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THE KERNEL OF A COOPERATIVE GAME

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Abstract

The kernel of a cooperative n-person game is defined. It is a subset of the bargaining set $\mathcal{M}_1^{(i)}$. Its existence and some of its properties are studied. We apply it to the 3-person games, to the 4-person constant-sum games, to the symmetric and n-quota games and to games in which only the n and the (n-1)-person coalitions are allowed to be non-flat.

In order to illustrate its merits and demerits as a predictor of an actual outcome in a real-life situation, we exhibit an example in which the kernel prediction seems frustrating. The opinions of other authors are quoted in order to throw some light on this interesting example.

1. Introduction.

In [1], R. J. Aumann and M. Maschler introduced the various bargaining sets for n-person cooperative games. These are meant to assign payoffs to the various coalition in structures which make them stable in some sense.

In [7], B. Peleg proved that for each coalition structure there exists at least one payoff which makes it stable in the sense of the bargaining set $\mathcal{M}_1^{(i)}$. Whether it is possible to single out a unique element in $\mathcal{M}_1^{(i)}$, for each coalition structure which has interesting strong stability properties is still unknown. Perhaps, a better approach to this would be to decompose the bargaining set into various subsets — each of which represents a specific "way of thinking" that may cause the players to end up within a particular set of outcomes.

In this paper (Section 2), we define a particular subset of the bargaining set $\mathcal{M}_1^{(i)}$, which we call the kernel of the game. We study some of its properties in Section 3 and prove its existence in Section 5. In Section 4 we study the kernel for the 3-person game and show that it can be given an interesting dynamic interpretation. These results are generalized in Section 7 to games in which all sets other than the n-1 and the n-person coalitions are flat.(1)

It is easier to compute the kernel than to compute the bargaining set of a game, although even for the kernel, the computation may be tedious and may require non-systematic short cuts. In Section 8 we characterize completely the 4-person constant-sum game. In Section 9 we treat the symmetric games and cite B. Peleg's analysis of the m-quota games.

Is the kernel interesting in itself as a predictor of possible outcomes? Is it in some sense more plausible that the players will end up with an outcome in the kernel instead of another outcome, say, in the bargaining set?

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(1) A coalition is called flat if its value is equal to the sum of the values of its members - considered as 1-person coalitions.

We do not think so, although, in Section 7 (and as a matter of fact, in all the other sections), we do our best to justify the kernel. We do so by showing that the outcomes of the kernel seem reasonable in some games, and we also try to justify the definition as a method of reaching a compromise. In this interpretation, however, we need an assumption of interpersonal comparison of utilities.

Nevertheless, we believe that the kernel throws light on a wealth of interesting compromise aspects of n-person games. We also believe that its mathematical properties deserve further study.

In Section 7 we describe an analysis of a certain weighted majority game according to which one arrives at outcomes which, at least at first glance, seem "unintuitive," although they are in the kernel. This analysis was presented to several experts on Game Theory, who were asked to express their opinions. It turned out that these opinions summarized various schools of thought that exist today in Game Theory, concerning its applicability to real life situations. We have therefore decided to quote the answers here. We wish to express our indebtedness and gratitude to the authors. The reader will undoubtedly benefit from their contributions.

2. The kernel of a cooperative game.

Let Γ be a cooperative n-person game, described by the ordered pair $(v; N)$. Here $N \equiv \{1,2, \ldots, n\}$ is the set of the players and $v = v(B)$ is the characteristic function of the game. We do not assume that $v(B)$ is a super-additive function.

The non-empty subsets of N are called coalitions and we require that they form the domain of $v(B)$, and that $v(B)$ satisfies

(2.1) $$v(B) \geq 0 \text{ for each coalition } B,$$

(2.2.) $$v(i) = 0 \text{ for}^{(1)} \text{ each 1-person coalition } \{i\}.$$

An outcome of the game will be denoted by

(2.3) $$(\mathcal{X}; \mathcal{B}) \equiv (x_1, x_2, \ldots x_n ; B_1, B_2, \ldots, B_m),$$

where x_i denotes the payoff to the ith player and $\mathcal{B} \equiv (B_1, B_2, \ldots, B_m)$ represents the coalition-structure which was formed.

Thus, $\mathcal{B}$ is a partition of N, hence it satisfies

(2.4) $$B_j \cap B_k = \emptyset \text{ if } j \neq k, \bigcup_{j=1}^{m} B_j = N,$$

and the payoff vector $\underline{x} \equiv (x_1, x_2, \ldots, x_n)$ is assumed to satisfy:

(2.5) $$x_i \geq 0, \; i = 1, 2, \ldots, n \text{ (individual rationality)}$$

(2.6) $$\sum_{i \in B_j} x_i = v(B_j), \; j = 1, 2, \ldots, m.$$

The symbol $(\mathcal{X}; \mathcal{B})$ will be called an individually rational payoff configuration (i.r.p.c.).

If we fix the coalition structure $\mathcal{B}$, then the set of all the payoffs $\mathcal{X}$, satisfying (2.5) and (2.6), is a cartesian product of m simplices:

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(1) We write $v(i)$ instead of $v(\{i\})$. It turns out that (2.1) and (2.2) cause no real loss of generality. (See the beginning of section 3 and Remark 3.1).

(2.7) $$X \equiv X(\mathcal{B}) \equiv S_1 \times S_2 \times \ldots \times S_m,$$

where

(2.8) $$S_j = \{\{x_i\}_{i \in B_j} \mid x_i \geq 0, \sum_{i \in B_j} x_i = v(B_j)\}, \; j = 1, 2, \ldots, m.$$

Definition 2.1 Let $(\mathcal{X} ; \mathcal{B})$ be an i.r.p.c. for a game Γ, and let D be an arbitrary coalition. The excess of D with respect to $(\mathcal{X} ; \mathcal{B})$ is

(2.9) $$e(D) \equiv v(D) - \sum_{i \in D} x_i .$$

The excess of D therefore represents the total amount that the members of D gain (or lose, if $e(D) < 0$), if they withdraw form $(\mathcal{X} ; \mathcal{B})$ and form the coalition D. Clearly,

(2.10) $$e(B_j) = 0, \; j = 1, 2, \ldots, m.$$

Let k and ℓ be two distinct players in a coalition B_j of $\mathcal{B}$; we denote by $T_{k,\ell}$ the set of all the coalitions which contain player k but do not contain player ℓ; i.e.,

(2.11) $$T_{k,\ell} \equiv \{D \mid D \subset N, \; k \in D, \; \ell \notin D\}$$

Definition 2.2 Let $(\mathcal{X} ; \mathcal{B})$ be an i.r.p.c. for a game Γ, and let k and ℓ be two distinct players in a coalition B_j of $\mathcal{B}$. The maximum surplus of k over ℓ with respect to $(\mathcal{X} ; \mathcal{B})$ is

(2.12) $$s_{k,\ell} \equiv \max_{D \in T_{k,\ell}} e(D).$$

The maximum surplus, therefore, represents the maximal amount player k can gain (or the minimal amount that he must lose), by withdrawing from $(\mathcal{X} ; \mathcal{B})$ and joining a coalition D which does not require the consent of ℓ (since $\ell \notin D$), with the understanding that the other members of D will be satisfied with getting the same amount they had in $(\mathcal{X} ; \mathcal{B})$.

Definition 2.3 Let $(\mathcal{X} ; \mathcal{B})$ be an i.r.p.c. for a game Γ, and let k, ℓ be two distinct players in a coalition B_j of $\mathcal{B}$. Player k is said to outweigh player ℓ with respect to $(\mathcal{X} ; \mathcal{B})$, and this is denoted by $k \gg \ell$, or, equivalently, by

$\ell \ll k$, if

(2.13) $$s_{k,\ell} > s_{\ell,k} \text{ and } x_\ell \neq 0.$$

If neither $k \gg \ell$ nor $\ell \gg k$, we say that k and ℓ are in equilibrium. For the sake of completeness we define each player to be in equilibrium with himself. Similarly, we also regard any two players, who belong to disjoint coalitions of $\mathcal{B}$, as being in equilibrium. We write $k \approx \ell$ if k and ℓ are in equilibrium.

Note the special role a player possesses if he gets 0 in $(\mathcal{X}; \mathcal{B})$. In this case, no player can outweigh him.

Definition 2.4 Let $(\mathcal{X}; \mathcal{B})$ be an i.r.p.c. for a game Γ. A coalition B_j of $\mathcal{B}$ is said to be balanced with respect to $(\mathcal{X}; \mathcal{B})$, if each two players of B_j are in equilibrium.

Clearly, a 1-person coalition in $\mathcal{B}$, if such occurs, is always balanced.

Definition 2.5 The kernel $\mathcal{K}$ of a game Γ is the set of all the i.r.p.c.'s having only balanced coalitions. Or, equivalently, $(\mathcal{X}; \mathcal{B}) \in \mathcal{K}$, if and only if each two players are in equilibrium w.r.t. $(\mathcal{X}; \mathcal{B})$.

Corollary 2.1 It is easy to verify that

(2.14) $$k \gg \ell \text{ if and only if } (s_{k,\ell} - s_{\ell,k})\, x_\ell > 0$$

and that

(2.15) $$k \approx \ell \text{ if and only if } (s_{k,\ell} - s_{\ell,k})\, x_\ell \leq 0 \text{ and } (s_{\ell,k} - s_{k,\ell})\, x_k \leq 0.$$

We shall study in this paper some of the properties of the kernel in an attempt to find out to what extent and in what context it may serve as a useful tool in "predicting" the outcome of a game.

3. Elementary properties of the kernel, the pure bargaining game.

In principle, one can compute the kernel of a game by solving systems of inequalities of the form (2.15). In practice, unless the number of the players is small, this may be an enormous task in view of the huge number of possibilities which have to be considered.

It is clear from the definition that the kernel does not depend on the labeling of the players.

If $(v; N)$ and $(w; N)$ are strategically equivalent games, and if $(\mathcal{x}; \mathcal{B})$ and $(\xi; \mathcal{B})$ are corresponding i.r.p.c.'s in $(\mathcal{x}; \mathcal{B})$ and $(\xi; \mathcal{B})$, respectively, then the corresponding excesses of the various coalitions, with respect to the two games, are proportional, (with a positive factor of proportion). Therefore $(\mathcal{x}; \mathcal{B})$ belongs to the kernel of $(v; N)$ if and only if $(\xi; \mathcal{B})$ belongs to the kernel[1] of $(w; N)$. Thus, the kernel is invariant under strategic equivalence.

The simplest game which we shall now consider is the pure bargaining game;[2] namely, a game in which all the coalitions, except, perhaps, the grand coalition N, are flat.[3] For reasons of future convenience, we shall not require the normalization (2.2), and therefore (2.1) will take the form:

(3.1) $$v(N) \geq v(1) + v(2) + \ldots + v(n).$$

It is easy to verify that $(\mathcal{x}; \mathcal{B}) \in \mathcal{K}$ for $\mathcal{B} \neq N$, if and only if $x_i = v(i)$, $i = 1, 2, \ldots, n$; and that $(\mathcal{x}; \mathcal{B}) \in \mathcal{K}$ if and only if

(3.2) $$x_i = v(i) + [v(N) - v(1) - v(2) - \ldots - v(n)]/n, \quad i = 1, 2, \ldots, n.$$

Remark 3.1 We shall dispose of the case when (3.1) is not satisfied by treating a more general case: If the value of a coalition is smaller than the sum of the value of its members, then this coalition cannot occur in any i.r.p.c. Also, such

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(1) Obviously, the 0's in (2.5) and (2.13) should be replaced by $v(i)$ and $v(\ell)$, respectively.

(2) The name is suggested from L. S. Shapley [8].

(3) A coalition B is called flat if $v(B) = \sum_{i \in B} v(i)$.

a coalition can be omitted from $T_{k,\ell}$ without changing $s_{k,\ell}$ (see (2.11) and (2.12)), because its excess is less than the excess of the 1-person coalition k. Its actual value has no influence on the kernel. Moreover, if we replace its value by the sum of the values of its members, we do not change $s_{k,\ell}$. Therefore, the following rule may be applied:

If a coalition B in a game Γ makes less than the total α gained by its members acting as 1-person coalitions, replace its value by α (i.e., make it a flat coalition) and compute the kernel for the new game. The kernel of the original game will then be obtained by removing from the new kernel all the i.r.p.c.'s in which B appears in the coalition structure.

