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A SPECTRUM ANALYSIS OF SEASONAL ADJUSTMENT

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Abstract

This paper applies the methods of cross-spectrum analysis to the problem of the seasonal adjustment of economic time series. In all, four methods are analyzed by means of empirical analysis of the results of applying the methods to artificially generated time series of known composition. Three of the four analyzed seasonal adjustment methods are known in the literature while the fourth method was developed on the basis of analysis of the other three. The three methods are: Hannan's method, Wald's method, and the Census Method. The fourth method is termed the rational-function method.

It is found that there exist considerable differences in performance between the various methods when the seasonal varies with time. Hannan's method estimates (in an efficient, unbiased way) a constant seasonal, and thus is unsuited to the estimation of variation in the seasonal. The Wald and rational-function methods are able to estimate relatively more rapid changes in the seasonal amplitude than is the case with the Census Method. However, the Census Method and the Rational-function method both estimate changes in seasonal pattern which Wald's method takes as constant.

Ultimately which method is termed best, or better, depends on one's view of the character of seasonal fluctuations and one's evaluation of what constitutes important information in the particular time series (or class of series) under study.

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A SPECTRUM ANALYSIS OF SEASONAL ADJUSTMENT*)

1.0 Introduction

The problem of dealing with the apparent seasonal variation in time series has historically been treated by a large number of research workers. [c.f. 16, 20, 27] One of the most interesting of these treatments has been that of Abraham Wald. [27] On the basis of criticisms put forward by Oskar Morgenstern of the then current methods, Wald first analyzed the statistical basis of the most important of these methods and then put forward a new method. This method has received relatively little attention and almost no practical utilization. [20, pp. 151-176] Therefore, we analize this method and present its derivation (Appendix A). In addition we treat the most widely used method, the Census Method, along with a newer method, as put forward by Hannan. [10] And finally we treat a fourth method which is developed on the basis of the analysis of the three previously mentioned methods.

This paper is principally concerned with the analysis of the comparative performance of various methods of seasonal adjustment. The approach taken is to generate the components of an additive seasonal model:

$$Y_t = C_t + S_t + I_t$$

where

Y_t - observed time series

C_t - "trend-cycle" components

S_t - "seasonal" components

I_t - "irregular" components .

*) While a number of people have commented on earlier drafts of this paper, the authors would particularly like to thank Professor John Tukey and Dr. David Brillinger for their many helpful suggestions.

The "trend-cycle" plus "irregular" $(C_t + I_t)$ and the "seasonal" (S_t) are generated separately in order that each seasonal adjustment method may be tested in terms of its effects upon a process, the components of which are known separately.

The experimental technique of actually generating series and applying the seasonal adjustment methods to them was adopted partly because of the analytic difficulty of fully analyzing the properties of these methods, some of which are intended to adjust non-stationary series. However it is also felt that this approach would help to give some further insight into the analytic problems of seasonal adjustment. It also may contribute in some way to a more direct understanding of the possible effects of seasonal adjustment as commonly applied to actual data of unknown composition.

In order to discuss the significance of the results of this analysis, it is first necessary to discuss the rationale of seasonal adjustment.

1.1 The Objectives of Seasonal Adjustment

An heuristic statement of the objective of seasonal adjustment might be given as: The elimination of variations in a time series which are attributable to predictable seasonal events in order to display more clearly the more important underlying variations. This statement, while not a rigorous definition, has two interesting implications. The first implication is that the seasonal variations are predictable and not of interest to analysis of the underlying system. The second is that these seasonal variations are separable from the rest of the series. The acceptance of these implications

is a matter of open debate. Their acceptance is normally related to the use to which the data are to be put. If the data are to be used for the estimation of econometric models, it is normally assumed that the seasonal variation is of interest and its contribution to the explanation of the variation of the dependent variables is provided for through the use of seasonal variables and the application of the model to unadjusted data. If on the other hand the data are to be used for single time series extrapolation one would be indifferent, within the context of linear theory, as to whether the seasonal were removed and treated separately or left in the data.

Finally, if the data are to be used to present a time series which "best" represents the realizations from some underlying process (which does not involve strictly periodic terms of period one year) for purposes of, say, government policy decisions, then seasonal adjustment may be justified.

It is from this last viewpoint that we approach the problem of determining criteria for measuring the performance of seasonal adjustment methods. In a verbal form our criterion for the methods may thus be stated loosely as follows: That method is judged "best" which, when it operates on a time series composed of "seasonal" and other variations, produces a series which most closely approximates the other variations in the original series. This criterion will be more accurately specified and elaborated upon in Section 5.0. However, a difficulty with this criterion should be introduced at the point. If we consider the spectrum of a seasonally adjusted series, it is intuitively clear that the subjective significance which may be attached to contributions to the spectrum of "errors" in the seasonal adjustment will change considerably depending on the frequency at which the error occurs. It is natural to

suppose that "errors" introduced by seasonal adjustment which occur at low frequencies would be considered as more undesirable than such "errors" occurring at high frequencies. Underlying this problem is the idea that information at certain frequencies is more important than information at certain other frequencies.

1.2 Scope of the Method of Analysis

Since we base our analysis on the analysis of spectra, we are restricted to the analysis of the linear information in the time series and linear dependence between two series. This restriction prevents the full analysis of non-linear operators. However, the severity of this restriction is reduced by the fact that nonlinear operators may be, at least partially, analyzed in terms of their effects on linear information; and by the fact that in some cases transformations of the processes involved may yield linear relationships.

Another restriction is that of stationarity. Strictly, the spectral estimates used only have meaning for second-order stationary processes. However, spectrum methods have been shown to give useful estimates under a fairly wide range of non-stationary conditions. [12] Thus we assume that our estimates have meaning if the time averaging property of the pseudo-spectrum as described in [12] is kept in mind.

2.0 The Generating Process for the Time Series Used

The underlying process for the time series used is defined by a second-order autoregressive scheme of the following form:

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \epsilon_t \qquad 2.1$$

where α_1, α_2 - parameters

ϵ_t - normally distributed random independent numbers.

This corresponds to the conventional trend-cycle and irregular components of the seasonal adjustment model. The coefficients of the autoregressive scheme may be chosen to yield time series which in terms of spectral analysis, appear similar to many economic time series. It has been found in the course of various analyses of economic data that the series may frequently be well represented by a low order autoregressive. It is most commonly found that a first- or second-order autoregressive may be used to approximate the estimated spectra quite closely. Interesting discussions of this point occur in [8 and 21] where it is mentioned that a characteristic root of the estimated autoregressive is frequently very close to unity. This point has further implications both in terms of the implied stability of the generating process and in terms of the sampling variance of the estimates of the process.

The seasonal component was generated by one of the two following processes:

The first process is defined by the following equations:

$$S_{i,k} = S_{1,k}(1 + v_{12i+k}) \qquad k = 1, 12;\ i = 1, \ell \qquad 2.2$$

$$v_j = v_{j-1} + \epsilon_j \qquad 2.3$$

where $S_{i,k}$ - seasonal factor of k^{th} month of i^{th} year

v_j - disturbance term

ϵ_j - normally distributed random numbers.

This process produces a seasonal with a constant pattern (given by $S_{1,k}$) but with a changing amplitude. The amplitude is determined by the first order autoregressive scheme given by equation 2.3. The variance of the random term, ϵ_i, determines the amplitude of the variations in the seasonal amplitude.

The second process is defined by the equations:

$$S_{i,k} = \beta S_{i-1,k} + \rho_{i,k} \qquad k = 1,\ldots,12;\ i = 1,\ldots,\ell \qquad 2.4$$

$$\rho_{i,k} = \rho_{i-1,k} + \sum_{j=1}^{12} C_j \epsilon_{12i+j+k} \qquad 2.5$$

where

$S_{i,k}$ - seasonal factor of k^{th} month of i^{th} year

$\rho_{i,k}$ - disturbance term

ϵ_n - normally distributed random numbers

C_j, β - parameters.

Seasonal series generated from this process vary both in amplitude and pattern. Clearly the values for the ϵ_j coefficients determine the amount of correlation between a change in one month's seasonal and a change in the seasonal of neighboring months. At one extreme one may have simply twelve independent first-order autoregressive processes for the twelve monthly factors; or, on the other hand, one may have a high degree of correlation between months so that the seasonal pattern changes very little.

It is felt that these processes fairly describe a large class of possible seasonal components. In particular the first process is in close agreement with Wald's assumptions, while the second seems to fit the assumptions of the Census Method. The assumptions of Wald's method are discussed below (Section 4.2 and Appendix A). The Census Method is fully described in [20].

3.0 Digital Filtering and Extrapolation for Seasonal Adjustment

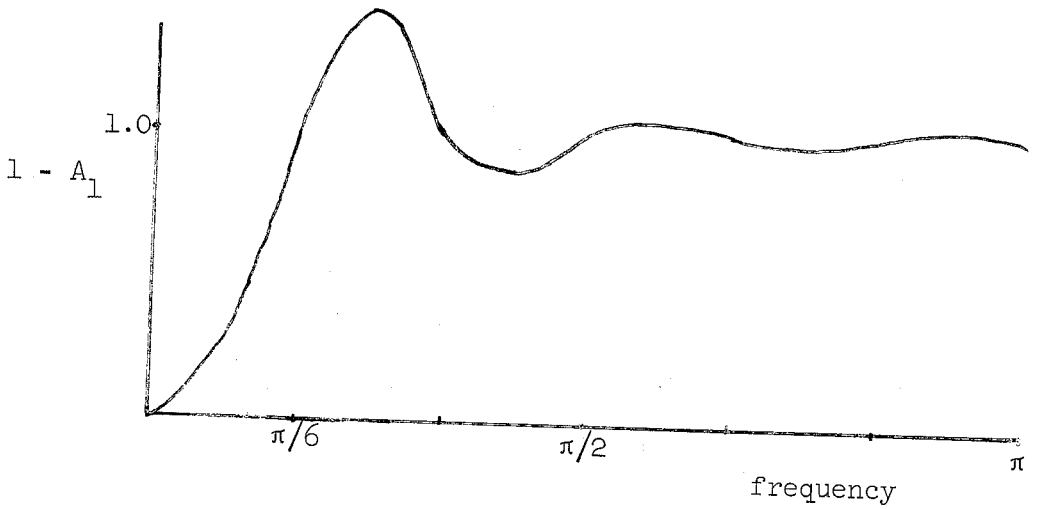
3.1 Digital Filtering

It has recently been recognized in the field of seasonal adjustment [10] that the application of moving averages is an example of the application of a time-invariant linear operator which may be characterized by its transfer function. From this observation stems Hannan's important observation that, since the effects of the operator are completely described by its transfer function, it is possible to correct for any unwanted effects when the operator is applied to a time series. Hannan specifically points out that the Spencer 15-point formula affects the amplitude of not only very low frequencies but also the seasonal and higher frequencies. Thus the simple estimate of the seasonal amplitude is biased by the attenuation introduced by the Spencer operator. Hannan goes on to show explicitly the calculations required to compensate for this attenuation.

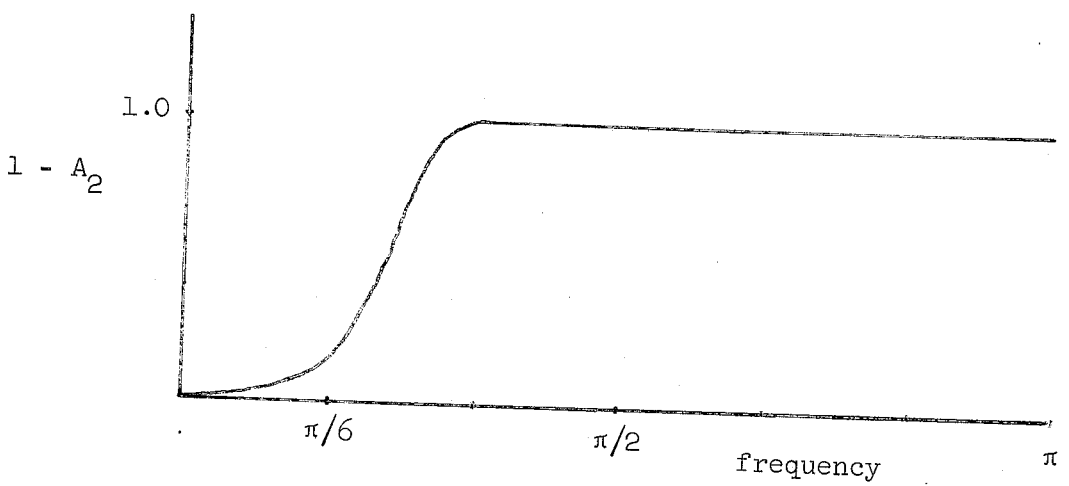
Rather than use simple unweighted moving averages or particular weighted values (such as Spencer's formula) we have, using the spectral approach, employed a somewhat more general filtering technique for the purpose of elimination of low frequencies. These techniques have been used in conjunction with the Hannan, Wald, and rational function methods which were all programmed by the authors. The Census Method was received in completely programmed form from the Bureau of the Census and no functional modifications were performed on it. (Since the Census Method has been so extensively used for routine adjustment of actual data it may be used as a standard of comparison.)

Inspection of the transfer functions of the simple 12 month moving average and the 15 point Spencer formula (Figure 3.1) shows clearly that the latter is to be preferred on the basis of elimination of frequencies below

Frequency response functions



A_1 - simple 12 month moving average



A_2 - 15 point Spencer formula

Figure 3.1

the fundamental seasonal. The fact that it eliminates some of the variance at 1 cycle per year is unimportant because this is easily corrected for in the final estimate. However, the Spencer formula does have the disadvantage that more observations are lost at the end of the series. Thus the problem of producing estimates for the last months of the series when the seasonal is allowed to vary is aggravated. In addition one feels that it would be desirable to have a filter whose properties were, in some sense, optimal for the problem at hand. For the sake of simplicity in handling phase information it was decided to restrict the class of filters that would be considered, for the initial high-pass filtering operation, to symmetric moving averages. This class of filters is, of course, characterized by transfer functions with phase identically equal to zero at all frequencies. On the basis of previous analysis it seemed natural to adopt the minimum mean square error criterion put forward by Parzen [22]. Thus we were led to use the convergence factor given by:

$$\lambda(v) = 1 - 6\left(\frac{v}{m}\right)^2 + 6\left(\left|\frac{v}{m}\right|\right)^3 \qquad 0 \leq v \leq \frac{m}{2}$$

$$= 2\left(1 - \left|\frac{v}{m}\right|\right)^3 \qquad \frac{m}{2} < v \leq m \qquad 3.1.1$$

which satisfies this criterion for a specified class of functions. This function is discussed by Parzen [22] in connection with spectral windows. It was arbitrarily decided to restrict the filter to a 12 month average. However, it was still possible to vary the bandwidth of the filter to arrive at a "best" value. After some experimentation it was found that the properties of the estimates were not extremely sensitive to small changes in filter bandwidth and the following transfer function was settled upon:

$$A(\omega) = I - \sum_{v=1}^{m} \frac{\sin kv}{v} \Lambda(v) \cos \omega v$$

where $k = \frac{\pi}{12}$. 3.1.2

The moving average coefficients corresponding to $A(\omega)$ are obtained by taking the inverse Fourier transform:

$$D_j = \frac{2}{\pi} \sum_{\omega=0}^{\pi} A(\omega) \cos j\omega \qquad j = 0,1,\ldots,6 \qquad 3.1.3$$

The gain of this filter is shown in Figure 3.2. The gain is about 0.57 at 1 cycle per year. Very little variation at frequencies below 1 cycle per year will pass through the filtering operation.

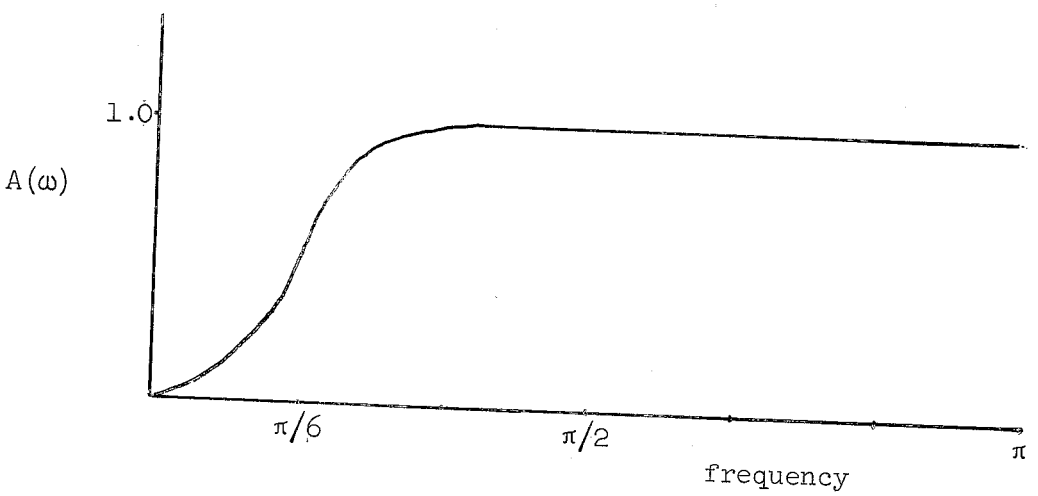


Figure 3.2

3.2 Extrapolation

After the filtering operation the resultant series is lacking points which correspond to the first and last six observations of the original series. In the context of seasonal adjustment this does not present a problem if only fixed seasonal factors are being estimated. However, if a moving seasonal is being estimated as in the Wald and rational-function methods, then the filtered series must be extended in some way to the last observation of the original series. Wald [26] analyzed this problem in connection with his seasonal adjustment method and incorporated extrapolation of the moving average into the method as it was used in Austria. The extrapolation method is fully described in [26]. However, like the monograph on seasonal adjustment [27], this paper has not been translated and no longer seems to be referred to in the literature. Therefore we will briefly outline the technique.

The extrapolation method is based upon three assumptions.

Define the 12 month moving average, $\psi(t)$, of a set of observations $\Phi(t)$ by:

$$\psi_{i,k} = \frac{\sum_{j=k-6}^{k+5} \Phi_{i,j} + \frac{1}{2}(\Phi_{i,k-6} + \Phi_{i,k+6})}{12} \quad . \qquad 3.2.1$$

Defining $s(t)$ as the seasonal component and $z(t)$ as the random component of $\Phi(t)$ we may define a function $f(t)$ by:

$$f(t) = \Phi(t) - s(t) - z(t) \quad .$$

The first assumption is then that:

$$\frac{\sum_{j=k-\ell}^{k+\ell} f_{i,k}}{2\ell + 1} \sim \psi_{i,k} \qquad \ell = 2,3,4,5 \; . \qquad 3.2.2$$

The second assumption defines the seasonal component as it has been defined in Wald's seasonal adjustment method. (See Section 4.2.) Therefore the seasonal is given by:

$$s(t) = \lambda(t)\, p(t) \qquad 3.2.3$$

where $p(t)$ is a strictly periodic function and $\lambda(t)$ varies only slowly over time. The third assumption restricts the random term by the following approximate equation:

$$\frac{\sum_{j=k+\ell}^{k+m} Z_{i,j}}{m} \sim 0 \; . \qquad 3.2.4$$

The values of m for which assumption three may hold will vary with the kind of data being used.

From these assumptions the following expression for the extrapolated values, $\psi^{*}(t)$, of $\psi(t)$ is derived:

$$\psi^{*}_{i,k+6-\ell} = \alpha_{i,\ell} - \frac{\sum_{j=k-5}^{k+6} |\Phi_{i,j} - \psi_{i,k}|}{\sum_{j=k-5}^{k+6} |\Phi_{i-1,j} - \psi_{i-1,k}|} (\alpha_{i-1,\ell} - \psi_{i-1,k+6-\ell}) \qquad 3.2.5$$

where $$\alpha_{i,\ell} = \frac{\sum_{j=k+6-2\ell}^{k+6} \Phi_{i,j}}{2\ell + 1} \quad .$$

This expression is valid under the three assumpsions (equations 3.2.2, 3.2.3, 3.2.4) for $\ell = 3,4,5$, if (3.2.4) is assumed valid for $m \geq 7$; and is valid for $\ell = 2,3,4,5$ if (3.2.4) holds for $m \geq 5$. For $\ell = 0,1$ and possibly 2 this equation cannot be used. To arrive at estimates for these values of ℓ Wald applies simple linear extrapolation of the last values of the series $\psi(t)$ and the extrapolated values given by equation 3.2.5. In the case of 3.2.4 being taken to be valid for $m = 7$ this leads to extrapolation using the 5 values

$$\psi_{i,k-1} \ , \ \psi_{i,k} \ , \ \psi^*_{i,k+1} \ , \ \psi^*_{i,k+2} \ , \ \psi^*_{i,k+3} \quad .$$

Thus, defining

$$\Delta = \frac{2(\psi^*_{i,k+3} - \psi_{i,k-1}) + (\psi^*_{i,k+2} - \psi_{i,k})}{10} \qquad 3.2.6$$

and μ = the arithmetic mean of the five values, the last three values of $\psi^*(t)$ are determined by:

$$\begin{aligned} \psi^*_{i,k+4} &= \mu + 3\Delta \\ \psi^*_{i,k+5} &= \mu + 4\Delta \qquad 3.2.7 \\ \psi^*_{i,k+6} &= \mu + 5\Delta \quad . \end{aligned}$$

4.0 The Analyzed Seasonal Adjustment Methods

4.1 Hannan's Method

The first method that has been examined is the one developed by Hannan[10]. Here, only its main features will be discussed. The basic model is given by the additive relation:

$$y(t) = p(t) + s(t) + x(t)$$

where

$p(t)$ - the trend-cycle component

$s(t)$ - the seasonal component

$x(t)$ - the residual.

