AN INTRODUCTION TO POLYSPECTRA

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ABSTRACT

Consider a multivariate stationary time series $\{X_1(t),\ldots,X_k(t)\}$ in which a number of the components are possibly identical. The k-th order polyspectrum of this series is defined to be the Fourier transform of the k-th order cumulant (which is effectively of order (k-1.) The polyspectrum may be estimated by three distinct techniques; (i) Fourier transforming an estimated cumulant with the use of convergence factors, (ii) forming an appropriate combination of band-passed versions of the series, (iii) forming an appropriate combination of the results of complex demodulating the series. Under appropriate conditions these estimates may be seen to be consistent and asymptotically normal.

Polyspectra would appear to be of use in the following two problems, (i) does a particular frequency component in one of the series under consideration appear to be linearly related to the product of frequency components in other series, (ii) is there some function of a series $X(t)$, say $F\{X(t)\}$ admitting a simpler harmonic analysis. These two questions lead to the definition of polyspectral coefficients which may be estimated from an observed stretch of series.

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1. Introduction and Summary

The intent of this paper is to study the higher-order spectra or polyspectra of multivariate time series. The definition of such spectra for the case of a single time series has been given in [3], [18], for example, where various properties of the spectra have been considered as well.

The polyspectrum, in the second-order case of a single time series, reduces to the power spectrum considered in [2], [9], [14] for example, whereas in the second-order case of two time series, it reduces to the cross-spectrum considered in [5], [10]. Finally in the third-order case of a single time series, the polyspectrum reduces to the bispectrum recently considered in [8], [16], [19].

The structure of this paper after the present section is as follows: Section Two is an attempt to motivate the consideration and estimation of polyspectra. The third section presents basic definitions and properties relating to polyspectra, while the fourth section is concerned with estimation problems relating to polyspectra. Section Five is concerned with elementary statistical properties of a variety of estimates. The final section contains an argument delayed from an earlier section of the paper to the effect that for a wide variety of processes it makes more sense to consider the Fourier transform of the cumulant of the process under consideration rather than the more elementary Fourier transform of the product-moment.

Reference should also be made to an early paper [11], related to the procedures described in this paper.

2. General Motivation

In a heuristic sense, the harmonic analysis of a time series X_t, means the consideration of a representation of the series of the form,

$$X_t = \Sigma R_k e^{i\omega_k t + i\Phi_k} \quad . \tag{2-1}$$

This consideration gains some validity from a theorem of Cramer's, [4], to the effect that any wide-sense stationary time series X_t, with mean 0, has a representation of the form,

$$X_t = \int e^{i\omega t}\, dZ(\omega) \quad , \tag{2-2}$$

where $Z(\omega)$ is a stochastic set function.

In the representation (2-1) one often thinks of the various terms in the sum as being independent, perhaps because the $Z(\omega)$ in (2-2) is such that,

$$E\{dZ(\omega)\, dZ^*(\omega')\} = 0 \quad ,$$

unless $\omega = \omega'$, implying independence in the Gaussian case. Or perhaps it is because one imagines the series as coming about as the result of a variety of linear operations on a Wiener process and one knows that linear operations do not allow components at different frequencies to influence one another (see [13], p. 83, for example).

If one persists in the above sort of thinking, one is led solely to the power spectrum in the case of a single series or the cross-spectrum in the case of two series, because there is need of a function of solely a single frequency to describe the behavior of the series.

In practice the various frequency components of a time series do not always appear to be completely independent of one another. For example, many economic time series appear to contain a seasonal effect of persistent non-cosinusoidal shape. Such an effect is possible only if the seasonal and its various harmonics remain in some sort of fixed relation to one another.