We shall later encounter the situation in which players, confronted with a game $(v; N)$, will "decide" to treat instead a pure bargaining game $(u; N)$ with $u(B) = 0$ for $B \neq N$, $u(N) = v(N)$, yet the decision on the split of $v(N)$ among the players will be based on a "recognition" that the "strength" of a player i is a certain real number w_i, $i = 1, 2, \ldots, n$, which is derived, in a certain way from v. The sum of the w_i's may be smaller, greater or equal to $v(N)$. To be precise we define:

<u>Definition 3.1</u> A <u>pseudo pure bargaining n-person game</u> is the triplet $(u; N; w_1, w_2, \ldots w_n)$, where N is the set of the players,

$$u(B) = 0 \text{ for } B \neq N,\ u(N) \geq 0, \tag{3.3}$$

and w_i, $i = 1, 2, \ldots, n$, are real numbers. A <u>share of $u(N)$, based on $w_1, \ldots w_n$</u> is defined to be $(\mathfrak{x}; N)$, where

$$x_i = w_i + [u(N) - w_1 - w_2 - \ldots - w_n]/n, \quad i = 1, 2, \ldots, n, \tag{3.4}$$

<u>provided that</u> $x_i \geq 0$ for each i, $i = 1, 2, \ldots, n$.

If the last condition is not satisfied, we shall define the share inductively as follows: Take a player i_o who has a smallest (and therefore negative) x_i and give him the amount 0. The share for the other players will then be based on the pseudo pure bargaining game[1] $(u^{i_o}; N, w_1, \ldots, w_{i_o-1}, w_{i_o+1}, \ldots, w_n)$.

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(1) The reasons for such a definition will become clear later. It is independent of the particular player i_o which we choose.

Intuitively, a player i acts as if he could make w_i by himself, (compare (3.2) with (3.4), but since this amount is fictitious, the players are not bound by the individual rationality condition $x_i \geq w_i$. They still respect the condition $x_i \geq 0$ which, when violated by a player i_o, forces him to leave the bargaining and permit the rest of the players to share $v(N)$ among themselves.

4. The 3-person game.

The 2-person game is a particular case of the pure bargaining game, discussed in the previous section.

The following lemma will be useful subsequently:

Lemma 4.1. Let $(\mathcal{X};\mathcal{B})$ be an i.r.p.c. for a game Γ, and suppose that the coalition B_1, $B_1 \in \mathcal{B}$, is balanced with respect to $(\mathcal{X};\mathcal{B})$. Let $(\mathcal{Y};\mathcal{B})$ be another i.r.p.c., having the same coalition structure $\mathcal{B}$, with the same payoffs to the players outside of B_1, and with only player in B_1 getting strictly less [more] than what he received in $(\mathcal{X};\mathcal{B})$. Under these conditions, the coalition B_1 will not be balanced with respect to $(\mathcal{Y};\mathcal{B})$.

Proof: Suppose that player k was the only player in B_1, who received in $(\mathcal{Y};\mathcal{B})$ less than he did in $(\mathcal{X};\mathcal{B})$. By (2.6), B_1 contains at least one other player, say, ℓ, who received in $(\mathcal{Y};\mathcal{B})$ more than he did in $(\mathcal{X};\mathcal{B})$. Using (2.6) once more, we find that the excess of each coalition in $T_{k,\ell}$ (see (2.9), (2.10)) increases in the transition from $(\mathcal{X};\mathcal{B})$ to $(\mathcal{Y};\mathcal{B})$ by the amount $y_\ell - x_\ell > 0$, at least. However, the excess of each coalition in $T_{\ell,k}$ decreases in this transition at least by the same amount, and by (2.5), $y_\ell > 0$; hence player k outweighs player ℓ with respect to $(\mathcal{Y};\mathcal{B})$, and B_1 cannot be balanced.

If player k was the only player in B_1 who received in $(\mathcal{Y};\mathcal{B})$ more than he did in $(\mathcal{X};\mathcal{B})$, assume that $(\mathcal{Y};\mathcal{B})$ is balanced, interchange the roles of $(\mathcal{X};\mathcal{B})$ and $(\mathcal{Y};\mathcal{B})$ in the first part of this proof, and you get a contradiction. This completes the proof.

We shall prove later that for each coalition-structure $\mathcal{B}$ in a game Γ, there exists at least one payoff vector $\mathcal{X}$, such that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$. Granting this, we shall now prove:

Theorem 4.1. For each coalition structure $\mathcal{B}$, in a 3-person game, there exists a unique payoff vector $\mathcal{X}$ such that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$.

Proof: Each coalition in $\mathcal{B}$ contains at most 3 players. Also, at most one coalition has more than one member — the others, if such exist, receive the fixed amount 0 . If $(x ; \mathcal{B}) \in \mathcal{K}$ and $(y ; \mathcal{B})$ is a different i.r.p.c., having the same coalition structure $\mathcal{B}$, then the payments to members of only one coalition could be changed and either only one player's payment increased or one player's payment decreased. By Lemma 4.1, $(y ; \mathcal{B})$ cannot belong to $\mathcal{K}$.

In general, it is not true that for each coalition structure $\mathcal{B}$ there exists a unique payoff vector x having the property that $(x ; \mathcal{B}) \in \mathcal{K}$.

Example 4.1. Let Γ be a 4-person game whose characteristic function is $v(12) = v(23) = v(34) = v(14) = 100$ and $v(B) = 0$ otherwise. Clearly, every p.c. of the form $(x, 100 - x, x, 100 - x ; 12, 34)$, $0 \leq x \leq 100$, belongs to the kernel of this game.

Having computed the kernel of the general 3-person game, treating the various possible cases, we were somewhat surprised to realize that the various results could be given a reasonable dynamic interpretation. The purpose of this section is to describe it. Proofs will not be given, since this game is a special case of a case which will be studied in Section 7 . Let us say at once that we do not regard this dynamic interpretation as a justification of the kernel. Other plausible procedures may lead to different outcomes. Yet, the fact that such a procedure exists (and can be generalized to other cases) seems interesting. In treating the general 3-person game we shall assume that

(4.1) $$v(12) \leq v(13) \leq v(23).$$

I. The coalition structure (1, 2, 3).

There can be only one i.r.p.c. having this coalition structure, namely (0, 0, 0 ; 1, 2, 3). It belongs to the kernel of the 3-person game.

II. The coalition structures (12, 3), (13,2), (23, 1).

Suppose e.g., that the coalition {12} is formed. Its members know that player 3 must get the amount 0 . In order to determine the share of v(12) among them, they consider the pseudo-pure bargaining game (see Definition 3.1) (u; 1, 2; v(13), v(23)), where u(12) = v(12), and share accordingly. The "reason" for choosing v(13), v(23) as a basis for their strength lies in the fact that a player i can make "almost" v(i 3), i = 1, 2, by joining player 3 and offering him a positive small amount. True, both players cannot achieve this value simultaneously, but they do not want to join player 3, since they want the coalition {12}. Under these circumstances they compromise by regarding v(13) and v(23) as the basis for negotiation. A similar situation occurs when {13} or {23} is formed. In computing the outcome, two cases should be distinguished:

A. If $v(12) + v(13) \geq v(23)$, then, in view of (4.1), the p.c.'s in the kernel are:

$$(\omega_1, \omega_2, 0 \; ; \; 12, 3)$$

(4.2) $$(\omega_1, 0 \;, \omega_3 ; \; 13, 2)$$

$$(0 \; ; \; \omega_2, \omega_3 ; \; 1, 23)$$

where $\omega_1 = [v(12) + v(13) - v(23)]/2$, $\omega_2 = [v(12) + v(23) - v(13)]/2$ and $\omega_3 = [v(13) + v(23) - v(12)]/2$.

Note that $(\omega_1, \omega_2, \omega_3)$ is the quota[1] of the game in the sense that $\omega_i + \omega_j = v(ij)$, $i, j = 1, 2, 3$, $i \neq j$.

B. If[2] $v(12) + v(13) \leq v(23)$, player 1 is deleted in the pseudo bargaining game, if {12} or {13} forms, and the outcome, which is in the kernel, is:

$$(0, \; v(12) \;, \; 0 \; ; \; 12, 3)$$

(4.3) $$(0, \; 0 \;, \; v(13) \; ; \; 13, 2)$$

$$(0, \; \omega_2 \;, \; \omega_3 \; ; \; 1, 23).$$

III. The coalition structure (123).

In this case, one of several procedures will determine the outcome.

(1) The term is suggested from L. S. Shapley [8], although its meaning here is somewhat different. See also [4].

(2) In this case player 1 is weak, i.e., has a negative quota.

These procedures will be described and we shall "rationalize" the choice from the various procedures.

Procedure A. If $v(123)$ is "large," the players will disregard the 2-person coalitions and will simply share $v(123)$ equally. Thus,

(4.4) $$(v(123)/3,\ v(123)/3,\ v(123)/3\ ;\ 123)$$

will result. This will be the situation, provided

(4.5) $$v(123) \geq 3v(23).$$

If (4.5) is not satisfied, the players 2 and 3 which form the strongest 2-person coalition (see (4.1)), will together make more by adopting the next procedure.

Procedure B. If $v(123) \leq 3v(23)$, but $v(123)$ is not "too small," then, at first, the strong coalition {23} will form and act as one player, playing against player 1 in the game $(u;\ 1,\ \{23\})$, where $u(1) = 0$, $u(\{23\}) = v(23)$, $u(1\{23\}) = v(123)$. This will determine the payoff $\alpha = [v(123) - v(23)]/2$ to player 1. Knowing player 1's payoff, the players 2 and 3 will determine their share of $[v(123) + v(23)]/2$ in accordance with the pseudo-pure bargaining game (see Definition 3.1), $(v^*;\ 2,\ 3;\ w_2,\ w_3)$, where $v^*(23) = [v(123) + v(23)]/2$, $w_2 = \text{Max}\ (0,\ v(12) - \alpha)$, $w_3 = \text{Max}\ (0,\ v(13) - \alpha)$. Thus the threat to join player 1 is used only if it is preferable to playing alone.

This procedure will be used as long as player 1 "has nothing to say;" otherwise, other procedures will arise (see Procedures C and D). To be more precise, the range for Procedure B is

(4.6) $$2v(12) + 2v(13) - v(23) \leq v(123) \leq 3v(23)$$

if $v(12) + v(13) \geq v(23)$, and

(4.7) $$v(23) \leq v(123) \leq 3v(23)$$

otherwise.[1]

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(1) Note that in either case both (4.6) and (4.7) hold.

The variants of this procedure are as follows: If

(4.8) $$2v(13) + v(23) \leq v(123) \leq 3v(23) ,$$

both w_2 and w_3 are equal to zero and the final outcome is

(4.9) $$([v(123) - v(23)]/2, [v(123) + v(23)]/4, [v(123) + v(23)]/4 ; 123).$$

If

(4.10) $$2v(12) + v(23) \leq v(123) \leq 2v(13) + v(23),$$

then $w_2 = 0$, $w_3 = v(13) - \alpha$, and the final outcome is

(4.11) $$([v(123) - v(23)]/2, [v(123) - v(13)]/2, [v(13) + v(23)]/2 ; 123).$$

Finally, if

(4.12) $$\text{Max } (v(23), 2v(12) + 2v(13) - v(23)) \leq v(123) \leq 2v(12) + v(23),$$

then $w_2 = v(12) - \alpha$, $w_3 = v(13) - \alpha$ and the final outcome is

(4.13) $$([v(123)-v(23)]/2,[v(123)+v(23)+2v(12)-2v(13)]/4,[v(123)+v(23)+2v(13)-2v(12)]/4;123).$$

Remark 4.1 We noted that if $v(123) < 3v(23)$, players 2 and 3 will make more together by using Procedure B instead of Procedure A. However, it is possible that one of them (player 2) will do worse!

If, e.g., $v(12) = 6$, $v(13) = 12$, $v(23) = 15$, $v(123) = 21$, then (4.12) is satisfied hence, by (4.13), the outcome (3, 6, 12; 123) is in the kernel. Here player 2 gets less than $v(123)/3 = 7$, which would have resulted if Procedure A was used. This strikes us as a flaw in the bargaining[1] of Procedure B.

Procedure C. This procedure occurs if and only if $v(12) + v(13) \geq v(23)$ and

(4.14) $$2v(23) - v(12) - v(13) \leq v(123) \leq 2v(12) + 2v(13) - v(23).$$

Unlike the situation in Procedure B, where player 1 stayed idle in the "first round," allowing for the coalition {23} to form first, he now enters the first round demanding that his voice be heard at the stage in which 2-person coalitions are formed. We have seen (see (4.2)), that the outcome of such a round assigns to

(1) The formation of the coalition {12} or the coalition {13} will not "work," since, again, one of the two members of the formed coalition will obtain eventually an amount smaller than 7.

player i a quota ω_i which he manages to get if he succeeds in entering a 2-person coalition. In this situation there is no priority of one coalition over another. The players "therefore" decide to regard the quotas as the basis of negotiations in the pseudo pure bargaining game $(u ; 1, 2, 3 ; \omega_1, \omega_2, \omega_3)$, where $u(123) = v(123)$. This yields the outcome

(4.15) $(\omega_1+[v(123)-\omega_1-\omega_2-\omega_3]/3, \omega_2+[v(123)-\omega_1-\omega_2-\omega_3]/3, \omega_3+[v(123)-\omega_1-\omega_2-\omega_3]/3 ; 123)$,

where ω_1, ω_2 and ω_3 are defined immediately after (4.2).

