$$h(p) = c(p) - \int I ;$$

$h(p)$ is the vector at which the minimum distance from $\int I$ to $\int\Gamma_p$ is attained.

[11] This section is an outline. Unsupported or unclear statements made here will be clarified in the sequel.

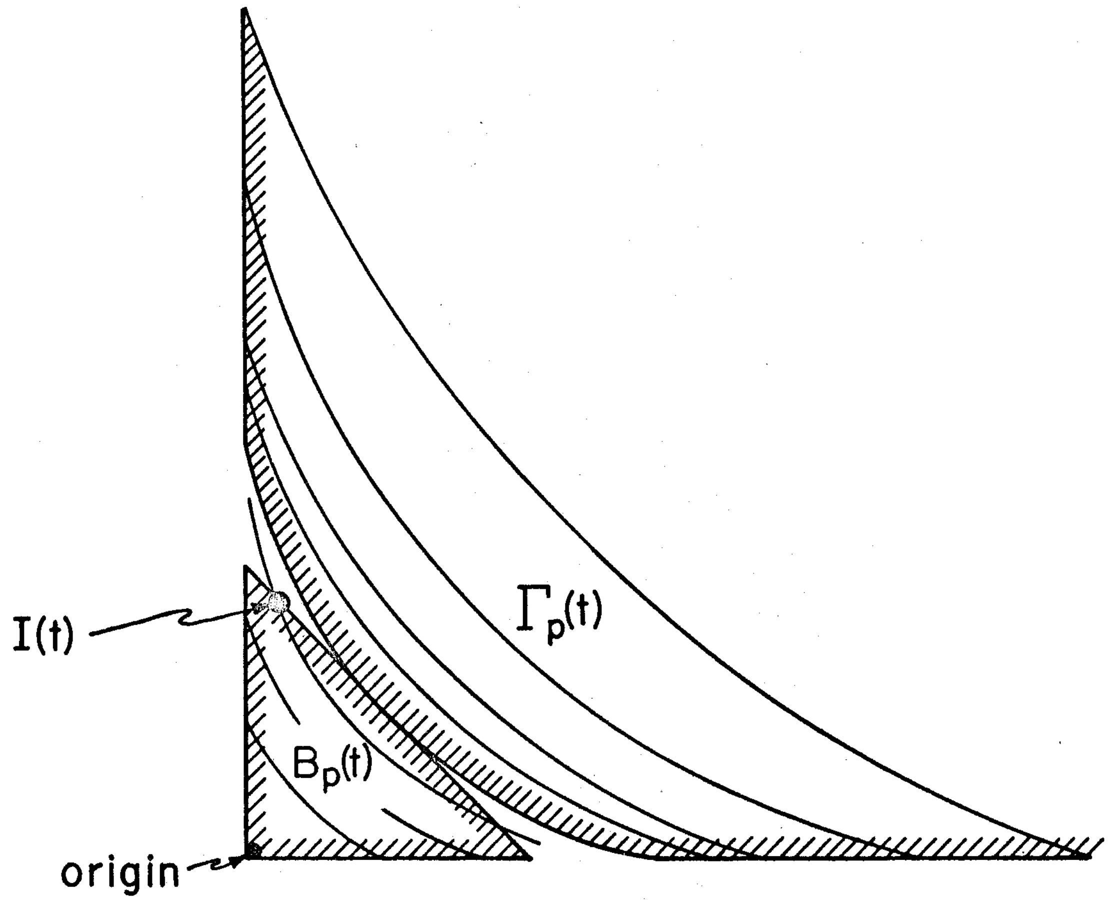

FIG. 3

Let Π be the simplex of price vectors normalized so that their sum is 1, i.e.

$$\Pi = \{p \in \Omega : \Sigma_{i=1}^{n} p^i = 1\} \quad .$$

The central idea of the proof is to use h to construct a continuous function f from Π to itself, and then to apply Brouwer's fixed point theorem;[12] the resulting fixed point--denoted q--turns out to be an equilibrium price. More precisely, f is defined by

[12] Brouwer's theorem asserts that every continuous single-valued function f from Π to itself has a fixed point, i.e. a point p such that $f(p) = p$. For a proof, see Dunford-Schwartz, Section V.12, p. 468.

$$f(p) = \frac{p + h(p)}{1 + \Sigma_{i=1}^{n} h^i(p)} .$$

We will show later that

$$h(p) \geq 0.$$

Therefore the denominator in the definition of f does not vanish and $f(p) \in \Pi$ for all $p \in \Pi$. Suppose q is a fixed point of f. Then

$$q(1 + \Sigma_{i=1}^{n} h^i(q)) = q + h(q) ,$$

i.e.

$$(3.1) \qquad h(q) = \alpha q ,$$

where because $h(p) \geq 0$,

$$\alpha = \Sigma_{i=1}^{n} h^i(q) \geq 0.$$

We wish to show that

$$(3.2) \qquad h(q) = 0.$$

Indeed, suppose (3.2) is false. From the definition of h and the convexity of $\int\Gamma_p$ it follows that for all p, the hyperplane through $h(p) + \int I$ perpendicular to $h(p)$ supports[13] $\int\Gamma_p$. Applying this for $p = q$, we obtain

$$(y - \int I)\cdot h(q) \geq h(q)\cdot h(q)$$

for all $y \in \int\Gamma_q$. Because (3.2) is false, $\alpha > 0$; so by (3.1), we obtain

$$(y - \int I)\cdot \alpha q \geq \alpha^2(q\cdot q),$$

---

[13] Indeed, this is a standard method of constructing a supporting hyperplane. An explicit proof is given by McKenzie (lemma 7(1), p. 61).

and hence

$$(3.3) \qquad (y - \int I)\cdot q \geq \alpha(q\cdot q) > 0 \quad \text{for all} \quad y \in \int \Gamma_q .$$

Now if for each t we let $X(t)$ be a point in the budget set $B_q(t)$ which is maximal with respect to t's preference order, then on the one hand we have $(X(t) - I(t))\cdot q \leq 0$, and on the other hand $X(t) \in \Gamma_q(t)$. Hence by integrating we obtain $(\int X - \int I)\cdot q \leq 0$, and $\int X \in \int \Gamma_q$; this contradicts (3.3), and establishes (3.2).

(3.2) says that $\int I \in \int \Gamma_q$, i.e. there is an assignment X such that $\int X = \int I$ and $X(t) \in \Gamma_q(t)$ for all t. So X is an allocation, and $X(t)$ is preferred or indifferent to all elements of $B_q(t)$. To complete the proof that (q,X) is a competitive equilibrium, it is only necessary to show that $X(t)$ is in $B_q(t)$ for all t. Suppose now that $q\cdot X(t) < q\cdot I(t)$ for some t. Then $X(t)$ is in $\Upsilon(t)$, and therefore does not saturate. Applying the weak form of (2.4), we find that $X(t) + (\delta, \ldots, \delta) \succ_t X(t)$ for $\delta > 0$. But for δ sufficiently small, we will still have

$$q\cdot(X(t) + (\delta,\ldots,\delta)) = q\cdot X(t) + \delta < q\cdot I(t) \quad ,$$

so $X(t) + (\delta,\ldots,\delta) \in B_q(t)$; this contradicts $X(t) \in \Gamma_q(t)$. So $q.X(t) < q.I(t)$ is impossible, and we conclude that $q\cdot X(t) \geq q\cdot I(t)$ for all t. If the $>$ sign would hold for some t, we could deduce $\int q\cdot X > \int q\cdot I$, contradicting $\int X = \int I$. So $q\cdot X(t) = q\cdot I(t)$ for all t, and it follows that $X(t) \in B_q(t)$ for all t. So (q,X) is a competitive equilibrium.

The foregoing is simply an outline of McKenzie's proof, with the notation changed and integration substituted for summation over the set of traders. However, there are some points in this outline that stand in need of verification, and it is in these points that the proofs for the finite and continuous cases diverge. Let us list the points in question.