A simple form of the tying together of a number of frequencies would occur if a number of independent frequency components, $R_k e^{i\omega_k t + i\Phi_k}$, instead of simply adding together to produce a series X_t as at (2-1), added together and also beat together in pairs to produce a time series,

$$X_t = \Sigma R_k e^{i\omega_k t + i\Phi_k} + \Sigma A_{k\ell} e^{i\alpha_{k\ell}} R_k R_\ell e^{i(\omega_k + \omega_\ell)t + i(\Phi_k + \Phi_\ell)} . \quad (2\text{-}3)$$

That is, we are moving away from an additive model to a model containing second-order product interactions. This is similar to the procedure in [6] and [17], where as an alternative to the additive two factor analysis of variance model,

$$y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} , \quad (2\text{-}4)$$

the model,

$$y_{ij} = \mu + \alpha_i + \beta_j + c\alpha_i\beta_j + \epsilon_{ij} , \quad (2\text{-}5)$$

is considered. The generalization being made is also similar to what Bartlett did in [1], p. 47, where instead of the usual additive factor analysis model,

$$x_i = m_{i1}f_1 + m_{i2}f_2 + m_{i0}s_i , \quad (2\text{-}6)$$

he considered the model,

$$x_i = m_{i1}f_1 + m_{i2}f_2 + m_{i3}f_1f_2 + m_{i4}(f_1^2 - 1) + m_{i5}(f_2^2 - 1) + m_{i0}s_i \quad . \tag{2-7}$$

(The two factor case was presented for simplicity.)

If desired, the second-order terms that have been added in each of the above models may be looked at as further terms in a Taylor series expansion involving some basic entities of interest.

A hint of what is to come in this paper may be provided by the observation that in the expression (2-3), the correlation between the product of the components at frequencies ω_k and ω_ℓ with the component at frequency $\omega_k + \omega_\ell$ is one, provided no other pair of frequencies present in the expansion add up to $\omega_k + \omega_\ell$.

One may naturally extend all of the above discussion to a situation in which one imagines a number of frequency components, from a variety of time series $X_1(t), \ldots, X_{k-1}(t)$ beating together to produce a component in some additional time series $X_k(t)$. Such a situation may be inquired into by means of polyspectra.

Another approach to the introduction of polyspectra can be made through the consideration of nonlinear operations on the series involved. This approach is analogous to the introduction of power spectra because of the ease with which the effect of certain linear operations upon the original time series may be described by their effect on the power spectrum. If one wishes to describe easily the effect of multilinear or polynomial (in the sense of [12]) operations, one finds oneself led to the polyspectrum. Tukey in [21] has commenced the development of a calculus relating polynomial operations to polyspectra.

3. Definitions

Consider a multivariate stochastic process $\{X_1(t),X_2(t),\ldots,X_k(t)\}$ in which a number of the components may be identical. Let this process be such that the product moment,

$$M_{1\ldots k}(t_1,\ldots,t_k) = E\{X_1(t_1)\ldots X_k(t_k)\} \quad , \qquad (3\text{-}1)$$

of order k and all lower orders exists.

In the remainder of this paper the following notation will be adhered to:

(i) $(\Pi_1,\Pi_2,\ldots,\Pi_\ell)$ corresponds to a grouping of the integers $1,2,\ldots,k$ into ℓ groups. (The word group is used to indicate that order does not matter.)

(ii) $t_\pi = (t_{i_1},t_{i_2},\ldots,t_{i_j})$ where π corresponds to the particular grouping $(i_1,i_2,\ldots,i_j)$. For example if $\pi = (1,8,9)$, $t_\pi = (t_1,t_8,t_9)$.

Making use of the above notation, the cumulant of order k may now be defined as,

$$C_{1\ldots k}(t_1,\ldots,t_k) = \sum_p (-1)^{p-1}(p-1)!\; m_{\pi_1}(t_{\pi_1})\ldots m_{\pi_p}(t_{\pi_p}) \; . \qquad (3\text{-}2)$$

The summation in this expression extends over all ways of grouping the subscripts $1\ldots k$.