$x(t)$ is assumed to be stationary, though this is not essential for the method.

The seasonal component is assumed to be unchanging and of the form:

$$s(t) = \sum_{k=1}^{6} [\alpha(k)\cos \lambda(k)t + \beta(k)\sin \lambda(k)t], \quad \lambda(k) = \frac{2\pi k}{12} \qquad 4.1.1$$

To remove the low-frequency components (trend-cycle) from the series Hannan considers, using the spectrum approach, several well-known operators. Using the notation I-A (where A is a moving average operator) for an operator which removes low frequencies, the trend-cycle removed series, $y'(t)$, is given by:

$$y'(t) = [I-A]\, y(t) \quad . \qquad 4.1.2$$

The preliminary estimates of the seasonal $\mu'(j)$ for $j = 1,2,\ldots,12$ are derived by calculating the mean for each calendar month:

$$\mu'(j) = \frac{1}{m} \sum_{t=1}^{m} y'_{12t+j} \quad \text{for} \quad j = 1,2,\ldots,12 \qquad 4.1.3$$

where m equals the number of (full) years for which the series $y'(t)$ is computed.

The 12 $\mu'(j)$'s are then adjusted to add to zero by subtracting their mean $\bar{\mu}$; the new estimates are called $\mu(j)$.

The final seasonal adjustments $\hat{a}(j)$ for $j = 1,2,\ldots,12$ are then estimated by taking a moving average, with weights $v(k)$, of the $\mu(k)$:

$$\hat{a}(j) = \sum_{k=1}^{12} \mu(k)\, v(k-j) \qquad j = 1,2,\ldots,12 \qquad 4.1.4$$

where $v(k) = v(k+12)$ for $k \leq 0$. $v(k)$ is given by:

$$v(k) = \frac{1}{12} \sum_{s=1}^{11} \frac{1}{1 - h(\omega_s)}\, e^{-is\omega_k}$$

where $h(\omega)$ = transfer function of the operator A. The function $v(k)$ is the transformed inverse of the transfer function of the operator I-A. Thus the application of $v(k)$ to the $\mu'(k)$ corrects the seasonal weights for any change in the amplitude of variation of the original series at the seasonal frequencies which the operator I-A may have introduced.

4.2 Wald's Method

The second seasonal adjustment method that has been analyzed is the one developed by Abraham Wald in 1936. Wald was at that time associated with

the Austrian Institute for Business Cycle Research, under the direction of Oskar Morgenstern. The method was published as Contribution No. 9 of that Institute, under the title "Berechnung und Ausschaltung von Saisonschwankungen." [27] There does not seem to be an English translation of this monograph[1], which might help to explain why this method is not well-known in English-speaking countries. The method is also known as the moving-amplitude method and has been characterized as being able to produce better results than other methods when the seasonal amplitude is thought to change relatively rapidly from year to year. Rapid amplitude changes have been notes in a certain number of agricultural crop series.[2]

The assumptions of Wald's method require that the seasonal pattern (i.e. the proportionality relationship between the seasonal at each month and the seasonal at each other month) remain constant over time. This constant pattern is used to estimate changes in the amplitude of the seasonal. This approach permits the estimation of more rapid changes in the seasonal amplitude than would be possible if no assumption about the stability of the pattern were made.

Because Wald's method does not appear to be well-known, we will give an outline of it here with a full description of the derivation given in Appendix A. The technique which Wald developed after the publication of [27] for extrapolation of the moving average series has been discussed separately in Section 3.2. This extrapolation method was incorporated in the adjustment method as it was employed in Vienna in order to improve the accuracy of the current estimates of the seasonal factors, and has also been used in our computations.

1) A very brief summary and application of the method can be found in G. Tintner, Econometrics, John Wiley & Sons, 1952, pp. 227-233.
2) [20] p. 64.

The basic model for Wald's method is that the functions of time which represent seasonal variations, the trend and business cycles, and the random terms are additive. In Wald's notation

$$\varphi(t) = f(t) + s(t) + z(t) \qquad \text{for } t = 1,2,\ldots,n \qquad 4.2.1$$

where $\varphi(t)$ - observed monthly series

$f(t)$ - "trend cycle" component

$s(t)$ - seasonal fluctuations

$z(t)$ - residual.

The first step is the removal of the "trend-cycle", $f(t)$. This is accomplished by subtracting the 12-month moving average of $\varphi(t)$ from the original series $\varphi(t)$. This yields the series $\psi(t)$:

$$\psi(t) = \varphi(t) - \varphi^*(t) \qquad \text{for } t = 7,8,\ldots,(n-6) \qquad 4.2.2$$

where: $$\varphi^*(t) = \frac{\varphi(t-6) + 2[\varphi(t-5) + \cdots + \varphi(t) + \ldots + \varphi(t+5)] + \varphi(t+6)}{24}$$

$$\text{for } t = 7,8,\ldots,(n-6) \; . \qquad 4.2.3$$

Replacing $\varphi(t)$ and $\varphi^*(t)$ in this expression by their components (where φ^* indicates the operation of taking the 12-month moving average), we find:

$$\psi(t) = f(t) - f^*(t) + s(t) - s^*(t) + z(t) - z^*(t) \; . \qquad 4.2.4$$

Since $f(t)$ represents the "trend-cycle", we may assume that $f(t)$ can be well approximated by a straight line over periods of 12 months. Therefore $f(t) \sim f^*(t)$.

With respect to the seasonal fluctuations $s(t)$, Wald rejects the assumption that it is merely a 12-month periodic function. The hypothesis that

it is a periodic function, which is multiplicative with the original observations $\varphi(t)$ or the "trend-cycle" $f(t)$ is also rejected. Thus the models:

$$s(t) = p(t) \cdot \varphi(t); \quad s(t) = p(t) \cdot f(t)$$

or

$$s(t) = p(t) \cdot \varphi(t) + q(t)$$

where $p(t)$ and $q(t)$ are 12-month periodic functions, are all considered to be unsatisfactory. Wald instead assumes that $s(t) = \lambda(t)\, p(t)$. $\lambda(t)$ is an arbitrary function, the value of which will slowly change over time, and $p(t)$ is a 12-month periodic function with mean = 0. In other words the intensity (amplitude) of the seasonal fluctuations, indicated by the function $\lambda(t)$, changes slowly with time, but is not systematically related to other variations in the series. The pattern of the seasonal fluctuations, indicated by the function $p(t)$, is however assumed to remain constant over time. The allowance for change in the intensity of the seasonal fluctuations is based on the observation that this intensity is influenced by the trend and the business cycle. However, since there is no a priori reason to expect that this influence follows a well-defined scheme, e.g. that the intensity of the seasonal fluctuations is proportional to the trend, the function $\lambda(t)$ is left arbitrary.

On the basis of the model $s(t) = \lambda(t)\, p(t)$ it is observed that $|s^*(t)|$, the absolute value of $s^*(t)$, will be the smaller, the smaller the fluctuation of $\lambda(t)$ within the period of 12 months, and that $s^*(t) = 0$ if $\lambda(t)$ is constant. This leads to the assumption that $s^*(t) \sim 0$. Equation 4.2.4 is now reduced to:

$$\psi(t) \sim s(t) + y(t) \qquad \text{for } t = 7,8,\ldots,(n-6) \qquad 4.2.5$$

where $y(t) = z(t) - z^*(t)$.

It is convenient at this point to introduce the matrix $\psi(i,k)$, the (i,k)th element of which refers to the k^{th} month of the i^{th} year. Let the corresponding values of $s(t)$ and $y(t)$ be similarly arranged in two matrices, the elements of which will be designated $s(i,k)$ and $y(i,k)$. Computing now the arithmetic mean of the values of the k-month of $\psi(i,k)$ as well as of $s(i,k)$ and $y(i,k)$ one obtains, after substituting $\lambda(i,k)\ p(i,k)$ for $s(i,k)$ in equation 4.2.4:

$$\frac{\sum_{i=1}^{m} \psi(i,k)}{m} \sim \frac{\sum_{i=1}^{m} \lambda(i,k)p(i,k)}{m} + \frac{\sum_{i=1}^{m} y(i,k)}{m} \qquad 4.2.6$$

$$\text{for } k = 1,2,\ldots,12$$

where: $m = \frac{n}{12} - 1.$

From these assumptions Wald arrives at the following expression for the estimated seasonal:

$$s(i,k) = a(k) \frac{\sum_{i=k-6}^{k+5} \psi(i,j)a(j)}{\sum_{\ell=1}^{12} [a(\ell)]^2}$$

where: $$a(k) = \frac{1}{m} \sum_{i=1}^{m} \psi(i,k)$$

for the m by 12 matrix of seasonal coefficients.

4.3 The Census Method

The third method that has been examined is Census Method II, which will only briefly be described here.[3] The trend-cycle, seasonal and irregular components are assumed to be combined multiplicatively. It is sometimes indicated that this is the commonest form of seasonal relationship for the broad mass of economic time series.[4] However, this model may be transformed into the additive model by taking logarithms. This transformation may introduce certain problems, particularly with respect to its effect on the distributions of the variables in the model. However, on the basis of computed comparisons of the performance of the Census Method with and without taking logarithms there was no evidence that, for the kind of series dealt with here, any difficulties would be caused by the logarithmic transformation.

An important characteristic of the so-called "moving seasonality" which is incorporated in the Census Method is that no restriction is placed on the nature of any relationships between the changes in amplitudes in successive months. The method can therefore take care of changes in the pattern of seasonal variation over successive periods of twelve months as well as changes im amplitude.[5] However, the changes in both amplitude and pattern of the seasonal ratios are assumed to be gradual and smooth. The method in its original version was not successful when applied to series with drastic changes in S-I (Seasonal-Irregular) ratios[6] as, for instance, total unemployment. Nor

3) For a more elaborate description, the reader is referred to Julius Shiskin's paper, "Test and Revisions of Bureau of the Census Methods of Seasonal Adjustments," Bureau of the Census Technical Paper No. 5, November 1960. This paper was incorporated (pp. 79-150) in [20].
4) Ref. [20], page 58.
5) Ref. [20], page 259, footnote.
6) Seasonal-irregular ratios are the ratios of the original observations to the 15-term Spencer trend-cycle curve.

could it satisfactorily adjust series with constant seasonal patterns but sharply varying amplitudes as, for example, agricultural stocks and farm employment series. However, later versions of the method contain devices which can take better care of series with extreme S-I ratios than could the original.[7] For this study version X-10 has been used as this was the version which was, according to our information, the most highly developed in early 1963.

4.4 A Rational Transfer-Function Method

The purpose of this method has been to develop a simple method, based on spectral concepts, for comparison with the other methods. The method involves the extension of Wald's method to treat a changing seasonal pattern, and the inclusion of the basic ideas of Hannan's method.

The first step of the method is the conventional one of applying a linear operator to remove low frequency variations. As with the Hannan and Wald methods the operators and notation used are discussed in Section 3.0.

Next it is assumed that the seasonal pattern may be represented in the form of a set of twelve mixed moving average autoregressive processes with identical coefficients. This is a natural assumption if, for reasons of simplicity, the processes which generate the seasonal coefficients are taken to be linear. Thus the seasonal pattern coefficients are given by:

$$S_{i,k} = \sum_{j=1}^{n} A_j S_{i,k-j} + \sum_{j=0}^{m} B_j \gamma_{i,k-j} \qquad 4.4.1$$

7) Suggestions for modification of the method were given in [20], pp. 257-311.

where $S_{i,k}$ - seasonal pattern coefficient of month k, year i

$\gamma_{i,k}$ - random disturbance term

A_j, B_j - coefficients.

Thus it is natural to attempt to estimate $S_{i,k}$ from the filtered series by:

$$S^*_{i,k} = \sum_{j=1}^{n} A_j S_{i,k-j} + \sum_{j=1}^{m} B_j \psi_{i,k-j} \qquad 4.4.2$$

where $S^*_{i,k}$ - estimated seasonal

$\psi_{i,k}$ - filtered series

A_j, B_j - coefficients.

If the Z-transform operator(defined by $Z(x_t) = x_{t-1}$) is applied to the above equation and the terms rearranged we have:

$$S^*_{i,k} = \frac{\sum_{j=0}^{m} B_j Z^{-j}}{1 - \sum_{j=1}^{n} A_j Z^{-j}} \psi_{i,k} . \qquad 4.4.3$$

From equation 4.4.3 it is clear that we are simply applying a time invariant, linear, rational function operator to the 12 series $\psi_{i,k}$ ($i = 1,\ldots,\ell$; $k = 1,\ldots,12$).

The coefficients A_j and B_j might be estimated for each series on the basis of a minimum mean square error criterion. However, in the interest of simplicity and generality the coefficients were in fact determined on the basis of more general, and in part heuristic, criteria. The transfer function

defined by equation 4.4.3 should have a gain characteristic which in some sense minimizes the error of the estimate $S^{*}_{i,k}$. One may assume that $S_{i,k}$ has relatively high spectral density at low frequencies, while the spectrum of $\gamma_{i,k}$ is relatively flat. Then the two spectral densities will be of the form given in Figure 4.4.1.

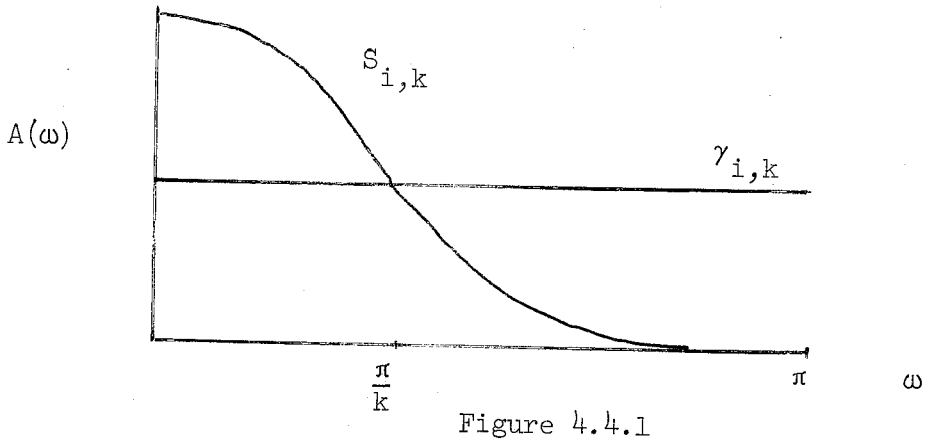


Figure 4.4.1

The transfer function given in equation 4.4.3 should then take the form:[8]

$$A(\omega) = \frac{f_s(\omega)}{f_s(\omega) + f_\upsilon(\omega)}$$

where $A(\omega)$ - transfer function of filter

$f_s(\omega)$ - spectrum of seasonal coefficients $S_{i,k}$ $i = 1,\ldots,\ell$ for all k

$f_\upsilon(\omega)$ - spectrum of random term $\upsilon_{i,k}$ $i = 1,\ldots,\ell$ for all k .

Given the above assumptions $A(\omega)$ will take the form indicated in Figure 4.4.2.

8) For an exact description of the optimal filter transfer function cf. The Extraction of Signals from Noise, L. A. Wainstein, V. D. Zubrakov, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

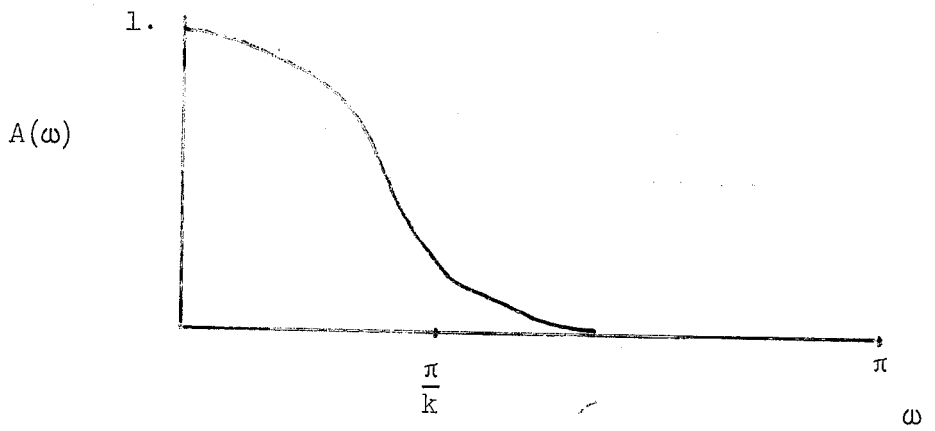


Figure 4.4.2

It seems reasonable for many economic series to assume that π/k falls in the range of about .2 to .3 cycles per year. This gain characteristic could be approximated by a symmetric moving average filter in a way similar to that applied in the Census Method. However, the rational function filter seems more natural given the assumptions made about the way in which the seasonal variation is generated.

In deciding on the transfer function given in equation 4.4.3 it would be desirable to apply general analytic criteria in terms of both the gain and phase characteristics. However, appropriate general methods based on some practical error of estimate concept are not yet available. Therefore the coefficients were simply chosen on the basis of inspection of the transfer function and experimentation.

To this point the method has been analogous to the Census Method in that the seasonal coefficients have been estimated from the twelve annual series. However, we now make the assumption that the seasonal coefficients are intercorrelated. We will, in fact, assume that for relatively high frequency variations the seasonal factors vary proportionally. This is analogous to Wald's assumption of a constant seasonal pattern. However, as

previously assumed, we allow the seasonal pattern (i.e. the proportionality factors) to vary relatively slowly. Thus we attempt to combine the assumption of the Census Method that each seasonal coefficient may change very slowly but independently of the others with the assumption of Wald's method that the amplitude of the seasonal pattern may vary relatively rapidly.

In order to derive the estimate for the seasonal amplitude we have simply paralleled Wald's derivation. This results in the following expression:

$$S^{*}_{i,k} = \frac{S_{i,k} \sum_{j=k-5}^{k+6} S_{i,j}\psi_{i,j}}{\sum_{j=k-5}^{k+6} (S_{i,j})^2} \quad . \qquad 4.4.4$$

To complete the adjustment procedure the estimated seasonal, $S^{*}_{i,k}$ is corrected for any bias introduced by the original filtering operation and is then subtracted from the original data.

The actual equations computed are as follows:

First the low frequencies are removed by:

$$\psi_t = \Phi_t - \sum_{j=-6}^{6} D_j \Phi_{t+j} \quad . \qquad 4.4.5$$

The last six values of ψ_t are extrapolated, and the D_j's are determined, as described in Section 3.0.

Then, starting values for the rational function filter are computed by averaging the first four available years of the series ψ_t

$$S_{1,k} = \frac{1}{4} \sum_{i=2}^{5} \psi_{i,k} \qquad k = 1,\ldots,12$$

$$S_{2,k} = S_{i,k} \quad . \qquad 4.4.6$$

Next the rational function filter is applied according to

$$S_{i,k} = A_1 S_{i-1,k} + A_2 S_{i-2,k} + B_1 \psi_{i,k} + B_2 \psi_{i-1,k} + B_3 \psi_{i-2,k} \qquad 4.4.7$$

$$i = 3,\ldots,\ell; \quad k = 1,\ldots 12$$

These estimates of the evolving seasonal pattern are then used to estimate the seasonal coefficients according to:

$$S^*_{i,k} = \frac{S_{i,k} \sum_{j=k-5}^{k+6} S_{i,j} \psi_{i,j}}{\sum_{j=k-5}^{k+6} (S_{i,j})^2} \quad . \qquad 4.4.8$$

Finally, the $S^*_{i,k}$ are corrected by:

$$S^{*'}_{i,k} = \sum_{j=1}^{12} \alpha_j S^*_{i,k-j} \qquad 4.4.9$$

where the α_j's are determined by the inverse of the transfer function of the D_j's given in equation 4.4.5. The equation 4.4.9 is extrapolated to the ends of the series.