(3.4) $h(p)$ <u>is uniquely defined for each</u> $p \in \Pi$, <u>and is continuous as a function of</u> p.

This is needed to show that Brouwer's fixed point theorem is applicable.

(3.5) $h(p) \geq 0$ <u>for each</u> $p \in \Pi$.

(3.6) <u>There is an assignment</u> X <u>whose integral</u> $\int X$ <u>contradicts</u> (3.3).

The point of (3.6) is that the function X defined above for this purpose was not shown to be measurable.

4. Properties of h

Let $\mathbb{N}$ be a commodity-wise saturating assignment. This section is devoted to the proof of (3.4) and (3.5); that is, the existence, uniqueness, continuity, and non-negativity of h. All these will be proved approximately simultaneously at the end of the section, but before we get there, we must develop a considerable amount of machinery. A major tool is the notion of "continuity" of a point-to-set function:

<u>Definition</u> A function Ξ from Π to subsets of Ω is <u>upper-semicontinuous</u> if for each convergent sequence $\{p_1, p_2, \ldots\}$ in Π, every limit point of every sequence $\{x_1, x_2, \ldots\}$ for which $x_1 \in \Xi(p_1)$, $x_2 \in \Xi(p_2), \ldots$, is in $\Xi(\lim_{k\to\infty} p_k)$. It is <u>lower-semicontinuous</u> if for each convergent sequence $\{p_1, p_2, \ldots\}$ in Π, every point in $\Xi(\lim_{k\to\infty} p_k)$ is the limit of a sequence $\{x_1, x_2, \ldots\}$ for which $x_1 \in \Xi(p_1)$, $x_2 \in \Xi(p_2)$, ... It is <u>continuous</u> if it is both lower- and upper-semicontinuous.

In principle, the existence, uniqueness and continuity (in p) of h follow from the closedness, convexity and continuity (in p) of $\int\Gamma_p$ respectively. Non-negativity follows from the weak form of (2.4). However, in carrying out the proofs, the unboundedness of the $\Gamma_p(t)$ and of $\int\Gamma_p$ causes difficulties. We therefore prefer to work with the bounded sets[14] $\Delta(t) \cap \Gamma_p(t)$, which we will denote by $\Lambda_p(t)$, and with

$$\int\Lambda_p = \{\int X : X \text{ is an assignment such that } X(t) \in \Lambda_p(t) \text{ for all } t\}.$$

We pass back to $\int\Gamma_p$ only at the very end of the section.

Lemma 4.1 *The $\Lambda_p(t)$ are closed and convex, and $\int\Lambda_p$ is convex.*

Proof The $\Gamma_p(t)$ are closed because of the continuity condition (2.2) for preferences, and convex because $B_p(t) \subset \Upsilon(t)$ and the convexity condition (2.6). $\Delta(t)$ is obviously closed and convex. Hence $\Lambda_p(t) = \Delta(t) \cap \Gamma_p(t)$ has the same properties. The convexity of $\int\Lambda_p$ follows from that of $\Lambda_p(t)$. Incidentally, $\int\Lambda_p$ is also closed; but the proof of this lies deeper, and is postponed until later.

Lemma 4.2 *For each t, $\Lambda_p(t)$ is a continuous function of p.*

The reader should have no trouble verifying this lemma. It is proved by McKenzie, (lemma 4, pp. 57 and 68), on condition that every $B_p(t)$ has a maximal element which is in $\Delta(t)$. But this holds here; for if x is maximal in $B_p(t)$, then by commodity-wise saturation so is $u_t(x)$, and this is surely

[14] $\Delta(t) = \{x : x \leq N(t)\}$. See section 2.

in $\Delta(t)$. In the proof of this lemma, McKenzie uses what we call the strong form of (2.1); this is our only important use of that assumption.

In the sequel we shall have occasion to construct functions and prove them measurable. In order not to have to digress at that place, let us now say a few words about the general technique of proving a set or a function measurable. Often the set in question is given in the form $\{t: \mathcal{P}(t)\}$, where $\mathcal{P}(t)$ is some "predicate"--i.e. a statement whose truth or falsity depends on the value of t. We will say that $\mathcal{P}$ itself is <u>measurable</u> whenever $\{t: \mathcal{P}(t)\}$ is measurable. Usually, of course, we do not know a priori whether or not $\mathcal{P}$ is measurable. $\mathcal{P}$ is often a rather complicated statement--built up from simpler component statements by means of connectives like "or", "and", "implies", "not", etc., and the quantifiers "there exists" and "for all". A basic principle is that <u>all predicates built up in this way are measurable, provided that the component predicates are measurable and the quantifiers have at most a denumerable range</u>. For example, suppose $\mathcal{P}_1, \mathcal{P}_2, \ldots$ and $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$ are measurable predicates; let $\mathcal{P}(t)$ be the statement "for no j do we have $\mathcal{P}_j(t)$ and not $\mathcal{Q}_j(t)$." Then

$$\{t: \mathcal{P}(t)\} = T - \cup_{j=1}^{\infty} [\{t: \mathcal{P}_j(t)\} \cap (T - \{t: \mathcal{Q}_j(t)\})] ,$$

where "-" denotes set-theoretic subtraction. Note that "and" transforms into intersection, "not" into complementation, "there exists" into union and "for all" into intersection. The reason that quantification must be over a denumerable range is that intersections and unions preserve measurability only if they are denumerable. Incidentally, note that "$\mathcal{P}$ implies $\mathcal{Q}$" is equivalent to "$\mathcal{Q}$ or not $\mathcal{P}$", so that also "implies" transforms measurable predicates into measurable predicates. Finally, recall that a real function φ on T is defined to be measurable if and only if $\{t: \varphi(t) \leq \alpha\}$ is measurable for each real α.

As an application of these ideas, we prove

Lemma 4.3 For a fixed $\omega \in \Omega$, the predicates "$\omega \in \Gamma_p(t)$" and "$\omega \in \Lambda_p(t)$" are measurable.

Proof "$\omega \in \Lambda_p(t)$" is equivalent to "$\omega \in \Gamma_p(t)$ and $\omega \leq N(t)$"; the second statement is measurable because $N(t)$ is a measurable function, so we need only establish the measurability of "$\omega \in \Gamma_p(t)$". Because of the continuity condition (2.2), this is equivalent to "For all rational points r in $B_p(t)$, $\omega \succsim_t r$" (a rational point is a point with rational coordinates). This can be restated in the form "For all rational points r in Ω, $r \in B_p(t)$ implies $\omega \succsim_t r$". There are only denumerably many rational points in Ω; "$r \in B_p(t)$" is equivalent to "$p \cdot r \leq p \cdot I(t)$", so its measurability follows from that of I; and "$\omega \succsim_t r$" is measurable because of the measurability condition (2.3) on the preferences. Therefore "$\omega \in \Gamma_p(t)$" is measurable, and the lemma is proved.

Lemma 4.4 $\int \Lambda_p$ is a lower-semicontinuous function of p.

Proof Let $p_1, p_2, \ldots$ be a sequence in Π with limit p, and let $x \in \int \Lambda_p$. Let X be an assignment such that $X(t) \in \Lambda_p(t)$ for each t, and $\int X = x$. For each k and t, let $X_k(t)$ be the point in $\Lambda_{p_k}(t)$ that is closest to $X(t)$; the existence and uniqueness[15] of this point follows from the closedness and convexity of $\Lambda_{p_k}(t)$ (lemma 4.1). By the lower semi-continuity of $\Lambda_p(t)$, (lemma 4.2), we have $X_k(t) \longrightarrow X(t)$. If X_k is measurable, then because

[15] For an explicit proof of uniqueness see McKenzie, lemma 6, p. 61.

it is bounded by N it is an assignment. For the same reason we can apply Lebesgue's dominated convergence theorem[16], and obtain $\int X_k \longrightarrow \int X = x$. Since $\int X_k \in \int \Lambda_{p_k}$, this establishes the lower-semicontinuity of $\int \Lambda_p$. It therefore remains only to verify the measurability of X_k.