The definition (3-2) may be inverted to obtain an expression for the product moment in terms of the cumulants, namely,

$$M_{1\ldots k}(t_1,\ldots,t_k) = \sum_p c_{\pi_1}(t_{\pi_1})\ldots c_{\pi_p}(t_{\pi_p}) \quad . \tag{3-3}$$

From this point forward, assume that the series in stationary through the k-th-order cumulant, that is the k-th-order and all lower-order cumulants are such that,

$$C_{1\ldots k}(t+t_1,t+t_2,\ldots,t+t_k) = C_{1\ldots k}(t_1,t_2\ldots,t_k) \quad , \tag{3-4}$$

for $-\infty < t < \infty$ in the continuous case or $t = 0,\pm1, \pm2, \ldots$ in the discrete case. This assumption of stationarity implies,

$$C_{1\ldots k}(t_1,t_2,\ldots,t_k) = C_{1\ldots k}(t_1-t_k,t_2-t_k,\ldots,0) \quad . \tag{3-5}$$

The k-th-order polyspectrum of the multivariate time series $\{X_1(t),\ldots,X_k(t)\}$ is now defined to be the (k-1)-th order Fourier transform $C_{1\ldots k}(\omega_1,\omega_2,\ldots,\omega_{k-1})$ of $C_{1\ldots k}(\tau_1,\tau_2,\ldots,\tau_{k-1},0)$. In order to maintain the symmetry of the series involved, consider instead the k-th-order Fourier transform of $C_{1\ldots k}(\tau_1,\tau_2,\ldots,\tau_k)$. This is easily seen to be

$$\delta(\omega_1 + \omega_2+\ldots+\omega_k) \; C_{1\ldots k}(\omega_1,\omega_2,\ldots,\omega_k) \quad , \tag{3-6}$$

$\delta(\omega)$ being the Dirac δ-function and ω_k a dummy argument added to $C_{1\ldots k}(\omega_1,\ldots,\omega_{k-1})$ such that $\omega_k = -(\omega_1 +\ldots+ \omega_{k-1})$.

At this point the reader is no doubt wondering why the polyspectrum was defined as the Fourier transform of the cumulant rather than of the product moment. Two reasons may be given for this choice. The first reason is identical to the reason why in the second-order case the means are often subtracted from the series, namely to avoid the occurrence of spikes. The argument demonstrating that for a wide class of processes spikes will occur in the Fourier transform of the product moment, but not in the Fourier transform of the cumulant, is so lengthy that it has been postponed until Section 6.

A second reason for considering the cumulant is the following. In the Gaussian case, all of the information is contained in the first two moments, consequently a k-th-order product moment, $k > 2$, has no new information to provide, nor does its Fourier transform. The k-th-order cumulant is that function of the product moments of order k and less which is zero in the Gaussian case. Consequently the consideration of the cumulant in the Guassian case is not liable to deceive one into believing that he has gained some information, while in the non-Guassian case it does provide an indication of the non-Gaussianity. To make this last statement quantitative, suppose that the process X_t is the sum of a Gaussian process G_t and an independent non-Guassian process Y_t. In this case the cumulants of the process X_t of order higher than two are identical with the cumulants of the process Y_t.

Independent of the above reasons, the reader may easily see that the cumulant and the product moment cannot be "nice". Consider the Fourier transform of $M_{1\ldots k}(t_1,t_2,\ldots,t_k)$. As derived from (3-3) it is,

$$\delta(\omega_1+\ldots+\omega_k)\, M(\omega_1,\ldots,\omega_k) = \Sigma\, \delta(\tilde{\omega}_{\pi_1})\ldots\delta(\tilde{\omega}_{\pi_p})\, C_{\pi_1}(\omega_{\pi_1})\ldots C_{\pi_p}(\omega_{\pi_p}) \,, \tag{3-7}$$

where $\tilde{\omega}_\pi = \omega_{i_1} + \omega_{i_2} + \ldots + \omega_{i_j}$ if $\pi = (i_1,i_2,\ldots,i_j)$. This expression is seen to contain many spikes if the lower-order polyspectra do not vanish (as the ordinary power spectrum must not).

