VERIFICATION OF DISARMAMENT BY INSPECTION:

A GAME THEORETIC MODEL

Morton Davis

Econometric Research Program
Research Memorandum No. 62
December 3, 1963

The research described in this paper was supported by The Carnegie Corporation of New York.

Princeton University
Econometric Research Program
92-A Nassau Street
Princeton, N.J.

ABSTRACT

This paper is a summary of part of a report written by several people for Mathematica, a subsidiary of Market Research Corporation of America, under a contract for the Arms Control and Disarmament Agency. The model was that of Professor Frank Anscombe. The results and comments of Professor Anscombe have been used so freely that it would be difficult to acknowledge in detail my indebtedness.

VERIFICATION OF DISARMAMENT BY INSPECTION: A GAME THEORETIC MODEL

Introduction

The problem that originally motivated the construction of the model below was to determine the "proper" behavior of two parties to a disarmament agreement: one, the inspector attempting to either inhibit evasion or to maximize the probability of detecting any evasion should it occur, the other, an evader or possible evader, attempting to evaluate the possible gains from evasion, the possible losses in being caught evading, and thus deciding whether to evade or not, and if deciding to evade, how to do so with the least chance of being caught. In an actual agreement, either party may find themselves in both the roles of inspector and (potential) evader, i.e., playing two games simultaneously.

The model will of course be a simplification of reality and one simplification we will make initially is to assume that the evader has decided to evade and has but to choose the "proper" manner. This obviates the necessity of balancing the disadvantages of being caught cheating with the gains from such activities (along with their respective probabilities) - variables very difficult to define and quantify. Perhaps such decisions are best made in a less formal manner. As an input to such decisions however, it is important to know the consequences of evading, so that the problems we consider logically precede the ones we will ignore. The model we consider will have an inspector and evader with diammetrically opposed interests, trying to raise and lower, the probability of detecting evasions, respectively. In the terminology of Game Theory, this is a zero-sum game which simplifies the mathematical analysis considerably.

General Description of the Model.

We assume there exist a number of individual units where both evasion and inspection may take place, their number and significant characteristics being known to both parties; each evasion will be assumed to take place in one such unit. The units themselves may be factories, geographically defined locations, power plants, or any part or union of any of the above, etc. It will be assumed that the inspector will have some restrictions on his ability to inspect these facilities: explicit restrictions spelled out by the terms of a treaty perhaps, financial restrictions on personnel and mobility, etc., but in any case in our model he will have some fixed allotment of inspections which will be all that he will be allowed and all of which he will be assumed to take. The "cost" (in terms of his inspection allotment) will vary from unit to unit. The evader will be assumed to be able to gain from each evasion a certain amount of evasion units again dependent upon which unit he chooses to evade. In a given unit he has only the choice to evade or not evade; quantitative variations of evasion within a given unit will not be considered. The various evasion units will be assumed to be comparable and additive for our purposes but not necessarily comparable to any other unit, e.g., money. The evader will be assumed to evade a fixed amount determined in advance and may evade neither more (perhaps to minimize the chance of his being detected) nor less (perhaps because evasion less than a certain minimum is pointless).

Although this model grew out of one particular situation it can be applied to a considerably wider class of problems.

It is essentially a type of Colonel Blotto game and the model may be used in determining the best strategies for smugglers vs. border police, police patrols vs. burglars, a spot checking auditor vs. an embezzler, etc.

We will assume that the choice of units to be inspected and the choice of units for evasion are chosen simultaneously or that one choice is made in a continuous random fashion and the other at a single point in time. (Police are continually patrolling various areas although not necessarily the same areas whereas a burglary will take place at a single point of time). The analysis is applicable in either case.