First note that $||X_k - X||$ is measurable, where $|| \ ||$ denotes the Euclidean norm in R^n (i.e. the distance from the origin). Indeed, the statement "$||X_k(t) - X(t)|| \geq \alpha$" is equivalent to "for no rational point $r \in \Omega$ do we have $||r - X(t)|| < \alpha$ and $r \in \Lambda_{p_k}(t)$"; the measurability of $||X_k - X||$ then follows from that of $X(t)$ and of "$r \in \Lambda_{p_k}(t)$" (lemma 4.3). Next, let $\{r_1, r_2, \ldots\}$ be an enumeration of the rational points in Ω, and for each positive integer j define $X_{kj}(t)$ to be the first rational number in $\Lambda_{p_k}(t)$ whose distance from $X(t)$ is at most $||X_k(t) - X(t)|| + \frac{1}{j}$. Let $\mathcal{P}_m(t)$ be the statement "$r_m \in \Lambda_{p_k}(t)$ and $||r_m - X(t)|| \leq ||X_k(t) - X(t)|| + \frac{1}{j}$"; then from the measurability of "$r_m \in \Lambda_{p_k}(t)$" (lemma 4.3), of X and of $||X_k - X||$, it follows that $\mathcal{P}_m$ is measurable. Now "$X_{kj}(t) = r_m$" is equivalent to "$\mathcal{P}_m(t)$ and for $\ell < m$, not $\mathcal{P}_\ell(t)$". Since X_{kj} takes only rational values, it follows that X_{kj} is measurable. But $\lim_{j\to\infty} X_{kj}(t) = X_k(t)$ for all t, and the limit of measurable functions is measurable. This completes the proof of lemma 4.4.

We must now prove that $\int \Lambda_p$ is an upper-semicontinuous function of p. Let $\mathcal{X}$ denote the set of all assignments X such that $X(t) \leq N(t)$ for all t. We are given a point-to-set mapping $\Xi : \Pi \longrightarrow \Omega$, which takes a given point p of Π into the subset $\int \Lambda_p$ of Ω. Actually, Ξ is the composition

---

[16] This asserts that if φ_j is a sequence of integrable functions that converges for every $t \in T$, and if for some integrable function ψ we have $|\varphi_j(t)| \leq \psi(t)$ for all t, then $\lim_j \int \varphi_j = \int \lim_j \varphi_j$. See, for example, Dunford-Schwartz, p. 151, II.6.16.

of two mappings: a point-to-set mapping $\Lambda:\Pi \longrightarrow \mathcal{X}$, which takes a given point p of Π into the subset $\{X:X(t) \in \Lambda_p(t)$ for all $t\}$ of $\mathcal{X}$; and a point-to-point mapping $\int: \mathcal{X} \longrightarrow \Omega$, which takes a given point X of $\mathcal{X}$ into the point $\int X$ of Ω. This raises the possibility of demonstrating the upper-semicontinuity of Ξ by proving separately that Λ is upper-semicontinuous and that $\int$ is continuous, and then proving an appropriate theorem about the composition of upper-semicontinuous point-to-set functions with continuous point-to-point functions. This will indeed be our method.

The basic ingredient of the definition of continuity is the notion of "convergence". We know what convergence means in Π and in Ω, but in $\mathcal{X}$ convergence can be defined in a number of ways. The most appropriate for our purposes is what is called "weak" convergence.[17]

<u>Definition</u> Call a measurable set of traders a coalition. A sequence $X_1, X_2, \ldots$ of assignments <u>converges weakly</u> to the assignment X, if for each coalition S, $\int_S X_k \longrightarrow \int_S X$. Henceforth we will assume that $\mathcal{X}$ is "endowed with the weak topology", that is, when we talk about convergence in $\mathcal{X}$ we shall mean weak convergence; in the definitions of continuity and upper-semicontinuity, weak convergence is meant wherever the concept of "convergence" occurs.

<u>Lemma 4.5</u> <u>Let</u> $\{\varphi_k\}$ <u>be a sequence of real-valued measurable functions on</u> T <u>which are bounded from below by an integrable function</u> ψ. <u>Then for each coalition</u> S,

$$\liminf_{k\to\infty} \int_S \varphi_k \geq \int_S \liminf_{k\to\infty} \varphi_k .$$

[17] For those familiar with functional analysis, we are working with weak convergence in the Banach space of n-dimensional vector-valued functions on T, with the L^1-norm $||X|| = \Sigma_{i=1}^n \int |X^i|$. One can think of this space as $L^1(nT)$, where nT is the disjoint union of n copies of T. (i.e., $nT = \{1,\ldots,n\} \times T$; the correspondence is given by $X^i(t) = X(i,t)$). $\mathcal{X}$ is a subset of this space.

Proof When $\psi \equiv 0$ this is Fatou's lemma (see Dunford-Schwartz, III.6.19, p. 152). In the general case, we have from Fatou's lemma that

$$\liminf_{k\to\infty} \int_S (\varphi_k - \psi) \geq \int_S \liminf_{k\to\infty} (\varphi_k - \psi) ,$$

and the result follows on adding $\int_S \psi$ to both sides.

Lemma 4.6 Let $\{X_1, X_2, \ldots\}$ be a weakly convergent sequence in $\mathcal{X}$, with weak limit X. For each t, let $\Phi(t)$ be the set of limit points of $X_k(t)$ as $k \to \infty$. Then $X(t)$ is in the convex hull $\Theta(t)$ of $\Phi(t)$ for all t.

Proof If the lemma is false, then $X(t) \notin \Theta(t)$ for t in a non-null coalition S. For each t, $\Phi(t)$ is a set of limit points, and is therefore closed; since it is bounded by $N(t)$ it is compact, and therefore its convex hull $\Theta(t)$ is also compact. So for $t \in S$, we may apply the separating hyperplane theorem to obtain a rational point $r(t) \in R^n$ such that for all $x \in \Theta(t)$, $r(t)\cdot x - r(t)\cdot X(t) > 0$. Since there are only denumerably many rationals and S is non-null, there must be some rational r and a non-null subset R of S such that $r(t) = r$ for all $t \in R$, i.e. such that

$$(4.7) \qquad r\cdot x - r\cdot X(t) > 0 \qquad \text{for all } t \in R \text{ and } x \in \Theta(t) .$$

Now from the weak convergence of X_k to X it follows that

$$\int_R r\cdot X_k \longrightarrow \int_R r\cdot X .$$

Since the X_k are bounded from above by an integrable function and from below by 0, it follows that $r\cdot X_k$ is bounded from below by an integrable function. So we may apply lemma 4.5, and obtain

$$(4.8) \qquad \int_R r \cdot X = \lim_k \int_R r \cdot X_k = \liminf_k \int_R r \cdot X_k \geq \int_R \liminf_k r \cdot X_k \quad .$$

Suppose now that for some $t \in R$, $\liminf_k r \cdot X_k(t) \leq r \cdot X(t)$. We can then find a sequence of k such that $r \cdot X_k(t)$ converges to a value $\leq r \cdot X(t)$, and a subsequence thereof such that $X_k(t)$ converges. The limit of $X_k(t)$ as k ranges over this subsequence is a member x of $\Phi(t)$, and we have $r \cdot x \leq r \cdot X(t)$, contrary to (4.7). Hence $\liminf_k r \cdot X_k(t) > r \cdot X(t)$ for all $t \in R$. Hence

$$\int_R \liminf_k r \cdot X_k > \int_R r \cdot X \quad ,$$

contradicting (4.8). This proves the lemma.

<u>Lemma 4.9</u> Λ <u>is upper-semicontinuous</u>.

