

Tests to Discriminate Between Linear
and Ratio Models

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ABSTRACT

Two exact tests are presented for testing the hypothesis that a linear [ratio] model explains the relationship between variables as opposed to the alternative that the ratio [linear] specification is correct. The first test is parametric and uses the F - statistic. The second test is nonparametric and uses the number of peaks in the ordered sequence of unsigned residuals. In conclusion, the results of some experimental calculations of the powers of the tests are discussed.

TESTS TO DISCRIMINATE BETWEEN LINEAR AND RATIO MODELS

1. Introduction

The formulation of economic hypotheses frequently involves the choice between two concrete models:

$$y_i = a_0 + a_1 x_{1i} + \dots + a_m x_{mi} + u_i \quad (1)$$

and

$$\frac{y_i}{x_{mi}} = a_0 \frac{1}{x_i} + a_1 \frac{x_1}{x_{mi}} + \dots + a_m + v_i \quad (2)$$

where the following assumptions are made irrespective of which model is true:¹

(a) The error term (u in (1) and v in (2)) is normally distributed with zero mean, constant variance and is serially independent.

(b) The independent variables are nonstochastic and identical in repeated samples, or if not, are distributed independently of the error term.

Three illustrations from an economic context will suffice to show the importance of the choice between (1) and (2).

(a) Assume that we wish to predict for some product the effect of advertising on sales on a cross-sectional basis by states. Let S_i be total sales in the i^{th} state, a_i the number of dollars expended on

1. A somewhat less general alternative to (1) is discussed by Valavanis, S., Econometrics (McGraw-Hill, 1959), pp. 152-153, who writes (2) with $a_0 = 0$.

advertising in the i^{th} state and P_i the population in the i^{th} state. We may then hypothesize either

$$S_i = a + bA_i + cP_i$$

or

$$\frac{S_i}{P_i} = a\frac{1}{P_i} + b\frac{A_i}{P_i} + c$$

(with a possibly equal to zero).

(b) Much attention has been devoted to the problem of predicting aggregate consumption on the basis of aggregate income on a time series basis. Let Y_i be dollar income in the i^{th} period, C_i consumption in dollars in the i^{th} period and P_i a measure of the price level in the i^{th} period. Investigators frequently face the choice between the following models:

$$C_i = a + bY_i + cP_i$$

or

$$\frac{C_i}{P_i} = a\frac{1}{P_i} + b\frac{Y_i}{P_i} + c$$

(with a possibly equal to zero), thus relating either consumption in money terms to money income or real consumption to real incomes.

(c) Recently a great deal of interest has centered on analyzing the portfolio behavior of various economic units. In particular, portfolios have been decomposed into broad asset categories and the changes in asset composition over time have been examined. In these models it is frequently assumed that the equilibrium holdings of all assets are homogeneous of degree one in dollar magnitudes. If A is the dollar magnitude of the asset in question, W the dollar magnitude of the total

portfolio, r the rate of interest and X_1, X_2 other variables, measured in dollars, affecting A , the hypothesis can be expressed as either

$$\frac{A}{W} = a + b \frac{X_1}{W} + c \frac{X_2}{W} + dr$$

or

$$A = W(a + b \frac{X_1}{W} + c \frac{X_2}{W} + dr) .$$

The purpose of this paper is to state two simple exact tests of the hypothesis that (1) [(2)] is the true model against the single alternative that (2) [(1)] is the true model. Section 2 is devoted to a discussion of the tests. Section 3 presents and discusses some experimental results concerning the power of the tests.