We alluded earlier to the term "proper" behavior with respect to the evader and the inspector and before going into a formal description of the model it might be well to discuss what the criteria might be for determining whether behavior is proper. Since we have assumed already that there will be evasion it seems that the goal of the inspector will be to inspect so as to maximize his effectiveness in detecting evasions and it will be the goal of the evader to elude as best he can the inspections that are made. "Proper" behavior then will be behavior that seems directed toward these goals. We have been deliberately vague in defining what we mean by effective inspections since there are many plausible ways this might be done. In our analysis we have chosen two possible ways of measuring effectiveness, i.e., the payoff function will have two different definitions giving rise to two different models and we will say a word about the situations the two models are meant to reflect. If one envisions a border patrol where there is constant sporadic smuggling, the expected number of evasions detected seems a reasonable criterion for judging effectiveness. If one is policing a test ban treaty, a disarmament treaty, etc. where the essential problem is to detect the good faith of the inspected, the number of violations detected is secondary; what is really important is that you determine that there are violations (or that there are not) and that you then may act in accordance with reality. It would seem then, that the probability of detecting at least one evasion would be a reasonable payoff in such a situation, and this is what was chosen in the second model. Finally, the probability of detecting an evasion will depend upon the manner in which the inspector and evader choose the units in which they will inspect, evade, respectively. The strategy of the inspector will depend upon what he thinks the evader will do which in turn depends upon what he thinks the evader will think he will do, etc. To avoid these circular traps we accept the standard procedure for solving zero-sum games: minimax strategies for the two players. We very briefly describe the properties of such strategies.

We allow first of all the inspector to inspect a definite set of units of each type and this we shall call a pure strategy. He may also choose to define some probability distribution over the set of sets of units that he may inspect (i.e., all aggregations of units whose total cost when added together equal his inspection allotment) and this we call a mixed strategy. If we insist that any strategy the inspector chooses, pure or mixed, must be ANNOUNCED TO THE EVADER, who will then act to minimize the payoff function, and if the inspector then chooses a strategy so as to maximize these minimums, we have a value for the payoff function which is a security level for the inspector, i.e., he can certainly ensure that he can get this much (it is assumed the higher the payoff, the better off the inspector, the worse off the evader). If we likewise insist the evader announce his strategy, we may calculate his security level; it turns out that BOTH SECURITY LEVELS ARE IDENTICAL; this common value for the security levels is called the value of the game (the equality is for the expected value of the outcome; not the outcome itself). These ideas will be illustrated in our subsequent analysis.

The Formal Model

Assume there are a total of N units which are broken down into T types with N_k units in the k^{th} type. Within each type all units will have the same pertinent characteristics A, C, f, which we will define below.

As we indicated earlier we associate with each unit an inspection cost, evasion cost and a probability of detection in the event that there is both an inspection and an evasion at a given unit. We define

A_i = the gain in evasion units to the evader if there is an evasion of a unit of the i^{th} type.

C_i = the cost to the inspector if he inspects at a unit of the i^{th} type.

f_i = the probability of an inspection detecting an evasion that occurred at a unit of the i^{th} type.

N_i = the number of units of type i.

T = the number of different types of units.

$$N = \sum_{i=1}^{T} N_i$$

Pure Strategies

For a single play, the evader's pure strategy will consist of choosing E_i units of type i in which he will evade where $i = 1,2,\ldots.T$. Similarly, the inspector's strategy will consist of choosing I_i units of type i which he will inspect. (Note: the strategies given above are not actually pure in that once the inspector (evader) chooses the number of units of type i in which he will inspect (evade), the actual units within the type will be chosen randomly; there is clearly no advantage to be gained, however, by doing otherwise.)

Although it is conceivable that the inspector not use all his inspections, we assume that in this model he will (he may not of course use more than his allotment of inspections). We also assume the evader will evade some fixed amount (in evasion units).

So we define,

A_o = The sum of all the evader's evasions (in evasion units).

C_o = The total number of inspection units alloted to the inspector.

This yields our boundary conditions:

$$\sum_{i=1}^{T} I_i C_i = C_o$$

$$\sum_{i=1}^{T} A_i E_i = A_o$$

Finally, we assume that all parameters are known to both parties and that both are using the same payoff function.

MINIMAX STRATEGIES AND VALUE

The thing to do if one wishes to determine which road to take, is to decide first where one wants to go (as the Cheshire cat pointed out to Alice). Similarly, if one wishes to choose a "good" strategy the first thing to do is to decide what one wants. As was mentioned earlier we will use two payoff functions as possible goals:

X_D = the probability of detecting at least one evasion, and

D = the expected value of the number of detected evasions.