A number of properties of $C(\omega_1,\ldots,\omega_k)$ may be written down. For example if the series involved are real,

$$C^*(\omega_1,\ldots,\omega_k) = C(-\omega_1,\ldots,-\omega_k) \quad . \tag{3-8}$$

If the series are all identical, then C is completely symmetric in the variables $(\omega_1,\omega_2,\ldots,\omega_k)$.

Consider now the following discussion relating to the heuristic model of the beating together of frequency components described earlier.

Let $G_i(\upsilon)$ denote the transfer function of a band-pass filter centered at the frequency ω_i, that is ideally

$$\begin{aligned} G_i(\upsilon) &= 1 |\omega_i-\upsilon| < \Delta, \quad |\omega_i+\upsilon| < \Delta \;, \\ &= 0 \;\; \text{otherwise.} \end{aligned} \tag{3-9}$$

Suppose that the W_i are such that $\Sigma\, \omega_i = 0$. Let $Y_i(t)$ denote the result of passing the series $X_i(t)$ through the filter $G_i(\upsilon)$.

If the question at issue is whether or not the component at frequency ω_k comes about in part as a result of the beating together of the components at frequencies $\omega_1,\ldots,\omega_{k-1}$ the imagined model is essentially,

$$Y_k(t) \sim \alpha Y_1(t)\ldots Y_{k-1}(t) \quad . \tag{3-10}$$

The usual estimate for α in such a model is

$$\alpha = \frac{E\{Y_1(t)\ldots Y_k(t)\}}{E\{|Y_1(t)|^2\ldots|Y_{k-1}(t)|^2\}} \quad . \tag{3-11}$$

Using the machinery of [18],

$$E\{Y_1(t)\ldots Y_k(t)\} = \int\ldots\int \delta(\upsilon_1+\ldots+\upsilon_k)G_1(\upsilon_1)\ldots G_k(\upsilon_k)\; M(\upsilon_1,\ldots,\upsilon_k)d\upsilon_1\ldots d\upsilon_k \; . \tag{3-12}$$

Assuming that M is effectively constant in the pass-band of the filters, (3-12) yields,

$$E\{Y_1(t)\ldots Y_k(t)\} \sim M(\omega_1,\ldots, \omega_k)\int\ldots\int \delta(\upsilon_1+\ldots\upsilon_k)\; G_1(\upsilon_1)\ldots G_k(\upsilon_k)d\upsilon_1\ldots d\upsilon_k \; . \tag{3-13}$$

Carrying out a similar computation for the denominator if it is assumed that the ω_i satisfy no relation of the form $\omega_{i_1}+\ldots+\omega_{i_j} = 0$, $j < k$, yields,

$$\alpha = \frac{M(\omega_1,\ldots,\omega_k)}{f_1(\omega_1)\ldots f_{k-1}(\omega_{k-1})} \cdot \frac{\int\ldots\int \delta(\upsilon_1+\ldots+\upsilon_k)\; G_1(\upsilon_1)\ldots G_k(\upsilon_k)d\upsilon_1\ldots d\upsilon_k}{\int |G_1(\upsilon_1)|^2 d\upsilon_1\ldots\int |G_{k-1}(\upsilon_{k-1})|^2 d\upsilon_{k-1}} \; , \tag{3-14}$$

where $f_i(\upsilon)$ denotes the power spectrum of the series $X_i(t)$.

The goodness of fit of the model (3-10) is often measured by the coefficient of determination,

$$\rho^2 = \frac{|E\{Y_1(t)\ldots Y_k(t)\}|^2}{E\{|Y_1(t)|^2\ldots|Y_{k-1}(t)|^2\}\ E\{|Y_k(t)|^2\}} \quad . \tag{3-15}$$