2. Some Exact Tests

We shall consider the two models in the following form:

$$Y = X\beta + U \tag{3}$$

where Y and X are respectively $n \times 1$ and $n \times (m+1)$ matrices of observations, β is the $(m+1) \times 1$ vector of coefficients and U is the $n \times 1$ vector of error terms and

$$[Y/x_m] = [X/x_m]\beta + V \tag{4}$$

where $[Y/x_m]$ and $[X/x_m]$ are matrices differing from Y and X only in that they contain the elements of Y and X each divided by the corresponding element of the last column of X . Of course the ratio hypothesis is meaningless if any X -value is zero; typically in economic applications the X -variable used as deflator is strictly

positive. The ratio hypothesis is obtained directly from the linear hypothesis by division by x_m ; hence if the linear model (3) is true and the ratio model (4) is fitted, the assumption of homoscedasticity of the residuals from the correct model implies that the residuals from the incorrectly specified model cannot be homoscedastic. To distinguish between (3) and (4) it appears natural to test both models for homoscedasticity. The following possibilities arise: (a) we cannot reject the hypothesis of homoscedasticity in either case. We shall then suspend judgment as to which model is preferable. (b) We reject the hypothesis of homoscedasticity for one but not the other case. We shall then accept the formulation leading to homoscedastic residuals as the true one. (c) We reject homoscedasticity in both cases. We are then again (as in (a)) unable to choose between the ratio and linear models but are forced to consider an enlargement of the set of possible alternatives.

To test for the homoscedasticity of (3) we may proceed in one of two ways:

A Parametric Test. On the assumption that the error terms are normally distributed a test can be constructed in the following manner.

(a) Order the observations by the values of the variable x_m which is the potential deflator; i.e., we renumber such that $x_{mi} \leq x_{mj}$ if and only if $i < j$ and we then index the remaining variables so that the index values correspond with those of the x_{mi} .

(b) Given some choice of the number of central observations, k , to be omitted, we fit separate regressions (by least squares) to the first $(n-k)/2$ and last $(n-k)/2$ observations, provided also that

$(n-k)/2 > m+1$, the number of parameters to be estimated.

(c) Denoting by S_1 and S_2 the sum of the squares of the residuals from the regressions based on the relatively small and relatively large values of x_m respectively, we form

$$R = \frac{S_2}{S_1} .$$

The quantity R clearly has the F -distribution with $\left(\frac{n-k-2m-2}{2}, \frac{n-k-2m-2}{2}\right)$ degrees of freedom under the null hypothesis. Under the alternative hypothesis values of R will tend to be large since, if the ratio hypothesis is true, $u = vx_m$ and $\text{Var}(u) = x_m^2 \text{Var}(v) = x_m^2 \text{constant}$.²

The following observations are relevant:

(a) Since the sum of the squares of the residuals can be expressed as a quadratic form in the true errors,³ the ratio R is homogeneous of degree zero in the true error terms; hence the ratio is independent of σ_u under the null hypothesis and σ_v if the alternative hypothesis is true.

(b) The ratio R is independent of the regression coefficients β .

(c) We normally cannot use the standard test for the homogeneity of several variances since we generally do not have repeated observations

2. An analogous procedure holds when we test for the homoscedasticity of (4). The ordering procedure, of course, is carried out with $1/x_{mj}$ rather than x_{mj} .

If the linear model is true, then $\text{Var}(v) = \text{Var}(u)/x_m^2$. However, since we order by $1/x_m$ for the ratio model, R would also tend to be large if the linear hypothesis is correct and we estimate a ratio model.

3. $\Sigma \hat{u}^2 = u'(I - X(X'X)^{-1}X')u$.

for given x 's.⁴ In addition, the number of observations is frequently too small for an analogous multiple division of a given sample.

(d) The power of this test will clearly depend upon the value of k ; for very large values of k the power will be small but it is not obvious that the power increases as k tends to 0.

(e) The power of the test will clearly depend on the nature of the sample of values for the variable which is the deflator. Thus, if the variance of x_m is small relative to the mean of x_m the power can be expected to be small and conversely.

A Nonparametric Test. A nonparametric test can be constructed in the following manner.

(a) Fit both the linear and the ratio models to the entire series of data. If the data were generated by the linear [ratio] model, the variances of residuals from the ratio [linear] model will be monotonically declining [increasing] as the value of the deflator variable increases.

(b) Confining our attention to the residuals from the linear model,⁵ let \hat{u}_i be the i th residual corresponding to the i th value of the deflator variable x_i . The set of residuals, $\{\hat{u}_j\}$, is assumed to be ordered in the following manner. First, we order the values of the deflator variable x_m so that $x_{mi} \leq x_{mj}$ if and only if $i < j$. We then index the residuals \hat{u}_i so that the index values correspond with those of x_{mi} . Thus if $i < j$ then \hat{u}_i appears to the left of

4. Goldberger, A. S., Econometric Theory (John Wiley and Sons, Inc., 1964), p. 243.

5. Analogous remarks are valid if we confine our attention to the residuals from the ratio model. For a full test both procedures are undertaken.