In this procedure, therefore, player 1 undermines the coalition {23}, causing its breakdown and a compromise on the quota as a basis for the pseudo pure bargaining is reached. Player 1 would not interfere if (4.6) holds because, by interfering, he would not gain in this range. On the other hand, it is to his advantage to interfere if $v(123) < 2v(12) + 2v(13) - v(23)$, as one can easily verify.

Procedure D. If, however,

(4.16) $$v(123) < \text{Min } (v(23), 2v(23) - v(12) - v(13)),$$

both Procedure B and Procedure C would yield player 1 a negative payment. This he can certainly avoid since he can always assure himself the 0 payment. The players 2 and 3 will then be faced with the pseudo pure bargaining game $(v^{**} ; 2, 3; v(12), v(13))$, where $v^{**}(23) = v(123)$. Note that this case has two variants:

If

(4.17) $$v(13) - v(12) \le v(123) \le \text{Min}(v(23), 2v(23) - v(12) - v(13)),$$

only player 1 is "deleted" (see Definition 3.1), and the outcome is

(4.18) $$(0, [v(123) + v(12) - v(13)]/2, [v(123) + v(13) - v(12)]/2 ; 123).$$

If

(4.19) $$v(123) \le v(12) - v(13),$$

player 2 would do at least as well by leaving the bargaining, and reaching the outcome

(4.20) $$(0, 0, v(123) ; 123).$$

Discussion. We do not claim that the above procedures justify the kernel for the 3-person game. In order to claim this we should have shown that these procedures are more natural than others. For instance, if we allow in Procedure B for the strong coalition {23} to act as single player, why should not one consider other coalitions acting similarly? Yet it is interesting to note that these procedures, which on first sight seem unrelated among themselves and would certainly be rejected as an *a priori* analysis of the 3-person game, stem, in fact, from the same kernel concept, and perhaps should not be rejected *a posteriori*.

The actual outcome of the 3-person game, however, seems to us intuitively reasonable and useful for the following reasons:

(i) The quota occupies a central role when a 2-person coalition is formed.

(ii) The "right" order of payoffs when the 3-person coalition is formed is preserved; namely, player 1 does not receive more than player 2 and player 2 does not receive more than player 3.

(iii) Two-person coalitions have no influence on the outcome if 123 is formed and $v(123)$ is relatively large, but their influence becomes more and more decisive if $v(123)$ becomes smaller.

5. Existence Theorems.

In this section we assume that the reader is acquainted with the Bargaining Set $\mathcal{M}_1^{(i)}$, which is studied in M. Davis and M. Maschler [2] and [5]. This and other bargaining sets were first introduced by R. J. Aumann and M. Maschler [1], where a discussion of some of their properties is given.

Theorem 5.1. The kernel of a game is contained in the bargaining set $\mathcal{M}_1^{(i)}$.

Proof: Let $(\mathcal{X} ; \mathcal{B}) \in \mathcal{K}$ for a game Γ . Let k and ℓ be any two players belonging to a coalition B of $\mathcal{B}$. Under these circumstances, it follows from (2.15) that $s_{k,\ell} = s_{\ell,k}$ or that $x_\ell = 0$ and $s_{k,\ell} > s_{\ell,k}$ or that $x_k = 0$ and $s_{\ell,k} > s_{k,\ell}$.

In the first and second cases player k has a counter objection against any objection of ℓ , if such exists, by joining a coalition D in $T_{k,\ell}$ which has a maximum surplus. Indeed, if there exists an objection of ℓ against k then there exists a coalition in $T_{\ell,k}$ with a positive surplus; hence, $s_{k,\ell} \geq s_{\ell,k} > 0$. Let S be the intersection of D with the coalition used by ℓ in his objection against k . The players of S , if they exist, get together in the objection less than $s_{\ell,k}$ above the amount they had in $(\mathcal{X} ; \mathcal{B})$. Player k can therefore offer them at least what they received in the objection and give the rest of the players in D at least what they had in $(\mathcal{X} ; \mathcal{B})$, because $s_{k,\ell} \geq s_{\ell,k}$ and because(1) $s_{k,\ell} > 0$.

In the third case, player k can counter object by acting as a 1-person coalition.

A similar argument shows that ℓ has always a counter objection against any objection of k . This completes the proof of the theorem.

Lemma 5.1. The relation $>>$ with respect to an i.r.p.c. of a game (see Definition 2.3) is a partial order relation.

---

(1) This last relation is important in case S is empty.

Proof: If $k >> \ell$ with respect to $(\mathcal{X};\mathcal{B})$ in a game Γ, then, by (2.14), $(s_{k,\ell} - s_{\ell,k})\, x_\ell > 0$; hence, since $x_k \geq 0$, $x_\ell \geq 0$, it follows that $(s_{\ell,k} - s_{k,\ell})x_k \not> 0$. Thus $\ell >> k$ cannot occur. Hence $>>$ is an antisymmetric relation.

We shall now prove that this relation is transitive. If this were not the case, then there would exist three players k, ℓ, m, such that $k >> \ell$, $\ell >> m$ and either $m >> k$, or $m \approx k$, with respect to $(\mathcal{X};\mathcal{B})$ in a game Γ. By (2.14) and (2.15) this would mean that k, ℓ and m belong to the same coalition B of $\mathcal{B}$, and that the following relations (5.1), (5.2) and (5.3) hold.

(5.1) $$x_\ell(s_{k,\ell} - s_{\ell,k}) > 0$$

(5.2) $$x_m(s_{\ell,m} - s_{m,\ell}) > 0$$

(5.3) $$x_m(s_{k,m} - s_{m,k}) \leq 0$$

Since $x_m > 0$ by (5.2), $s_{k,m} - s_{m,k} \leq 0$.

If we take the set of coalitions containing at least one but not all of the players k, ℓ and m, and consider those coalitions in the set that have the largest surplus, we find that player k is in each of them and m is in none of them. Indeed, if D is one such coalition having a maximum surplus and $\ell \in D$, then $k \in D$ by (5.1); and if $m \in D$, then $\ell \in D$ by (5.2). Therefore, $s_{k,m} > s_{m,k}$, a contradiction. This proves that the relation $>>$ is transitive.

Example 5.1. The relation $\approx$ is certainly symmetric but it need not be transitive. Let $(v\,;N)$ be a 4-person game, where $v(13) = v(23) = v(24) = 80$, $v(1234) = 100$ and $v(B) = 0$ otherwise. Clearly $1 \approx 2$, $2 \approx 3$ but $3 >> 1$ with respect to the p.c. $(25, 25, 25, 25; 1234)$.

Lemma 5.2. Let $E_\upsilon \equiv E_\upsilon(\{x_i\}_{i \notin B_j};\mathcal{B})$, $\upsilon \in B_j \in \mathcal{B}$ be the set of points $\mathcal{X}^*$, $\mathcal{X}^* \in S_j$ (see (2.8)), for which player υ either outweighs or is in equilibrium with each of the other players, with respect to the i.r.p.c. $(\hat{\mathcal{X}};\mathcal{B})$, where $\hat{x}_i = x_i^*$; $i \in B_j$; then E_υ is a closed set in the simplex S_j, and contains the face $x_\upsilon = 0$.

Proof: It follows from the relations (2.14), that $x^* \in E_\upsilon$ if and only if $(\hat{x};\mathcal{B})$, with $\hat{x}_i = x_i^*$ for $i \in B_j$, satisfies

(5.4) $$(s_{\upsilon,\ell} - s_{\ell,\upsilon})\ \hat{x}_\upsilon \geq 0 \text{ for each } \ell,\ \ell \in B_j .$$

Thus E_υ is a union of a finite number of closed convex polyhedra, each of which contains the face $x_\upsilon = 0$.

Theorem 5.2. The kernel of a game is a union of a finite number of closed convex polyhedra.

Proof: This follows from the relations (2.15).

Theorem 5.3. Let $(\mathcal{x};\mathcal{B})$ be an i.r.p.c. for a game Γ, and let B_j be a fixed coalition in $\mathcal{B}$. It is possible to modify the payoffs to the players in B_j, while changing neither the payoffs of the other players nor the coalition structure, in such a way that B_j will become balanced (see Definition 2.4) with respect to the modified p.c.

The proof follows from Lemmas 5.1, 5.2 and it is completely analogous to the proof given for Theorem 4.1 in [2].

Theorem 5.4. Let $\mathcal{B}$ be a coalition structure for a game Γ, then there exists a payoff vector $\mathcal{x}$ such that $(\mathcal{x};\mathcal{B}) \in \mathcal{K}$.

One can prove this theorem in a way completely analogous to the proof given by B. Peleg in [7], which concerns the bargaining set $\mathcal{M}_1^{(i)}$. We shall present here a somewhat different proof, which like Peleg's proof, is based on his

Lemma 5.3 (B. Peleg [7]). Let $c^1(\mathcal{x}), c^2(\mathcal{x}), \ldots, c^m(\mathcal{x})$ be non-negative continuous real functions defined for $\mathcal{x} \in X(\mathcal{B})$, where B is a coalition structure for an n-person game Γ, and $X(\mathcal{B})$ is defined by (2.7) and (2.8). If for each $\mathcal{x}$ in $X(\mathcal{B})$, and for each coalition B_j in $\mathcal{B}$, there exists a player in B_j, such that $c^i(\mathcal{x}) \geq x_i$, then there exists a point $\xi \equiv (\xi_1, \xi_2, \ldots, \xi_n)$ in $X(\mathcal{B})$ such that $c^k(\xi) \geq \xi_k$, for all k, $k = 1, 2, \ldots, n$.

Proof of Theorem 5.4: Since the kernel is invariant under strategic equivalence, there is no loss of generality in assuming that $v(B) < \frac{1}{2}$ for each coalition B. It follows that $|e(B)| < \frac{1}{2}$ for each coalition B.

Let

$$(5.5) \qquad c^i(\mathcal{X}) = \underset{\substack{j \\ j \in B_\ell \ni i,\ j \neq i}}{\text{Min}} \quad x_i(s_{i,j} - s_{j,i}) + x_i\ , \quad i = 1, 2, \ldots, n.$$

Obviously $c^i(\mathcal{X})$ are continuous non-negative real functions for $\mathcal{X} \in X(\mathcal{B})$. By (5.4), $c^i(\mathcal{X}) \geq x_i$ if and only if player i is not outweighed by any other player, with respect to $(\mathcal{X}\ ;\mathcal{B})$. It follows[1] from the transitive property of the relation $>>$, that for each i.r.p.c. $(\mathcal{X}\ ;\mathcal{B})$ and for each coalition B_j in $\mathcal{B}$, there exists a player i in B_j who is not outweighed by any other player, with respect to $(\mathcal{X}\ ;\mathcal{B})$. Thus, $c^i(\mathcal{X})$ satisfy the conditions of Lemma 5.2, and there exists a p.c. $(\xi\ ;\mathcal{B})$ such that $c^k(\xi) \geq \xi_k$ for all k, $k = 1, 2, \ldots, n$. Thus no player is outweighed by another player with respect to $(\xi\ ;\mathcal{B})$, and therefore $(\xi\ ;\mathcal{B}) \in \mathcal{K}$. This completes the proof.

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(1) The proof is indirect. See also Proof of Theorem 4.1 in [2].

6. Me and my Aunt

It seems that the kernel has at least one merit, namely, in many cases it is easier to compute than to compute the wider bargaining set $\mathcal{M}_1^{(i)}$. Indeed, in order to compute the bargaining set of a game, one has to solve a system of inequalities, while, in general, equalities of the type $s_{ij} = s_{ji}$ occur in computing the kernel, as long as in the final outcome, $x_i > 0$, $x_j > 0$. Thus, in cases in which it is hard to compute the bargaining set, one may still hope to obtain some points in it, by computing the kernel.

In this section, we consider whether the kernel is interesting for its own sake, as a predictor of realistic situations. One way to decide this is to examine the definition of the kernel in order to judge whether it renders an appropriate "translation" of real life situations. If this fails, one may try to examine the outcomes to see if, and in what sense, they are intuitively reasonable. If this turns out to be the case, one may say that the kernel is justified since it yields reasonable outcomes. Of course, even one counter example, which yields unintuitive outcomes, will, in this case, lead to reject the kernel. Such a situation is not uncommon in the sciences (where "intuitive" is to be replaced by "observable").

Examining the definition of the kernel, one observes that the maximum surplus is given as a measure of a player's strength. Indeed, each of any two players in such a coalition is credited with the maximum amount he can hope to gain or the minimum amount he must lose, in a coalition which does not contain the other player, and the payoff is distributed in such a way that both players have equal gains (losses).(1) Of course, regardless of his maximum surplus strength, a player may not be deprived of what he could obtain by himself.