<u>Proof</u> Let $p_k \longrightarrow p$, $X_k \in \Lambda(p_k)$, $X_k \longrightarrow X$ weakly. By lemma 4.6, $X(t)$ is in the convex hull of the limit points of $\{X_k(t)\}$ for all t. By upper-semicontinuity of $\Lambda_p(t)$ (lemma 4.2), all limit points of $\{X_k(t)\}$ are in $\Lambda_p(t)$. Since $\Lambda_p(t)$ is convex, the convex hull of these limit points is also in $\Lambda_p(t)$, and so $X(t) \in \Lambda_p(t)$ for each t. But this means $X \in \Lambda(p)$, and the lemma is proved.

<u>Lemma 4.10</u> <u>Let</u> A, B, C <u>be topological spaces</u>[18] <u>with</u> B <u>compact</u>[19], $\Lambda: A \longrightarrow B$ <u>an upper-semicontinuous point-to-set mapping</u>, $f: B \longrightarrow C$ <u>a continuous point-to-point mapping</u>. <u>Then</u> $f\Lambda$ <u>is upper-semicontinuous</u>.

[18] Spaces in which a notion of convergence is defined.

[19] Every sequence has a convergent subsequence.

Proof Let $\{x_k\}$ and $\{z_k\}$ be sequences in A and C respectively, $x_k \longrightarrow x$, $z_k \longrightarrow z$, $z_k \in f\Lambda(x_k)$; we wish to show $z \in f\Lambda(x)$. Let $y_k \in \Lambda(x_k)$ be such that $f(y_k) = z_k$. By the compactness of B, the y_k have a limit point y. Then from the upper-semicontinuity of Λ it follows that $y \in \Lambda(x)$. If $\{y_{k_j}\}$ is a subsequence of $\{y_k\}$ that approaches y, then from the continuity of f we have

$$z = \lim_k z_k = \lim_j z_{k_j} = \lim_j f(y_{k_j}) = f(y) \in f\Lambda(x) .$$

This completes the proof of the lemma.

Lemma 4.11 $\mathcal{X}$ is compact in the weak topology.

Proof This follows from the fact that all elements of $\mathcal{X}$ are bounded in absolute value by the integrable function N. See for instance Dunford-Schwartz, IV.8.9, p. 292.

Lemma 4.12 $\int\Lambda_p$ is an upper-semicontinuous function of p.

Proof We must establish the upper-semicontinuity of the composition Ξ of $\int$ and Λ. $\int$ is clearly continuous. The lemma now follows from lemmas 4.9, 4.10, and 4.11.

Lemma 4.13 $\int\Lambda_p$ is closed.

Proof We must show that the limit of a convergent sequence in $\int\Lambda_p$ is in $\int\Lambda_p$. This follows from upper-semicontinuity (lemma 4.12) if we set $p_1 = p_2 = \cdots = p$.

For each p in Π, let $d(p)$ be the point in $\int\Lambda_p$ that is closest to $\int I$. Such a point exists because $\int\Lambda_p$ is non-empty (it contains $\int N$) and closed (lemma 4.13); it is unique because $\int\Lambda_p$ is convex (lemma 4.1). To prove that it is continuous in p, we need the following lemma:

<u>Lemma 4.14</u> <u>Let</u> Ξ <u>be a continuous point-to-set mapping from</u> Π <u>to subsets of</u> Ω, <u>such that</u> $\Xi(p)$ <u>is uniformly bounded for</u> $p \in \Pi$. <u>Let</u> $\omega \in \Omega$ <u>be arbitrary but fixed</u>. <u>Then the point in</u> $\Xi(p)$ <u>that minimizes the distance to</u> ω <u>is a continuous function of</u> p.

The reader may verify this without difficulty. It is also proved in McKenzie, lemma 10, p. 62.

Now $\int\Lambda_p$ is uniformly bounded (by $\int N$) and continuous in p (lemmas 4.4 and 4.12), and so we may apply lemma 4.14 and deduce that $d(p)$ is continuous in p. Next, we have

<u>Lemma 4.15</u> <u>For each</u> p <u>in</u> Π, $d(p) \geq \int I$.

<u>Proof</u> If not, then $d(p)$ has a coordinate--without loss of generality let it be the first--such that $d^1(p) < \int I^1$. Now $d(p) = \int X$, where $X(t) \in \Lambda_p(t)$ for all t. Let $Y(t) = (N^1(t), X^2(t), \ldots, X^n(t))$. Then $Y(t) \geq X(t)$ and $Y(t) \leq N(t)$; therefore $Y(t) \in \Lambda_p(t)$ for all t. Therefore

$$(\int N^1, d^2(p), \ldots, d^n(p)) = \int Y \in \int\Lambda_p.$$

Now $d^1(p) < \int I^1$ and $\int N^1 > \int I^1$; so there is an α with $0 < \alpha < 1$ such that $\alpha \int N^1 + (1 - \alpha) d^1(p) = \int I^1$. Setting $Z = \alpha Y + (1 - \alpha) X$ and $z = \int Z$, we obtain $z \in \int\Lambda_p$ (by the convexity of $\int\Lambda_p$), and

$$z = (\int I^1, d^2(p), \ldots, d^n(p)) .$$

Denoting the distance between points x and y in R^n by $||x-y||$ and noting that $(d^1(p) - \int I^1)^2 > 0$, we have

$$||z - \int I||^2 = \Sigma_{i=2}^n (d^i(p) - \int I^i)^2$$

$$< \Sigma_{i=1}^n (d^i(p) - \int I^i)^2 .$$

Thus z is closer to $\int I$ than $d(p)$, a contradiction. This proves the lemma.

Let $g(p) = d(p) - \int I$. We have established for $g(p)$ all the properties that we set out to establish for $h(p)$: existence, uniqueness, continuity, and non-negativity (the last by lemma 4.15). So with the following lemma we achieve our aim:

Lemma 4.16 $g(p) = h(p)$

Proof Fix p, and write $g = g(p)$, $h = h(p)$, $c = c(p)$, $d = d(p)$. If $g = 0$ there is nothing to prove. Otherwise, by the definition of g, the hyperplane through d perpendicular to g supports $\int \Lambda_p$ (see footnote 13).

This means that

$$(4.17) \qquad x \cdot g \geq ||g||^2 \quad \text{for all } x \in \int \Lambda_p - \int I .$$ [20]

(Of course (4.17) also holds when $g = 0$.) Suppose there is a point in $\int \Gamma_p$ that is nearer than d to I. This means that there is a point y in $\int \Gamma_p - \int I$ that is nearer than g to 0. Then

[20] Here "-" means algebraic subtraction, not set-theoretic.

(4.18) $$||y||^2 < ||g||^2 \ .$$

Furthermore $$||y||^2 - 2y\cdot g + ||g||^2 = ||y-g||^2 > 0 \ .$$

Hence $$||y||^2 > y\cdot g + [y\cdot g - ||g||^2] \ .$$

If $y\cdot g - ||g||^2 \geq 0$, then it follows that $||y||^2 > y\cdot g \geq ||g||^2$, contradicting (4.18). Hence

(4.19) $$y\cdot g < ||g||^2 \ .$$

(4.19) expresses the geometrically obvious fact that any point nearer than d to $\int I$ must be on the near side of the hyperplane through d perpendicular to g.

Now $y = \int X - \int I$, where $X(t) \in \Gamma_p(t)$ for all t. Then by commodity-wise saturation, $u_t(X(t)) \in \Gamma_p(t)$ for all t. Furthermore $u_t(X(t)) \leq X(t)$, and $u_t(X(t)) \leq N(t)$. Setting $Z(t) = u_t(X(t))$, we obtain $\int Z \in \int \Lambda_p$ and $\int Z - \int I \leq y$. Since $g \geq 0$ (lemma 4.15), it follows that $(\int Z - \int I)\cdot g \leq y\cdot g$. Hence by (4.19), $(\int Z - \int I)\cdot g < ||g||^2$. But since $\int Z - \int I \in \int \Lambda_p - \int I$, it follows from (4.17) that $(\int Z - \int I)\cdot g \geq ||g||^2$, and this is the contradiction that proves our lemma.