\hat{u}_j in the ordered list of residuals.

(c) Define a peak in the ordered residuals at observation i to be an instance where $|\hat{u}_i| \geq |\hat{u}_j|$ for all $j < i$.

(d) If the residuals are heteroscedastic such that the variance increases with x_m , the number of observed peaks will tend to be large.

(e) Otherwise, under the null hypothesis of homoscedasticity we can calculate the probability of $0, 1, \dots, n-1$ peaks in a sequence of n residuals as follows.

Define $N(n, k)$ as the number of permutations of n absolute values of residuals yielding k peaks. For convenience we define $N(1, 0) = 1$. We make the observation that adjoining an n^{th} residual to $n-1$ others can create a total of k peaks in two ways: (i) if the preceding $n-1$ residuals yielded k peaks and the last one creates no new one, and (ii) if the preceding $n-1$ residuals yielded $k-1$ peaks and the i^{th} one does create an additional peak. This yields immediately the following recursions:

$$\begin{aligned}
N(n, n-1) &= 1 \\
N(n, n-2) &= (n-1)N(n-1, n-2) + N(n-1, n-3) \\
&\vdots \\
N(n, k) &= (n-1)N(n-1, k) + N(n-1, k-1) \\
&\vdots \\
N(n, 1) &= (n-1)N(n-1, 1) + N(n-1, 0) \\
N(n, 0) &= (n-1)N(n-1, 0).
\end{aligned}$$

Since n absolute values of residuals can appear in a total of $n!$ permutations the probability $P(n, k)$ of n absolute values of residuals yielding exactly k peaks is

$$P(n,k) = \frac{1}{n!}N(n,k) .$$

The values of $P(n,k)$ and of the cumulative probabilities $\sum_{i=0}^k P(n,i)$ are displayed for selected values of n and k in Table 1.⁶

The concrete process for applying this test is then as follows:

(a) We fit least squares regression lines to the data according to both the linear and ratio models; (b) we order the absolute values of the residuals from each regression such that the number of peaks will tend to be large if the hypothesis employed in fitting the particular relation (linear or ratio) is false; (c) we count the number of peaks in both series of residuals and compare them with Table 1; depending on the outcome of the comparisons we accept one or the other or neither hypothesis analogously with the procedure discussed under the F -test.

It should be noted that this peak test is not a general test of homoscedasticity. It is such a test only under an alternative hypothesis which states that σ_u is monotonic in x_m .

3. An Experimental Calculation of Power

Sampling experiments were performed on a simple model with one independent variable in order to obtain (experimental) estimates of the powers of the two tests. Four parameters were varied: (a) the total number of observations was either 30 or 60; (b) the number of (central) observations omitted was 0, 4, 8, 12, or 16 for the parametric test; the omission of central observations is not relevant for the nonparametric test;

6. It is noteworthy that the recursions above also yield the number of permutations of n objects with $k+1$ cycles. The numbers generated are therefore the unsigned Stirling numbers of the first kind. See Riordan, J., An Introduction to Combinatorial Analysis (John Wiley and Sons, New York, 1958), pp. 66-72.

(c) the independent variable was identical in repeated samples and each particular sample of x 's was chosen from the uniform distribution with mean $\mu_x = 10, 20, 30, 40, 50$, and (d) standard deviation $\sigma_x = 5, 10, 15, 20, 25, 30$. Only those cases were used in which the range of the actual sample of x 's generated did not overlap the interval $(-\infty, 1)$. For each μ_x, σ_x combination one sample of x 's was generated and for each such sample 100 samples of 30 (or 60) u -values were generated from the normal distribution with zero mean and unit variance. Corresponding samples of y -values were calculated from

$$y_i = a_0 + a_1 x_i + u_i$$

and the resulting samples of x and y values were used to calculate the F -ratio for both