In both models it will be assumed that the inspector (evader) is trying to maximize (minimize) the payoff function.

First Model - Payoff X_D

In this model we denote the pure strategies of the inspector, evader, by T-dimensional vectors of the form $(I_1, I_2, \dots I_T)$, $(E_1, E_2, \dots E_T)$, respectively. For the two such pure strategies just given we will derive the value of X_D.

We first note that in type k the probability that there are exactly j units in $E_k \cap I_k$ is

$$\frac{\binom{I_k}{j}\binom{N_k - I_k}{E_k - j}}{\binom{N_k}{E_k}}$$

and so the probability of there being j units in common with no detection is the above multiplied by $(1 - f_k)^j$. The probability of no detections in the k^{th} type is therefore

$$\frac{\sum_{j=0}^{M}\left[\binom{I_k}{j}\binom{N_k - I_k}{E_k - j}(1 - f_k)^j\right]}{\binom{N_k}{E_k}}$$

where $M = \text{Min.}\ [\ I_k\ ,\ E_k\ ]$; Hence, the probability of detection in at least one unit,

$$X_D = 1 - (\text{the probability of detection in no unit})$$

$$= 1 - \prod_{k=1}^{T} (\text{the probability of no detection in the } k^{th} \text{ unit})$$

$$= 1 - \prod_{k=1}^{T}\left[\sum_{j=0}^{M} \frac{\binom{I_k}{j}\binom{N_k - I_k}{E_k - j}(1 - f_k)^j}{\binom{N_k}{E_k}}\right]$$

where the probabilities of detection in two different units are assumed to be independent.

The X_D as given is exact but unfortunately is in a form with which it is difficult to work. Consequently, we will make some assumptions about X_D and on the basis of these assumptions make an approximation to X_D and use this approximation for further analysis.

First Approximation: The Linear Model.

The first and crudest approximation is obtained by assuming

$\frac{I_k}{N_k}$, $\frac{E_k}{N_k}$, and $\frac{Tf_k E_k I_k}{N_k}$ are all $<< 1$ for all k . Since the E_k , I_k , depend upon the strategy chosen and we don't know in advance what they are, we must assume that the amounts A_0 , C_o are sufficiently small to ensure the approximating assumptions.

$$\binom{N_k - I_k}{E_k - j} = \binom{N_k - I_k}{E_k} \cdot \prod_{i=1}^{j} \left\{ \frac{E_k + i - j}{N_k - I_k - E_k + i} \right\} ;$$

by assuming $\frac{I_k}{N_k}$, $\frac{E_k}{N_k}$ are small we may neglect

$$\prod_{i=1}^{j} \frac{E_k + i - j}{N_k - I_k - E_k + i} \quad \text{for } j \geq 2 .$$

Also, under the same assumption,

$$\frac{\binom{N_k - I_k}{E_k}}{\binom{N_k}{E_k}} = \prod_{i=1}^{E_k} \left[1 - \frac{I_k}{N_k - i + 1} \right] \cong 1 - \frac{I_k E_k}{N_k}$$

$$X_D \cong 1 - \prod_{k=1}^{T} \left(\frac{\binom{N_k - I_k}{E_k} + \binom{N_k - I_k}{E_k - 1} \binom{I_k}{1} (1 - f_k)}{\binom{N_k}{E_k}} \right)$$

$$\cong 1 - \prod_{k=1}^{T} \left(\frac{\binom{N_k - I_k}{E_k}}{\binom{N_k}{E_k}} \right) \left(1 + \frac{I_k E_k}{N_k} \{1 - f_k\} \right)$$

$$\cong 1 - \prod_{k=1}^{T} \left(1 - \frac{I_k E_k}{N_k} \right) \left(1 + \frac{I_k E_k}{N_k} - \frac{f_k E_k I_k}{N_k} \right)$$

$$\cong 1 - \prod_{k=1}^{T} \left(1 - \frac{f_k E_k I_k}{N_k} \right) \cong \sum_{k=1}^{T} \frac{f_k E_k I_k}{N_k} = \sum_{k=1}^{T} Q_k E_k I_k$$

$$\text{since} \quad \frac{T f_k E_k I_k}{N_k} << 1$$

To summarize, the probability of detection is approximated by

$$X_D \cong \sum_{k=1}^{T} Q_k E_k I_k$$

where $Q_k = \dfrac{f_k}{N_k}$.