(1) One might think that a competition element is absent from the definition. This is not the case, because if two coalitions which render maximum surplusses to two players overlap, then any amount given by one player to the intersection of the two coalitions could be matched by the other player, without changing the difference of the corresponding excesses.

This principle of strength assumes that utilities are interpersonally compared. Indeed, we assume that an outcome will be rejected, if one player is in a position to show another player that if they part his maximum gains (minimum losses) are less than (greater than) the other's. This approach may be criticized on the ground that a theory of interpersonal comparison of utilities is not yet developed, and therefore, it is meaningless to compare such utilities.

However, if the values of the various coalitions represent money, then the outcome of the kernel may be achieved in real life situations, since in many respects, money does serve as a common utility worth for a group of people.

One might criticize the kernel theory on the ground that only pairs of players are considered and not, say, triplets. It seems to us that taking accounts of more sets of players should result in obtaining a smaller set of predictions. We have not succeeded so far in narrowing satisfactorily the kernel to exactly one point in each coalition-structure.

The outcomes for the 3-person game seem to us quite reasonable. Moreover they reveal some interesting dynamic procedures which are a pleasant bonus to the theory. However, if one examines other games one encounters situations where the predictions of the kernel seem to be unintuitive at least on a first impression. An example of such a game will subsequently be given. We have asked the opinion of several experts concerning this game and we inted to quote briefly their opinions and analyze the example beyond the general scope of this paper.(1)

Let us say at once — we defend here an outcome to which very few would agree. This outcome looks strange even to us, but we do not see any flaw in our intuitive interpretation. We hope that at least this discussion will reveal interesting aspects of Game Theory.

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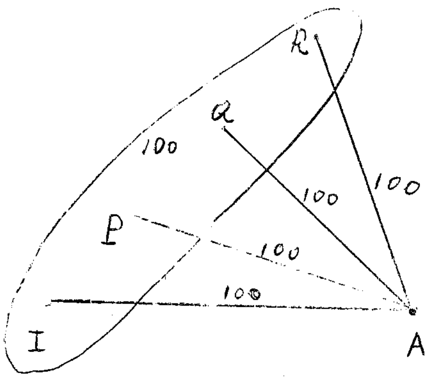
(1) Unfortunately, not every author quoted here was told the same story, and therefore we may be wrong in attributing the quotations to the story told here. Before submitting this paper for publication, we shall ask the authors to revise, if necessary, their opinions.

My aunt (player A) and I (player I) can enter a partnership in which we shall both win 100 units. "In principle," we agree to form the partnership, provided that we reach an agreement on the split. Each of us have other alternatives, which are shown in the figure. One can see that my aunt need convince one, and any one, of three players P, Q and R, while I need the agreement of all these players as my only alternative. Intuitively, my aunt is stronger than I and it seems that she should get more than 50, if we both form a coalition. If so — how much more?



The bargaining set $\mathcal{M}_1^{(i)}$ yields $(x, 100-x, 0, 0, 0; AI, P, Q, R)$, for the requested coalition structure, with $50 \leq x \leq 75$. The kernel yields $(50, 50, 0, 0, 0; AI, P, Q, R)$.

We shall try to defend the 50:50 split, assuming[1] that communication is perfect!

Indeed, if I is offered less than 50, I can convince P, Q and R to join him with a (50, 50/3, 50/3, 50/3) split. Each of them knows that A wants more than 50, and, therefore, would offer one of them less than 50. Since they are three players, each has an expectation of less than 50/3 from A . Since I offers more, each of them would agree to join him (instead of getting 0 if I joins A). This shows that perhaps I can protect his 50. We shall now show that A cannot protect more than 50. In fact, suppose that she wants 60 (any amount above 50 would do), then she may try to convince R to accept a 60:40 split.

(1) We also assume that the coalition AI has no inner value, due to family relations — it just happens to occur during the negotiations. Also, we assume that if AI forms, it would not pay outsiders. (Otherwise, we would say that a bigger coalition was formed.)

Realizing this, P and Q would try to save something for themselves. They would therefore agree with I on, say, a (50, 3, 3, 43) split. Thus A seems to be strong enough to determine who, if I P Q R forms, will get more than the others, but the result is that she gets 0 ! Trying to flip a coin with an equal probability, in order to decide which player to bribe, would not help her, as we saw previously. Declaring that she would take the first player who agrees on a 60:40 split would not help, since all of them, at best, would rush to her — and so where do we go from here? As a last resort, suppose A approaches R with a take it or leave it firm, irrevocable promise to split 60:40, threatening to give the same offer to player Q — if R does not agree, hoping that R will not believe the (50, 3, 3, 44) split, since Q will have a better offer. This will not help, since I, P, Q and A will approach R with a similar firm, irrevocable promise to (50, 3, 3, 44).

Against this interpretation, we would like to quote briefly other experts.

H.W.Kuhn. "If every one wants to maximize his own profits, A will be able to force I to give her 75."

B. Peleg. "I think that A is much stronger than I , and I am glad that the bargaining set reflects this."

R.J. Aumann. "I believe that the 50:50 split is justified. There are reasons which indicate that A is stronger, but there exists also an argument which shows that I is stronger: if the players P, Q, R join $\mathcal{M}$, then this coalition is more stable, because none of them will feel deprived, and everyone knows this. However, if one joins A , he himself is not sure that this coalition will last, because he will be afraid that the others will feel deprived."

In a second letter R.J. Aumann writes: "On a second thought, perhaps A is stronger, since each of the remaining players will only receive 25, if I P Q R forms. Therefore, it will be difficult for I to demand more from A."

J. C. Harsanyi. "I think that A should be able to obtain 75. If A and I exclude all other players from sharing in the payoffs, then we may speak of a discriminatory agreement. If the players reach an agreement giving symmetric roles to all players and to all subsets of the players, then we may speak of a non-discriminatory agreement. I wish to argue that in the case of a non-discriminatory agreement each player will obtain his Shapley value for the game. In contrast, under a discriminatory agreement, the members of a discriminating coalition are usually able to obtain more than their Shapley value. I want to argue that in this case each member of the discriminating coalition will receive an equal premium above his Shapley value. In this example, the Shapley value of the 5-person game is 60 to A and 10 to each other player. If A and I reach a discriminatory agreement so as to divide the whole 100 between them, then they "should" split equally the surplus of 30. Thus I will get 10 + 15 = 25 while A will get 60 + 15 = 75. (If we regard the Shapley values as the main alternative to a discriminatory agreement, then this equal-premium rule follows from Nash's bargaining solution.)

The case for a 25:75 split between I and A, if they discriminate against the other players, is further reinforced by considering what would happen if I, P, Q and R reached a discriminatory agreement against A. The four of them together could get 100, and since the game is completely symmetric among I, P, Q and R, each of them would presumably get 25. Thus I should get the same payoff of 25 whether he joined A alone or joined P, Q and R."

Answering our question — why should the players regard the Shapley value as the main alternative, and not, for example, other discriminatory solutions, J.C. Harsanyi answers that this would make no difference if the new alternative would include A and I in a winning coalition. If, however, one player would argue in favor of a discriminatory solution in which he is in a winning coalition and the other is in a losing coalition, then the other player would argue in favor of a discriminatory solution in which the roles are interchanged.

L. S. Shapley. "The issue, as I see it, turns on the relationship between two kinds of utility that appear in the problem. The first is the utility that makes the units of payoff desirable. The second is implicit in the statement: "...but since we are relatives, we both want the partnership.[1]

Thus, at one extreme, if the blood ties are so strong that no other partnership is thinkable, then offers by A and I to the other players will not be credible. The symmetry between A and I will then disappear and the 50-50 split is the only fair solution (if a solution must be unique).

At the other extreme, if the disutility of failing to form the coalition AI were negligible compared to 100 units of payoff at stake, it would appear that players I, P, Q and R are on an equal footing. One merely wishes to know what A's "proper" share is, if it happens that she manages to form a partnership. (Note: A good deal depends here on the extensive form of the game, i.e., on whether the game is actually presented to the players as a pure coalition-forming exercise (as you seem to have assumed), or whether there is a structure of moves and strategies which just happens to yield the indicated characteristic function. The passage from extensive (or normal) form to characteristic function form is not without pitfalls; its validity depends to some extent on the nature of the solution-concept that is applied to the characteristic function.)

But let me assume, that the "standard of behavior" in this situation dictates that the players shall negotiate until a single winning set of players declares itself, having settled how the proceeds are to be split among its members. Also, we must evidently assume that the standard of behavior prevents the ultimately winning coalition from making any payments to outside parties.

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(1) L.S.Shapley did not see the remark on p. 20 . Moreover, the above statement is replaced in this paper by the statement: "In principle, we agree to form the partnership, provided...". Nevertheless, we quote L. S. Shapley's remark since it may be relevant even with this formulation.

Under these two assumptions about the "standard of behavior," the Von Neumann-Morgenstern concept of main simple solution applies quite naturally. Since the game in question is the homogeneous weighted majority game $[3, 1,1,1,1]_h$ players I and A will split 25:75 if their partnership forms.

In summary, if the coalition AI is very valuable in itself, apart from the payoff it obtains, then the 50:50 split seems right. If it is valueless in itself (but, perhaps, "very likely" to form, in some sense), then the 25:75 split seems right. One can imagine intermediate cases between these extremes. But if the bargaining process can include side payments, enforced penalties, conditional agreements, etc., then it's hard to say -- we need more specifications of the model. One possibility is certainly the (Shapley - Harsanyi) value, giving a 10:60 split with side payments of 10 each to players P, Q, R."

R.D.Luce. "I find that I am slightly perplexed by your formulation of the problem. It is clearly phrased in an asymmetric fashion, in which I appears to have a special relationship to A different from that of P, Q and R, whereas in terms of payoffs to coalitions, they are completely equivalent. The asymmetry seems to be introduced via the suggestion that I and A have a bias to form a coalition (although this seems to be denied by your footnote). Thus, for example, your argument for the 50-50 split depends on the assumption that I and A want to form a coalition, but also rests upon the power each exerts by threatening to form other coalitions. It seems to me that this argument does not take sufficiently seriously A's threat not to form a coalition with I at all, in which case he may get zero. My feeling is that the problem is not really the bargaining between I and A, which is only a limited part of the total situation, but rather, whether or not that coalition is at all stable. I would guess that a "reasonable" analysis would lead to the conclusion that I, P, Q and R would form a coalition against A and split the proceeds equally. It is not that this is really stable — for A can offer any of them more — but that any coalition involving A is equally unstable and quite asymmetric. Each of the "weaker" players will see this and ultimately agree to the symmetric solution.

I do not know how to formalize this sort of process, but I suggest it sometimes occurs in real life. If one does not introduce an "I give up; lets divide it equally" type of mechanism, I don't see how such a problem can possibly be solved. The bare structure is inherently unstable.

I doubt that this helps much, but I have always felt that these problems, when cast in such abstract form, eliminate some of the "glue" that makes real situations work."

R.M. Thrall. "Your game presents an interesting challenge. My intuitive feeling is that your aunt is entitled to somewhat more than half of the proceeds in any two-person collection. On the other hand, the five way split with 50 per cent for auntie and the balance divided equally among the remaining gamesters seems not unreasonable."

M. Shubik refused to commit himself, declaring that he does not want to fall into the trap. M. Shubik demanded more information about the rules of the game, holding to the position that one cannot predict anything on a game given solely in terms of the characteristic function form.

The issues, as we see them, boil down to the following items:

(i) Is it meaningful to ask how a coalition will split <u>if</u> it forms? If it is meaningless since this coalition may not form, then how can we decide whether a coalition may form, (i.e., is stable) without knowing if, and how much, the players may expect out of it "if it does form"? It is hard for us to accept R.D. Luce's view that any 2-person coalition involving A will not form since it is unstable. Such an argument, we feel, may at most force A to offer his potential partner a large payoff.

(ii) Is it legitimate, in a case of perfect information, to single out one coalition (in a more involved game, one coalition structure), and decide on the splits by comparing the power exerted by each player threatening to depart from the coalition? We believe that this is the case in real life situations. The players may negotiate <u>until</u> eventually a coalition "starts" to form, its members refuse to listen to other offers, and yet they decide on the split according to their relative threat-possibilities.

(iii) If AI happened to start forming — will this fact place player I in a better bargaining position, compared to P, Q and R ? On this point we do not wish to commit ourselves. This phenomenon is certainly observed in real life situations, but it may be due to communication difficulties and to solidarity feeling; these we purposely try to avoid in this model.

If AI decide on the split basing their arguments on their relative power <u>prior</u> to the formation of AI, then the 75:25 seems to follow (if one accepts the Von Neumann-Morgenstern intuitive feeling that player I should get the same amount in this simple game, if he succeeds in participating in a winning coalition, or if one accepts Harsanyi's argument to view the Shapley value as the main alternative).

If, however, the decision on the split is based on the relative powers of the players <u>after</u> I has been put in the asymmetric role, then the argument for the 50:50 split seems to hold.