This completes the proof of (3.4) and (3.5).

5. Completion of the Proof of Theorem A

In this section we prove (3.6), thus completing the proof of Theorem A.

To prove (3.6), we first show that for all t, there is a maximal element in $B_q(t)$, which eo ipso is a member of $\Gamma_q(t)$. Indeed, $\Delta(t) \cap B_q(t)$ is compact, and therefore from the continuity condition (2.2) for preferences, it easily follows[21] that it has a maximal element y. Then because of commodity-

[21] An explicit proof is given by McKenzie. See Lemma 1, pp. 57 and 67.

wise saturation, y is also maximal in $B_q(t)$. Indeed, suppose $z \in B_q(t)$ is such that $z \succ_t y$. Now $z \in B_q(t)$ means that $q \cdot z \leq q \cdot I(t)$; therefore $q \cdot u_t(z) \leq q \cdot z \leq q \cdot I(t)$, and therefore also $u_t(z) \in B_q(t)$. But by definition of u_t, $u_t(z) \in \Delta(t)$; therefore $u_t(z) \in \Delta(t) \cap B_q(t)$. Finally, $u_t(z) \sim_t z \succ_t y$. Thus $u_t(z)$ contradicts the maximality of y in $\Delta(t) \cap B_q(t)$. So the existence of a maximal element in $B_q(t)$ is proved.

In section 3, we said that the integral of an assignment that assigns to each t a maximal element of $B_q(t)$ will contradict (3.3). However, the construction that was used there involves the axiom of choice, and thus may lead to a non-measurable function. It is possible to avoid the axiom of choice by singling out a specific maximal element of each $B_q(t)$, but even then the proof of measurability is quite complicated. We will circumvent these difficulties by defining an assignment X such that $X(t) \in \Gamma_q(t)$ and $X(t)$, though not actually in $B_q(t)$, is very close to it. Then $\int X$ will still contradict (3.3) as desired.

Let $(r_1, r_2, \ldots)$ be an enumeration of the rational points in Ω. For each t, let $X(t)$ be the first rational point r in this enumeration such that

(5.1) $$r \in \Gamma_q(t) \quad ,$$

(5.2) $$(r - I(t)) \cdot q < \alpha(q \cdot q), \quad \text{and}$$

(5.3) $$r \leq 2N(t) \; .$$

There is such a point: for if $y \in \Delta(t)$ is maximal in $B_q(t)$, then $y \in \Gamma_q(t)$, $(y - I(t)) \cdot q \leq 0$, and $y \leq N(t)$.

So we can find a rational point $X(t) \geq y$ sufficiently close to y to satisfy (5.2) and (5.3), which by the weak form of (2.4) will also satisfy (5.1). To show that the function $X(t)$ so defined is measurable, we first note that (5.1) is measurable (lemma 4.3); that (5.2) is measurable (because I is measurable); and that (5.3) is measurable (because $N(t)$ is). Then since X takes only rational values, its measurability follows as in the proof of measurability of the X_{kj} at the end of the proof of lemma 4.4.

From (5.3) we get $X(t) \leq qN(t)$, so X is an assignment. From (5.1) and (5.2) we get $X(t) \in \Gamma_q(t)$ and $(X(t) - I(t)).q < \alpha(q.q)$. Integrating, we get $\int X \in \int \Gamma_q$, and $(\int X - \int I).q < \alpha(q.q)$. So (3.3) is contradicted, and the proof of (3.6) is complete.

This completes the proof of Theorem A.

6. Proof of Theorem B

We now assume the weak form of (2.1) and the strong form of (2.4). Our procedure will be to define a sequence of markets $\mathcal{M}_1, \mathcal{M}_2, \ldots$, with the following properties: If I_k is the initial assignment in $\mathcal{M}_k$, then

(6.1) $I_k \longrightarrow I$ uniformly for all t;

(6.2) $I_k(t) > 0$ for all t;

There is a sequence of assignments $\{N_k\}$ such that for all t,

(6.3) $N_k(t) \longrightarrow \infty$,

and (6.4) $N_k(t)$ commodity-wise saturates t's desire in $\mathcal{M}_k$;

and (6.5) The preferences in $\mathcal{M}_k$ coincide with the preferences in the original market $\mathcal{M}$, for all x and y such that $x,y \leq N_k(t)$ and $y \in \Upsilon_k(t)$.

Intuitively, the $\mathcal{M}_k$ are "approximations" to $\mathcal{M}$, which satisfy the conditions of Theorem A. Application of Theorem A yields competitive equilibria (q_k, X_k) for the $\mathcal{M}_k$; it will be shown that the (q_k, X_k) have a (weak) limit point (q,X) that is a competitive equilibrium in $\mathcal{M}$.

For each t, let $\Upsilon'(t)$ be the set all unsaturated bundles. By continuity (2.2), $\Upsilon(t)$ is open. By the saturation restriction (2.5), $\Upsilon(t) \subset \Upsilon'(t)$.

The $\mathcal{M}_k$ are defined as follows: Let $\{\gamma_1, \gamma_2, \ldots\}$ be a sequence of positive numbers for which

$$(6.6) \qquad \gamma_k \longrightarrow \infty .$$

Define the N_k by

$$(6.7) \qquad N_k(t) = I(t) + \{\gamma_k, \ldots, \gamma_k\} .$$

The $\Delta_k(t)$ and $u_{k_t} : \Omega \longrightarrow \Delta_k(t)$ are defined in the obvious way. Now $\Upsilon(t) \cap \Delta_k(t)$ is compact, and is contained in the open set $\Upsilon'(t)$. So every sufficiently small neighborhood of $\Upsilon(t) \cap \Delta_k(t)$ is still in $\Upsilon'(t)$; in particular, if we choose $\delta_k(t) > 0$ sufficiently small and set

$$(6.8) \qquad I_k(t) = I(t) + \delta_k(t)$$

and

$$(6.9) \qquad \Upsilon_k(t) = \{y : \text{not } y > I_k(t)\} ,$$

then

$$(6.10) \qquad \Upsilon_k(t) \cap \Delta_k(t) \subset \Upsilon'(t) .$$

We define I_k by (6.8), so as to satisfy (6.1), (6.10), and

$$(6.11) \qquad I_k(t) < N_k(t).$$

Then (6.2) is also satisfied. Also the δ_k can be chosen to be measurable; indeed the statement "$w \in \Upsilon'(t)$" is measurable for each $w \in \Omega$, because it means that there is a rational that is preferred to w; and $\delta_k(t)$ can be chosen to be the first positive rational (in a fixed enumeration of the rationals in Ω) which is, say, $< 1/\gamma_k$ and satisfies (6.10) and (6.11).

The preference relations $\succsim_t^k$ in $\mathcal{M}_k$ are defined by

$$(6.12) \qquad x \succsim_t^k y \quad \text{if and only if} \quad u_{k_t}(x) \succsim_t u_{k_t}(y) .$$

This completes the definition of the markets $\mathcal{M}_k$. These markets satisfy the conditions of Theorem A; in order not to disrupt the continuity of the proof, we put off the verification of these conditions to the following section.

By Theorem A, each market $\mathcal{M}_k$ has a competitive equilibrium (q_k, X_k). Because of the compactness of Π, the sequence $\{q_k\}$ has a convergent subsequence, and we may suppose w.l.o.g. (without loss of generality) that this subsequence is the original sequence. Let $q = \lim_k q_k$. The following is the crucial lemma:

<u>Lemma 6.13</u> $\qquad q > 0$

<u>Proof</u> Suppose, on the contrary, that some coordinate of q vanishes, say $q^1 = 0$. Since $q \in \Pi$ and $\int I > 0$ (2.1 weak form), $q \cdot \int I = \int q \cdot I > 0$.