If we let the strategy for the inspector be $\bar{I}_i = \dfrac{C_o A_i}{Q_i \sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}$ for i=1 to T

then the resulting probability of detection $X_D = \sum_{j=1}^{T} \bar{I}_j E_j Q_j$

$$= \sum_{i=1}^{T} \left(\frac{C_o A_i}{Q_i \sum_{j=1}^{T} \frac{A_j C_j}{Q_j}} \right) E_i Q_i = C_o \frac{\sum_{j=1}^{T} A_j E_j}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}$$

$$= \frac{A_o C_o}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}$$ (by the boundary conditions) independent of what

the evader does; similarly, if $\bar{E}_i = \dfrac{A_o C_i}{Q_i \sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}$

$$X_D = \sum_{i=1}^{T} I_i \bar{E}_i Q_i = A_o \frac{\sum_{i=1}^{T} C_i I_i}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}} = \frac{A_o C_o}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}$$

so $\bar{I}_j$, $\bar{E}_j$ are minimax strategies and

$$X_D = \frac{A_o C_o}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}$$ is the value of the game.

Finally, $$\sum_{j=1}^{T} \bar{I}_j C_j = C_o \frac{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}} = C_o$$

and $$\sum_{j=1}^{T} \bar{E}_j A_j = A_o \frac{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}}{\sum_{j=1}^{T} \frac{A_j C_j}{Q_j}} = A_o$$

so the two boundary conditions are satisfied.

It should be noted in the linear model we have just examined that there exist pure (as we have defined the term) strategies that are minimax for both parties. This will not in general be the case as will be seen in what follows.

Second approximation: The Non-Linear Model

In the Linear Model we assumed that the number of inspections and evasions were both small compared to the total number of units within any type. We will relax these assumptions somewhat in that we will alow the number of inspections to be large - the number of evasions being limited as before.

We first observe that $\frac{\binom{N-E}{I}}{\binom{N}{I}} \approx \left(1-\frac{I}{N}\right)^E$ (the derivation of this and what follows is very similar to the derivation of the Poisson distribution from the Binonial distribution). Also we note:

$$\binom{N-E}{I-k} = \binom{N-E}{I} \prod_{j=1}^{k} \left(\frac{I-j+1}{N-E-I-j+1+k}\right)$$

$$\cong \binom{N-E}{I} \left(\frac{I}{N-I}\right)^k \quad \text{so,}$$

$$\frac{\binom{N-E}{I-k}}{\binom{N}{I}} \approx \left(1 - \frac{I}{N}\right)^{E} \left(\frac{I}{N-I}\right)^{k} = (N-I)^{E-k} \frac{I^{k}}{N^{E}}$$

Finally, noting that

$$\frac{\binom{I_k}{j}\binom{N_k - I_k}{E_k - j}}{\binom{N_k}{E_k}} = \frac{\binom{E_k}{j}\binom{N_k - E_k}{I_k - j}}{\binom{N_k}{I_k}}$$

we have

$$X_D \cong 1 - \prod_{k=1}^{T} \left[\sum_{j=0}^{E_k} \binom{E_k}{j} (N_k - I_k)^{E_k - j} \frac{I_k^{j}}{N_k^{E_k}} (1 - f_k)^{j} \right]$$

$$\cong 1 - \prod_{k=1}^{T} \left[\sum_{j=0}^{E_k} \binom{E_k}{j} \frac{(N_k - I_k)^{E_k - j} (I_k\{1-f_k\})^{j}}{N_k^{E_k}} \right]$$

$$= 1 - \prod_{k=1}^{T} \left[\frac{(N_k - I_k) + I_k(1-f_k)}{N_k} \right]^{E_k}$$

$$= 1 - \prod_{k=1}^{T} \left[1 - \frac{I_k f_k}{N_k} \right]^{E_k} = 1 - \prod_{k=1}^{T} (1 - I_k Q_k)^{E_k}$$

So X_D will be assumed to be

$X_D \cong 1 - \prod_{k=1}^{T} (1 - I_k Q_k)^{E_k}$ in our subsequent analysis in this model.