Then the seasonally adjusted series is formed by

$$\Phi^s_t = \Phi_t - S^{*'}_t \quad . \qquad 4.4.10$$

5.0 Outline of Computations

As shown in the flow chart given in Figure 5.1 the typical computation consisted of the following steps:

5.1 The computation of the underlying and seasonal series for a set of specified parameter values.

5.2 The seasonal adjustment of the sum of the underlying series and the seasonal.

5.3 The computation of the spectrum and cross-spectrum properties of the following pairs of series:

5.3.1 The adjusted series and the underlying process.

5.3.2 The adjusted series and the sum of the underlying process and the seasonal.

5.3.3 The adjusted series and the seasonal.

5.3.4 The seasonal series and the difference between the sum of the underlying process and the seasonal on the one hand and the adjusted series on the other. This computation is, then, a comparison of the actual seasonal and the seasonal estimated by the seasonal adjustment method.

On the basis of the general statement that the quality of the seasonal adjustment method may be measured by its ability to accurately recover the underlying process from the sum of the seasonal and this process the following ideal results may be identified in terms of these calculations. For calculation 5.3.1 the perfect adjustment method would produce spectra with identical shapes, and a cross-spectrum with unit gain and zero phase at all frequencies. The coherence should be one at all frequencies. For calculation 5.3.2 the cross-spectrum will

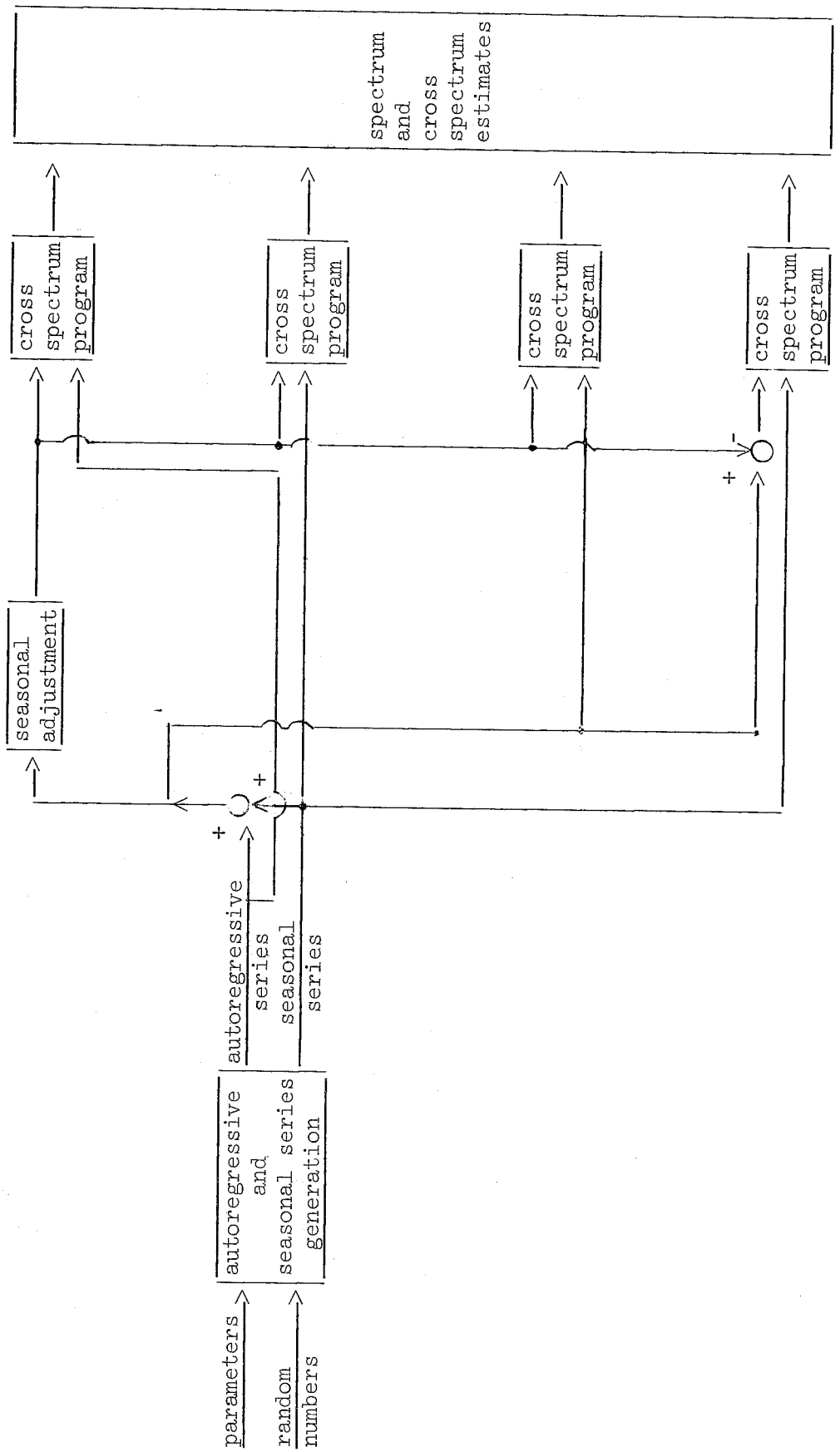

parameters
random numbers
autoregressive and seasonal series generation
autoregressive series
seasonal series
seasonal adjustment
cross spectrum program
cross spectrum program
cross spectrum program
cross spectrum program
spectrum and cross spectrum estimates
+
+
+
-

Figure 5.1

depend on the seasonal component. The reason for including this calculation is not to provide a direct test of the seasonal adjustment method, but to present the cross-spectrum for comparison with the cross-spectrum which may be computed from actual series with unknown seasonals. The perfect adjustment method would produce for calculation 5.3.3 a coherence consistent with the hypothesis of independence of the series. Therefore, the cross-spectrum should not be significant. Finally, in calculation 5.3.4 the perfect method should produce a unit correlation, as in calculation 5.3.2.

Imperfections in the various methods may show up in a considerable variety of ways as will be discussed in the next section.

6.0 Experimental Results

In determining the experimental procedure for these computations it was necessary to decide on the values of two sets of parameters. One set involved the values of the autoregressive scheme while the other was the correlation vector for the seasonal pattern. Through the computation of several pilot runs it was found that the seasonal adjustment methods were not very sensitive to changes in the autoregressive parameter values over a considerable range (i.e. those yielding characteristic periods from about 2 years to ∞). Therefore the calculations were mainly performed with only one set of values. The values of the parameters used in equation 2.1 are:

$$\alpha_1 = 1.4$$

$$\alpha_2 = -.40 \quad .$$

It is interesting to note that the roots of the characteristic equation of equation 2.1, for these parameter values, are both real with one equal to 1.0 and the other equal to 0.4. Thus, the solution of the equation is not oscillatory. However, the time series generated by this process appear to be better approximations for a wide class of economic time series than series generated using parameter values which give an oscillatory solution.

For the seasonal processes it was, however, found that the methods were sometimes differentially sensitive to the process and parameter values used. Therefore the computations were performed with several sets of values.

The first process (equations 2.2, 2.3) gives a seasonal with a constant pattern. The only variable parameter is the variance of the error term. Several values for the variance were used. This process will be referred to as Type 1.

The second process (equations 2.4, 2.5) allows the seasonal pattern to vary slowly but may preserve some correlation between months. Two sets of values for the C_j were used. The first set is:

$$C(1) = 0.6$$
$$C(2) = 0.2$$
$$C(3) = 0.1$$
$$C(4) = 0.05$$
$$C(5) = 0.05$$
$$C(i) = 0 \qquad i = 6,\ldots,12 \ .$$

This process will be referred to as Type 2. The second set of values allows each month to vary independently of the other months. These values are:

$$C(1) = 1.0$$

$$C(i) = 0.0 \qquad i = 2,\ldots,12 .$$

This process will be referred to as Type 3. For each of the sets of values for the vector $C(i)$ several values of the variance of the term ϵ_j were used in order to give several different levels of amplitude change in the seasonal. In addition values of β other than one were used in order to introduce trends in the seasonal amplitude.

The results of these computations show that all of the methods performed quite well when the generating process conformed to the assumptions on which the particular method was based. In the simple case of the constant seasonal (Type 1 with no error term) Hannan's method was definitely superior. Wald's method performed well for a constant seasonal pattern but changing amplitude. The Census Method performed well when both amplitude and pattern were changing if the changes in amplitude did not become large or relatively rapid. The rational function method tended to combine the ability of the Census Method to adjust for very slow changes in pattern with the ability of Wald's method to estimate changes in amplitude. As the characteristics of each method are considerably different we will now discuss each method separately in greater detail.

First, however, it is necessary to explain the general form of the graphical results. Each page of graphs (containing either four or six graphs) displays the results of the analysis of two time series. Viewing the page of graphs with the figure label at the bottom, the top row of three graphs displays the two time series used in the computation. Each graph is divided in half horizontally. The upper half of the graph shows the series resulting

from the seasonal adjustment process while the lower half shows the original generated series. Nine years (or 108 observations) are shown on each graph. Thus the second graph is a continuation of the first for observations 109 through 217 of each series. The third graph completes the series of 300 observations. The lower row of graphs gives the spectra of the two series and the cross-spectrum between them. The first graph in this row gives the spectra of the two series. The spectrum of the series from the seasonal adjustment process is labelled Y while the other series is labelled X. In some cases, as in Figure 6.01.1, these two spectra lie on top of each other and are not distinguishable. If, in addition, the two series were not significantly different at any frequency in terms of the cross-spectrum, as for Figure 6.01.1, then the graphs of the cross-spectrum are not shown. The deletion of the two graphs following the graph of the spectra implies that the coherence between the two series was not significantly different from unity at any frequency. In the cases where the two series were significantly different the graph following the graph of the spectra gives the coherence between the two series. The last graph shows the transfer function of the two series where the original series is taken as the input and the series from the seasonal adjustment process as the output. The upper half of the graph gives the gain while the lower half displays the phase.

The following index lists each of the figures and gives the two series used in each case as well as the type of seasonal and the name of the seasonal adjustment method.

Index to Figures

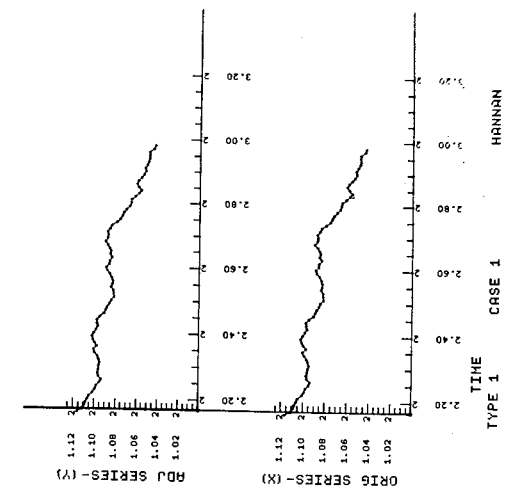
Figure No.	Series	Seasonal	Method
6.01.1	Adjusted and autoregressive	Type 1 (error term =0)	Hannan's method
6.01.2	Adjusted and sum of autoregressive and seasonal	Type 1	Hannan's method
6.01.3	Seasonal and estimated seasonal	Type 1	Hannan's method
6.02.1	Adjusted and autoregressive series	Type 1	Wald's method
6.02.2	Seasonal and estimated seasonal	Type 1	Wald's method
6.03.1	Adjusted and autoregressive	Type 2	Wald's method
6.03.2	Seasonal and estimated seasonal	Type 2	Wald's method
6.04.1	Adjusted and autoregressive	Type 3	Wald's method
6.04.2	Seasonal and estimated seasonal	Type 3	Wald's method
6.05.1	Adjusted and autoregressive	Type 1	Census method
6.05.2	Seasonal and estimated seasonal	Type 1	Census method
6.06.1	Adjusted and autoregressive	Type 2	Census method
6.06.2	Seasonal and estimated seasonal	Type 2	Census method
6.07.1	Adjusted and autoregressive	Type 3	Census method
6.07.2	Seasonal and estimated seasonal	Type 3	Census method

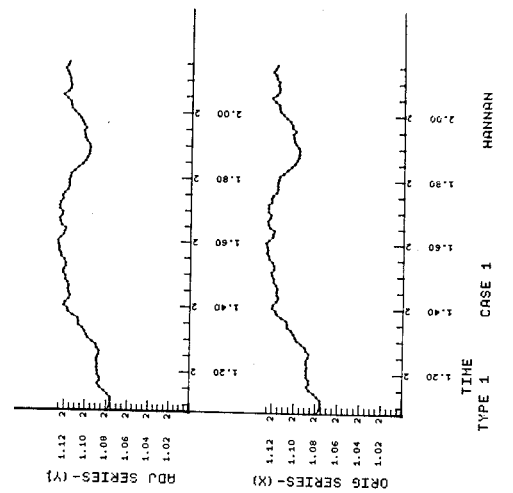
Figure No.	Series	Seasonal	Method
6.08.1	Adjusted and autoregressive	Type 1	Rational function method
6.08.2	Seasonal and estimated seasonal	Type 1	Rational function method
6.09.1	Adjusted and autoregressive	Type 2	Rational function method
6.09.2	Seasonal and estimated seasonal	Type 2	Rational function method
6.10.1	Adjusted and autoregressive	Type 3	Rational function method
6.10.2	Seasonal and estimated seasonal	Type 3	Rational function method

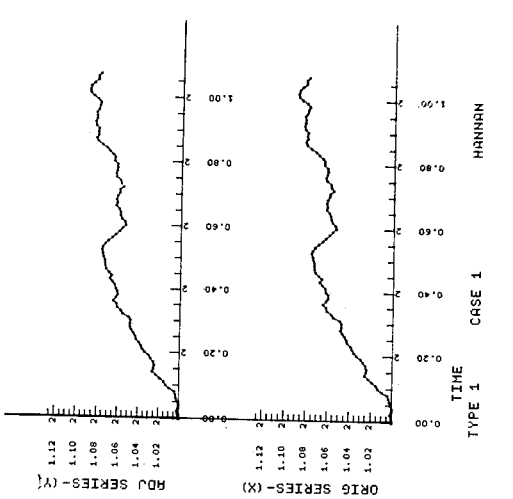
6.1 Hannan's Method

As was remarked above, Hannan's method provides the "best" seasonal adjustment of a series which contains a constant seasonal. The seasonally adjusted series is nearly identical to the original autoregressive series as is clearly indicated in Figure 6.01. When the seasonal changes, Hannan's method simply takes an arithmetic average of each month's values and estimates this constant average seasonal. Thus Hannan's method continues to perform satisfactorily only as long as the variations in the seasonal can be adequately represented by their time averages.

In this context it is important to mention the effect of the Hannan method on a non-stationary seasonal. To take the simplest case of a linear trend in amplitude, it is clear that the estimated seasonal has a constant


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 1 HANNAN


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 1 HANNAN


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 1 HANNAN

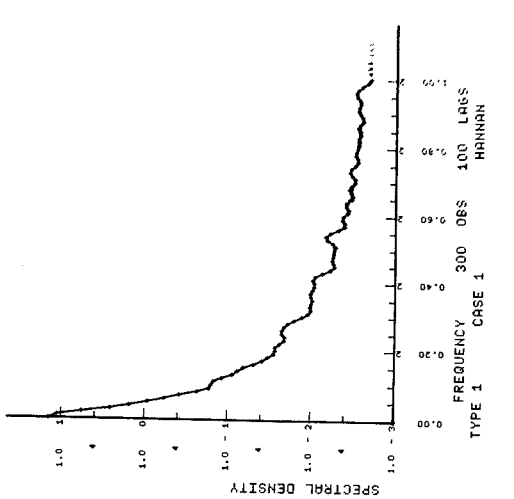
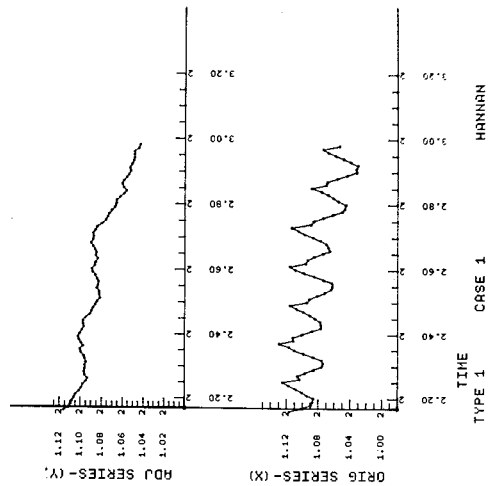
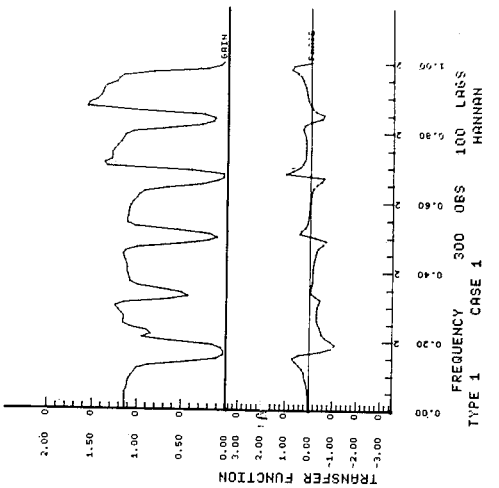
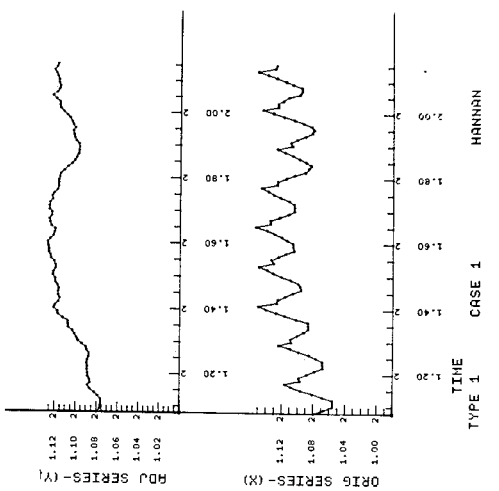
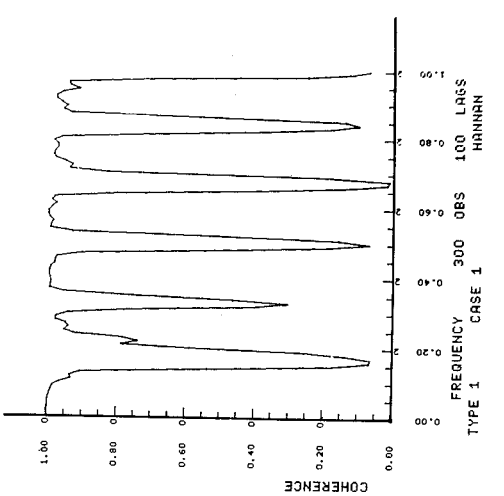

SPECTRAL DENSITY
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE 1 HANNAN

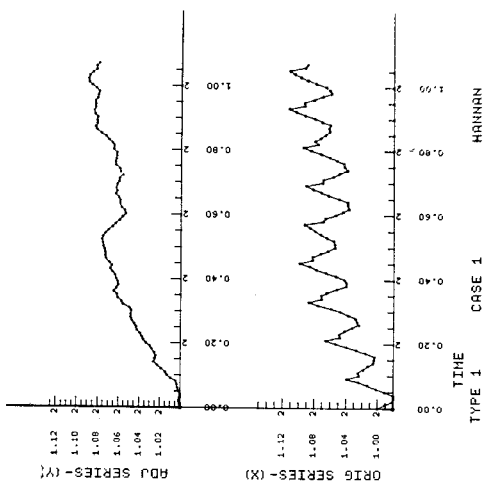
FIGURE 6.01.1


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 1 HANNAN


TRANSFER FUNCTION
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE 1 HANNAN