This is approximately,

$$\frac{|M(\omega_1,\ldots,\omega_k)|^2}{f_1(\omega_1)\ldots f_k(\omega_1)} \cdot \frac{\left|\int\ldots\int \delta(\upsilon_1+\ldots+\upsilon_k)\ G_1(\upsilon_1)\ldots G_k(\upsilon_k)d\upsilon_1\ldots d\upsilon_k\right|^2}{\int |G_1(\upsilon_1)|^2\ d\upsilon_1\ldots\int |G_k(\upsilon_k)|^2\ d\upsilon_k} \cdot \tag{3-16}$$

The expressions (3-14), (3-16) lead one to the consideration of the two quantities,

$$\frac{M(\omega_1,\ldots,\omega_k)}{f_1(\omega_1)\ldots f_{k-1}(\omega_{k-1})} \quad , \tag{3-17}$$

and

$$\frac{|M(\omega_1,\ldots,\omega_k)|^2}{f_1(\omega_1)\ldots f_k(\omega_k)} \quad . \tag{3-18}$$

The first of these measures the gain and phase involved in the imagined model (3-10) relating the components at frequencies $\omega_1,\ldots,\omega_{k-1}$ in series $X_1(t),\ldots,X_{k-1}(t)$ to the component at frequency ω_k in the series $X_k(t)$. The second may be thought of as measuring in some sense the relative appropriateness of the model (3-10) at various frequency combinations or polyfrequencies. In this connection it could equally well be thought of as relating to the beating together of the components at frequencies $\omega_1,\ldots,\omega_j$ to produce a result linearly related to the result of the beating together of the components at frequencies $\omega_{j+1},\ldots,\omega_k$.

In an attempt to avoid spikes we are led to modify the definitions (3-17), (3-18) and to define the polyspectral regression coefficient of the series $X_1(t),\dots,X_{k-1}(t)$ on the series $X_k(t)$ at frequencies $(\omega_1,\dots,\omega_k)$ as

$$\frac{C_{1\dots k}(\omega_1,\dots,\omega_k)}{f_1(\omega_1)\dots f_{k-1}(\omega_{k-1})}\quad, \tag{3-19}$$

and the polyspectral coefficient,

$$\frac{|C_{1\dots k}(\omega_1,\dots,\omega_k)|^2}{f_1(\omega_1)\dots f_k(\omega_k)}\quad. \tag{3-20}$$

That this change in the definition has no effect provided the ω's satisfy no relation of the form $\omega_{i_1} + \dots + \omega_{i_j} = 0$, $j < k$, may be seen from (3-7). That in many situations the change produces a more reasonable function to be attempting to estimate will be seen later.

As regards the secondary factors in the expressions (3-14), (3-16), the first approaches 1 for the case of a set of ideal band-pass filters as the band-width shrinks, while the second approaches 0. (This latter fact may be used to demonstrate the approximate normality of a class of narrow band signals.)

Another coefficient which may prove useful in certain situations is the following,

$$\frac{C_{1\dots k}(\omega_1,\dots,\omega_k)}{(k-1)\,\Sigma f_{i_1}(\omega_{i_1})\dots f_{i_{k-1}}(\omega_{i_{k-1}})}\quad, \tag{3-21}$$

where the summation in the denominator extends over the indices $1,\ldots,k$ taken $(k-1)$ at a time. This coefficient results from the following considerations; suppose that one is interested in a time series $X(t)$. Is one wise to carry out a harmonic analysis of $X(t)$ or does some function of $X(t)$, say $\log X(t)$, have a simpler analysis? This question may be answered to a certain extent by noting that many functional relations may be approximated by a relationship of the form,

$$x = y + \alpha y^{k-1} \quad , \tag{3-22}$$

where α is small. Consequently consider the relationship,

$$X(t) = Y(t) + \alpha Y^{k-1}(t) \quad , \tag{3-23}$$

where $Y(t)$ is a simpler series than $X(t)$, simpler in the sense that cumulants of order j, $2 < j \leq k$ are negligible. Taking the k-th-order polyspectrum of (3-23) leads to,

$$C_{1\ldots k}(\omega_1,\ldots,\omega_k) \sim \alpha(k-1)\, \Sigma f_1(\omega_{i_1})\ldots f_1(\omega_{i_{k-1}}) \quad , \tag{3-24}$$

which in turn leads to the coefficient (3-21).