(A similar approximation for X_D may be obtained in the case where E is large and I is small.)

Solution of the Non-Linear Model

In this model we have the same boundary conditions as before but now

$$X_D = 1 - \prod_{k=1}^{T} \{1 - I_k Q_k\}^{E_k}$$

We will prove the following:

(a) The minimax strategy for the inspector is to choose the vector $\overline{I} = (\overline{I}_1, \overline{I}_2, \ldots, \overline{I}_T)$ (where $\overline{I}_j$ means we choose $\overline{I}_j$ units at random from the N_j units of type j) satisfying the T equations:

(i) $(1 - I_j Q_j)^{1/A_j} = (1 - I_1 Q_1)^{1/A_1} \quad j = 2$ to T

(ii) $\sum_{k=1}^{T} C_k I_k = C_o$

(b) The minimax strategy for the evader is $\overline{E}$ where $\overline{E}$ is a mixture of T pure strategies where the j^{th} pure strategy is to put all of the evasions in one type of facility, i.e., there will be $\frac{A_o}{A_j}$ evasions in the j^{th} facility and no evasions in any other facility. We denote this pure strategy by E_j , and the minimax strategy, $\overline{E}$, uses E_j, k_j^{th} of the time where the k_j's satisfy the following T equations:

$$\sum_{j=1}^{T} k_j = 1$$

$$k_j = k_1 \frac{(1 - \overline{I}_j Q_j) A_j C_j Q_1}{(1 - \overline{I}_1 Q_1) A_1 C_1 Q_j} \quad \text{for } j = 2 \text{ to } T$$

(c) The value of the game is

$$1 - (1 - \overline{I}_1 Q_1)^{A_o/A_1}$$

Proof:

Let the inspector choose $\bar{I}$ as his strategy. Then the probability of detection given any arbitrary strategy for the evader: $E = (E_1 E_2, \ldots, E_T)$ is given by

$$X_D = 1 - \prod_{k=1}^{T} (1 - \bar{I}_k Q_k)^{E_k}$$

$$= 1 - \prod_{k=1}^{T} (1 - \bar{I}_1 Q_1)^{E_k A_k / A_1}$$

$$= 1 - (1 - \bar{I}_1 Q_1)^{\left[\sum_{k=1}^{T} E_k A_k\right]/A_1}$$

$$= 1 - (1 - \bar{I}_1 Q_1)^{A_o/A_1}$$

which is independent of the evader's strategy.

In other words if the inspector plays what purports to be his minimax strategy he has a probability of detection of precisely

$1 - (1 - \bar{I}_1 Q_1)^{A_o/A_1}$ regardless of what the evader does.

Now suppose the evader chooses the mixed strategy $\bar{E}$. We have again, for an arbitrary inspector strategy I ,

$$X_D = \sum_{i=1}^{T} k_i \left[1 - (1 - Q_i I_i)^{A_o/A_i}\right]$$

$$= 1 - \sum_{i=1}^{T} k_i (1 - Q_i I_i)^{A_o/A_i}$$

where X_D is the expected probability of detection and $I = (I_1, I_2, \ldots, I_T)$ is some arbitrary pure strategy chosen by the inspector.

Let

$\vec{m}_i$ be a unit vector in the direction of increasing I_i

$$U_i = 1 - Q_i I_i$$

$$\bar{U}_i = 1 - Q_i \bar{I}_i .$$

$\vec{V} = \sum_{i=1}^{T} a_i \vec{m}_i$ be an arbitrary unit vector in the hyperplane

$$\sum_{i=1}^{T} C_i I_i = C_o .$$

We shall outline the proof.

We will show that if we take the point in the hyperplane $\sum_{i=1}^{T} C_i I_i = C_o$ (i.e., the set of I's the inspector is allowed to choose if he is to restrict himself to the boundary condition limitations) where the directional derivatives are zero in every direction, we find that the point is $I = \bar{I}$, we find the second derivative is less than zero in all directions so $I = \bar{I}$ yields a maximum expectation for the probability of detection and X_D for $I = \bar{I}$ is $1 - (1 - \bar{I}_1 Q_1)^{A_o/A_1}$ for any E , and so for $E = \bar{E}$ in particular. This means the evader may be assured that the probability of detection is at least as high as $1 - (1 - \bar{I}_1 Q_1)^{A_o/A_1}$ so this is the value of the game.