The bargaining set seems to reflect the range of these extremes, whereas the kernel yields one of them.

If the asymmetry, caused by the fact that AI are currently considering a split, is accepted, we can present a set of "rules" as requested by M. Shubik.(1)

In order to decide on the split between A and I , these players are allowed to threaten as follows: Each of them sends messages to P, Q and R in

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(1) We apply M. Shubik's ideas on how these rules should be given, though we modify them slightly.

which he states any offer he wishes, and also states what his alternatives are, if the other player makes a certain offer. The messages can name special players, or only specify a "choosing by lot" procedure. They can also contain alternatives for the cases in which the players refuse an offer.

These messages are given simultaneously to the players P, Q, and R, who are allowed to negotiate until they reach a conclusion, and can also make binding agreements. The offers, however, must be such that each of the two players will demand for himself that amount he wishes to protect. With these rules of the game, we feel that our arguments show that I will be able to protect 50. Note: This will not be the case if communication is not perfect. At one extreme, if A can approach P, Q and R before I can get hold of them, then it seems that 75:25 split would result. On the other extreme, if I can approach P, Q and R before player A can get hold of any one of them, then reasons similar to ours show that Player I can secure 50 for himself.

For other coalition structures, the kernel yields other outcomes, the most important of which are:

$$(25, 25, 25, 25, 0 \; ; \; I\,P\,Q\,R \; , \; A)$$

$$\left(\frac{3 \cdot 100}{7}, \frac{100}{7}, \frac{100}{7}, \frac{100}{7}, \frac{100}{7} \; ; \; A\,I\,P\,Q\,R\right)$$

These outcomes are unique for the corresponding coalition-structures, and are not unreasonable.

7. Games in which only the n-1 and the n-person coalitions are not flat.

In this section we shall consider n-person games whose characteristic function may take positive values only for the n-1 and the n-person coalitions, the value of the other coalitions being 0 . These games include the general 3-person games.

Clearly, only coalition structures of the form $(N-\{i\}, i)$ and (N) may possess more than one payoff, and therefore only these will be studied. Moreover, it is clear from the definition of the kernel that the value of the coalition N is immaterial when one studies which p.c.'s having a coalition structure of the form $(N-\{i\},i)$, are in the kernel.

The bargaining set for such games was studied in [5], by a method of deleting players. We shall make use of this method also in studying the kernel.

Definition Let $\Gamma \equiv (v;N)$ be an n-person cooperative game, $n \geq 3$, and let $(\mathcal{X};\mathcal{B})$ be a p.c. in this game. A game $\Gamma^{(*)} \equiv (v^*;N-\{i\})$ is said to be generated from Γ and $(\mathcal{X};\mathcal{B})$ by deleting player i , if its set of players is $N-\{i\}$ and its characteristic function v^* satisfies for $B \subset N-\{i\}$:

(7.1)
$$v^*(B) = \begin{cases} v(B) & \text{if } B \in \mathcal{B}, \\ v(B) - X_i & \text{if } B \cup \{i\} \in \mathcal{B} \\ \text{Max}\,(v(B),\ v(B \cup \{i\}) - x_i) & \text{if } B,\ B \cup \{i\} \notin \mathcal{B} \end{cases}$$

The expression $(\mathcal{X}^*;\mathcal{B}^*)$ with respect to Γ^* will be said to correspond to $(\mathcal{X};\mathcal{B})$ under this deletion, if, for $B \neq \phi$, $i \notin B$, $B \in \mathcal{B}^*$, whenever $B \in \mathcal{B}$, or $B \cup \{i\} \in \mathcal{B}$, and if $\mathcal{X}^* = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Note that, in general, Γ^* will neither satisfy the normalizations (2.1), and (2.2) nor will the value of a coalition be greater than or equal to the sum of the values of its members. We shall call such game - a pseudo game.

Similarly, $(\mathcal{X}^*,\mathcal{B}^*)$ is not a p.c. It satisfies the analogue of (2.6), it also satisfies (2.5) literally, but this latter condition is not individual rationality. Fur such pseudo payoff configurations, we shall still use literally, Definition 2.3, where 0 in (2.13) would mean zero and not $v^*(\ell)$. For games that will be treated in this section, these remarks will be pertinent only if $n = 3$.

Lemma 7.1. *If Γ^* is obtained from Γ and $(\mathcal{X};\mathcal{B})$ by deleting player i, if $k >> \ell$ $[k \approx \ell]$ with respect to $(\mathcal{X};\mathcal{B})$, and Γ, $k, \ell \neq i$, then $k >> \ell$ $[k \approx \ell]$ also with respect to the corresponding pseudo p.c. $(\mathcal{X}^*;\mathcal{B}^*)$ and Γ^*.*

Proof. The deletion of player i neither changes the maximum surplus of k over ℓ, nor does it effect the occurrence or non-occurrence of these players in the same coalition.

If there exists a unique payoff vector $\mathcal{X}$, such that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$ in Γ, and if we know the value x_i, then we can define the game Γ^* obtained by deleting player i with respect to Γ and $(\mathcal{X};\mathcal{B})$, and the problem of computing the other elements of $\mathcal{X}$ reduces to the problem of computing the "pseudo" kernel of Γ^*. We can use this method inductively on the games defined in the beginning of this section, since a deletion of a player in such games, for $n \geq 4$ will reduce such n-person games to (n-1)-person games of the same type. For coalition structures of the type $(N - \{i\}, i)$, the reduction is immediate, since we know that player i must obtain 0. The case $n = 3$ can be easily checked (see the outcome in (4.2) and (4.3)). We shall therefore be concerned only with a payoff to one player, when the coalition structure is N.

Theorem 7.1. *Let Γ be a game in which only the n-1 and the n-person coalitions may be non-flat. Then there exists a unique payoff vector $\mathcal{X}$, such that $(\mathcal{X};\mathcal{B})$ belongs to the kernel of the game.*

Proof. Let $(\mathfrak{X};N)$ and $(\hat{\mathfrak{X}};N)$ be two distinct i.r.p.c.'s in $\mathcal{K}$; then there exists players k and ℓ, such that $\hat{x}_k < x_k$ and $\hat{x}_\ell > x_\ell$, $\mathfrak{X} \equiv (x_1, x_2, \ldots, x_n)$, $\hat{\mathfrak{X}} \equiv (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$. Clearly, $x_k > \hat{x}_k \geq 0$. Consequently, by (2.15), $s_{k,\ell} \geq s_{\ell,k}$ with respect to $(\mathfrak{X};N)$. Therefore, $s_{k,\ell} > s_{\ell,k}$ with respect to $(\hat{x};N)$, since the maximal surplusses are always achieved via either the 1 or the (n-1)-person coalitions.

On the other hand, $\hat{x}_\ell > x_\ell \geq 0$. Therefore, by (2.15), $\ell << k$ with respect to $(\hat{x};N)$, which contradicts our assumption.

The existence has been proved in Theorem 5.4

Without loss of generality we can assume that

$$v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(n)}, \quad v^{(i)} = v(N - \{i\}), \; i = 1, 2, \ldots, n. \tag{7.2}$$

Lemma 7.2. Let Γ be a game in which all coalitions other than the n-1 and the n-person coalitions are flat. Let $(x_1, x_2, \ldots, x_n; N) \in \mathcal{K}$, and (7.2) hold. Then $x_1 \leq x_2 \leq \ldots \leq x_n$.

Proof.[1] Let k, ℓ be two players, $k < \ell$, then

$$s_{k,\ell} = \mathrm{Max}\,(-x_k,\; v^{(\ell)} - v(N) + x_\ell) \tag{7.3}$$

$$s_{\ell,k} = \mathrm{Max}\,(-x_\ell,\; v^{(k)} - v(N) + x_k) \tag{7.4}$$

We distinguish four cases (which may overlap):

A. $\quad s_{k,\ell} = -x_k \qquad s_{\ell,k} = -x_\ell$

It follows from (2.15) and (2.5) that $x_k = x_\ell$

B. $\quad s_{k,\ell} = -x_k, \qquad s_{\ell,k} = v^{(k)} - v(N) + x_k$

By (7.3) and (7.4), this case can occur only if

$$v(N) - v^{(k)} \leq x_k + x_\ell \leq v(N) - v^{(\ell)}. \tag{7.5}$$

(1) A somewhat shorter proof could be given. We have chosen this one since the arguments are needed later on.

By (2.15), four subcases occur:

(B_1) $x_k = [v(N) - v^{(k)}]/2$, which, by (7.5), implies $x_\ell \geq x_k$.

(B_2) $x_k = 0$ and $v^{(N)} - v^{(k)} < 0$, which implies $x_\ell \geq x_k$ since $x_\ell \geq 0$.

(B_3) $x_\ell = 0$ and $x_k < [v(N) - v^{(k)}]/2$, which, by (7.5) implies $x_\ell > x_k$, whence $x_k < 0$, a contradiction.

(B_4) $x_k = x_\ell = 0$.

C. $s_{k,\ell} = v^{(\ell)} - v(N) + x_\ell \qquad s_{\ell,k} = - x_\ell$

By (7.3) and (7.4), this case can occur only if

(7.6) $$x_k + x_\ell \leq v(N) - v^{(k)} \leq v(N) - v^{(\ell)} \leq x_k + x_\ell .$$

Therefore, in this case

(7.7) $$v^{(k)} = v^{(\ell)} \quad \text{and} \quad x_k + x_\ell = v(N) - v^{(k)} .$$

By interchanging the names of k and ℓ, we find ourselves in Case B. Examining the subcases, we find that $x_k = x_\ell$.

D. $s_{k,\ell} = v^{(\ell)} - v(N) + x_\ell \qquad s_{\ell,k} = v^{(k)} - v(N) + x_k$.

By (2.15), four subcases occur:

(D_1) $x_k + v^{(k)} = x_\ell + v^{(\ell)}$, which implies $x_k \leq x_\ell$

(D_2) $x_k = 0$ and $x_\ell + v^{(\ell)} < x_k + v^{(k)}$, which implies $x_\ell \geq x_k$, since $x_\ell \geq 0$.

(D_3) $x_\ell = 0$ and $x_k + v^{(k)} < x_\ell + v^{(\ell)}$ which contradicts $v^{(\ell)} \geq v^{(k)}$

(D_4) $x_k = x_\ell = 0$.

This completes the proof.

In analyzing the kernel of the games of this section, it will be relevant to consider the (n-1)-quotas;(1) i.e., the n-triple $(\omega_1, \omega_2, \ldots, \omega_n)$, satisfying

(1) The term is suggested by G.K. Kalish [3]. We use it here in the sense of B. Peleg's [6].

(7.8) $$\sum_{\nu \neq i} \omega_\nu = v^{(i)}, \quad v^{(i)} = v(N-\{i\}), \quad i = 1, 2, \ldots, n.$$

One easily verifies that

(7.9) $$\omega_i = [v^{(1)} + v^{(2)} + \ldots + v^{(n)} - (n-1)v^{(i)}]/(n-1),$$

$$i = 1, 2, \ldots, n,$$

and therefore, in view of (7.2), there are no <u>weak players</u> in the $(n-1)$-quota of the game, namely players having negative quotas, if and only if the "generalized" triangle inequalities (see [5])

(7.10) $$(n-1)\, v^{(1)} \leq v^{(1)} + v^{(2)} + \ldots + v^{(n)}$$

hold. Clearly, in view of (7.2),

(7.11) $$\omega_1 \leq \omega_2 \leq \ldots \leq \omega_n .$$

The payoff vectors $\mathfrak{x}$, such that $(\mathfrak{x}; N) \in \mathcal{K}$, are obtained in a similar fashion to the corresponding payoffs studied in Section 4. We shall, therefore, refrain from repeating much of the verbal discussions which describe the various procedures.

<u>Procedure A</u>. If

(7.12) $$v(N) \geq nv^{(1)}/(n-2),$$

then all the $(n-1)$-person coalitions are discredited and the p.c. $(v(N)/n, v(N)/n, \ldots, v(N)/n; N) \in \mathcal{K}$.

<u>Proof</u>. The maximum surplus of k against ℓ may be obtained either via $\{k\}$ or via $N-\{\ell\}$.

$e(N-\{\ell\} = v^{(\ell)} - (n-1)\, v(N)/n \leq v^{(1)} - (n-1)\, v(N)/n \leq [(n-2)v(N)-(n-1)v(N)]/n = -v(N)/n = e(k)$. Therefore the maximum surplus of any player k against any other player ℓ is equal to $-v(N)/n$.

<u>Procedure B</u>. If

$$(7.13) \qquad \operatorname{Max}\left(2\sum_{i=1}^{n} v^{(i)} - nv^{(1)},\ (n-2)v^{(1)}\right) < (n-2)v(N) < nv^{(1)},$$

then players 2, 3, ..., n, acting as a single player, first play against player 1 in the game $(u\ ;\ 1,\ \{23\ldots n\})$, where $u(1) = 0$, $u(\{23\ldots n\}) = v^{(1)}$ $u(1\{23\ldots n\}) = v(N)$. Player 1 receives $[v(N)-v^{(1)}]/2$, and the rest of the players then participate in the game obtained by deleting player 1 and his payoff from the original game. (See Lemma 7.1.)