Let $S = \{t: q\cdot I(t) > 0\}$; then S is non-null, and we denote its measure by $\mu(S)$. Define

$$\Delta' = \{x: \text{for all } i,\ x^i \leq 2\int \Sigma_{j=1}^n I^j/\mu(S)\}\ .$$

We claim

(6.14) <u>For all</u> $t \in S$, <u>there is a</u> k_0 <u>such that</u> $X_k(t) \notin \Delta'$ <u>for all</u> $k \geq k_0$.

Indeed, if (6.14) is false, then by the compactness of Δ', there is a trader t in S such that $\{X_k(t)\}$ has a limit point x in Δ'. W.l.o.g. assume that x is actually the limit of $\{X_k(t)\}$. Now because (q_k, X_k) is a competitive equilibrium in $\mathcal{M}_k$, we have

(6.15) $$q\cdot x = \lim_k q_k\cdot X_k(t) = \lim_k q_k\cdot I_k(t) = q\cdot I(t).$$

Hence $x \in T(t)$, and so does not saturate. So (2.4) (strong form) applies. Hence $x + \{1,0,\ldots,0\} \succ_t x$. By (6.15) and $t \in S$, there is a coordinate j such that $x^j > 0$ and $q^j > 0$; since $q^1 = 0$, we may assume w.l.o.g. that $j = 2$. If for sufficiently small $\delta > 0$ we define $y = x + \{1,-\delta,0,\ldots,0\}$, then $y \in \Omega$ and by continuity, $y \succ_t x$. Again using continuity, we deduce $y \succ_t X_k(t)$ for k sufficiently large. Since (q_k, X_k) is a competitive equilibrium in $\mathcal{M}_k$, we obtain $q_k\cdot y > q_k\cdot I_k(t)$. Letting $k \to \infty$ and applying (6.15), we deduce

$$q\cdot y = \lim_k q_k\cdot y \geq \lim q_k\cdot I_k(t) = q\cdot x\ .$$

But since $q^1 = 0$ and $q^2 > 0$, we have

$$q\cdot y = q\cdot x + q^1 - \delta q^2 = q\cdot x - \delta q^2 < q\cdot x\ ,$$

contradicting $q\cdot y \geq q\cdot x$. This proves (6.14).

From (6.14) it follows that for $t \in S$ and $k \geq k_0(t)$, there is an i such that $X^i_k(t) > 2\int \Sigma_j I^j / \mu(S)$. Hence $\Sigma_i X^i_k(t) > 2\int \Sigma_j I^j / \mu(S)$ for $k \geq k_0(t)$. Hence

$$\text{(6.16)} \qquad \liminf \Sigma_i X^i_k(t) \geq 2\int \Sigma_j I^j / \mu(S) \quad \text{for } t \in S .$$

Hence

$$\begin{aligned} \int \Sigma_i I^i &= \int \lim_k \Sigma_i I^i_k = \lim_k \int \Sigma_i I^i_k = \lim_k \int \Sigma_i X^i_k \geq \int \liminf_k \Sigma_i X^i_k \\ &\geq \int_S \liminf \Sigma_i X^i_k \geq \int_S [2\int \Sigma_j I^j / \mu(S)] = (2\int \Sigma_j I^j)(\int_S 1)/\mu(S) \\ &= 2\int \Sigma_j I^j . \end{aligned}$$

The justifications for these inequalities are as follows: (6.1); (6.1)--uniform convergence; X_k is an allocation in $\mathcal{M}_k$; Fatou's lemma (lemma 4.5); $S \subset T$; (6.16); the integrand is a constant; $\int_S 1 = \mu(S)$.

Since $\int \Sigma_i I^i \geq 2\int \Sigma_j I^j$ contradicts $\int I > 0$, we have proved lemma 6.13.

Since $q_k \longrightarrow q > 0$, there is a $\delta > 0$ such that $q^i_k \geq \delta$ for k sufficiently large and all i. W.l.o.g. assume $q^i_k \geq \delta$ for all i and k. Furthermore, by (6.1) assume w.l.o.g. that $I^i_k(t) \leq I^i(t) + \delta$ for all i, k, and t. Hence for all i, k, and t,

$$\begin{aligned} \delta \cdot X^i_k(t) &\leq q_k \cdot X_k(t) \leq q_k \cdot I_k(t) \leq q_k \cdot I(t) + \delta \\ &\leq \Sigma^n_{j=1} I^j(t) + \delta . \end{aligned}$$

Hence

$$X^i_k(t) \leq \Sigma^n_{j=1} I^j(t)/\delta + 1 .$$

Here the right side is integrable, so the X_k have a weak limit point, which we call X (compare lemma 4.11). W.l.o.g. $X_k \longrightarrow X$ weakly. Then

$$(6.17) \qquad \int X = \lim_k \int X_k = \lim_k \int I_k = \int I$$

(weak convergence, X_k is an allocation in $\mathcal{M}_k$, (6.1)), and so X is an allocation. Next, for given t, let x be a limit point of $\{X_k(t)\}$, say $x = \lim_m X_{k_m}(t)$. Since

$$q_{k_m} \cdot X_{k_m}(t) \leq q_{k_m} \cdot I_{k_m}(t) ,$$

we deduce by letting $m \longrightarrow \infty$ that

$$q \cdot x \leq q \cdot I(t) .$$

This holds for all limit points x of $\{X_k(t)\}$, so by lemma 4.6, $q \cdot X(t) \leq q \cdot I(t)$ for all t, i.e.

$$(6.18) \qquad X(t) \in B_q(t) \quad \text{for all } t.$$

Finally, suppose that for some t, there is a $y \in B_q(t)$ such that $y \succ_t X(t)$. Then there is also a limit point x of $\{X_k(t)\}$ such that

$$y \succ_t x \ ;$$

for if $x \succsim_t y$ for all such limit points x, then by convexity (2.6), any point in the convex hull of all these x is also $\succsim_t y$, and so by lemma 4.6, $X(t) \succsim_t y$, contrary to our supposition. Clearly $y \neq 0$; suppose w.l.o.g. that $y^1 > 0$. If for $\delta > 0$ sufficiently small we define $y_\delta = y - (\delta,0,\ldots,0)$, then we still have

$$(6.19) \qquad y_\delta \succ_t x .$$

Moreover, since

$$\lim_k q_k \cdot y_\delta \;=\; q\cdot y - q^1\delta \;<\; q\cdot y \;\le\; q\cdot I(t) \;\le\; \lim_k q_k\cdot I_k(t)\ ,$$

it follows that

$$q_k\cdot y_\delta \;\le\; q_k\cdot I_k(t)$$

for all sufficiently large k, say for $k > k_0$. Now since x is a limit point of $\{X_k(t)\}$, there is a subsequence $\{X_{k_m}(t)\}$ converging to x; hence for m sufficiently large,

$$y_\delta \succ_t X_{k_m}(t)\ ,$$

(by (6.19)). If we also pick m so large that $k_m \ge k_0$, then y_δ contradicts the maximality of X_{k_m} in $\{y: q_{k_m}\cdot y \le q_{k_m}\cdot I_{k_m}(t)\}$. So the supposition $y \succ_t X(t)$ has led to a contradiction, and we conclude that $X(t)$ is maximal in $B_q(t)$ for all t.

Combining this with (6.17) and (6.18), we deduce that (q,X) is a competitive equilibrium. All that is now necessary for the completion of the proof of Theorem B is the verification that the markets $\mathcal{M}_k$ satisfy the hypotheses of Theorem A.

7. Verification of the Preference Conditions in $\mathcal{M}_k$

We show here that the $\mathcal{M}_k$ satisfy the conditions of Theorem A.