Proof: We note first that if $\vec{V} = \sum_{i=1}^{T} a_i \vec{m}_i$ is in the hyperplane $\sum_{i=1}^{T} C_i I_i = C_o$, it follows that $\sum_{i=1}^{T} a_i C_i = 0$;

$$\left.\frac{\partial X_D}{\partial s}\right|_{\vec{V}} = \sum_{i=1}^{T} a_i \frac{\partial X_D}{\partial I_i}$$

$$= - \sum_{i=1}^{T} a_i \left[k_i \frac{A_o}{A_i} (- Q_i) \right] U_i^{A_o/A_i - 1}$$

(since $\frac{\partial U_i}{\partial I_j} = \begin{cases} 0 & \text{if } i \neq j \\ -Q_j & \text{if } i = j \end{cases}$)

$$= A_o \sum_{i=1}^{T} \frac{a_i Q_i k_i U_i^{A_o/A_i}}{U_i A_i}$$

Now at $I = \bar{I}$, (recalling that $\bar{U}_j^{1/A_j} = \bar{U}_1^{1/A_1}$)

$$\left.\frac{\partial X_D}{\partial s}\right|_{\vec{V}} = \frac{A_o k_1 Q_1}{A_1 C_1 U_1} \sum_{i=1}^{T} C_i a_i \bar{U}_i^{A_o/A_i}$$

(since $\frac{k_i Q_i}{\bar{U}_i A_i} = \frac{k_1 Q_1}{A_1 C_1 \bar{U}_1} C_i$)

$$= \frac{A_o k_1 Q_1 \bar{U}_1^{A_o/A_1}}{A_1 C_1 \bar{U}_1} \left(\sum_{i=1}^{T} C_i a_i \right) = 0 .$$

Also since $\frac{k_i Q_i}{A_i} = \frac{k_1 Q_1 C_i \bar{U}_i}{\bar{U}_1 A_1 C_1}$,

$$\left.\frac{\partial X_D}{\partial s}\right|_{\vec{V}} = \frac{A_o k_1 Q_1}{A_1 C_1 \bar{U}_1} \sum_{i=1}^{T} a_i \bar{U}_i C_i U_i^{A_o/A_i - 1}$$

and since a_i, $\bar{U}_i$, C_i are constants and U_i is only a function of I_i, we have

$$\left.\frac{\partial^2 X_D}{\partial s^2}\right|_{\vec{V}} = \frac{A_o k_1 Q_1}{A_1 C_1 \bar{U}_1} \sum_{i=1}^{T} a_i^{\,2}\, \bar{U}_i (-Q_i) C_i \; (A_o/A_i - 1) U_i^{\,A_o/A_i - 2}$$

If we assume

$0 < f_i$ (so $Q_i > 0$)

$k_i > 0$

$A_o > A_i$ — for all i

$\bar{U}_i > 0$

$A_i, C_i > 0$

and $A_o, C_o > 0$ (so $\sum_{i=1}^{T} a_i^{\,2} > 0$)

$I = \bar{I}$ is a maximum for X_D which was to be shown.

Note: All the above assumptions are very natural, e.g., $f_i = 0$ implies there is no chance of getting caught. $A_o = A_i$ means that the evader may use every facility of a type to evade which was ruled out by earlier hypothesis, $k_i = 0$ means you never evade in a given type facility which with a little thought is shown to be absurd for if this is really minimax the inspector should never inspect in his minimax which means the evader should evade after all, etc.

Second Model: Payoff D

We consider the same basic game with the following changes: the payoff function is D (the expected number of evasions discovered), and we consider a population of U individual units rather than a population divided into types (this is of course no loss in generality). A strategy for the inspector will consist of his defining a U-dimensional vector $\overline{I}_1, \overline{I}_2, \ldots, \overline{I}_U$ where the j^{th} component denotes the probability of choosing the j^{th} unit to evade at a given play. These probabilities are of course not independent; one may think of a wheel with the units marked off along the circumference with lengths proportional to the probability of choosing the j^{th} unit (or component size) twirling a pointer choosing a unit, deleting it, starting again, and continuing until the quota of inspections is exhausted. The expected number of inspections is then $\sum_{j=1}^{U} C_j \overline{I}_j = C_o$.