<u>Proof</u>. Let $(x_1, x_2, \ldots, x_n; N) \in \mathcal{K}$, then, by Lemma 7.2, $x_1 \leq x_n$. We examine the alternatives used in the proof of Lemma 7.2, for $k = 1$, $\ell = n$.

A. This case implies $x_1 = x_n$, and therefore $x_1 = x_2 = \ldots = x_n = v(N)/n$. Hence, by (7.4), $v(N)/n \leq [v(N) - v^{(1)}]/2$ which contradicts (7.13).

C. By (7.7), $v^{(1)} = v^{(2)} = \ldots = v^{(n)}$, which contradicts (7.13).

D. Clearly, by (7.13) and (2.1), $v(N) > 0$; hence, by Lemma 7.2, $x_n > 0$. This leaves us with subclasses D_1 and D_2. It follows from (7.3) that we can assume

$$(7.14) \qquad x_1 + x_n > v(N) - v^{(n)},$$

since, if equality occurs in (7.14), we may consider the case to be B.

If $x_1 = 0$, then (7.14) implies $e(N-\{n\}) > 0$, and therefore $s_{i,n} \geq 0$ for each i, $i \neq n$. Consequently, by (2.15), $s_{n,i} > 0$ for each i, $i \neq n$. In particular, $s_{n1} > 0$, i.e., $v^{(1)} - x_2 - x_3 - \ldots - x_n = v^{(1)} - v(N) > 0$, contrary to (7.13).

If $x_1 > 0$, then $x_i > 0$ for each i, $i = 1, 2, \ldots, n$. Relation (7.14) implies $e(N - \{n\}) > -x_1 \geq -x_i$, hence $s_{i,n} > -x_i$, $i \neq n$. By (2.15) also $s_{n,i} \geq s_{i,n} > -x_i \geq -x_n$, and $s_{i,n} = s_{n,i}$, $i \neq n$. Thus, the maximum surplusses are equal and are obtained via the $(n-1)$-person coalitions; hence, $v^{(i)} - v(N) + x_i = v^{(1)} - v(N) + x_1$, each i, $i \neq 1$. Thus,

$$nx_i = v(N) + v^{(1)} + v^{(2)} + \dots + v^{(n)} - nv^{(i)}, \quad i = 1,2,\dots,n, \tag{7.15}$$

and therefore, by (7.14),

$$2v(N) + 2\sum_{i=1}^{n} v^{(i)} - nv^{(1)} - nv^{(n)} > nv(N) - nv^{(n)}, \tag{7.16}$$

contrary to (7.13).

B. Exactly as in the analysis of case D, we find that $x_n \neq 0$. Clearly, subcase (B_2) cannot occur for $k = 1$, since $v^{(N)} > v^{(1)}$. Therefore (B_1) is the only subcase which can happen; i.e., $x_1 = [v(N) - v^{(1)}]/2$.

Remark 7.1. It follows from (7.10), that condition (7.13) can be stated as:

$$2 \Sigma v^{(i)} - nv^{(1)} < (n-2)\, v(N) < nv^{(1)}, \text{ if no player is weak,} \tag{7.17}$$

$$(n-2)\, v^{(1)} < (n-2)\, v(N) < nv^{(1)}, \text{ if there exists a weak player,} \tag{7.18}$$

Procedure C. If

$$nv^{(1)} - \sum_{i=1}^{n} v^{(i)} \leq v(N) \leq [2 \sum_{i=1}^{n} v^{(i)} - nv^{(1)}]/(n-2), \tag{7.19}$$

then the players decide to regard the quota as the basis of negotiations in the pseudo pure bargaining game $(u; N; \omega_1, \omega_2, \dots, \omega_n)$. Thus, the final outcome is

$$x_i = \omega_i + c, \qquad i = 1, 2, \dots, n, \tag{7.20}$$

where ω_i is defined by (7.9), and

$$c = v(N)/n - \sum_{i=1}^{n} v^{(i)}/n(n-1). \tag{7.21}$$

Proof. It follows from (7.9), (7.20) and (7.21) that

(7.22) $$x_i = [v(N) + \sum_{i=1}^{n} v^{(i)}]/n - v^{(i)} \qquad i = 1, 2, \ldots, n.$$

Clearly, $x_1 + x_2 + \ldots + x_n = v(N)$ and by (7.19), $x_i \geq 0$, $i = 1, 2, \ldots, n$. Let k and ℓ be two distinct players, then

(7.23) $$e(k) = v^{(k)} - [v(N) + \sum_{i=1}^{n} v^{(i)}]/n ,$$

(7.24) $$e(N - \{\ell\}) = v^{(\ell)} - x_1 - x_2 - \ldots - x_n + x_\ell = v^{(\ell)} - v(N) + x_\ell = [\sum_{i=1}^{n} v^{(i)} - (n-1)\, v(N)]/n.$$

By (7.19) and (7.2), $e(k) \leq e(N - \{\ell\})$, and therefore $s_{k,\ell} = e(N-\{\ell\})$. This expression, as shown in (7.24), is independent of k and ℓ , hence $(x_1, x_2, \ldots, x_n; N) \in \mathcal{K}$.

Remark 7.2. Relation (7.19) implies relation (7.10); therefore, this procedure can occur only if no player is weak in the $(n-1)$-quota.

Procedure D. If

(7.25) $$v(N) \leq \text{Min}\ (v^{(1)} , nv^{(1)} - \sum_{i=1}^{n} v^{(i)}),$$

then player 1 obtains the 0 payoff, and the rest of the players then participate in a game obtained by deleting player 1 and his payoff from the original game. (See Lemma 7.1.)

Proof. Let $(x_1, x_2, \ldots, x_n; N) \in \mathcal{K}$, and suppose that $x_1 > 0$, then $v(N) > 0$. By Lemma 7.2, $x_i > 0$, $i = 1, 2, \ldots, n$. We shall examine all the cases treated in the proof of Lemma 7.2, for $k=1$, $\ell=n$, and arrive at a contradiction each time.

A. In this case $x_1 = x_n$ and therefore $x_1 = x_2 = \ldots = x_n = v(N)/n > 0$. Therefore, $e(n) = - v(N)/n$ and $e(N - \{1\}) = [nv^{(1)} - (n-1)v(N)]/n$. By (7.4), $- v(N) \geq nv^{(1)} - (n-1)\, v(N)$, which contradicts (7.25).

B. Only subcase (B_1) need be considered. In this case $x_1 = [v(N)-v^{(1)}]/2$, and, by (7.25), this amount can be non-negative only if $x_1 = 0$. A contradiction.

C. It follows from (7.6) and (7.25) that this case can occur only if $x_1 = x_n = 0$, a contradiction.

D. Only subcase (D_1) need be considered. We prove that relation (7.15) holds, in the same fashion as it was proved in the case of Procedure B. By (7.25), it then follows that x_1 can be non-negative only if $x_1 = 0$. A contradiction.

Remark 7.3. It follows from (7.10), that condition (7.25) can be stated as

(7.26) $$v(N) \leq nv^{(1)} - \sum_{i=1}^{n} v^{(i)}, \quad \text{if no player is weak,}$$

(7.27) $$v(N) \leq v^{(1)}, \quad \text{if there exists a weak player,}$$

therefore, all the possibilities were examined.

The following "motivations" can easily be verified.

(i) In the domain of Procedure A, the players 2, 3, ..., n, would have received together a smaller[(1)] amount, had they acted along the rules of Procedure B.

(ii) In the domain of Procedure B, the players 2, 3, ..., n, would have received together a smaller amount, had the rules of Procedure A been adopted. Moreover, player 1 would have received in this domain a smaller amount, had he interfered as the rules of Procedure C or D require.

(iii) In the domain of Procedure C, player 1 would have lost, had he acted "passively" as described either by Procedure B or by Procedure D.

(iv) In the domain of Procedure D, the rules of Procedures B and C would have yielded player 1 a negative amount.

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(1) We ignore now and further on the border cases, in which neither a loss nor a gain occurs.

8. The 4-person constant-sum game.

We have seen in Example 4.1, that in a 4-person game it may well happen that a payoff $\mathcal{X}$ such that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$ is not unique. We shall show that it is unique in the case of a constant-sum game.

Let Γ be a 4-person general-sum game. By Lemma 7.1 and Theorem 4.1, there, if $\mathcal{B}$ contains a 1-person coalition, there is a unique payoff $\mathcal{X}$ such that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$, since the player in this coalition must get a 0 payoff.[1] We shall therefore be interested only in coalition structures of the form

(8.1) $$\mathcal{B} \equiv (ij,\ k\ell),\quad i,j,k,\ell \text{ mutually distinct, or}$$
$$\mathcal{B} \equiv (1\ 2\ 3\ 4).$$

Let $\mathcal{B}$ satisfy (8.1), and let $(\mathcal{X};\mathcal{B})$ and $(\mathcal{Y};\mathcal{B})$ belong to $\mathcal{K}$, $\mathcal{Y}$ and $\mathcal{X}$ being distinct payoff vectors. Let $\xi_i = y_i - x_i$, we can name the players in such a way that

(8.2) $$\xi_1 \leq \xi_2 \leq \xi_3 \leq \xi_4 .$$

By (2.6),

(8.3) $$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0 .$$

We can assume that players 1 and 4 belong to the same coalition in $\mathcal{B}$. This is trivially true if $\mathcal{B} = (1\ 2\ 3\ 4)$. If i is player 1, then, since $\xi_i + \xi_j = 0$, and $\xi_k + \xi_\ell = 0$, ξ_j cannot be smaller than any other ξ_ν so that without violating (8.2), we can name him player 4.

We denote by $s_{\nu,\mu}(\mathcal{X})$ and $s_{\nu,\mu}(\mathcal{Y})$ the maximum surplus of ν over μ with respect to $(\mathcal{X};\mathcal{B})$ and $(\mathcal{Y};\mathcal{B})$, respectively. (See (2.12)). Similarly,

(1) The proof of Theorem 4.1 (and Lemma 4.1) applies also to pseudo games.

we denote by $e_{\mathcal{X}}(D)$ and $e_{\mathcal{Y}}(D)$ the excesses of D with respect to these payoff configurations, respectively. (See (2.9).)

It follows from (8.2), (8.3) and $\mathcal{X} \neq \mathcal{Y}$, that

(8.4) $$\xi_{\circ} < 0 < \xi_4 ,$$

and therefore, by (2.5),

(8.5) $$x_1 > 0 \quad \text{and} \quad y_4 > 0 .$$

Thus, by (2.15),

(8.6) $$s_{1,4}(\mathcal{X}) \geq s_{4,1}(\mathcal{X}) \quad \text{and} \quad s_{1,4}(\mathcal{Y}) \leq s_{4,1}(\mathcal{Y}).$$

In view of (8.3) and (8.4), $e_{\mathcal{X}}(D) \leq e_{\mathcal{Y}}(D)$ for $D \in T_{1,4}$ (see (2.11)), since $\xi_1 + \xi_2 < \xi_3 + \xi_4$ and $\xi_1 + \xi_3 \leq \xi_2 + \xi_4$. Moreover, $e_{\mathcal{X}}(D) = e_{\mathcal{Y}}(D)$, $D \in T_{1,4}$, if and only if $D = \{1,3\}$, and

(8.7) $$\xi_1 + \xi_3 = 0 \quad \text{and therefore} \quad \xi_2 + \xi_4 = 0 .$$

Consequently,

(8.8) $$s_{1,4}(\mathcal{Y}) \geq s_{1,4}(\mathcal{X}) ,$$

and equality occurs if and only if (8.7) is satisfied and the coalition $\{1,3\}$ has a maximum excess among all the coalitions in $T_{1,4}$, both with respect to $(\mathcal{X};\mathcal{B})$ and to $(\mathcal{Y};\mathcal{B})$.