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 1 HANNAN


COHERENCE
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE 1 HANNAN


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 1 HANNAN

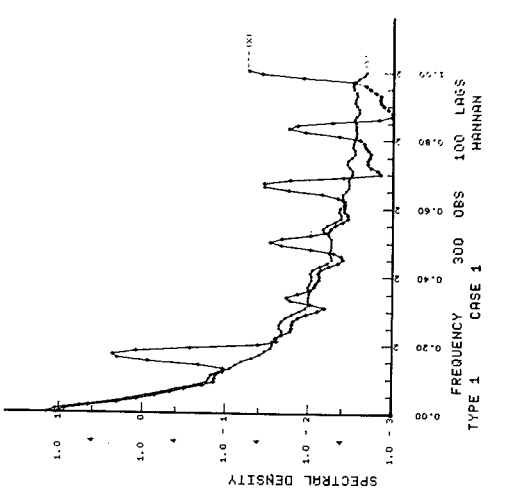
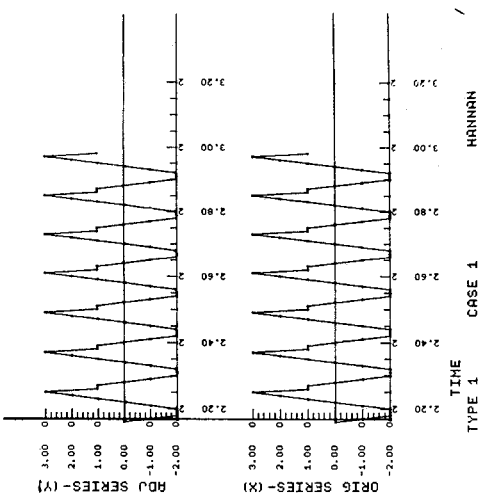
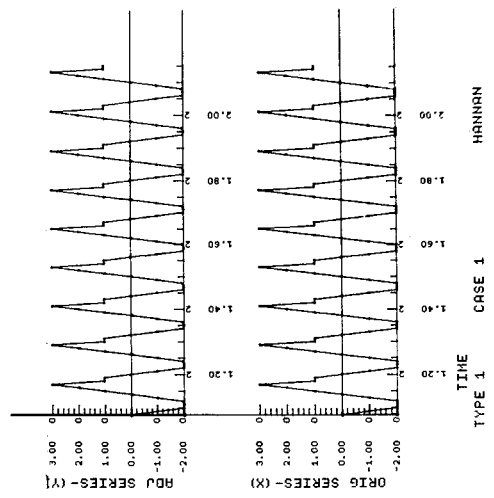
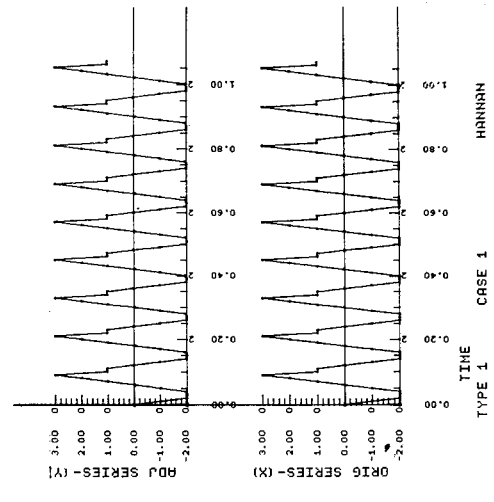

SPECTRAL DENSITY
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE 1 HANNAN

FIGURE 6.01.2


ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE 1 HANNAN


ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE 1 HANNAN


ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE 1 HANNAN

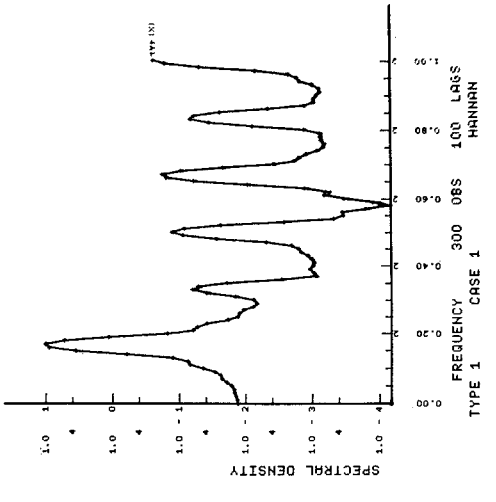

SPECTRAL DENSITY
FREQUENCY
TYPE 1 CASE 1 300 OBS 100 LAGS HANNAN

FIGURE 6.01.3

amplitude equal to the average amplitude of the actual seasonal. Thus, for the case of a linear trend, the error of the estimate reaches a maximum at both ends of the series and a minimum at the mid-point. In this case the performance indicated by cross-spectrum analysis may also be somewhat misleading due to the time averaging property of the spectrum estimates. The cross-spectrum shows the average error of the method over the sample. If, however, one is not only interested in the average error but also in the expected error of the estimate for the last observation (or final set of 12 observations) then the cross-spectrum alone does not give a complete measure of performance. As was mentioned previously this problem, and others relating to non-stationarity, are problems where the concept of the pseudo-spectrum may be applied. It has not been possible in the present paper to pursue this analysis in detail.

The practical limitations of Hannan's method are obvious. First, it is not often felt to be the case that the seasonal can adequately be represented by a set of constant monthly coefficients. Second is the fact that the use of stationarity leads in this case to the result that if the method is sequentially applied from year to year to a set of data to which each year 12 new observations are added the seasonally adjusted series will in general have different values for corresponding months not only in the current or recent years but from the beginning of the series. As has been frequently observed this leads to the necessity of continuing revisions.

Clearly, the major contribution of Hannan's method is not to be given in terms of its potential application to actual data but rather in its explicit treatment of effects of linear operators in terms of frequency decomposition. Given a constant seasonal, Hannan's technique of adjusting the seasonal

factors for any effects of the initial filtering operation leads to better estimates in terms of bias. This technique, obviously, is not restricted to Hannan's method. It is applicable to any method which employs a linear operator before estimation of the seasonal. In fact this adjustment technique, as mentioned previously, was experimentally applied to Wald's method and was used in the rational transfer function method. For a recent development by Hannan, which arrived too late for analysis in the present paper, see [11].

6.2 Wald's Method

Wald's method, since it employs certain nonlinear operators, affects the linear information in the series even when the actual seasonal is constant. However, these effects are typically quite small and are evident only at relatively high frequencies.

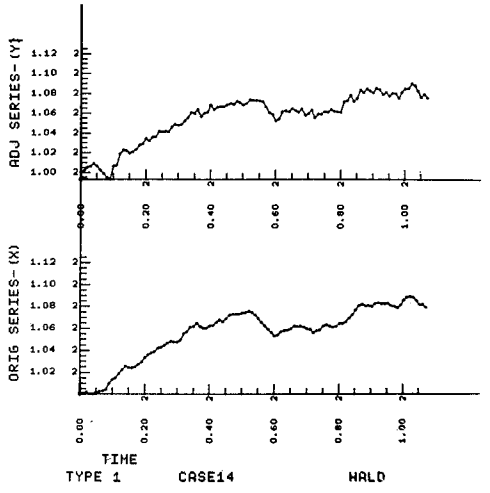
At the low frequency end of the spectrum Wald's method produces, for a constant seasonal, an estimate of the autoregressive series very nearly as good as Hannan's method. As is well known, the point at which Wald's method breaks down is when the pattern of the seasonal is allowed to change. Wald's method estimates a constant average pattern (as does Hannan's method) and uses this constant pattern to produce a "best" estimate of the change in amplitude of the seasonal for each year. This approach provides relatively very good estimates of the change in amplitude (particularly for rapid changes in amplitude) of the seasonal when the assumption of the constant pattern is met. In comparing Wald's method with a method which does not rely on the relationship of the seasonal in one month to the seasonal in each other month it is clear that (again assuming the constant pattern) the use of the information contained in all twelve observations for a given year will lead to a better estimate than

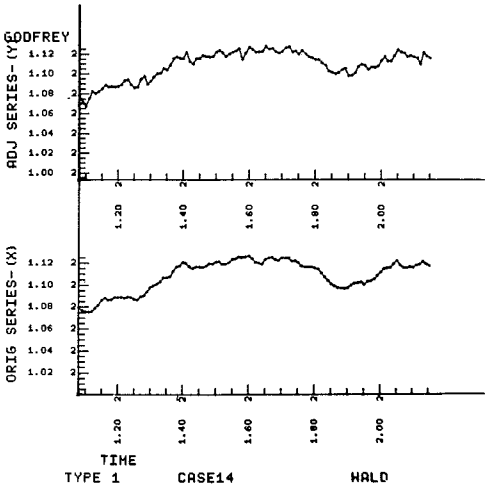
the use of only the series of annual observations for each month independently. Figure 6.02 shows the results of the application of Wald's method to a series containing a seasonal of constant pattern but varying amplitude. Figures 6.03 and 6.04 show the results of the use of Wald's methods when both the amplitude and pattern are allowed to change.

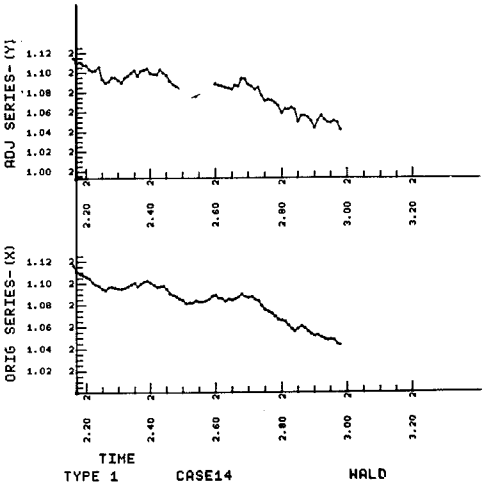
6.3 The Census Method

The Census method is more general than either of the two previously discussed methods in that it does not make use of the assumption of either a constant seasonal amplitude nor a constant seasonal pattern.

However, there are two respects in which the Census Method appears to be comparatively inferior to the two methods above. First no use is made of Hannan's technique of correcting for the bias introduced by the application of a linear operator to the original data. Since the Census Method uses the Spencer 15-point formula for this operation, the 12-month component is attenuated to 20% of its original value. This bias is not clearly evident in the results because Spencer's formula is applied only after most of the seasonal has been removed using a simple moving average. The use of Hannan's correction procedure eliminates the need for a two (or more) stage process as used in the Census Method. The second weakness of the Census Method is its poor response, when compared to Wald's method, to relatively high frequency variations of the seasonal amplitude. Part of the reason for this relatively poor frequency response characteristic is the large number of observations for a single month which are required in order to form a stable estimate (in the sense that the estimate contains very little spectral power at high frequencies) of each monthly factor.


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE14 HALD


GODFREY
ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE14 HALD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ORIG SERIES-(X)
TIME
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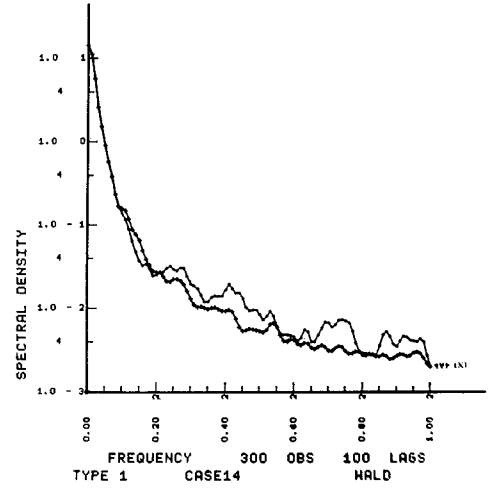
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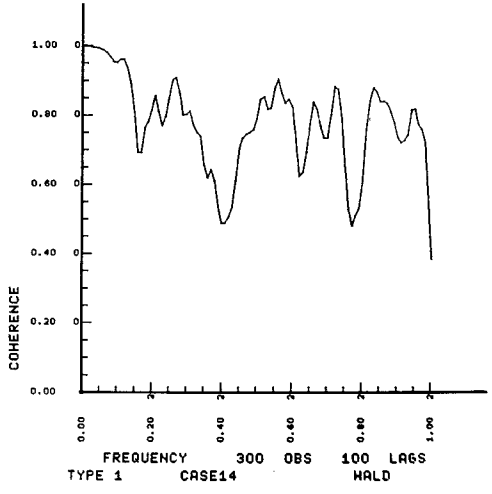
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE14 HALD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FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE14 HALD

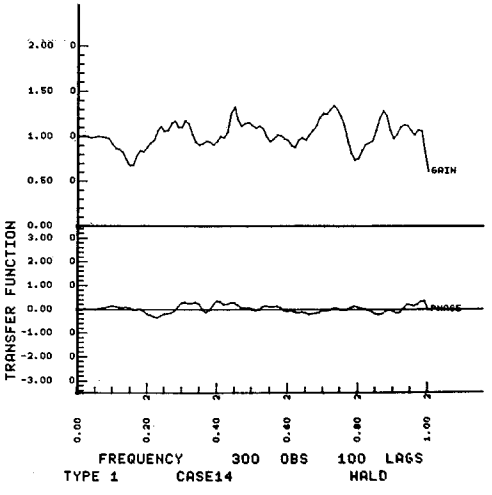

TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE14 HALD

FIGURE 6.02.1

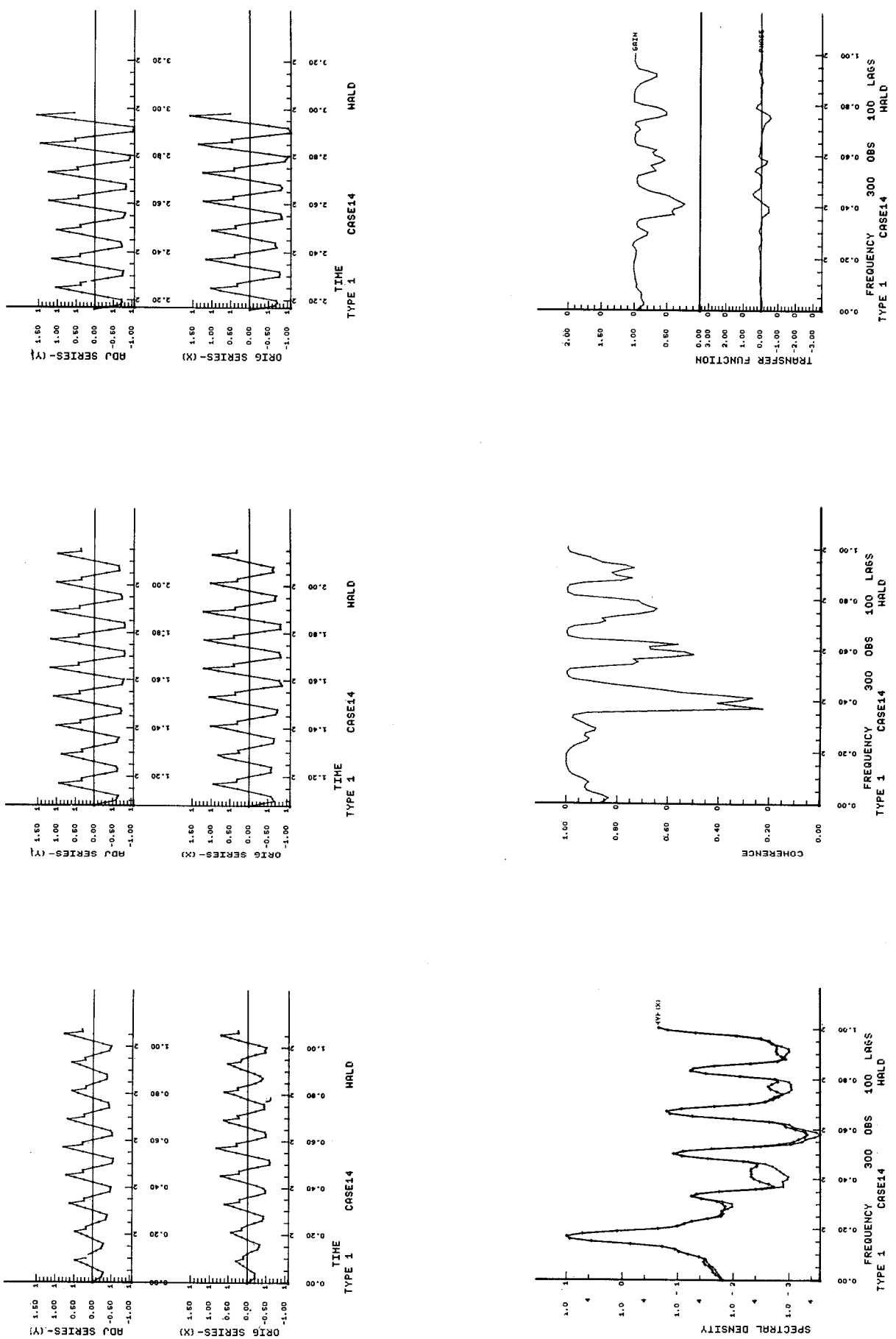
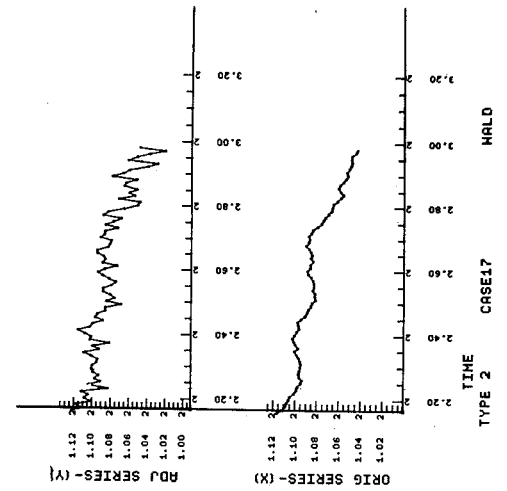
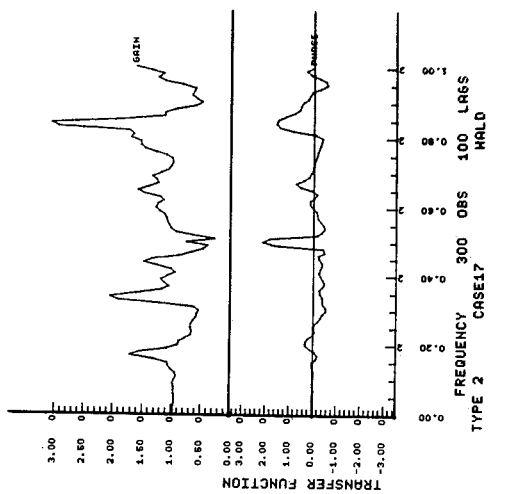
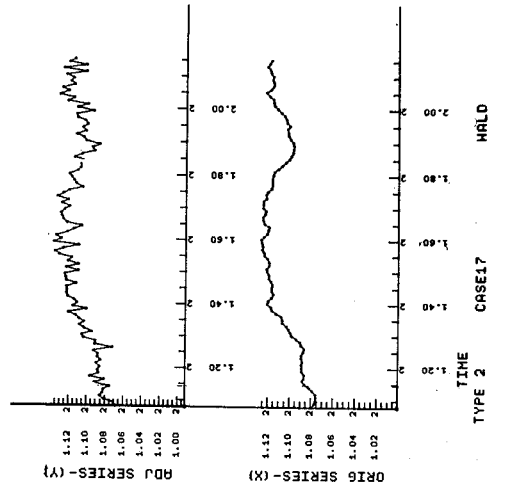

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE14 HALD
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE14 HALD
COHERENCE
SPECTRAL DENSITY

FIGURE 6.02.2


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 2 CASE17 HALD


TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 2 CASE17 300 OBS 100 LAGS HALD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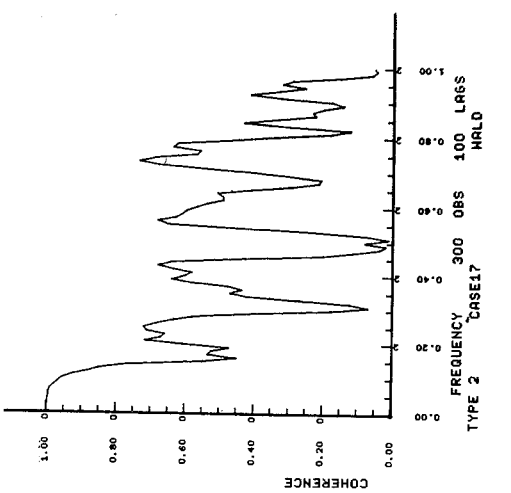
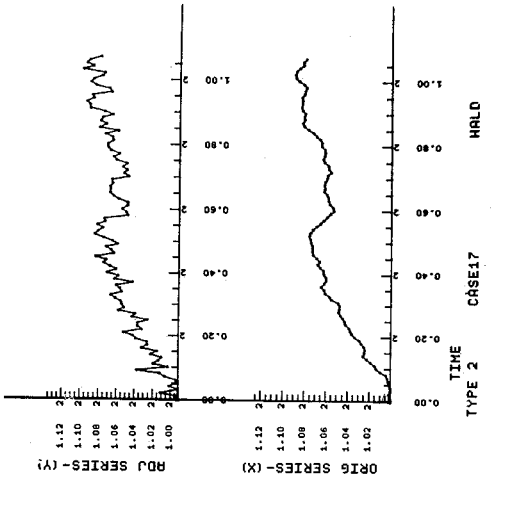
ORIG SERIES-(X)
TIME
TYPE 2 CASE17 HALD