$$\hat{y}_i = \hat{a}_0 + \hat{a}_1 x_i \quad (3-1)$$

and

$$\frac{\hat{y}_i}{x_i} = \hat{a}_0 \frac{1}{x_i} + \hat{a}_1 \quad (3-2)$$

Two statistics are of particular relevance: (a) the relative frequency (in 100 trials) of cases in which the false hypothesis (3-2) is rejected, this being an estimate of the power of the test and (b) the relative frequency of cases in which the correct statistical decision is reached, i.e., (3-1) is not rejected and (3-2) is, this being also an estimate of the power of the test. Since in the present experiments there is virtually no difference between the two statistics, we shall concentrate our attention on (a).

Tables 2 and 3 contain experimental estimates of the power of the

parametric test for $n = 30$ and $n = 60$ respectively. Each case — cases being numbered on the left — represents 100 replications of the experiment for a particular choice of a sample of x 's. For each case we rank the powers for the five values of k , the number of omitted central observations. If Kendall's coefficient of concordance W is calculated for each table, duly taking into account the number of ties, we find $W = .379$ for Table 2 which is significantly different from zero on the .05 level and $W = .170$ for Table 3 which is not. Most of the significance of W for Table 2 is due to the last column. The average powers over all cases are displayed in the last line of each table; if we pretended that we could use the ordinary test of the hypothesis that two percentages calculated from two independent samples are the same, we would find that the largest actual difference is significantly different from zero in Table 2 but not in Table 3. On the whole we cannot reject the null hypothesis that — within the ranges examined — the number of omitted central observations leaves power unaffected with the possible exception of the case where the number of omitted observations becomes very large compared to the total number of observations. It is nevertheless interesting to note that the highest mean power figures occur for $k = 8$ when $n = 30$ and $k = 16$ for $n = 60$.

We thus concentrate now on $k = 8$ for $n = 30$ and $k = 16$ for $n = 60$. Tables 4 and 5 display the experimentally calculated power of the parametric test tabulated by the parameters of the distribution from which a sample of x 's was chosen for each set of 100 replications.⁷

7. The two tables do not contain entries in all corresponding positions since no calculations were performed when a sample of x 's chosen happened to contain an x value less than one.

The tables confirm the conjecture that an increase in σ_x relative to μ_x improves the power of the test. If we calculate the sample ratio of the mean of x to the standard deviation of x and plot it against the power we obtain Figure 1 which further confirms the result. Tables 4 and 5 and Figure 1 also provide an estimate of the increase in power due to an increase in n .

Analogous results hold for the nonparametric peak test. Estimates of the power are displayed in Table 6. As expected, power generally increases with n and σ_x/μ_x . Perhaps more unexpectedly, the powers compare quite favorably with the parametric test, particularly when σ_x/μ_x and n are both large. In fact, over a comparable subset of cases computed for $n = 60$ the power of the peak test is 63 per cent of the power of the F -test. It appears thus as a sensible alternative to the F -test when the distribution of residuals is not known.

4. Conclusion

Two tests have been proposed for testing the hypothesis that a linear rather than a ratio model explains the relation between variables. A parametric F -test and the nonparametric peak test both rely upon testing residuals for homoscedasticity as against the alternative that residual variances are monotonic in the independent variable. It should be noted that the peak test can also be used as a two-tailed test when one examines whether the residual variances are monotonically increasing or decreasing in the independent variable. Finally, it should be mentioned that we have not dealt with the case treated by Theil where the variance of the residuals is proportional to the square of the mean of

the dependent variable.⁸ The power of both tests is estimated from sampling experiments and found satisfactory.