In a similar fashion, (the excesses decrease this time), we prove that

(8.9) $$s_{4,1}(\mathcal{Y}) \leq s_{4,1}(\mathcal{X}) ,$$

and equality occurs if and only if (8.7) is satisfied and the coalition $\{2,4\}$ has a maximum excess among all the coalitions in $T_{4,1}$, both with respect to $(\mathcal{X};\mathcal{B})$ and to $(\mathcal{Y};\mathcal{B})$. By (8.6), (8.8) and (8.9) we find that

(8.10) $$s_{1,4}(\mathcal{Y}) = s_{1,4}(\mathcal{X}) = s_{4,1}(\mathcal{X}) = s_{4,1}(\mathcal{Y}) .$$

Therefore, (8.7) is satisfied and, in addition,

(8.11) $$s_{1,4}(\mathcal{X}) = s_{1,4}(\mathcal{Y}) = e_{\mathcal{X}}(13) = e_{\mathcal{Y}}(13),$$

(8.12) $$s_{4,1}(\mathcal{X}) = s_{4,1}(\mathcal{Y}) = e_{\mathcal{X}}(24) = e_{\mathcal{Y}}(24).$$

Consequently,

(8.13) $$e_{\mathcal{X}}(13) = e_{\mathcal{X}}(24).$$

It also follows from (8.2) and (8.7) that $\xi_4 = \xi_3 = - \xi_2 = - \xi_1$, and, therefore, $\mathcal{Y}$ must be of the form:

(8.14) $$y_1 = x_1 - \delta , \; y_2 = x_2 - \delta , \; y_3 = x + \delta, \; y_4 = x_4 + \delta; \;\; \delta > 0.$$

This must hold in the 4-person general-sum game. Suppose now that Γ is a constant-sum game, then $e_{\mathcal{X}}(13) + e_{\mathcal{X}}(24) = v(13) + v(24) - x_1 - x_2 - x_3 - x_4 = v(1\ 2\ 3\ 4) - v(1\ 2\ 3\ 4) = 0$. By (8.3) it follows that $e_{24}(\mathcal{X}) = 0$. But $e_{24}(\mathcal{X}) = s_{4,1}(\mathcal{X}) \geq e(234) = v(1\ 2\ 3\ 4) - x_2 - x_3 - x_4 = x_1$, contrary to (8.5).

We have thus proved:

<u>Theorem 8.1.</u> <u>Let Γ be a 4-person constant-sum game, and let $\mathcal{B}$ be an arbitrary coalition-structure in Γ ; then, there exists a unique payoff vector $\mathcal{X}$, such that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$</u> .

In order to analyze the structure of the kernel for the 4-person constant-sum game, some observations should be made.

Only coalition structures of the form (8.1) should be discussed since, balancing other coalition structures reduces, by Theorem 4.1, to the case of a 3-person (psuedo-) game.

Let Γ be a 4-person constant-sum game.

We shall assume that $v(N) > 0$, since otherwise the computation of the kernel is trivial. Without loss of generality, we can assume that

(8.15) $$v(34) \geq v(24) \geq v(14), \quad v(23) \geq v(13) \geq v(12).$$

Clearly, $v(ij) + v(k\ell) = v(ijk) = v(N)$, for every mutually distinct i,j,k,ℓ. Thus, if $\mathcal{B}$ contains two 2-person coalitions or one 4-person coalition, then for any i.r.p.c. $(x;\mathcal{B})$,

(8.16) $$e(ij) + e(k\ell) = 0$$

(8.17) $$e(ijk) = x_\ell \geq 0, \quad e(i) = -x_i \leq 0,$$

(8.18) $$e(ijk) - e(ij) = v(1234) - v(ij) - (x_i + x_j + x_k - x_i - x_j) = v(k\ell) - x_k.$$

(8.19) $$e(1234) = 0.$$

Here, and in the rest of the section, i,j,k,ℓ will denote mutually disjoint players. It follows from (8.17) that only 2 and 3-person coalitions need be considered when computing the maximal surplusses.

A. <u>The coalition structure $\mathcal{B} = (1\ 2\ 3\ 4)$</u>.

A_1 If

(8.20) $$v(34)/3 \geq v(13) + v(23) - v(12),$$

then

(8.21) $$\begin{aligned}(x;\mathcal{B}) \equiv (&[3v(14) - 2v(24) - 2v(34) + 2v(1234)]/5,\\ &[3v(24) - 2v(14) - 2v(34) + 2v(1234)]/5,\\ &[3v(34) - 2v(14) - 2v(24) + 2v(1234)]/5,\\ &[v(14) + v(24) + v(34) - v(1234)]/5; 1234) \in \mathcal{K}.\end{aligned}$$

<u>Proof</u>. Clearly, $x_1 + x_2 + x_3 + x_4 = v(N)$. It follows from (8.15) that

$x_3 \geq x_2 \geq x_1$. Also, by (8.20), $5(x_4 - x_3) = 3v(14) + 3v(24) - 2v(34) - 3v(1234) = 3[-v(23) + (v(24) - v(34)) + v(34)/3] = 3[-v(23) + (v(12) - v(13)) + v(34)/3] \geq 0$. Finally, by (8.20), $5x_1 = (3v(14) - 3v(34)) + v(34) + 2v(13) = 3v(12) - 3v(23) + v(34) + 2v(13) \geq 5v(13) \geq 0$. Thus, $(\mathcal{X};\mathcal{B})$ is an i.r.p.c. Let us now check the excesses of the various relevant coalitions. By (8.20), $e(14) = e(24) = e(34) = e(123) = x_4 \geq 0$; hence, by (8.16), $e(23) = e(13) = e(12) \leq 0$. Also, by (8.17), $e(123) \geq e(124) \geq e(134) \geq e(234)$. Consequently, $s_{1,2} = s_{1,3} = s_{4,2} = s_{4,3} = e(14)$, $s_{2,1} = s_{2,3} = s_{4,1} = e(24)$, $s_{3,2} = s_{3,1} = e(34)$, $s_{1,4} = s_{2,4} = s_{3,4} = e(123)$, and $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$.

A_2. If

(8.22) $$v(23) \leq v(34)/3 \leq v(23) + v(13) - v(12)$$

then

(8.23) $$(\mathcal{X};\mathcal{B}) \equiv (\tfrac{1}{2}[v(12)+v(14)-v(24)] + v(34)/6,\ \tfrac{1}{2}[v(12)+v(24)-v(14)] + v(34)/6,\ v(34)/3,\ v(34)/3;\ 1\ 2\ 3\ 4) \in \mathcal{K} .$$

Proof. Clearly, $x_1 + x_2 + x_3 + x_4 = v(1\,2\,3\,4)$, and by (8.15), (8.22), $x_4 = x_3 \geq x_2 \geq x_1 \geq 0$; hence, $(\mathcal{X};\mathcal{B})$ is an i.r.p.c. Moreover, $e(34) = e(123) = e(124) \geq e(134) \geq e(123) \geq 0$, and, by (8.15), $e(14) = e(24) = \frac{1}{2}[v(24) + v(14) - v(12) - v(34)] = \frac{1}{2}[v(24) - v(13)] \geq 0$; hence, $e(12), e(23), e(13) \leq 0$, by (8.16).
Finally, by (8.18) and (8.22), $e(124) - e(24) = v(13) - x_1 = [6v(13) + 3v(24) - v(34) - 3v(12) - 3v(14)]/6 = [3v(23) + 3v(13) - 3v(12) - v(34)]/6 \geq 0$, and $e(14) - e(134) = x_3 - v(23) = v(34)/3 - v(23) \geq 0$; hence, checking all the relevant excesses, we find that $(\mathcal{X};\mathcal{B}) \in \mathcal{K}$.

In a similar fashion we prove:

A_3. If

(8.24) $$v(13) \leq v(34)/3 \leq v(23),$$

then

(8.25) $$([v(12) + v(13)]/2,\ [2v(24) + v(12) - v(13)]/6,\ v(34)/3,\ v(34)/3\ ;\ 1\ 2\ 3\ 4) \in \mathcal{K}.$$

Proof:[1] $s_{1,3} = s_{2,3} = s_{4,3} = e(124)$, $s_{1,4} = s_{2,4} = s_{3,4} = e(123)$, $s_{3,1} = s_{3,2} = s_{4,1} = s_{4,2} = e(34)$, $s_{2,1} = e(24)$, $s_{1,2} = e(134)$, $e(34) = e(124) = e(123)$, $e(24) = e(134)$.

A_4. If

(8.26) $$v(12) \leq v(34)/3 \leq v(13),$$

then

(8.27) $$(v(34)/6 + v(12)/2,\ v(34)/6 + v(12)/2,\ v(34)/3,\ v(34)/3;\ 1\ 2\ 3\ 4) \in \mathcal{K}.$$

Proof: $s_{1,3} = s_{2,3} = s_{4,3} = e(124)$, $s_{1,4} = s_{2,4} = s_{3,4} = e(123)$ $s_{3,1} = s_{3,2} = s_{4,1} = s_{4,2} = e(34)$, $s_{1,2} = e(134)$, $s_{2,1} = e(234)$ $e_{34} = e_{124} = e_{123}$, $e(234) = e(134)$.

A_5. If

(8.28) $$v(34)/3 \leq v(12),$$

then

(8.29) $$(v(1234)/4,\ v(1234)/4,\ v(1234)/4,\ v(1234)/4;\ 1234) \in \mathcal{K}.$$

Proof: $s_{i,j} = e(ik\ell) = v(1234)/4$ for each mutually distinct i, j, k and ℓ.

Clearly, these cases exhaust all the possibilities.

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(1) From now on, only outlines of the proofs will be given.

B. The coalition structure (14, 23).

B_1. If

$$2v(12) \geq \text{Max } (v(14), v(23)) \tag{8.30}$$

then

$$(v(14)/2, v(23)/2, v(23)/2, v(14)/2; \; 14, 23) \in \mathcal{K}. \tag{8.31}$$

Proof: $s_{1,4} = e(123) = v(14)/2$, $s_{4,1} = e(234) = v(14)/2$, $s_{2,3} = e(124) = v(23)/2$, $s_{3,2} = e(134) = v(23)/2$.

B_2. If

$$2v(12) \leq v(23) - 2v(14), \tag{8.32}$$

then

$$(0, v(23)/2, v(23)/2, v(14); \; 14, 23) \in \mathcal{K}. \tag{8.33}$$

Proof: $s_{4,1} = \text{Max } (e(24), e(34))$, $s_{1,4} = e(123) = e(34)$; hence, $1 \approx 4$, since $x_1 = 0$. Also, $s_{2,3} = e(124) = v(23)/2$ and $s_{3,2} = e(134) = v(23)/2$, hence $2 \approx 3$.

B_3. If

$$\text{Max}(2v(14) - v(23), v(23)-2v(14)) \leq 2v(12) \leq v(23), \tag{8.34}$$

then

$$([2v(14) + 2v(12) - v(23)]/4, \; v(23)/2, v(23)/2, [2v(34)-v(23)]/4; \; 14,23) \in \mathcal{K}. \tag{8.35}$$

Proof: $s_{4,1} = e(34) = [2v(34) - v(23)]/4$, $s_{1,4} = e(123) = [2v(34) - v(23)]/4$; hence $1 \approx 4$. Similarly, $s_{2,3} = e(124) = v(23)/2$, $s_{3,2} = e(134) = v(23)/2$; hence hence $2 \approx 3$.

B_4. If

$$v(23) \geq v(12) + v(13) \text{ and } v(34) \leq 3(v(14) - v(13)), \tag{8.36}$$

then

$$(8.37)\qquad ([3v(14) + v(12) - v(24)]/4,\ [v(24) + v(23) - v(34)]/2,\ [v(34) + v(23) - v(24)]/2,\ [v(34) + v(24) - v(23)]/4;\ 14,\ 23) \in \mathcal{K}.$$

Proof: $e(24) = e(34) = e(123)$, $s_{1,4} = e(123)$, $s_{4,1} = e(34)$; hence $1 \approx 4$. $s_{2,3} = e(24)$, $s_{3,2} = e(34) = e(24)$; Hence $2 \approx 3$.

B_5. If

$$(8.38)\qquad \mathrm{Max}(2v(12),\ 2v(34)/3) \leq v(14) \leq 2v(13),$$

then

$$(8.39)\qquad (v(14)/2,\ [2v(23) + 2v(12) - v(14)]/4,\ [2v(34) - v(14)]/4,\ v(14)/2;\ 14,\ 23) \in \mathcal{K}.$$

Proof: $s_{1,4} = e(123)$, $s_{4,1} = e(234)$, $e(234) = e(123)$; hence $1 \approx 4$.
Also, $s_{2,3} = e(124)$, $s_{3,2} = e(34)$, $e(124) = e(34)$; hence $2 \approx 3$.

B_6. If

$$(8.40)\qquad v(12) + v(34)/3 \leq \mathrm{Min}(v(23),\ v(14)),\quad v(14) \leq v(13) + v(34)/3,$$

then

$$(8.41)\qquad (v(14) - v(34)/3,\ v(23) - v(34)/3,\ v(34)/3,\ v(34)/3;\ 14, 23) \in \mathcal{K}.$$

Proof: $s_{1,4} = e(123)$, $s_{4,1} = e(34)$, $s_{2,3} = e(124)$, $s_{3,2} = e(34)$, $e(123) = e(124) = e(34)$; hence $1 \approx 4$, $2 \approx 3$.