4. Estimation

In this section three distinct techniques for the estimation of polyspectra will be proposed.

The first technique follows from the definition (3-6) given in Section 3. Estimate the product moment $M_{1\ldots k}(t_1,\ldots,t_k)$ and all lower-order product moments from an observed stretch of the series $\{X_1(t),\ldots,X_k(t)\}$ by the use of formulas similar to,

$$\frac{1}{T} \sum_t X_1(t + t_1)\ldots X_k(t + t_k) \quad , \tag{4-1}$$

where the summation is over all possible time points and is replaced by an integral in the continuous case (and is carried out for a number of lags). Next estimate the cumulant $C_{1\ldots k}(t_1,\ldots,t_k)$ by simply substituting into the expression (3-2). (Some workers will perhaps wish to substitute into the formulas for Fisher's k-statistics. This wish is in some ways analogous to the question of dividing by N or $N-k$ in the two-dimensional case.) If the means have previously been subtracted from the original series, the expression (3-2) will involve fewer terms.

An estimate of the k-th-order polyspectrum may now be obtained by Fourier transforming this estimated cumulant. Doing this alone however gives an estimate analogous to the periodogram in the two-dimensional case, and consequently we are led to the insertion of convergence factors into the Fourier transform. The proposed estimate is consequently,

$$\Sigma\ldots\Sigma\ \lambda_{j_1\ldots j_{k-1}}\ e^{i(\omega_1 j_1+\ldots+\omega_{k-1}j_{k-1})}\ \hat{C}_{1\ldots k}(j_1,\ldots,j_{k-1}) \quad , \tag{4-2}$$

in which the $\lambda_{j_1\ldots j_{k-1}}$ are the inserted convergence factors and $\hat{C}$ is the estimate of the cumulant. The simplest choice for $\lambda_{j_1\ldots j_{k-1}}$ would appear to be to take it as the product $\lambda_{j_1},\lambda_{j_2}\ldots\lambda_{j_{k-1}}$ of a number of one-dimensional convergence factors, e.g., Hanning; however it may also be desirable to take it to be spherically symmetric.

A second technique that may be employed is the following; consider a time series $X(t)$. Band-pass filter this series to a narrow frequency band about ω_0 (> 0) to obtain the series $X(t;\omega_0)$. Also obtain an estimate of the Hilbert transform of this latter series, denoted by $X^H(t,\omega_0)$. (A computational technique for estimating $X(t,\omega_0)$, $X^H(t,\omega_0)$ will be proposed later.) Define $X(t,\omega_0) = X(t, -\omega_0)$ and $X^H(t,\omega_0) = -X^H(t,-\omega_0)$ if ω_0 is negative.

To carry out the estimation of $C_{1\ldots k}(\omega_1,\ldots,\omega_k)$ where $\Sigma\, \omega_i = 0$, first estimate $X_i(t,\omega_i)$, $X_i^H(t,\omega_i)$ for each of the series $X_i(t)$. Now form the product,

$$\{X_1(t,\omega_0) + X_1^H(t,\omega_0)\}\ldots\{X_k(t,\omega_k) + iX_k^H(t,\omega_k)\} \quad . \qquad (4\text{-}2)$$

If this product and similar lower-order products are averaged for a stretch of time, an estimate of the required polyspectrum is obtained by forming the appropriate combination of the averages, i.e., the combination (3-2).

This technique has the advantage over the preceding technique that a running estimate of the polyspectrum is obtained and so the presence of nonstationarities may be investigated. In addition, once the band-pass filtered series have been obtained, they may be put to a variety of uses, for example, in the estimation of polyspectra of varied orders involving later series that have not yet been collected. The series has to be band-pass filtered only once provided enough foresight has been shown in the selection of the bandwidths of the filters employed.