TABLE 1. Cumulative Probabilities for the Distribution of Peaks

n	P (number of peaks $\leq x$)										
	x = 0	x = 1	x = 2	x = 3	x = 4	x = 5	x = 6	x = 7	x = 8	x = 9	x = 10
5	.2000	.6167	.9083	.9917	1.0000						
10	.1000	.3829	.7061	.9055	.9797	.9971	.9997	1.0000			
15	.0667	.2834	.5833	.8211	.9433	.9866	.9976	.9997	1.0000		
20	.0500	.2274	.5022	.7530	.9056	.9720	.9935	.9988	.9998	1.0000	
25	.0400	.1910	.4441	.6979	.8705	.9559	.9879	.9973	.9995	.9999	1.0000
30	.0333	.1654	.4001	.6525	.8386	.9395	.9815	.9953	.9990	.9998	1.0000
35	.0286	.1462	.3654	.6144	.8098	.9234	.9745	.9929	.9984	.9997	.9999
40	.0250	.1313	.3373	.5818	.7837	.9078	.9674	.9903	.9975	.9995	.9999
45	.0222	.1194	.3138	.5536	.7600	.8930	.9601	.9874	.9966	.9992	.9998
50	.0200	.1096	.2940	.5288	.7383	.8788	.9530	.9844	.9956	.9989	.9998
55	.0182	.1014	.2769	.5068	.7184	.8653	.9456	.9813	.9944	.9986	.9997
60	.0167	.0944	.2620	.4871	.7001	.8524	.9384	.9780	.9932	.9982	.9996

TABLE 2. Relative Frequency of Cases in Which False Hypothesis is Rejected for $n = 30$

Case	k = 0	k = 4	k = 8	k = 12	k = 16
1	.410	.450	.450	.400	.390
2	.240	.190	.240	.210	.210
3	.820	.840	.810	.820	.730
4	.990	.990	.990	.990	.980
5	.980	.990	.990	.990	.990
6	.140	.140	.100	.130	.080
7	.470	.490	.520	.530	.430
8	.860	.920	.920	.900	.800
9	.980	.990	1.000	1.000	.990
10	.150	.150	.150	.150	.060
11	.370	.360	.330	.240	.190
12	.450	.490	.510	.510	.500
13	.920	.900	.900	.870	.810
14	.940	.950	.950	.960	.930
15	.990	.990	1.000	1.000	.980
Average	.647	.656	.657	.647	.605

8. Theil, H. "Estimates and Their Sampling Variance of Parameters of Certain Heteroscedastic Distributions," Revue de l'Institut International de Statistique, 19 (1951), 141-147.

TABLE 3. Relative Frequency of Cases in Which False Hypothesis is Rejected for $n = 60$

Case	$k = 0$	$k = 4$	$k = 8$	$k = 12$	$k = 16$
1	1.000	1.000	1.000	1.000	1.000
2	.700	.720	.690	.710	.780
3	1.000	1.000	1.000	1.000	1.000
4	.500	.480	.510	.480	.500
5	.960	.970	.970	.960	.960
6	1.000	1.000	1.000	1.000	1.000
7	.290	.290	.300	.320	.340
8	.740	.750	.740	.770	.770
9	.970	.970	.980	.970	.970
10	1.000	1.000	1.000	1.000	1.000
11	.310	.290	.290	.310	.310
12	.550	.560	.550	.610	.580
13	.920	.920	.920	.940	.960
14	.960	.960	.960	.970	.970
15	1.000	1.000	1.000	1.000	1.000
Average	.793	.794	.794	.803	.809

TABLE 4. Power for $k = 8$ and $n = 30$ Tabulated by the Parameters Characterizing the Independent Variables

σ_x \ μ_x	10	20	30	40	50
5		.450	.240	.100	.150
10			.810	.520	.330
15			.990	.920	.510
20			.990	1.000	.900
25					.950
30					1.000

TABLE 5. Power for $k = 16$ and $n = 60$ Tabulated by the Parameters Characterizing the Independent Variables

σ_x \ μ_x	10	20	30	40	50
5	1.000	.780	.500	.340	.310
10		1.000	.960	.770	.580
15			1.000	.970	.960
20				1.000	.970
25					1.000

TABLE 6. Power of the Nonparametric Test
for $n = 30$ and $n = 60$

σ_x	μ_x	10	20	30	40	50
5	$n = 30$..	.27	.16	.16	.10
	$n = 60$.75	.37	.24	.18	.21
10	$n = 30$..	.33	.30	.18
	$n = 60$.86	.38	.29	.22
15	$n = 30$.56	.45	.24
	$n = 60$.84	.54	.54
20	$n = 30$.53	.54	.44
	$n = 60$..	.78	.69
25	$n = 30$.52
	$n = 60$.78
30	$n = 30$.64
	$n = 60$.86