To demonstrate continuity, recall that a quasi-order $\succsim$ on Ω satisfies (2.2) if and only if it has a utility function, i.e. a real function υ on Ω such that $x \succsim y$ if and only if $\upsilon(x) \ge \upsilon(y)$ (cf. Debreu (1959), 4.6 (1), p. 56). So there is a utility function υ_t for $\succsim_t$. Then $\upsilon_t\mu_{kt}$

is[22] a utility function for $\succsim_t^k$, and therefore $\succsim_t^k$ is also continuous. The proof of measurability (2.3) of $\succsim_t^k$ requires a number of straightforward manipulations that use the measurability of $\succsim_t$ and of $N(t)$, and the following slight strengthening of the measurability assumption (2.3) for $\succsim_t$:

<u>Lemma 7.1</u> <u>For all assignments</u> X <u>and</u> Y , <u>the set</u> $\{t:X(t) \succ_t Y(t)\}$ <u>is measurable</u>.

<u>Proof</u> Because of continuity (2.2), the statement "$X(t) \succ_t Y(t)$" is equivalent to "There is a rational point r such that for all positive integers m there is a rational point s such that

$$X(t) \succ_t r , \quad r \succ_t s , \quad \text{and} \quad \|s - Y(t)\| \leq 1/m ."$$

The lemma then follows from (2.3) and from the measurability of X and Y.

Lemma 7.1 coincides with the statement of assumption 2.3 in MCT I, II, and III. Here we make use of the fact that $\succsim_t$ is a quasi-order, which was not available there; also of two-sided continuity, which was not available in MCT III.

To prove the weak form of (2.4) for $\succsim_t^k$, let $x > y$, where y does not saturate in $\succsim_t^k$. Since y does not saturate, at least one of its components y^i is $< N_k^i(t)$. Then $u_{kt}^i(y) = y^i < \min(x^i, N_k^i(t)) = u_{kt}^i(x)$. Furthermore for all j we clearly have

$$u_{kt}^j(y) = \min(y^j, N_k^j(t)) \leq \min(x^j, N_k^j(t)) = u_{kt}^j(x) .$$

---

[22] Composition is meant, not multiplication. That is, $(\upsilon_t u_t)(x) = \upsilon_t(u_t(x))$.

So

$$(7.2) \qquad u_{kt}(x) \geq u_{kt}(y) \quad \text{and} \quad u_{kt}(x) \neq u_{kt}(y) .$$

Next, the non-saturation of y in $\succsim_t^k$ means that there is a z such that $u_{kt}(z) \succ_t u_{kt}(y)$. So $u_{kt}(y)$ does not saturate in $\succsim_t$. Hence by (7.2) and the strong form of (2.4) for $\succ_t$, $u_{kt}(x) \succ_t u_{kt}(y)$. Hence $x \succ_t^k y$, and the weak form of (2.4) is established for $\succ_t^k$.

To prove the saturation restriction for $\succsim_t^k$, let $y \in \Upsilon_k(t)$. Then

$$u_{kt}(y) \in \Upsilon_k(t) \cap \Delta_k(t) \subset \Upsilon'(t) \quad ;$$

so $u_{kt}(y)$ does not saturate for $\succsim_t$. Since $y \in \Upsilon_k(t)$, we have $y^i \leq I^i(t)$ for some i. Hence $u_{kt}^i(y) \leq y^i \leq I_k^i(t) < N_k^i(t)$, by (6.11). So by adding a small amount to the $i^{\underline{th}}$ coordinate of $u_{kt}(y)$, and nothing to the other coordinates, we obtain a z such that $z \geq u_{kt}(y)$, $z \neq u_{kt}(y)$, and $z \in \Delta_k(t)$. Since $u_{kt}(y)$ does not saturate for $\succsim_t$, we conclude from the strong form of (2.4) for $\succ_t$ that $z \succ_t u_{kt}(y)$. From $z \in \Delta_k(t)$ we conclude $u_{kt}(z) = z$, so $u_{kt}(z) \succ_t u_{kt}(y)$, i.e. $z \succ_t^k y$. So y does not saturate, and the saturation restriction is proved.

Finally, to prove convexity (2.6) for $\succ_t^k$, let $x \succ_t^k y$. Then $u_{kt}(x) \succ_t u_{kt}(y)$, and hence

$$(7.3) \qquad \alpha u_{kt}(x) + (1-\alpha)u_{kt}(y) \succ_t u_{kt}(y) \quad .$$

Now clearly $\alpha x + (1-\alpha)y \geq \alpha u_{kt}(x) + (1-\alpha)\, u_{kt}(y)$; hence

$$\begin{aligned} u_{kt}(\alpha x + (1-\alpha)y) &\geq u_{kt}(\alpha u_{kt}(x) + (1-\alpha)u_{kt}(y)) \\ &= \alpha u_{kt}(x) + (1-\alpha)u_{kt}(y) \, , \end{aligned}$$

since the right side is in $\Delta_k(t)$. Therefore, because of the strong form of (2.4) for $\succ_t$, we deduce

$$u_{kt}(\alpha x + (1-\alpha)y) \succsim_t \alpha u_{kt}(x) + (1-\alpha)\, u_{kt}(y) \quad ,$$

where the $\sim$ sign is there because of the possibility of equality. Combining this with (7.3), we deduce

$$u_{kt}(\alpha x + (1-\alpha)y) \succ_t u_{kt}(y) \; ,$$

and hence $\alpha x + (1-\alpha)y \succ_t^k y$, as was to be proved.

That N_k saturates commodity-wise in η_k follows at once from the definition.

8. Comparison with McKenzie's Proof

Since these proofs are based on McKenzie's (1959) for the finite case, it is worthwhile to point out some of the differences. Perhaps the most important are in the passages from properties proved for individual traders to the corresponding properties for the aggregate of all traders. These passages are almost trivial in the finite case, but in ours they require the use of function spaces and their properties. Lemmas 4.6 and 4.11 are the principal tools in this connection. An example is the continuity of the aggregate preferred set as a function of the price vector; this is a trivial consequence of the continuity of the individual preferred sets in McKenzie's paper, but in ours it requires all of section 4.

Another significant difference is in the matter of boundedness. In the proof of Theorem A the set of bundles under consideration must be in some sense bounded in order to establish the continuity--and indeed the existence--of the individual preferred sets. McKenzie does this by noting that no individual trader can have more goods than the whole market. This is no longer available here, because anyway each trader's bundle is infinitesimal compared with the whole market. We therefore need the notion of commodity-wise saturation, which does the job of bounding for us. In Theorem B we don't have commodity-wise saturation; here we first deduce from strong desirability that all prices must be non-vanishing, and this bounds the bundles under consideration to a finite simplex.

A relatively minor difference is that the construction of allocations for various purposes (e.g. lower semi-continuity) involves somewhat laborious measurability proofs here, whereas there is of course no such problem in McKenzie's work.

9. Counter-Examples

Counter-examples to the existence of a competitive equilibrium were given in section 4 of MCT III for the following cases:

(a) Assumption (2.1) is dropped entirely, even though the strong form of (2.4) holds.

(b) The weak forms of both (2.1) and (2.4) hold, but the strong form of neither holds.

10. Existence of a Competitive Equilibrium under the Conditions of MCT II[23]

It is possible to establish the existence of a competitive equilibrium under the conditions of MCT II, if in addition to the conditions assumed there

[23] Readers not familiar with MCT II should omit this section, as the definitions will not be repeated.

one assumes convexity and that the preferences are quasi-orders. The advantage of this is that it allows saturation for bundles that are "very large" (precisely, greater than $\lambda M(t)$) in at least one coordinate, without demanding that they be $>I(t)$ in all coordinates. The precise theorem is:

Theorem C Assume the conditions of this paper, with the weak form of (2.1) but without any form of (2.4). Instead, assume that there is a λ such that for all t, the relation $\;_t$ obeys λ-desirability at all $y \leq M(t)$. Then there is a competitive equilibrium.

Proof The basic tools in the proof are two lemmas from MCT I and MCT II. Both were proved under hypotheses considerably weaker than those of Theorem B; therefore they hold under the hypotheses of Theorem B.