A useful technique for obtaining series $X(t,\omega_0)$, $X^H(t,\omega_0)$ is presented in [7], pp. 77-78. The series $a_m(t)$, $b_m(t)$ derived therein may be used in place of $X(t,\omega_0)$, $X^H(t,\omega_0)$ where $w_0 = \frac{\pi m}{N}$ and the notation of [7] is being adhered to. The advantage of using this technique lies in the fact that $a_m(t)$, $b_m(t)$ are generated by means of recurrence relationships. This reduces the number of operations from the number required in a straightforward filtering procedure.

Estimates better, in a certain sense, than the $a_m(t)$, $b_m(t)$ above, may be obtained if some sort of convergence factors are employed, for example if,

$$.23\ a_{m-1}(t) + .54\ a_m(t) + .23\ a_{m-1}(t) \quad , \tag{4-3}$$

$$.23\ b_{m-1}(t) + .54\ b_m(t) + .23\ b_{m-1}(t) \quad , \tag{4-4}$$

are employed.

The simplest of the three proposed estimation techniques, so far as programming goes (the method was in fact programmed very quickly by Michael Godfrey) appears to be the following one based upon the procedure of complex demodulation [20].

Consider the time series X_t. Let $U(t,\omega_0)$, $U^H(t,\omega_0)$ denote the result of complex demodulating X_t at the frequency ω_0; that is U and U^H come about as the result of the following steps;

(i) the series $X_t \cos \omega_0 t$, $X_t \sin \omega_0 t$ are formed,

(ii) these two series are smoothed giving U and U^H respectively.

The desired polyspectrum may now be estimated by carrying out the steps of the previous technique with $X(t,\omega_0)$, $X^H(t,\omega_0)$ replaced by $U(t,\omega_0)$, $U^H(t,\omega_0)$ respectively.

In connection with the last two techniques mentioned above, the following deviation from the procedures described may prove useful in certain problems. Suppose that it is felt that the component at frequency ω_k in the k-th time series comes about as the result of the beating together of the components at frequencies $\omega_1, \omega_2, \ldots, \omega_{k-1}$ in other series where $\Sigma\, \omega_i = 0$. The bandwidth of the product series,

$$\{X_1(t,\omega_0) + iX^H(t,\omega_0)\}\ldots\{X_{k-1}(t,\omega_{k-1}) + i\, X^H_{k-1}(t,\omega_{k-1})\} \quad (4\text{-}5)$$

is $(k-1)$ times the bandwidth of the component,

$$\{X_k(t,\omega_k) + iX^H_k(t,\omega_k)\} \quad (4\text{-}6)$$

if the series have all been filtered with the same bandwidth. Consequently it may well be desirable to arrange for the k-th series to be filtered with a larger bandwidth.

In respect of the polyspectral regression coefficient and the polyspectral coefficient, these quantities may be estimated by inserting estimates of the relevant polyspectrum and power spectrum into the expressions (3-19), (3-20).

5. Some Statistical Properties

Consider the estimation of $M(\omega_1,\ldots,\omega_k)$ by time averaging the product of the outputs of a number of band-passed series, namely the results of band-passing $X_i(t)$ to a frequency ω_i , where the various ω_i are such that $\Sigma\, \omega_i = 0$. (Band-passed in this section refers to the passing of the series through a filter with transfer function = 1 if $|\omega_i - \omega| < \Delta$ and 0 otherwise.)

If $g_i(t)$ (which is complex-valued) denotes the impulse response of the i-th filter, the output of the filter may be written as,

$$Y_i(t) = \int g_i(\tau)\, X_i(t-\tau)\, d\tau \quad . \tag{5-1}$$

(Throughout the present section integrals in the time domain may be replaced by summations if discrete time series are involved.)