In a similar manner we define a U-dimensional vector for the evader with an analogous interpretation. Again $\sum_{i=1}^{U} A_i \overline{E}_i = A_o$.

We then have

(a) $$\overline{I}_i = \frac{C_o A_i}{f_i \sum_{j=1}^{T} \frac{A_j C_j}{f_j}}$$

(b) $$\overline{E}_i = \frac{A_o C_i}{f_i \sum_{j=1}^{T} \frac{A_j C_j}{f_j}}$$

are the minimax strategies (or weights) for the inspector, and the evader, respectively, and

(c) $$D = \frac{C_o A_o}{\sum_{j=1}^{T} \frac{C_j A_j}{f_j}}$$

is the value of the game.

Proof: We first note that

$$\sum_{i=1}^{T} C_i \overline{I}_i = C_o \quad \text{and}$$

$$\sum_{i=1}^{T} \overline{E}_i A_i = A_o \quad .$$

Suppose the inspector plays

$$\overline{I}_i = \frac{C_o A_i}{f_i \sum_{j=1}^{T} \frac{A_j C_j}{f_j}} \quad ; \text{ then}$$

$D = \sum_{i=1}^{T} f_i \overline{I}_i E_i$ ($f_i E_i I_i$ is the expected number of detections at unit i if I_i, E_i are the respective weights given by the inspector, evader)

$$\text{or } D = \sum_{i=1}^{T} \left[f_i E_i \left(\frac{C_o A_i}{f_i \sum_{j=1}^{T} \frac{A_j C_j}{f_j}} \right) \right] = C_o \frac{\sum_{i=1}^{T} E_i A_i}{\sum_{j=1}^{T} \frac{A_j C_j}{f_j}}$$

$$= \frac{C_o A_o}{\sum_{j=1}^{T} \frac{A_j C_j}{f_j}} \quad ;$$

so that

$$D = \frac{C_o A_o}{\sum_{j=1}^{T} \frac{A_j C_j}{f_j}}$$

which is independent of what the evader does.

Similarly, if the evader plays

$$\overline{E}_i = \frac{A_o C_i}{f_i \sum_{j=1}^{T} \frac{A_j C_j}{f_j}} \quad \text{we have}$$

$$D = \sum_{i=1}^{T} \bar{E}_i I_i f_i = \sum_{i=1}^{T} \left[\left(\frac{A_o C_i}{f_i \sum_{j=1}^{T} \frac{A_j C_j}{f_j}} \right) \right] I_i f_i$$

$$= \frac{A_o \sum_{i=1}^{T} C_i I_i}{\sum_{j=1}^{T} \frac{A_j C_j}{f_j}} = \frac{A_o C_o}{\sum_{j=1}^{T} \frac{A_j C_j}{f_j}}$$

which is independent of what the inspector does.

Since either the inspector or the evader may enforce

$$D = \frac{A_o C_o}{\sum_{j=1}^{T} \frac{A_j C_j}{f_j}}$$

this is clearly the value of the game and the strategies given are minimax.

Summary:

We have tried to give the first approximation of the solution of a complex problem, or rather to part of a complex problem, above. It is hoped that this will yield some qualitative insight into the actual problem -- little else should be expected at this stage. There were assumptions that were uneasily made and which might be relaxed in any subsequent extension. We mention a few.

While it was assumed that at a particular unit there was or was not an evasion or an inspection, it might be more realistic to let the intensity of evasion vary within a unit (and possibly the inspection as well) and let f , for a particular unit, depend on the intensity of the evasion and inspection therein rather than be fixed. Perhaps we may also assume that some parameters are not known to both parties, e.g., that the inspector know only the lower bound to the amount of evasion (anything below which he would be willing to ignore). Finally, if we are dealing with an unorganized opposition, e.g., police vs. burglars, or individual smugglers, one might consider solutions other than the pessimistic minimax ones.