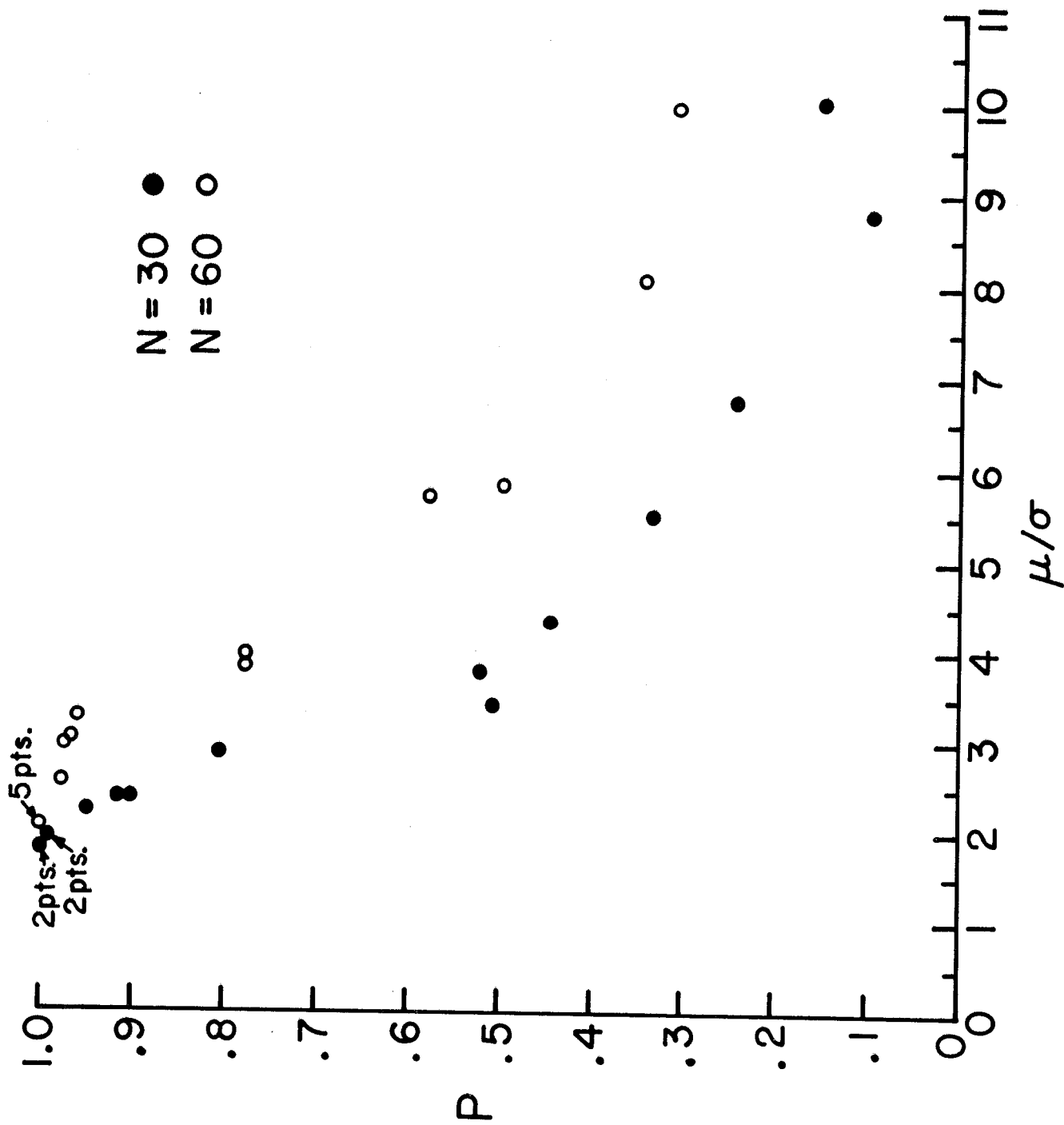


FIGURE 1