COHERENCE
FREQUENCY
TYPE 2 CASE17 300 OBS 100 LAGS HALD


ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 2 CASE17 HALD

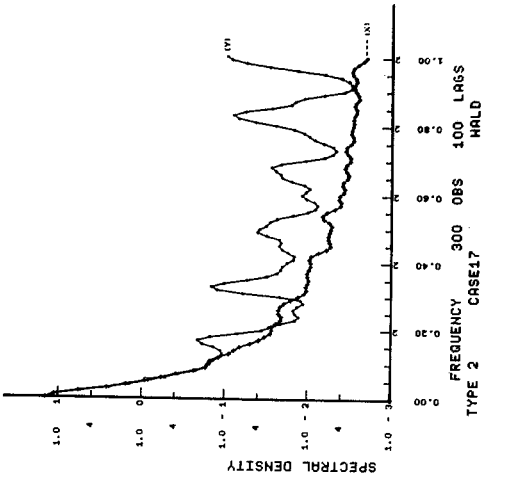

SPECTRAL DENSITY
FREQUENCY
TYPE 2 CASE17 300 OBS 100 LAGS HALD

FIGURE 6.03.1

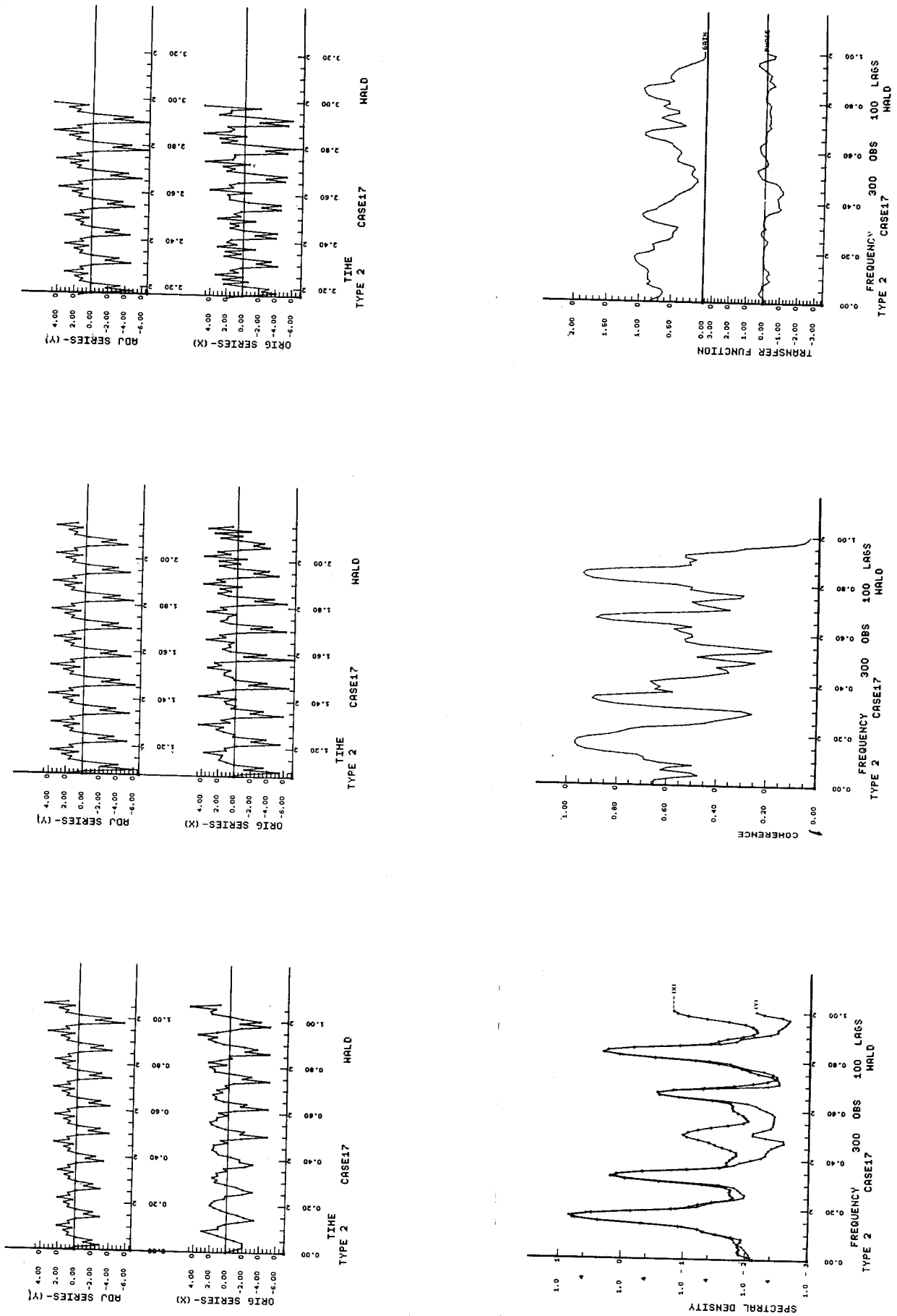

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 2 CASE17 HALD
TRANSFER FUNCTION
FREQUENCY 300 OBS 100 LAGS
COHERENCE
SPECTRAL DENSITY

FIGURE 6.03.2

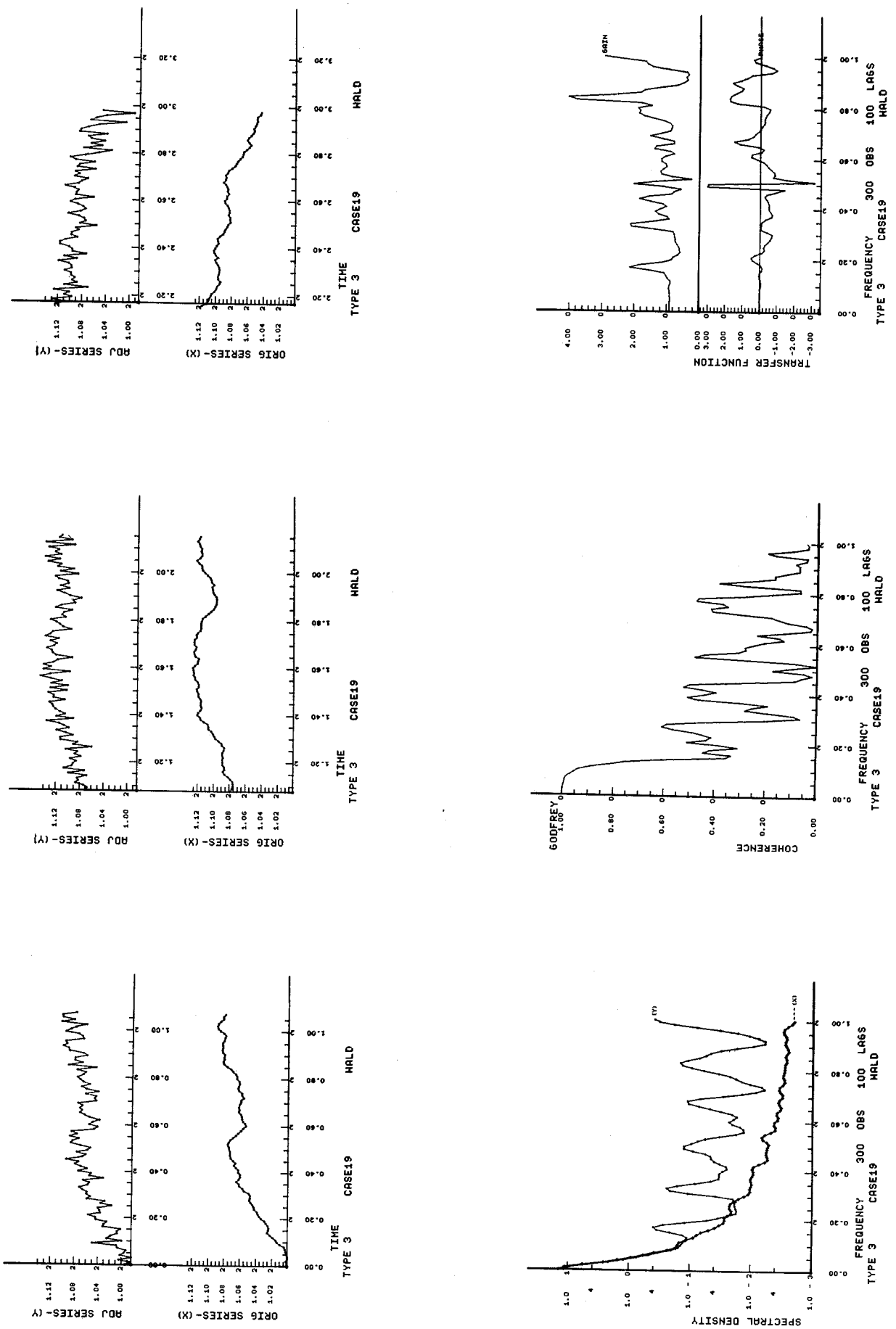

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 3 CASE19 HALD
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 3 CASE19 300 OBS 100 LAGS HALD
COHERENCE
GODFREY
SPECTRAL DENSITY

FIGURE 6.04.1

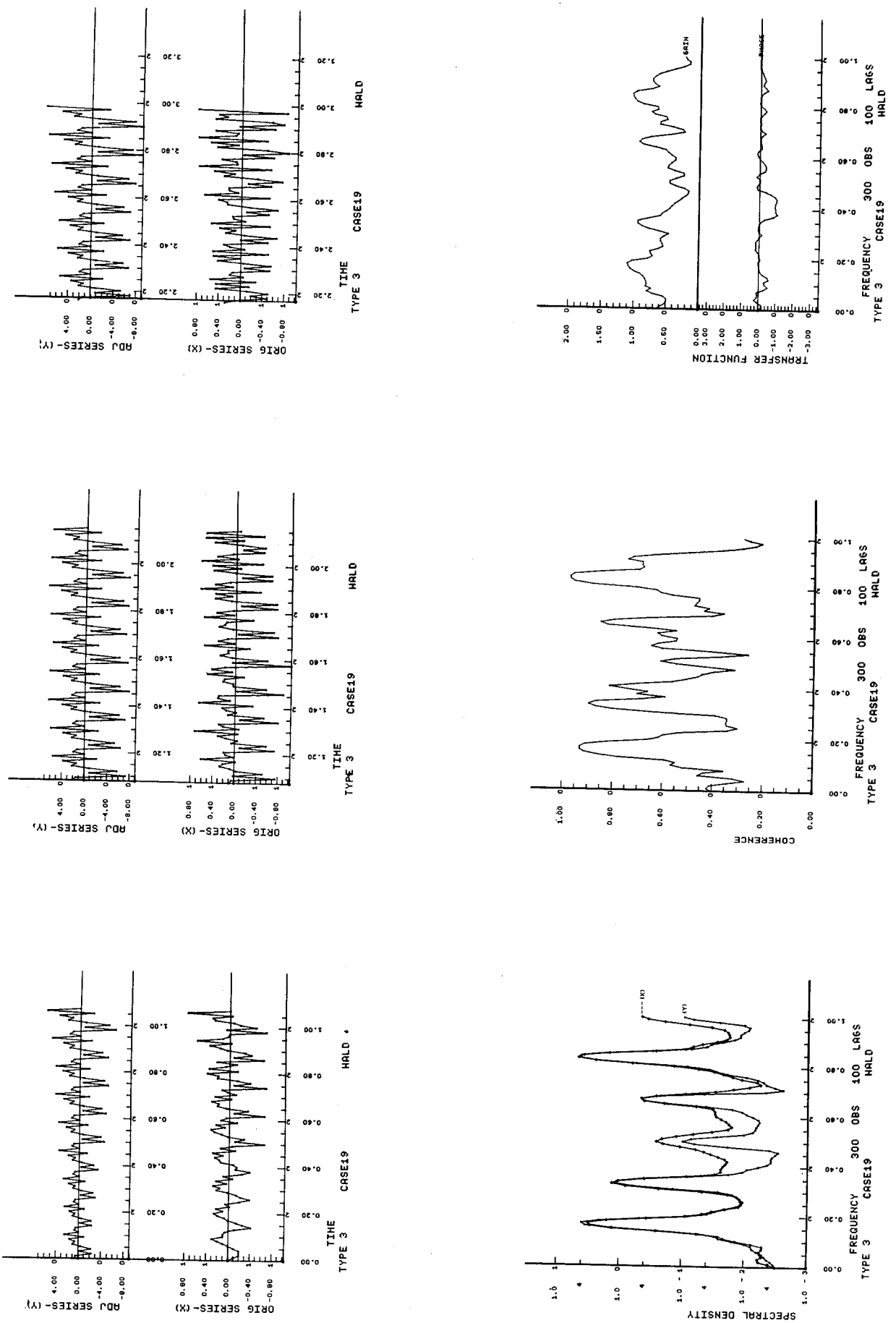

ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 3 CASE19 HALD
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 3 CASE19 300 OBS 100 LAGS HALD
COHERENCE
SPECTRAL DENSITY
(X)
(Y)

FIGURE 6.04.2

The observation of overriding importance which must be made about the Census Method is, however, that it performs reasonably well under a wide range of conditions. Only when the seasonal amplitude changes very rapidly is the seasonal not completely removed at least in the sense of removal of the peaks in the estimated spectrum. However, the method does allow sharp changes in the seasonal to appear in the adjusted series. In addition, under certain conditions, the method alters the series at frequencies other than the seasonal. Thus the coherence between the adjusted and the autoregressive series is sometimes quite low at low (but non-seasonal) frequencies. The gain is also sometimes considerably different from 1 at low frequencies. However, the phase is uniformly close to zero even under extreme conditions. The fact that the phase of the autoregressive series is not significantly altered by the seasonal adjustment method is of greatest importance for the interpretation and use of the adjusted series. Since the series may be used by the government for stabilization policy measures, it may be important to the stability of the system that a phase lag not be introduced. Figure 6.05 shows the performance of the Census Method for a constant seasonal pattern, while Figures 6.06 and 6.07 show the result of the same method when both the amplitude and pattern of the seasonal change considerably.

6.4 The Rational-Function Method

This method produces results that are quite naturally similar to the results obtained by Wald's method. While the method is in many ways similar to Wald's method, it represents an inovation in the use of a rational-function

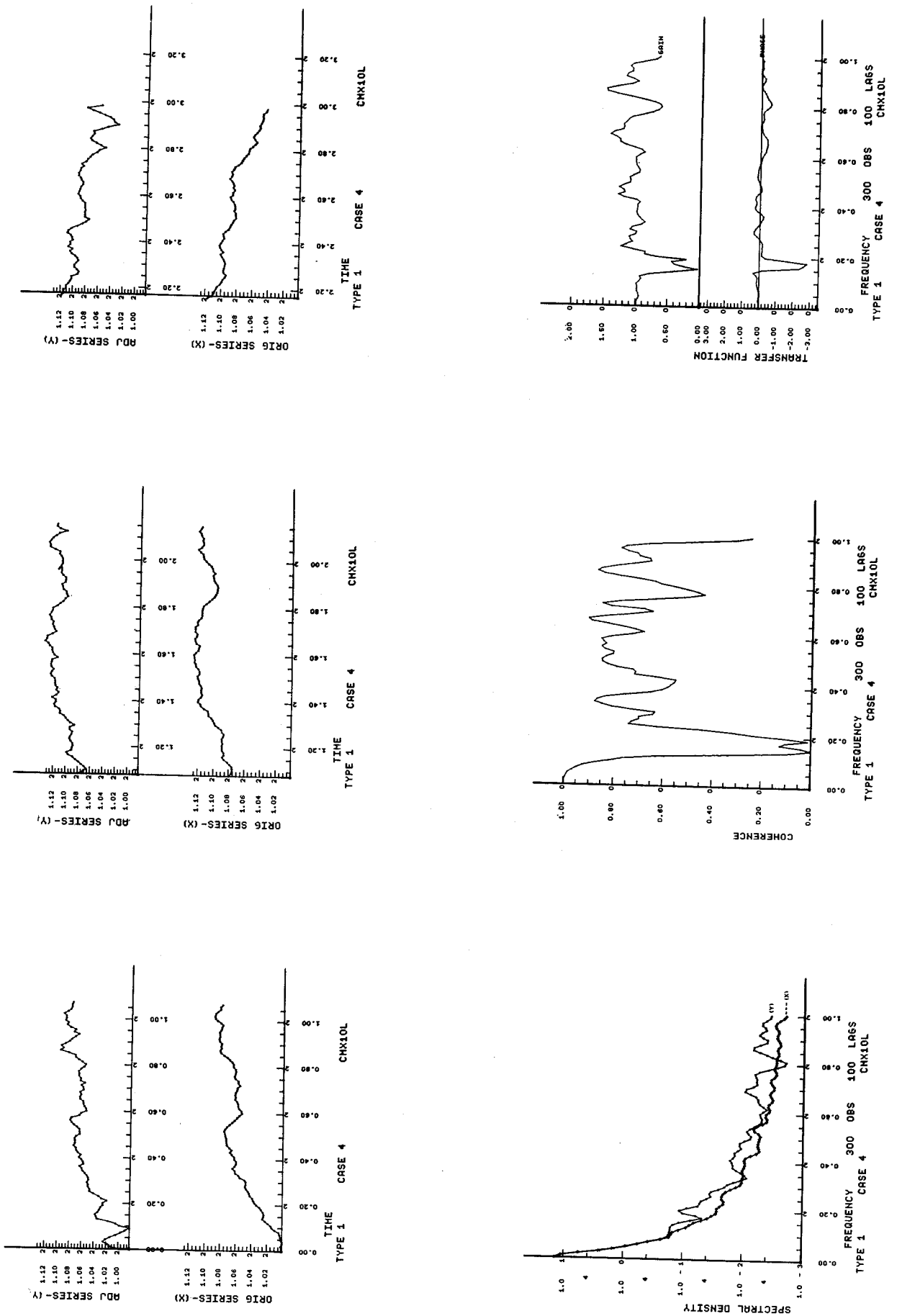

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE 4 CHX10L
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY 300 OBS 100 LAGS
TYPE 1 CASE 4 CHX10L
COHERENCE
SPECTRAL DENSITY

FIGURE 6.05.1

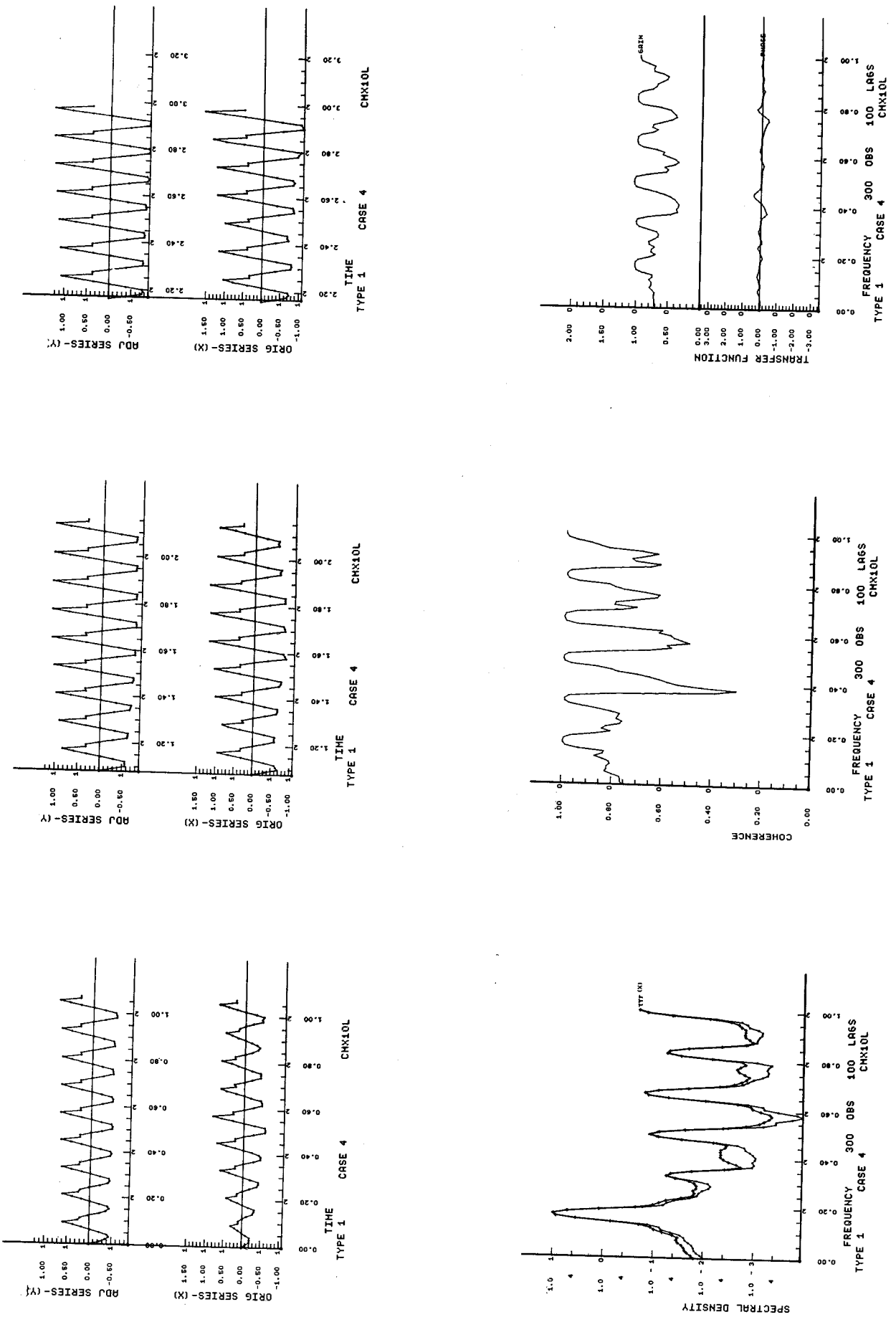

ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 1 CASE 4 CMX10L
TRANSFER FUNCTION
FREQUENCY
TYPE 1 CASE 4 300 OBS 100 LAGS CMX10L
COHERENCE
SPECTRAL DENSITY