Lemma 10.1 Every equilibrium allocation[24] is in the core.

Lemma 10.2 If X is in the core, then $X(t) \leq \lambda M(t)$ for all t.

Lemma 10.1 is the contents of section 3 of MCT I. Lemma 10.2 is lemma 5.5 of MCT II.

First assume that $I(t) \neq 0$ for all t. The procedure is to modify the given preference orders so as to satisfy the strong form of (2.4) as well as the other conditions of Theorem C. Then the conditions of Theorem B are satisfied. In the course of this modification, care is taken not to change the given preference orders in or near the hypercube

[24] See the introduction.

$\Delta^*(t) = \{x : x \leq \lambda M(t)\}$. Applying Theorem B, we obtain an equilibrium point (q,X) for the modified preference orders. Because of lemmas 10.1 and 10.2, we have $X(t) \in \Delta^*(t)$. If (q,X) were not a competitive equilibrium for the original preference orders, there would be a trader t who could do better within his budget than $X(t)$. But then by the convexity condition (2.6), he could also do better by choosing a commodity bundle in his budget set quite close to $X(t)$, sufficiently close to be in the region where the modified order coincides with the original order. But then (q,X) would not be a competitive equilibrium in the modified order either, contrary to our construction.

The original orders will be denoted, as always, by $\succsim_t$. Define $E(t) = \{x : \lambda M(t) \succsim_t x\}$; $E(t)$ is the corner of Ω that is "cut off" by the indifference surface on which $\lambda M(t)$ lies (see Figure 4). The modified orders $\succsim_*$ are defined as follows: Within $E(t)$, $\overset{*}{\succsim_t}$ coincides with $\succsim_t$.

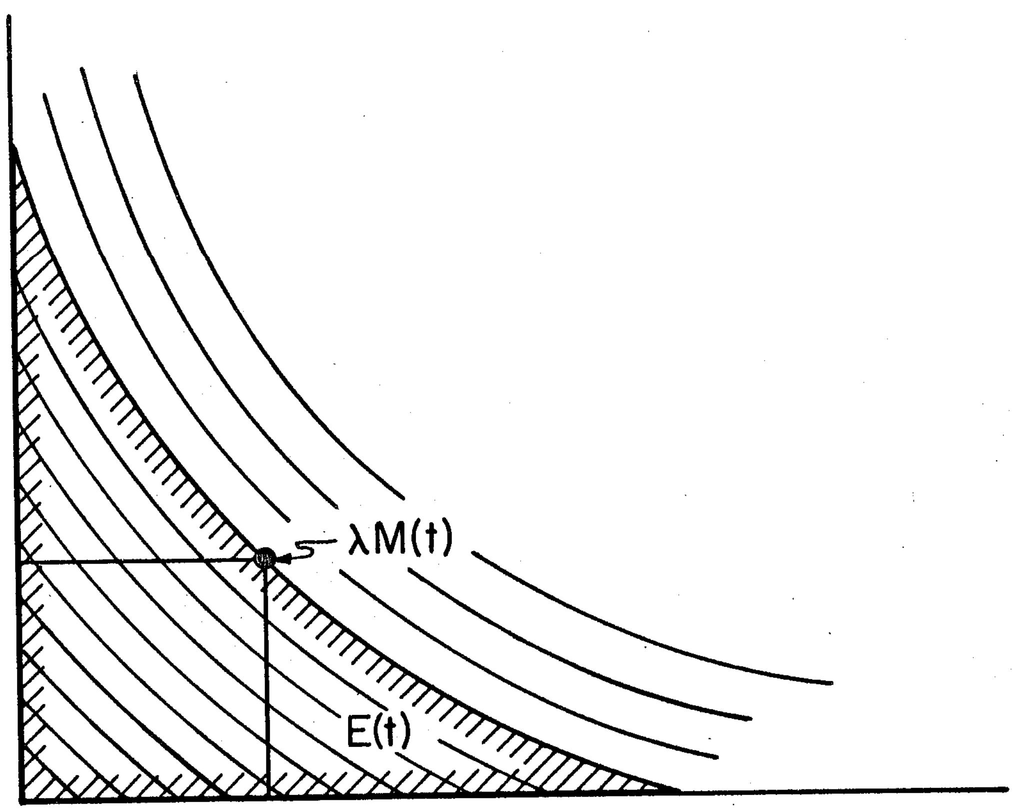

FIG. 4

In $\Omega - E(t)$, the indifference surfaces of $\underset{\sim t}{\succ}^{*}$ are scalar multiples of the "last" indifference surface in $E(t)$, i.e. the one that passes through $\lambda M(t)$. Formally: if $x,y \in \Omega - E(t)$, let x', y' be such that $x' \sim_t \lambda M(t)$, $y' \sim_t \lambda M(t)$, $x = \alpha x'$, $y = \beta y'$ for some scalars α and β. The existence of such x', y', α and β follows from the continuity condition (2.2), since $x \succ_t \lambda M(t) \succ_t 0$, and so some scalar multiple of x must be indifferent with $\lambda M(t)$. Furthermore x', y', and hence α, β are unique, for otherwise the convexity conditon (2.6) is violated. We now define: $x \underset{\sim t}{\succ}^{*} y$ <u>if and only if</u> $\alpha \geq \beta$. If $x \in \Omega - E(t)$ and $y \in E(t)$, we <u>define</u> $x \succ_t^{*} y$.

The straightforward verifications that $\underset{\sim t}{\succ}^{*}$ satisfies (2.2) through (2.6) are left to the reader. The remainder of the proof is as in the outline above, except for the following point: The proof depends on the fact that for any equilibrium allocation X and all t, $X(t)$ can be surrounded by a neighborhood in which the modified order and the original order coincide. This follows from $X(t) \in \Delta^{*}(t)$. For if $X(t) \neq \lambda M(t)$, then from λ-desirability it follows that $\lambda M(t) \succ_t X(t)$, so that $X(t)$ is in the interior of $E(t)$. Moreover, $X(t) = \lambda M(t)$ is not possible for any t. For since $I(t) \neq 0$, it follows that $M(t) > 0$, and hence $\lambda M(t) > M(t) \geq I(t)$. Therefore $\lambda M(t)$ cannot be in the budget set $B_q(t)$. Since (X,q) is a competitive equilibrium for the modified orders, it follows that $X(t) \in B_q(t)$, and so $X(t) \neq \lambda M(t)$.

Finally, if $I(t) = 0$ for some t, we first obtain a competitive equilibrium $(q, X|S)$ restricted to the set $S = \{t : I(t) \neq 0\}$. This is extended to all of T by defining $X(t) = 0$ when $I(t) = 0$. Clearly X is an allocation and $X(t) \in B_q(t)$ for all t; that $X(t)$ is maximal in $B_q(t)$ when $t \notin S$ follows from $q > 0$, since this implies $B_q(t) = \{0\}$ for $t \notin S$.

This completes the proof of Theorem C.

REFERENCES

Arrow, K. J., and G. Debreu, "Existence of an equilibrium for a competitive economy", Econometrica 22 (1954), pp. 265-290.

Aumann, R. J., "Markets with a continuum of traders I, II, III", Econometric Research Program Research Memoranda 39 (5 July 1962), 40 (16 July 1962), and 56 (23 October 1962).

Debreu, G., Theory of Value, John Wiley, 1959.

Dunford, N., and Schwartz, J. T., Linear Operators, Part I, Interscience, 1958.

Gale, D., "The law of supply and demand", Mathematica Scandinavica 3 (1955), pp. 155-169.

Karlin, S., Mathematical Methods and Theory in Games, Programming, and Economics, Addison-Wesley, 1959.

McKenzie, L. W., "On the existence of general equilibrium for a competitive market", Econometrica 27 (1959), pp. 54-71.

Nikaido, H., "On the classical multilateral exchange problem", Metroeconomica 8 (1956), pp. 135-145.