B_7. If

$$(8.42)\qquad v_{14} \geq 2v(13),\quad v(12) + v(13) + v(14) \geq v(1234),$$

then

$$(8.43)\qquad (v(14)/2,\ [v(23) + v(24) - v(34)]/2,\ [v(24) + v(34) - v(24)]/2,\ v(14)/2;\ 14,\ 23) \in \mathcal{K}.$$

Proof: $s_{1,4} = e(123)$, $s_{4,1} = e(234)$, $e(123) = e(234)$; hence $1 \approx 4$. $s_{2,3} = e(24)$, $s_{3,2} = e(34)$, $e(24) = e(34)$; hence $2 \approx 3$.

It remains to show that these seven categories exhaust all the possibilities. Let us assume that there exists a characteristic function v which is not in any of the above categories. We shall consider three cases.

I. If $v(23) \geq v(14)$, then

(i) $2v(12) < v(23)$, since v is not in B_1.

(ii) $2v(12) > v(23) - 2v(14)$, since v is not in B_2.

(iii) $2v(12) < 2v(14) - v(23)$, since v is not in B_3, and the other relations are satisfied by (i) and (ii).

It follows from (iii) that $v(23) < 2v(14) - 2v(12)$; hence, since $v(23) = v(1234) - v(14) = v(12) + v(34) - v(14)$,

(iv) $v(34)/3 < v(14) - v(12)$.

(v) $3(v(14) - v(13)) > v(34)$, since v is not in B_6, and the other relation is satisfied by (iv).

(vi) $v(23) < v(12) + v(13)$, since v is not in B_4, and the other relation is satisfied by (v).

Clearly, $v(23) \geq v(1234)/2 \geq v(14)$. Relations (vi) and (i) imply $v(13) > v(23) - v(12) > v(23)/2$; hence, $3v(14) - 2v(13) < 3v(14) - v(23) \leq 2v(14) \leq v(1234)$. On the other hand, by (v) and (8.15), $3v(14) - 2v(13) > v(34) + v(13) \geq v(34) + v(12) = v(1234)$, and a contradiction is established.

II. If $v(14) > v(23)$ and $v(23) < v(12) + v(13)$, then

(i) $v(14) > 2v(12)$, since v is not in B_1.

(ii) $v(14) < 2v(13)$, since v is not in B_7 and the other relation is satisfied.

By our assumption, $-v(13) < v(12) - v(23) = v(14) - v(34)$; hence

(iii) $v(14) > v(34) - v(13)$.

(iv) $v(13) > v(34)/3$ follows from (ii) and (iii).

(v) $v(14) < 2v(34)/3$, since v is not in B_5 and the other relations are satisfied by (i) and (ii). Therefore, by (iv) and (v), $v(13) > v(34)/3 + (v(14) - 2v(34)/3)$, or

(vi) $v(13) > v(14) - v(34)/3$.

(vii) $v(23) < v(12) + v(34)/3$, since v is not in B_6, and the other relation is satisfied by (vi).

It follows from (vii) that $v(34) < v(14) + v(34)/3$, or $2v(34)/3 < v(14)$, contrary to (v).

III. If $v(14) > v(23)$ and $v(23) \geq v(12) + v(13)$, then

(i) $v(34) > 3(v(14) - v(13))$, since v is not in B_4 and the other relation is satisfied by our assumption.

(ii) $v(23) < v(12) + v(34)/3$, since v is not in B_6 and the other relation is satisfied by (i). Relation (ii) implies

(iii) $v(14) > 2v(34)/3$, because $v(34) - v(14) = v(23) - v(12) < v(34)/3$; and relation (i) implies $2/3 \cdot v(34) > 2(v(14) - v(13))$; hence

(iv) $2v(13) > v(14)$.

(v) $v(13) > v(34)/3$ follows from (iii) and (iv).

It follows from our assumption that $v(34) - v(14) = v(23) - v(12) \geq v(13)$; hence

(vi) $v(34) \geq v(14) + v(13)$.

However, by (v) and (iii), $v(13) + v(14) > v(34)/3 + 2v(34)/3 = v(34)$, and this contradicts (vi).

Thus, every characteristic function must belong to (at least) one of the above categories.

C. The coalition structure (13, 24).

C_1. if

(8.44) $$v(24) \leq 2v(12),$$

then

(8.45) $$(v(13)/2,\ v(24)/2,\ v(13)/2,\ v(24)/2;\ 13,\ 24) \in \mathcal{K}.$$

Proof: $e(ijk) \geq e(ij)$ for $\{i,j\} \neq \{1,3\}$, $\{3,1\}$, $\{2,4\}$, $\{4,2\}$.

C_2. If

(8.46) $$v(24) \leq 2v(34)/3,$$

then

(8.47) $$(v(13) - v(34)/3,\ v(24) - v(34)/3,\ v(34)/3,\ v(34)/3;\ 13,\ 24) \in \mathcal{K}.$$

Proof. $s_{1,3} = e(124)$, $s_{3,1} = e(34)$, $e(124) = e(34)$; hence $1 \approx 3$. $s_{2,4} = e(123)$, $s_{4,2} = e(34)$, $e(123) = e(34)$; hence $2 \approx 4$.

C_3. If

(8.48) $$\text{Max}\ (2v(12),\ 2v(34)/3) \leq v(24) \leq \text{Min}\ (2v(23),\ 2[v(12) + v(13)]),$$

then

(8.49) $$([4v(13) - 2v(34) + v(24)]/4,\ v(24)/2,\ [2v(34) - v(24)]/4,\ v(24)/2;\ 13,24) \in \mathcal{K}.$$

Proof: $s_{1,3} = e(124)$, $s_{3,1} = e(34)$, $e(124) = e(34)$; hence $1 \approx 3$. $s_{2,4} = e(123)$, $s_{4,2} = e(134)$, $e(123) = e(134)$; hence $2 \approx 4$.

C_4. If

(8.50) $$v(12) + v(14) \geq v(24),\quad v(24) \geq 2v(23),$$

then

(8.51) $$([v(13) + v(14) - v(34)]/2,\ v(24)/2,\ [v(13) + v(34) - v(14)]/2,\ v(24)/2;\ 13,24) \in \mathcal{K}.$$

Proof: $s_{1,3} = e(14)$, $s_{3,1} = e(34)$, $e(14) = e(34)$; hence $1 \approx 3$.

$s_{2,4} = e(123)$, $s_{4,2} = e(134)$, $e(123) = e(134)$; hence $2 \approx 4$.

C_5. If

$$v(24) \geq \text{Max}\ (2[v(12)+v(13)],\ v(12)+v(14)), \tag{8.52}$$

then

$$(0,\ v(24)/2,\ v(13),\ v(24)/2;\ 13,\ 24) \in \mathcal{K}. \tag{8.53}$$

Proof: $e(34) \geq e(12), e(14), e(124)$; hence $s_{3,1} \geq s_{1,3}$. Thus $1 \approx 3$ since $x_1 = 0$. $s_{2,4} = e(123)$, $s_{4,2} = e(134)$, $e(123) = e(134)$; hence $2 \approx 4$.

In order to show that the five categories exhaust all the possibilities, we shall assume that a characteristic function v does not belong to them. We shall distinguish two cases:

I. If $v(24) \geq v(12) + v(14)$, then

(i) $v(24) > 2v(12)$, since v is not in C_1.

(ii) $2v(34)/3 < v(34)$, since v is not in C_2.

(iii) $v(24) < 2[v(12) + v(13)]$, since v is not in C_5.

(iv) $v(24) > 2v(23)$, since v is not in C_3 and the other relations are satisfied by (i), (ii), (iii).

By (iii) (iv) $v_{23} < v_{12} + v_{13}$ or $v_{24} < v_{12} + v_{14}$, contrary to the assumption.

II. If $v(24) < v(12) + v(14)$, then

(i) $v(24) < 2v(23)$, since v is not in C_4.

(ii) $v(24) > 2v(12)$, since v is not in C_1.

(iii) $2v(34)/3 < v(24)$, since v is not in C_2.

(iv) $v(24) > 2v(12) + 2v(13)$, since v is not in C_3, and the other relations are satisfied by (i), (ii) and (iii).

Our assumption can be written in the form

(v) $v(23) < v(12) + v(13)$,

and, therefore, by (iv), $v(23) < v(24)/2$, contrary to (i).

Thus, there is no characteristic function which does not belong to one of the five categories.

D. <u>The Coalition structure (12, 34)</u>.

Let $(x_1, x_2, x_3, x_4; 12, 34) \in \mathcal{K}$. We shall show that $x_3 = x_4 = v(34)/2$. Indeed, $x_1 + x_2 = v(12)$, hence $x_1, x_2 \leq v(12) \leq v(13), v(14), v(23), v(24)$. By (8.18), $e(123) \geq e(23), e(13)$ and $e(124) \geq e(24), e(14)$. Thus, $s_{3,4} = e(123) = x_4$ and $s_{4,3} = e_{124} = x_3$.

Clearly, $v(34) > 0$, since $v(1234) > 0$ and $v(34) \geq v(12)$; therefore, $x_3 + x_4 > 0$. By (2.15), $(x_4 - x_3)\, x_4 \leq 0$, $(x_3 - x_4)\, x_3 \leq 0$, hence $x_3 \neq 0$, $x_4 \neq 0$ and consequently $x_3 = x_4$.

Applying now Lemma 7.1 twice, we obtain the following cases:

D_1. If

(8.54) $$v(34) \leq 2v(13),$$

then

(8.55) $$(v(12)/2,\ v(12)/2,\ v(34)/2,\ v(34)/2;\ 12,\ 34) \in \mathcal{K}.$$

D_2. If

(8.56) $$v(34) \geq 2v(23),\ v(24) \leq v(12) + v(14),$$

then

(8.57) $$([v(12) + v(14) - v(24)]/2,\ [v(12) + v(24) - v(14)]/2,\ v(34)/2,\ v(34)/2;\ 12, 34) \in \mathcal{K}.$$

D_3. If either

(8.58) $$v(34) \geq 2v(23),\ v(24) \geq v(12) + v(14),$$

or

(8.59) $$v(34) \leq 2v(23),\ 3v(12) + v(13) \leq v(24),$$

then

(8.60) $$(0,\ v(12),\ v(34)/2,\ v(34)/2;\ 12,\ 34) \in \mathcal{K}$$

D_4. If

(8.61) $$2v(13) \leq v(34) \leq 2v(23),\ v(24) \leq 3v(12) + v(13),$$

then

(8.62) $$([4v(12) - 2v(24) + v(34)]/4,\ [2v(24) - v(34)]/4,\ v(34)/2,\ v(34)/2;\ 12,\ 34) \in \mathcal{K}.$$

9. Symmetric and Quota Games

A game is *symmetric* if the characteristic function satisfies

(9.1) $v(B_1) = v(B_2)$ whenever B_1 and B_2 have the same number of players.

Theorem 9.1[1]. *Let* $(v; N)$ *be a summetric game then a p.c.* $(\mathcal{X}; \mathcal{B}) \in \mathcal{K}$ *if and only if*

(9.2) $$x_i = x_j \quad \text{whenever} \quad i, j \in B \in \mathcal{B}$$

Proof. Obviously, $(\mathcal{X}; \mathcal{B}) \in \mathcal{K}$ if (9.2) is satisfied. If $(\mathcal{X}; \mathcal{B}) \in \mathcal{K}$, let $k, \ell \in B \in \mathcal{B}$, $k \neq \ell$. If $x_k > x_\ell$, then $e(D \cup \{k\} < e(D \cup \{\ell\})$, whenever $k, \ell \notin D$; consequently $s_{k,\ell} < s_{\ell,k}$. By (2.15), $x_k = 0$, which is impossible since $x_\ell \geq 0$. This completes the proof.

A game is an *m-quota game*[2], $1 < m < n$, if there exists an n-tuple $(\omega_1, \omega_2, \ldots, \omega_n)$ such that

(9.3) $$v(B) = \sum_{i \in B} \omega_i \geq 0 \quad \text{whenever } B \text{ contains } m \text{ players,}$$
$$v(B) = 0 \quad \text{otherwise.}$$

A player i whose *m-quota* ω_i is negative is called a *weak player*. A coalition structure which contains a maximal amount of m-person coalitions will be called *maximal*.

B. Peleg reports that the following theorem is true:

Theorem 9.2. *Let* Γ *be an m-quota game and let*[3] $n \geq 2m$. *Let* $\mathcal{B}$ *be a maximal coalition structure, and let* $(\mathcal{X}; \mathcal{B}) \in \mathcal{K}$. *Under these conditions,* $x_i = 0$ *if* i *is a weak player, and none of the non-weak players receives more than his quota.*

Under the condition of the theorem there exists a unique payoff $\mathcal{X}$, such that $(\mathcal{X}, \mathcal{B}) \in \mathcal{K}$. It is explicitly characterized by B. Peleg.

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(1) This theorem was independently proved also by B. Peleg (written communication).

(2) The term was first introduced by L. S. Shapley [8] and G. K. Kalish [3]. We use here the definition given in B. Peleg [6].

(3) This assumption is not required if no player is weak.

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