FIGURE 6.05.2

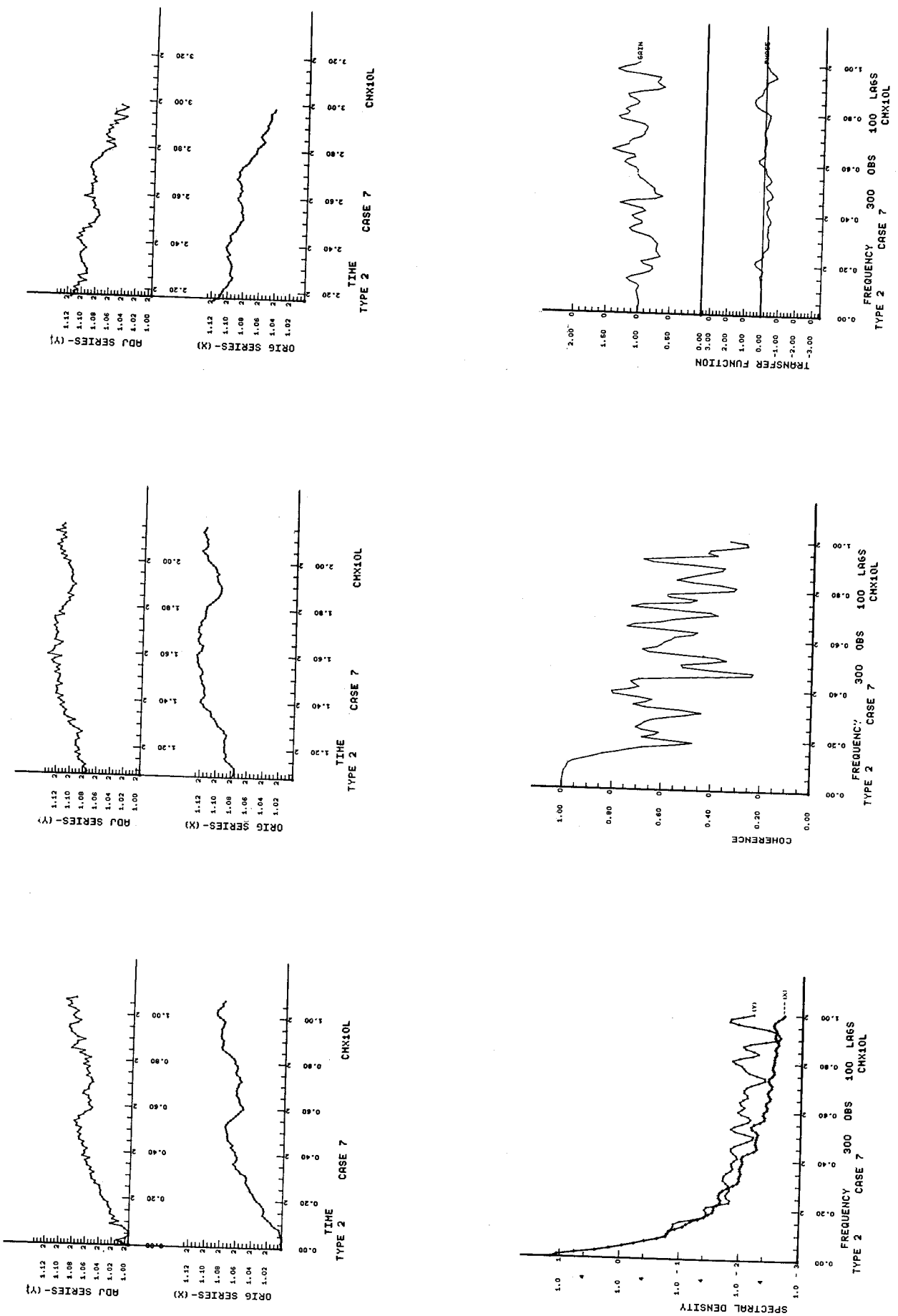

ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 2 CASE 7 CHX10L
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 2 CASE 7 300 OBS 100 LAGS CHX10L
COHERENCE
SPECTRAL DENSITY

FIGURE 6.06.1

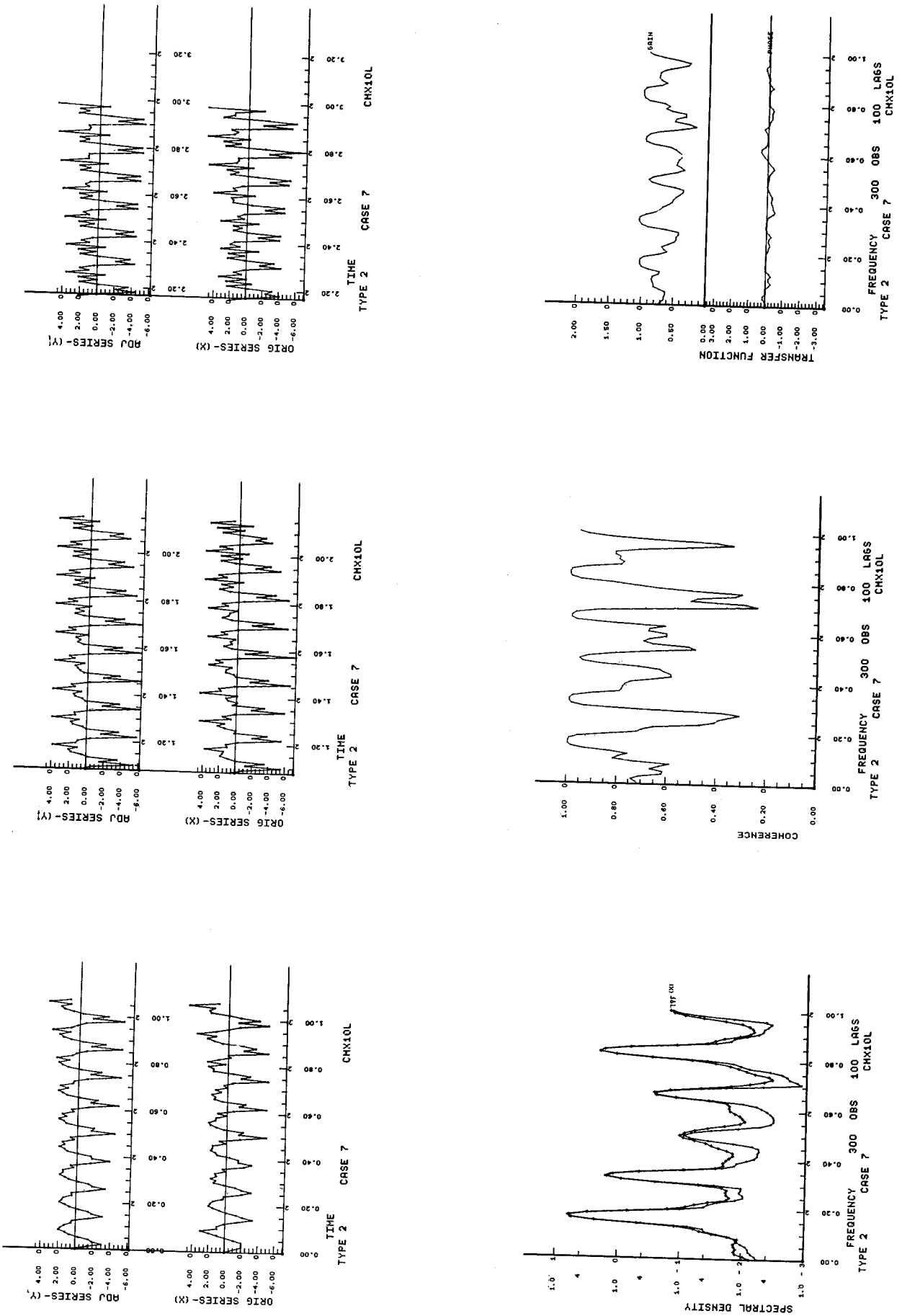

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 2 CASE 7 CMX10L
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 2 CASE 7 300 OBS 100 LAGS CMX10L
COHERENCE
SPECTRAL DENSITY

FIGURE 6.06.2

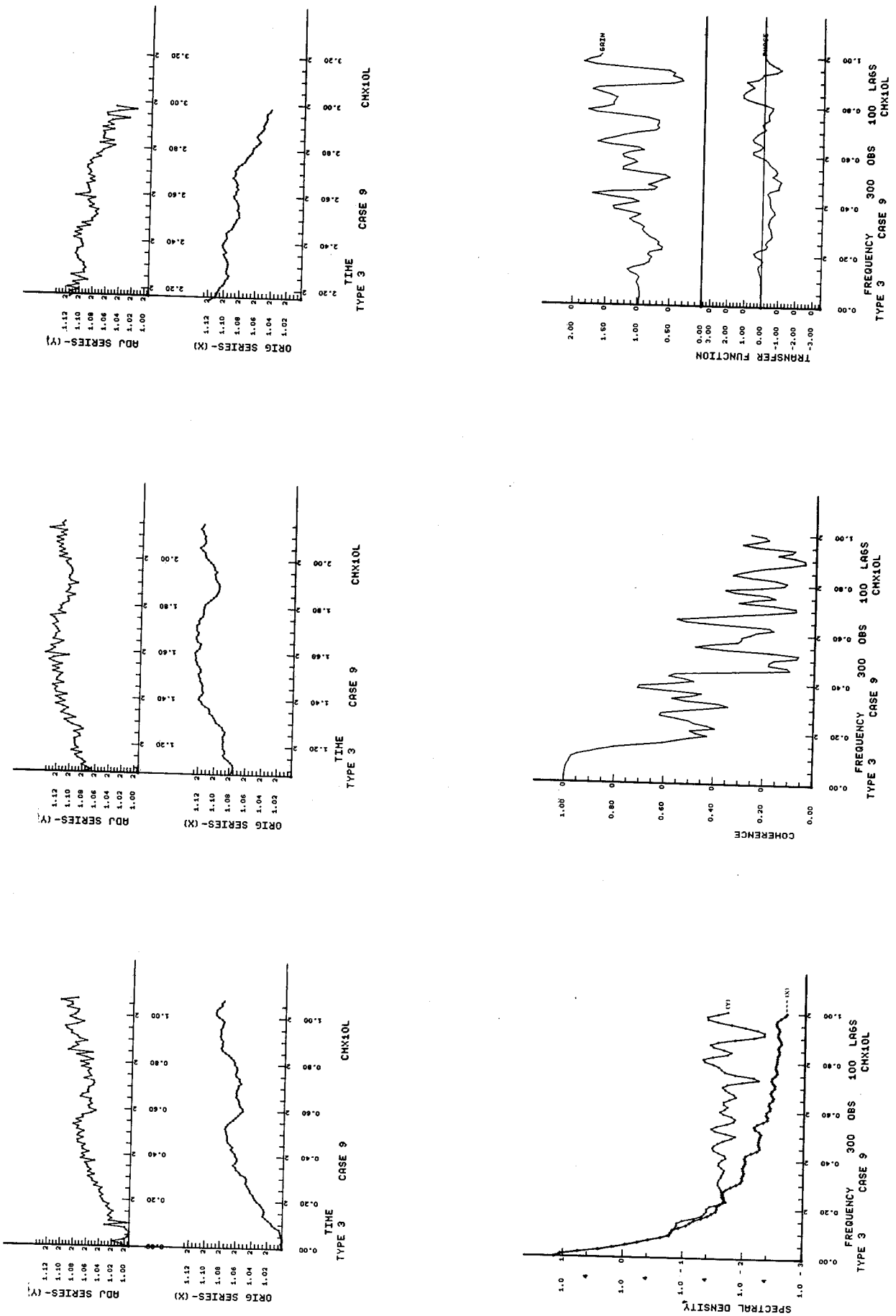

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 3 CASE 9 CHX10L
TRANSFER FUNCTION
FREQUENCY
TYPE 3 CASE 9 300 OBS 100 LAGS CHX10L
COHERENCE
SPECTRAL DENSITY

FIGURE 6.07.1

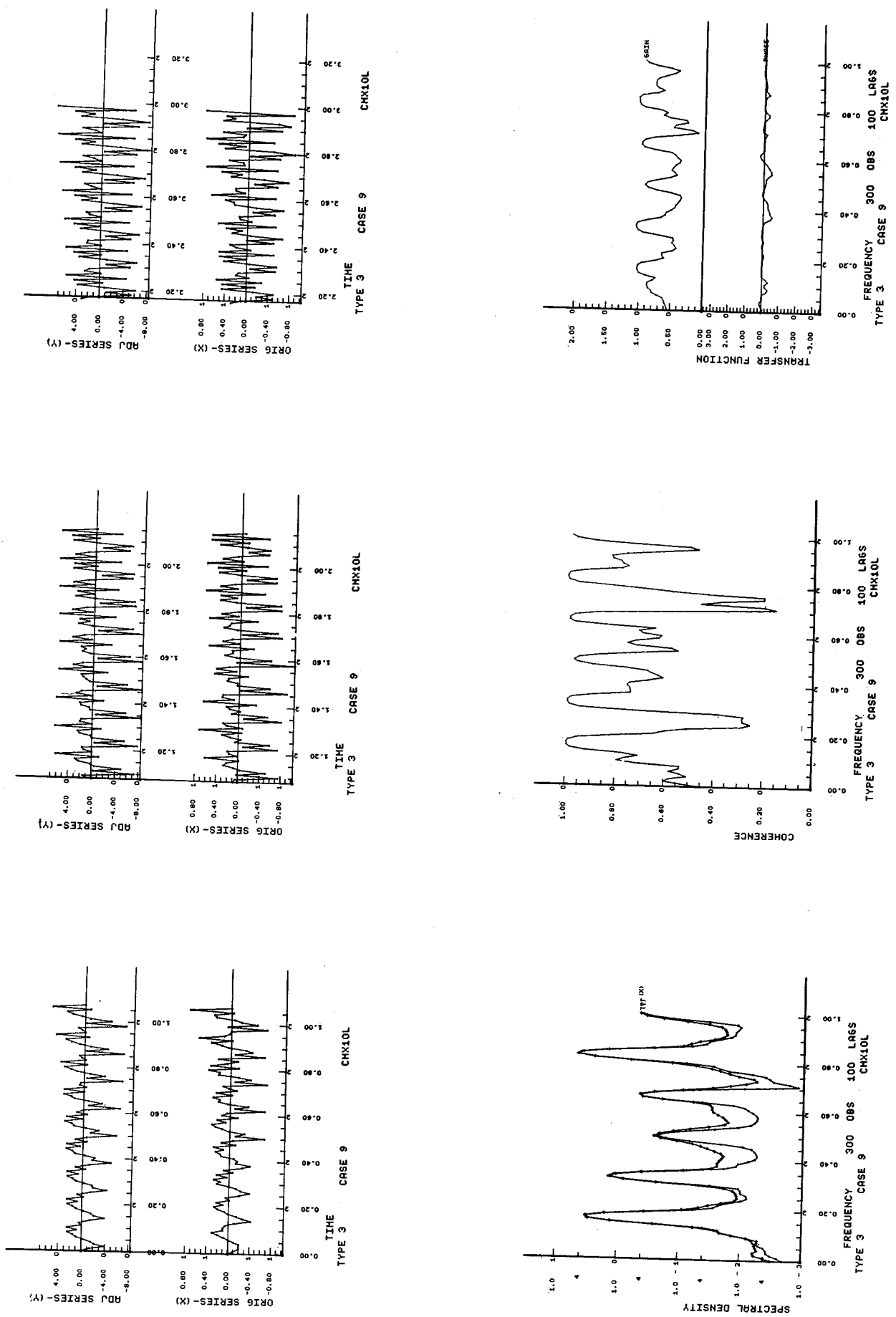
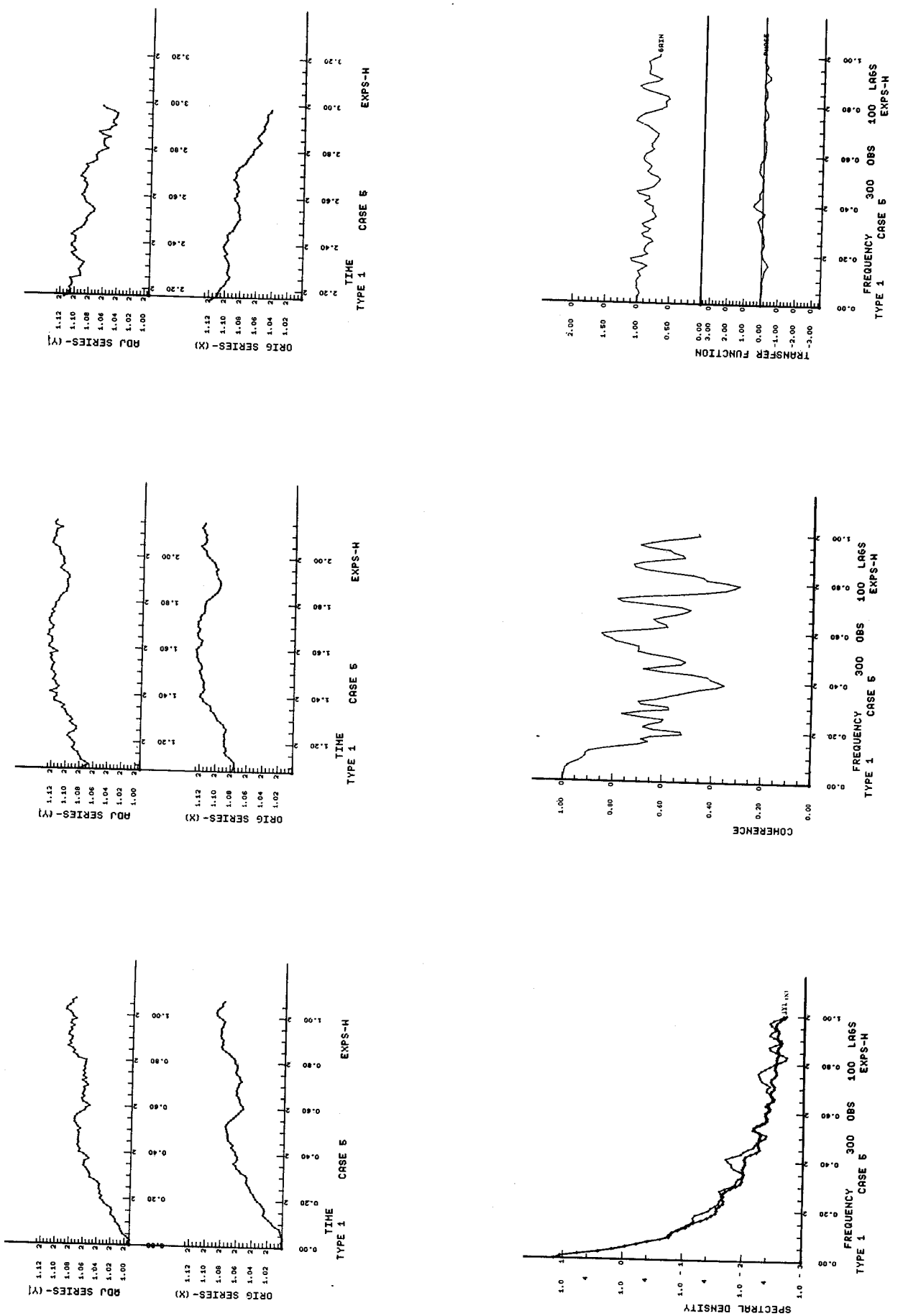

ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 3 CASE 9 CHX10L
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 3 CASE 9 300 OBS 100 LAGS CHX10L
COHERENCE
SPECTRAL DENSITY

FIGURE 6.07.2

filter for the estimation of the monthly seasonal factors. This removes the need for the use of a long moving average of the observations for each month and also reduces the requirement of extrapolation of the final values at this stage. The rational-function filter which has been used requires only observations which are coincident with or precede the estimate in time. Thus the estimates for the last twelve months in the series are produced exactly as the previous ones. Therefore no correction of the estimates is needed as more data become available, except the correction required by the initial filtering of the data. In addition only the last three years of data and the initial values for the filter estimates are required for the estimation of the adjusted series for the current year. After the adjustment method has been applied to a series then the final values of the rational-function filter estimates may be saved so that as new data become available it is only necessary to use these estimated values and the last three years of data for the new estimation. The only figures that will be revised in the entire computation are the last 12 months of each estimation. These are the estimates which are based upon extrapolations of the 12-month moving average which was applied to the original observations.

The use of Wald's least squares technique for estimation of the seasonal amplitude makes the performance of the rational function method comparable to Wald's method when the seasonal pattern is constant but the amplitude is undergoing rapid change. Figure 6.08 shows the performance of the method under this condition. Figures 6.09 and 6.10 show the performance of the method when both the seasonal pattern and amplitude are changing.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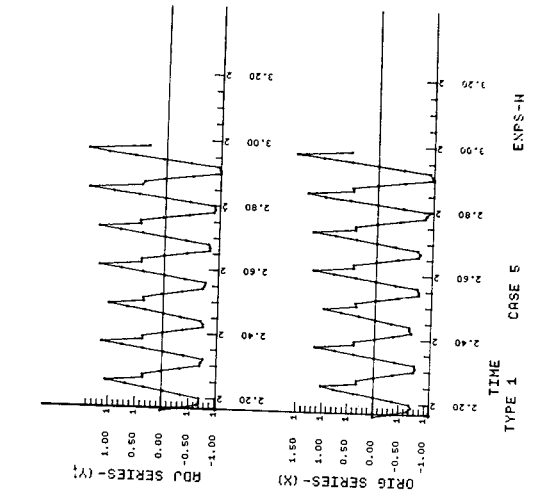
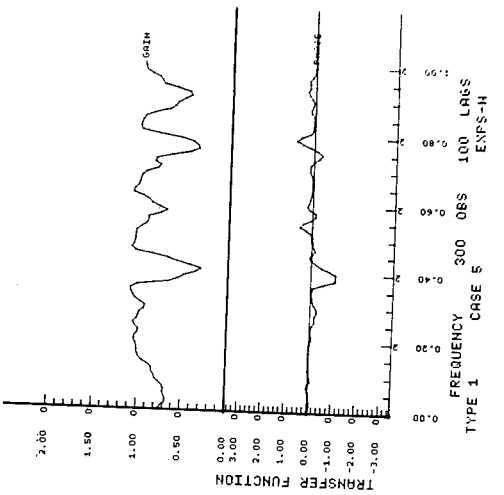
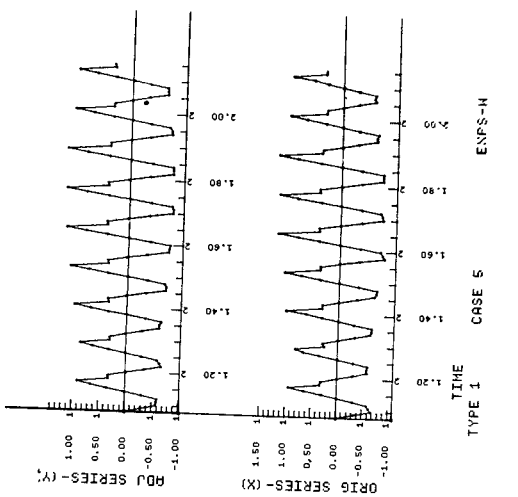
ORIG SERIES-(X)
TIME
TYPE 1 CASE 6 EXPS-H
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 1 CASE 6 300 OBS 100 LAGS EXPS-H
COHERENCE
SPECTRAL DENSITY

FIGURE 6.08.1


ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE 5 ENPS-H


TRANSFER FUNCTION
GAIN
FREQUENCY
TYPE 1 CASE 5 300 OBS 100 LAGS ENPS-H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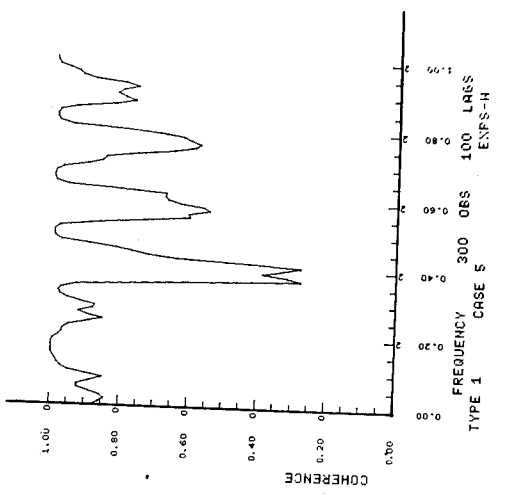
TIME
TYPE 1 CASE 5 ENPS-H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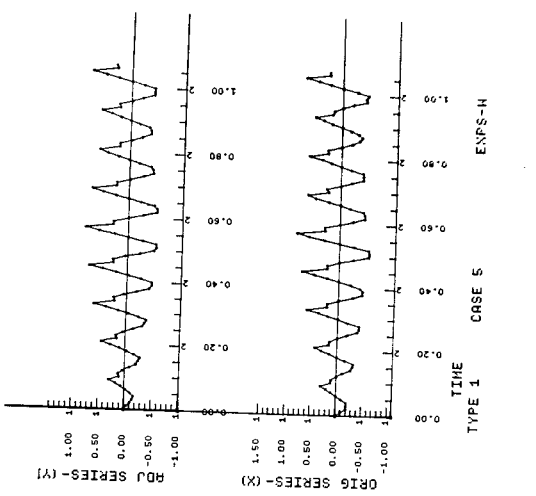
FREQUENCY
TYPE 1 CASE 5 300 OBS 100 LAGS ENPS-H


ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 1 CASE 5 ENPS-H

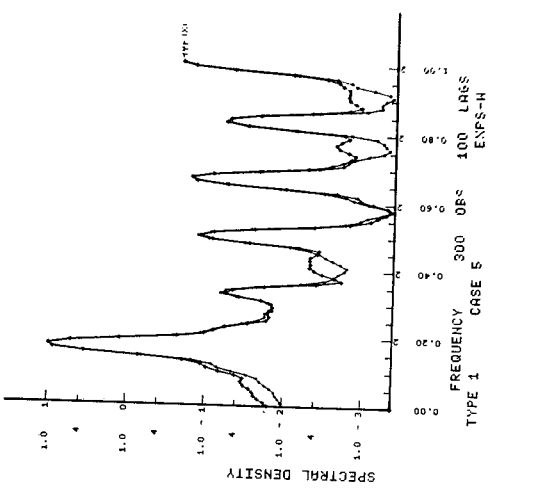

SPECTRAL DENSITY
FREQUENCY
TYPE 1 CASE 5 300 OBS 100 LAGS ENPS-H

FIGURE 6.08.2

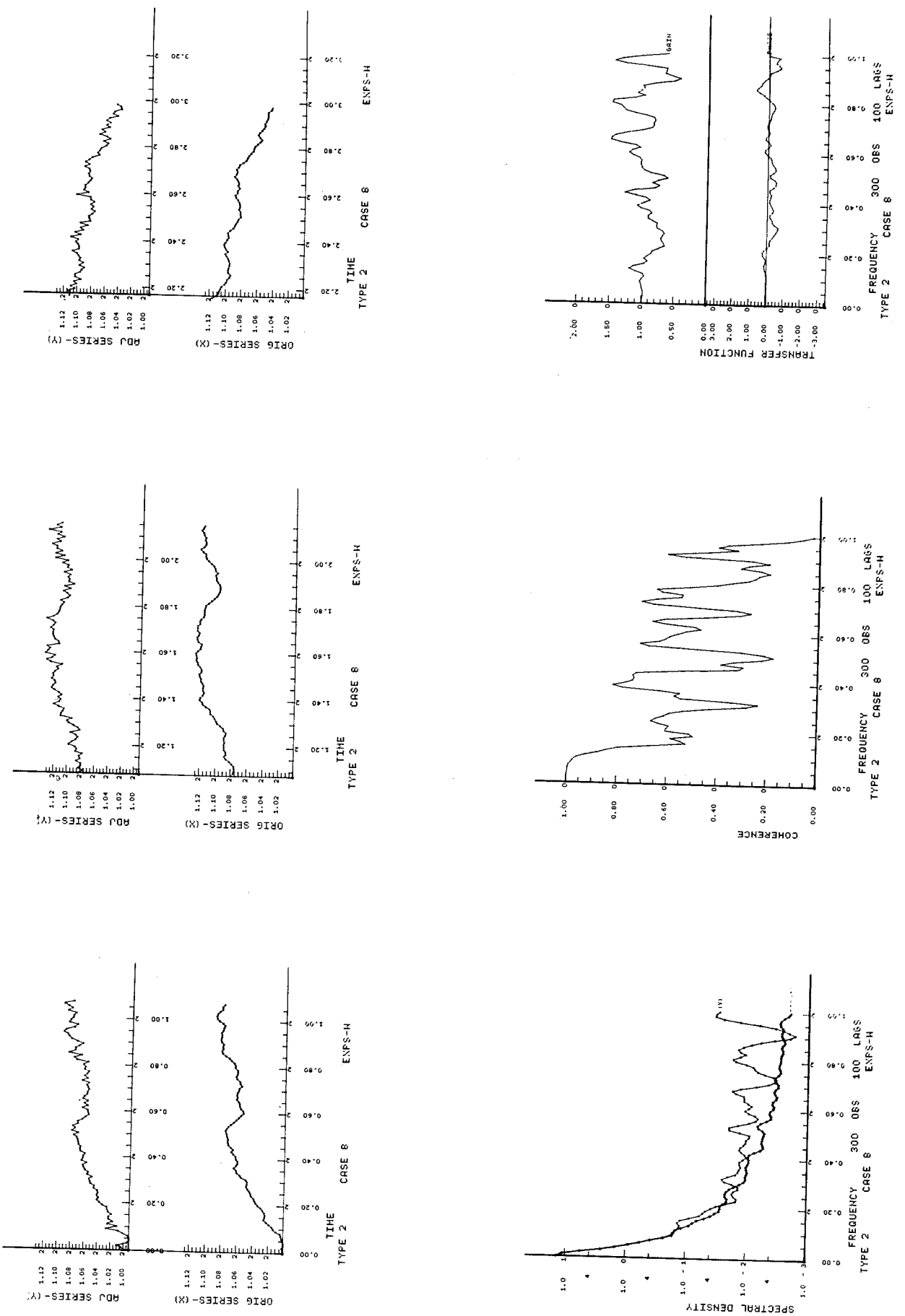
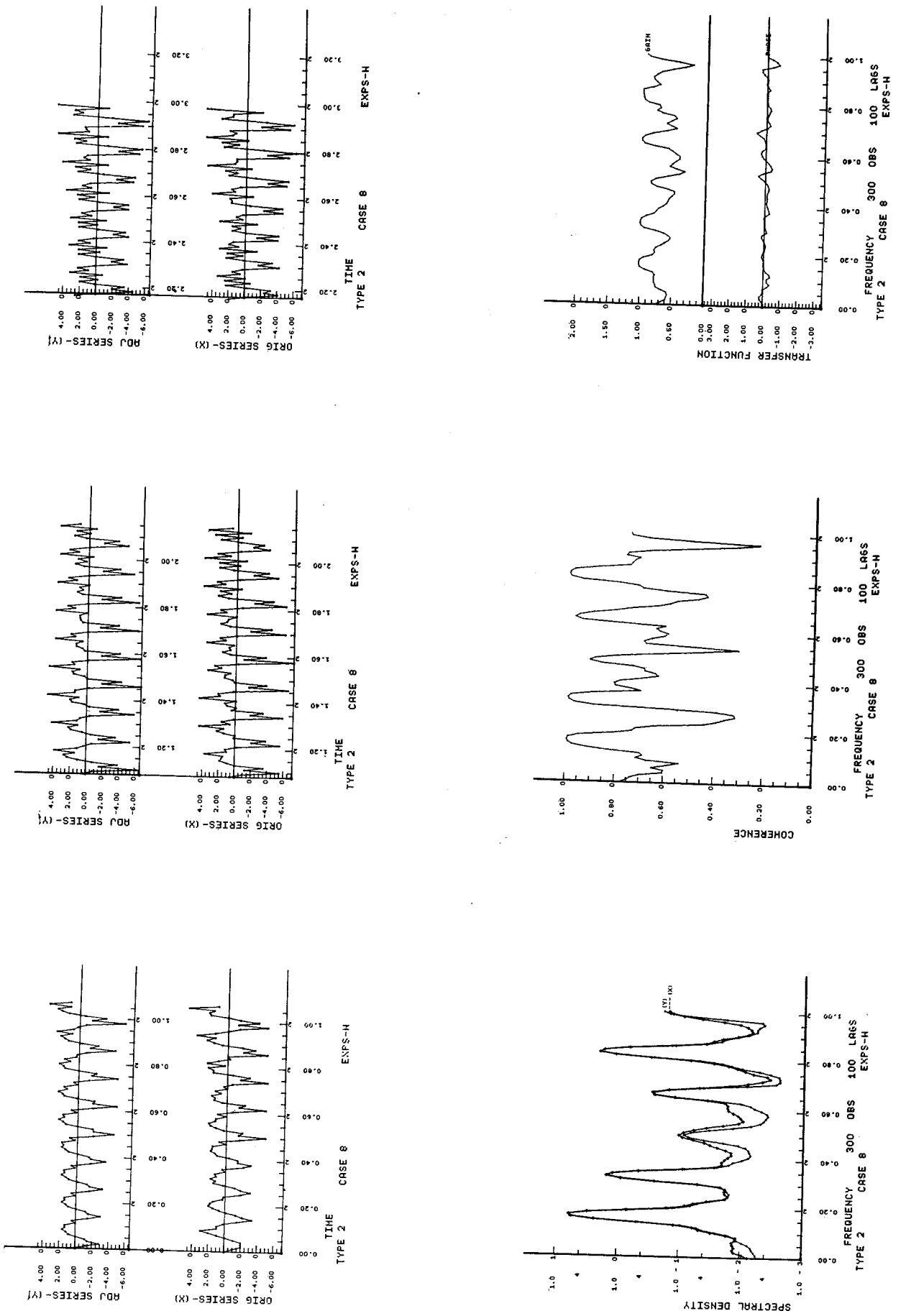

ORIG SERIES-(X)
ADJ SERIES-(Y)
TIME
TYPE 2 CASE 8 ENPS-H
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY
TYPE 2 CASE 8 300 OBS 100 LAGS ENPS-H
COHERENCE
SPECTRAL DENSITY

FIGURE 6.09.1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ORIG SERIES-(X)
TIME
TYPE 2 CASE 8 EXPS-H
TRANSFER FUNCTION
FREQUENCY
TYPE 2 CASE 8 300 OBS 100 LAGS EXPS-H
COHERENCE
SPECTRAL DENSITY

FIGURE 6.09.2

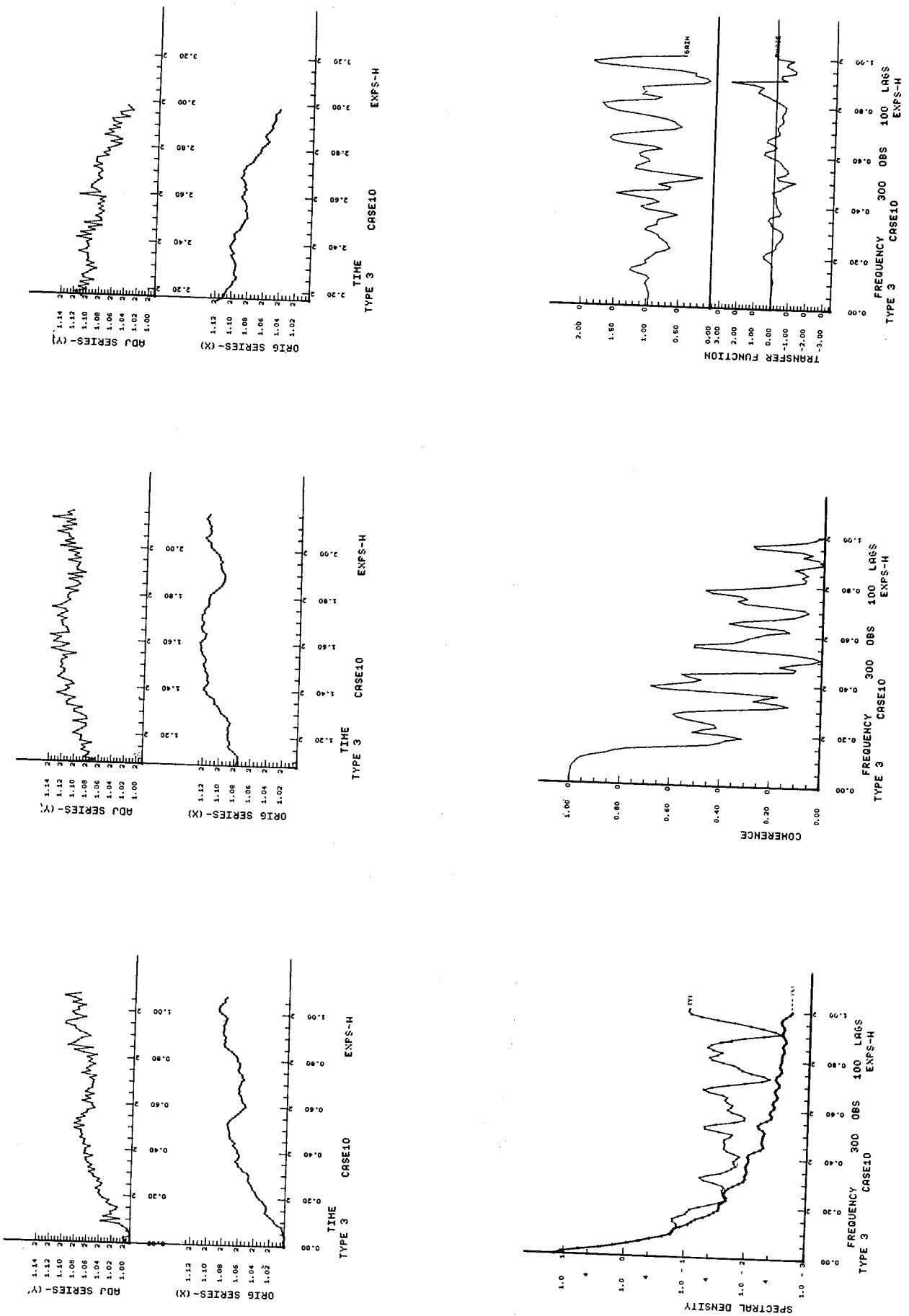
ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 3 CASE10 EXPS-H
TRANSFER FUNCTION
GAIN
PHASE
FREQUENCY 300 OBS 100 LAGS
TYPE 3 CASE10 EXPS-H
COHERENCE
SPECTRAL DENSITY

FIGURE 6.10.1

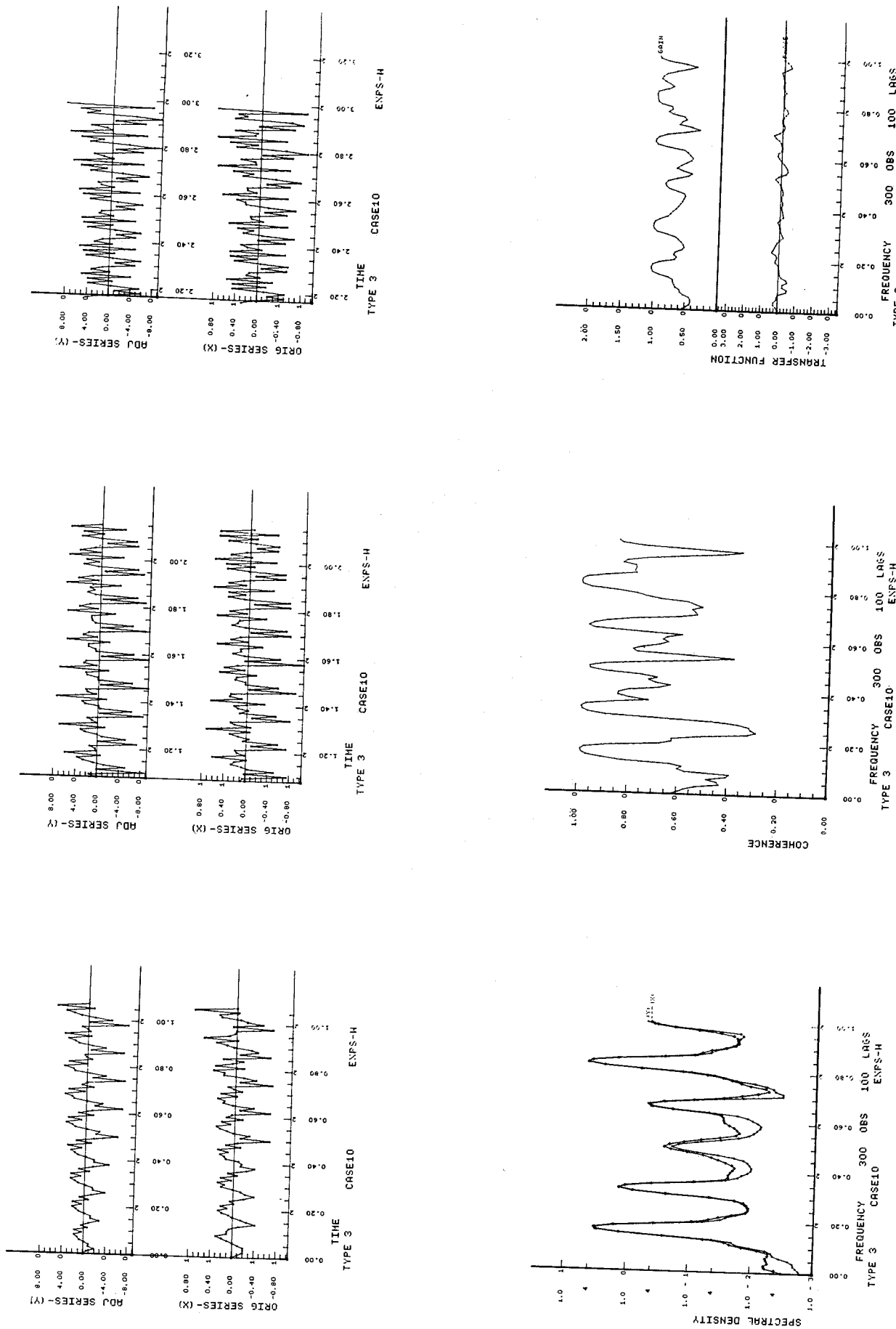
ADJ SERIES-(Y)
ORIG SERIES-(X)
TIME
TYPE 3 CASE10 EXPS-H
TRANSFER FUNCTION
GAIN
FREQUENCY 300 OBS 100 LAGS
TYPE 3 CASE10 EXPS-H
COHERENCE
SPECTRAL DENSITY

FIGURE 6.10.2

7.0 Conclusions

On the basis of the criteria put forward in the Introduction to this paper and the computations presented in Section 6.0 we have been able to compare the relative performances of various seasonal adjustment methods. The most important conclusions to be drawn from this analysis are as follows:

7.1 For the case of a constant strictly periodic seasonal it is difficult to see how it would be possible to improve upon Hannan's method. In addition, the use of transfer function analysis as applied in Hannan's method is of general applicability to seasonal adjustment methods which use moving averages.

7.2 Wald's method, which is unique in its use of the intercorrelation of the monthly seasonal factors, definitely displays the value of the use of this intercorrelation when the assumption of a constant seasonal pattern is met. When the amplitude (but not the pattern) of the seasonal varies relatively rapidly, Wald's method produces the best estimate of the underlying autoregressive series.

7.3 The value of the Census Method is that it fulfills the stated objectives reasonably well under all conditions. It continues to provide an adjusted series which closely approximates the autoregressive series even when both the amplitude and the pattern of the seasonal vary considerably. This result is not at all surprising given that the method uses a moving average estimate of the change in the seasonal and that it does not rely in any way upon intercorrelations between the monthly seasonal factors. However,

there are two points at which the Census Method is relatively weak. The first is that no correction is made for the bias introduced by the use of the Spencer 15-point formula. As mentioned previously, this correction could easily be made in the Census Method without any change in the basic method. Therefore the presence of this bias in the current version of the method is not a fundamental criticism. The other weakness of the Census Method is that the frequency response of the moving average used to estimate the moving monthly seasonal factors is very low even at quite low frequencies. This characteristic produces a very stable estimate of the seasonal factors but a poor estimate of relatively rapid variations in the amplitude or pattern of the seasonal.

7.4 The rational-function method--basically an extension of Wald's and Hannan's methods--is an attempt, shown to be at least partially successful, to remove the two weaknesses of the Census Method mentioned above, while maintaining the generality of the assumptions on which the Census Method is based. The method is very simple both conceptually and computationally. In addition some of the problems of treatment of the end values of the series are avoided through the use of an asymmetric, single-sided, filter function. This method is not intended as a complete new method of seasonal adjustment, but is simply presented to show the potential value of the application of simple ideas of frequency decomposition to seasonal adjustment.

Finally, mention must be made of the cost involved in the use of a seasonal adjustment method which estimates relatively rapid changes in the seasonal. It is not possible to estimate rapid changes in the seasonal without, in some way, affecting the information in the series at frequencies near to the seasonal. Thus the more "flexible" the seasonal estimator, the greater the disturbance to the series at non-seasonal frequencies. In attempting to estimate a changing seasonal, it would generally seem to be desirable to use a method that does not completely remove the seasonal in order to reduce the amount of distortion at non-seasonal frequencies. The exact trade-off between removal of a varying seasonal and distortion of the series is something which needs ultimately to be determined in the context of the intended use of the information in the series.

Appendix A: Derivation of Wald's Method

This appendix presents the derivation of equation 4.2.7 which defines the estimated seasonal coefficients. We continue the notation established in Section 4.2 and presuppose the statement of the assumptions given in that section.

The values of $p(i,k)$ for the k^{th} month are the same for all i since $p(t)$ is a periodic function. Replacing these values by a common value $p(k)$, we obtain from equation 4.2.6:

$$\frac{\sum_{i=1}^{m} \psi(i,k)}{m} \sim p(k) \frac{\sum_{i=1}^{m} \lambda(i,k)}{m} + \frac{\sum_{i=1}^{m} y(i,k)}{m} \qquad \text{A.1}$$

$$\text{for } k = 1,2,\ldots,12.$$

Now let

$$\frac{\sum_{i=1}^{m} \lambda(i,k)}{m} = \lambda(k) \qquad \text{for } k = 1,2,\ldots,12. \qquad \text{A.2}$$

Also, let

$$\frac{\sum_{k=1}^{12} \lambda(k)}{12} = \lambda(0) \ .$$

Then substituting in equation A.2:

$$\frac{\sum_{k=1}^{12} \sum_{i=1}^{m} \lambda(i,k)}{12m} = \lambda(0) \quad . \qquad \text{A.3}$$

Replacing the $\lambda(i,k)$ in A.2 by $\frac{1}{12} \sum_{k=1}^{12} \delta(i,k) + \delta(i,k)$

where: $\delta(i,k)$ - the deviation of $\lambda(i,k)$ from its mean,

results in:

$$\frac{\sum_{i=1}^{m} \left[\frac{\sum_{k=1}^{12} \lambda(i,k)}{12} + \delta(i,k) \right]}{m} = \lambda(k)$$

or

$$\frac{\sum_{i=1}^{m} \sum_{k=1}^{12} \lambda(i,k)}{12m} + \frac{\sum_{i=1}^{m} \delta(i,k)}{m} = \lambda(k) \quad .$$

Using A.3 this becomes

$$\lambda(0) - \lambda(k) = - \frac{\sum_{i=1}^{m} \delta(i,k)}{m} \quad . \qquad \text{A.4}$$

Let us assume that the maximum deviation of $\lambda(t)$ within a year is $\upsilon\%$ of the mean of that year:

$$|\delta(i,k)| \leq \frac{\upsilon}{100} \frac{\sum_{k=1}^{12} \lambda(i,k)}{12} \quad . \qquad \text{A.5}$$

Taking the absolute values of the left and right sides of A.4 one finds:

$$|\lambda(0) - \lambda(k)| = \left| \frac{\sum_{i=1}^{m} \delta(i,k)}{m} \right| \leq \frac{\upsilon}{100} \frac{\sum_{i=1}^{m} \sum_{k=1}^{12} \lambda(i,k)}{12m} = \frac{\upsilon}{100} \lambda(0) \quad . \qquad \text{A.6}$$

In general, the $\delta(i,k)$ will have alternating signs and $\left| \frac{\sum_{i=1}^{m} \delta(i,k)}{m} \right|$ will therefore be considerably smaller than $\frac{\upsilon}{100} \lambda(0)$. This justifies the assumption that:

$$\lambda(k) \sim \lambda(0) \qquad \text{for} \quad k = 1,2,\ldots,12 \quad . \qquad \text{A.7}$$

Consequently, one can write for A.1:

$$\frac{\sum_{i=1}^{m} \psi(i,k)}{m} \sim p(k)\,\lambda(0) + \frac{\sum_{i=1}^{m} y(i,k)}{m} \quad . \qquad \text{A.8}$$

With respect to the residuals $Z(t)$ the following two assumptions are made:

$$\frac{\sum_{i=1}^{m} \sum_{k=1}^{12} Z(i,k)}{12m} \sim 0 \qquad \text{A.9}$$

and further

$$\frac{\sum_{i=1}^{m} Z(i,k)}{m} \sim 0 \quad .$$

By definition $y(i,k) = Z(i,k) - Z^*(i,k)$. Therefore:

$$\sum_{i=1}^{m} y(i,k) = \sum_{i=1}^{m} Z(i,k) - \sum_{i=1}^{m} Z^*(i,k) \quad . \qquad \text{A.10}$$

Since

$$\sum_{i=1}^{m} Z^*(i,k) = \frac{\sum_{k=1}^{12} \sum_{i=1}^{m} Z(i,k)}{12} \quad ,$$

equation A.10, after dividing by m, becomes:

$$\frac{\sum_{i=1}^{m} y(i,k)}{m} = \frac{\sum_{i=1}^{m} Z(i,k)}{m} - \frac{\sum_{i=1}^{m} \sum_{k=1}^{12} Z(i,k)}{12m} \quad .$$

From A.9:

$$\frac{\sum_{i=1}^{m} y(i,k)}{m} \sim 0 \quad .$$

Hence, A.8 can be replaced by:

$$\frac{\sum_{i=1}^{m} \psi(i,k)}{m} \sim \lambda(0)\, p(k) \qquad \text{for } k = 1,2,\ldots,12. \qquad \text{A.11}$$

The 12 sums, one for each month, on the left side of A.11 can be obtained from the $\psi(i,k)$ series. What is left to be done is to split these sums into the two components $\lambda(0)$ and $p(k)$, where $\lambda(0)$ is a constant, being the arithmetic

mean of all $\lambda(i,k)$. The $\lambda(i,k)$ and $p(k)$ are, however, still to be determined. From 4.2.5, we can write, after replacing $s(t)$ by $\lambda(t)\ p(t)$:

$$\psi(i,k) \sim \lambda(i,k)\ p(i,k) + y(i,k) \qquad \text{for } i = 1,2,\ldots,m \quad k = 1,2,\ldots,12 \ ,$$

or, since the function $p(t)$ is periodic:

$$\psi(i,k) \sim \lambda(i,k)\ p(k) + y(i,k) \ .$$

Then

$$\psi(i,k) \sim \frac{\lambda(i,k)}{\lambda(\theta)} \cdot \lambda(0)\ p(k) + y(i,k) \ ,$$

or, from A.11

$$\psi(i,k) \sim \frac{\lambda(i,k)}{\lambda(0)} \cdot \frac{\sum_{i=1}^{m} \psi(i,k)}{m} + y(i,k)$$

and

$$\psi(i,k) \sim \mu(i,k) \cdot a(k) + y(i,k) \qquad \text{A.12}$$

where

$$\mu(i,k) \sim \frac{\lambda(i,k)}{\lambda(0)} \qquad \text{and} \qquad a(k) = \frac{\sum_{i=1}^{m} \psi(i,k)}{m} \ .$$

Now the $\mu(i,k)$ remain to be determined.

A.12 may also be written:

$$y(i,k) \sim \psi(i,k) - \mu(i,k)\ a(k) \qquad \text{for } i = 1,2,\ldots,m \quad k = 1,2,\ldots,12 \ . \qquad \text{A.13}$$

Since $y(t)$ may be assumed to be a normally distributed random variable, the $\mu(i,k)$ may be determined so as to minimize:

$$\sum_{i=1}^{m} \sum_{k=1}^{12} [\psi(i,k) - \mu(i,k)\, a(k)]^2 \quad .$$

The additional condition to be imposed on the function $\mu(t) = \frac{\lambda(t)}{\lambda(0)}$ is that its value will change only slowly over time. This leads to the assumption that the $\mu(i,k)$ may be approximated by minimizing:

$$\sum_{j=k-6}^{j=k+5} [\psi(i,j) - \mu(i,k)\, a(j)]^2 \qquad j = 1,2,\ldots,12 \qquad \text{A.14}$$

subject to the condition that for each point t in time, the value of $\mu(t)$ will be constant in each period $(t-6,\ t+5)$ for $t = 7,8,\ldots,(n-5)$.[1)]

Rather than minimizing the expression $[\psi(i,k) - \mu(i,k)\, a(k)]^2$ with respect to $\mu(i,k)$ over the whole period $t = 7,8,\ldots,(n-5)$, it is minimized over consecutive 12-month periods. The first period includes the time points $7,8,\ldots,18$, the second one $8,9,\ldots,19$ etc., the last one $(n-16)$, $(n-15),\ldots,$ $(n-5)$; there are, therefore, n-6 periods all together. The condition imposed on the $\mu(t)$ assures that the 12 μ's related to the first period, $\mu(1)$, $\mu(2)$, ..., $\mu(12)$ will all have the same value, say $C(1)$. Similarly, the 12 μ's related to the j^{th} period, $\mu(j)$, $\mu(j+1),\ldots,$ $\mu(j+11)$, will have the same value $C(j)$, where, in general, $C(1) \neq C(2) \neq \cdots \neq C(j)$. A particular μ, say $\mu(i)$, is determined when the minimization of equation A.14 has been performed for the 12-month period i.

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1) It should be noted that if, in A.15 the index j of $\psi(i,j)$ and $a(j)$ becomes ≤ 0, the former are replaced by $\psi(i-1,\ j+12)$ and $a(j+12)$ and if $j \geq 12$, by $\psi(i=1,\ j-12)$ and $a(j-12)$.

Replacing the 12 $\mu(i,k)$'s in A.14 by a single $\mu(i,k)$ and differentiating that expression with respect to that $\mu(i,k)$ we have:

$$\mu(i,k) = \frac{\sum_{j=k-6}^{k+5} \psi(i,j)\, a(j)}{\sum_{j=k-6}^{k+5} [a(j)]^2} \quad . \qquad \text{A.15}$$

The seasonal fluctuations $s(i,k)$ can now directly be derived from these $\mu(i,k)$'s:

$$s(i,k) = a(k)\, \mu(i,k) = a(k)\, \frac{\sum_{j=k-6}^{k+5} \psi(i,j)\, a(j)}{\sum_{j=k-6}^{k+5} [a(j)]^2} \quad . \qquad \text{A.16}$$

Since the same 12 $a(k)$'s appear in the denominator repeatedly, formula A.16 can be still further simplified to:

$$s(i,k) = a(k)\, \frac{\sum_{j=k-6}^{k+5} \psi(i,j)\, a(j)}{\sum_{\ell=1}^{12} [a(\ell)]^2} \quad . \qquad \text{A.17}$$

Now, the $s(i,k)$ can be obtained from the original observations $\varphi(t)$ via the series of differences $\psi(t)$ and the $a(k)$ derived from the $\psi(t)$.

Wald also includes in his monograph the following scheme for carrying out the computational steps:

1. First $\varphi^*(t)$, the 12-month moving average of $\varphi(t)$, is computed.
2. Then the differences $\psi(t) = \varphi(t) - \varphi^*(t)$ are formed and put

in the form of a matrix with 12 columns, one for each month. At this point, it should be remarked that if in actual situations the values of some of the $\psi(i,k)$ are too extreme, due to special circumstances (strikes, for example), then these $\psi(i,k)$ may be excluded from the rest of the computations.

3. For each month $k(k = 1,2,\ldots,12)$ the arithmetic mean $a(k)$ of the values of $\psi(i,k)$ which appear in the k^{th} column of the matrix is then computed.

4. This is followed by an adjustment of the $a(k)$ according to the formula:

$$a'(k) = a(k) - |a(k)| \frac{a(1) + a(2) + \cdots + a(12)}{|a(1)| + |a(2)| + \cdots + |a(12)|} \qquad \text{A.18}$$

so that $\sum_{k=1}^{12} a'(k) = 0.$

5. Then the series $F(i,k) = b'(k)\,\psi(i,k)$ is formed where

$$b'(k) = \frac{a'(k)}{\sum_{\ell=1}^{12} [a'(\ell)]^2} .$$

6. Adding the first 12 values of the series $F(i,k)$ will give $\mu(1)$, the 1st term of the $\mu(t)$ series. Subtracting from $\mu(1)$ the 1st term of $F(i,k)$ and adding to it the 13th term of $F(i,k)$ will give $\mu(2)$, the 2nd term of the $\mu(t)$ series. This procedure is continued till the last term of $F(i,k)$ has been added and the $(\text{last-12})^{th}$ term of $F(i,k)$ has been subtracted from $\mu(t-1)$ to give $\mu(t)$. After the $\mu(t)$ series has been computed, it is arranged in the form of a matrix with 12 columns, one column for each month.

7. The seasonal fluctuations $s(i,k)$ are then computed according to the formula:

$$s(i,k) = a'(k)\,\mu(i,k)\ . \qquad \text{A.19}$$

It is not possible to obtain values for $\mu(t)$ and $s(t)$ for the last 11 months this way, which is a serious drawback in practical applications. Wald suggests in [27] that the simplest solution to this problem is to add 11 more terms to the $\mu(t)$ series, all equal to the last computed $\mu(t)$. This will then make it possible to compute the $s(t)$ for the last 11 months on the basis of A.19. However, the extrapolation technique described in Section 3.0 is shown by Wald to be a better procedure.

8. Finally the seasonal series $s(t)$ is subtracted from the series $\varphi(t)$ to give the seasonally adjusted series.

Appendix B: Details of Computations

While analysis using the computation of spectra and cross-spectra is becoming more widespread in economics, the technique is neither completely accessible to all mathematical economists nor is it fully standardized. Therefore the following definitions and equations will be stated.

The correlation coefficient for two series, x_t and y_t, (which may be identical) is defined by:

$$R_{xy}(s) = \frac{\sum_{t=1}^{N-S} (x_t - \bar{x})(y_{t+s} - \bar{y})}{\left[\sum_{t=1}^{N} (x_t - \bar{x})^2 \sum_{t=1}^{N} (y_t - \bar{y})^2\right]^{1/2}} \qquad s = 0,\ldots,m$$

where: N - number of observations.

$$\bar{x} = \frac{1}{N} \sum_{t=1}^{N} x_t$$

$$\bar{y} = \frac{1}{N} \sum_{t=1}^{N} y_t$$

$$R_{xy}(s) = R_{yx}(-s)$$

$$R_{xx}(s) = R_{xx}(-s) \quad .$$

The spectrum is then given by:

$$F_{xy}(\omega) = \sum_{s=-m}^{m} R_{xy}(s)\, \lambda(s)\, e^{i\omega s}$$

where:

$$\lambda(s) = 1 - 6\left(\frac{s}{m}\right)^2 + 6\left(\left|\frac{s}{m}\right|\right)^3 \qquad s \leq \frac{m}{2}$$

$$= 2\left(1 - \left|\frac{s}{m}\right|\right)^3 \qquad s \geq \frac{m}{2} \quad .$$

Due to $R_{xx}(s) = R_{xx}(-s)$,

$$F_{xx}(\omega) = R_{xx}(0) + 2 \sum_{s=1}^{m} R_{xx}(s)\, \lambda(s) \cos \omega s \quad .$$

Writing $F_{xy}(\omega)$ in terms of its real and imaginary parts we have the real part, the co-spectrum:

$$C_{xy}(\omega) = R_{xy}(0) + \sum_{s=1}^{m} [R_{xy}(s) + R_{yx}(s)]\, \lambda(s) \cos \omega s$$

and the imaginary part, the quadrature spectrum:

$$Q_{xy}(\omega) = \sum_{s=1}^{m} [R_{xy}(s) - R_{yx}(s)]\, \lambda(s) \sin \omega s \quad .$$

From these we define:

1. Coherency

$$S^2_{xy}(\omega) = \frac{|F_{xy}(\omega)|^2}{F_{xx}(\omega) F_{yy}(\omega)}$$

2. Gain

$$G_{xy}(\omega) + \frac{|F_{xy}(\omega)|}{F_{xx}(\omega)}$$

3. Phase

$$\Phi_{xy}(\omega) = \tan^{-1} \frac{Q_{xy}(\omega)}{C_{xy}(\omega)} \quad .$$

All of the series computed for this paper were made up of 300 observations. 100 lags were used. Thus $N = 300$, $m = 100$.

All of the computations were Fortran programmed for the Princeton University IBM 7090. Extensive use was made of the on-line graphic display and recording facilities on the computer. In fact only summary statistics and identifying comments were output on tape for subsequent printing. All other results, of which the figures in the paper are examples, were recorded directly on 35 mm microfilm. Without this graphic output facility, the development of the various programs would have been much slower, the analysis of results vastly slower and more laborious, and the analysis and development of the rational-function method nearly impossible within the time available.

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