The time average under consideration may now be written as,

$$\frac{1}{2T}\int_{-T}^{T} Y_1(t)\ldots Y_k(t)\, dt \tag{5-2}$$

$$= \int\cdots\int g_1(\tau_1)\ldots g_k(\tau_k)\Big\{\frac{1}{2T}\int_{-T}^{T} X_1(t-\tau_1)\ldots X_k(t-\tau_k)dt\Big\}\, d\tau_1\ldots d\tau_k \tag{5-3}$$

$$= \int\cdots\int g_1(\tau_1)\ldots g_k(\tau_k)\hat{m}_{1\ldots k}(\tau_1,\ldots,\tau_k)d\tau_1\ldots d\tau_k \quad . \tag{5-4}$$

The expected value of (5-4) may be written (subject to regularity conditions) as,

$$\int\cdots\int g_1(\tau_1)\ldots g_k(\tau_k)m_{1\ldots k}(\tau_1,\ldots,\tau_k)d\tau_1\ldots d\tau_k \quad , \tag{5-5}$$

or in terms of the Fourier transforms as,

$$\int \cdots \int \delta(X_1+\cdots+X_k)\ G_1(X_1)\cdots G_k(X_k)\ M(X_1,\ldots,X_k)\ dX_1\cdots dX_k\ . \qquad (5\text{-}6)$$

Consequently if the filters are normalized such that,

$$\int \cdots \int \delta(X_1+\cdots+X_k)\ G_1(X_1)\cdots G_k(X_k) dX_1\cdots dX_k = 1 \ , \qquad (5\text{-}7)$$

and M is approximately constant in the pass-band of the filters, then the expected value of (5-2) is approximately the desired $M(\omega_1,\ldots,\omega_k)$.

However, in spite of this result, the reason for the concern evidenced in this paper over the presence of delta functions in $M(\omega_1,\ldots,\omega_k)$ is the expression (5-6). This expression indicates that if one proceeds via the indicated technique of estimation, one must be content with a weighted average of the expression of interest. Consequently, if delta functions are present, they will influence, possibly greatly, the estimates at all frequency combinations.

Consider next another estimate of the term (5-2),

$$\frac{1}{2T} \int_{-T}^{T} Z_1(t)\cdots Z_k(t)\ dt \quad , \qquad (5\text{-}8)$$

where $Z_i(t) = \int h_i(t-\tau)\ X_i(\tau)d\tau$. The product of (5-2) and (5-8) may be written,

$$\int \cdots \int g_1(\tau_1)\cdots g_k(\tau_k)\ h_1(\sigma_1)\cdots h_k(\sigma_k)\{\frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T} X_1(t-\tau_1)\cdots X_k(t-\tau_k)$$

$$X_1(s-\sigma_1)\cdots X_k(s-\sigma_k)dtds\}d\tau_1\cdots d\tau_k\ d\sigma_1\cdots d\sigma_k$$

(5-9)

for which the expected value is,

$$\int\cdots\int g_1(\tau_1)\cdots g_k(\tau_k)\ h_1(\sigma_1)\cdots h_k(\sigma_k)\{\frac{1}{(2T)^2}\int\int m_{11\ldots kk}$$

$$(t-\tau_1,s-\sigma_1,\ldots,t-\tau_k,s-\sigma_k)dtds\}\ d\tau_1\ldots d\tau_k\ d\sigma_1\ldots d\sigma_k \quad . \tag{5-10}$$

In terms of the Fourier transforms this may be written as,

$$\int\cdots\int G_1(X_1)\cdots G_k(X_k)H_1(\zeta_1)\cdots H_k(\zeta_k)\ \frac{\sin(X_1+\ldots+X_k)T}{(X_1+\ldots+X_k)T}\ \frac{\sin(\zeta_1+\ldots+\zeta_k)T}{(\zeta_1+\ldots+\zeta_k)T}$$

$$M_{11\ldots kk}(X_1,\zeta_1,\ldots,X_k,\zeta_k)\ \delta(X_1+\zeta_1+\ldots+X_k+\zeta_k)dX_1\ldots dX_k d\zeta_1\ldots d\zeta_k \quad . \tag{5-11}$$

The covariance of (5-2) and (5-7) may consequently be found by subtracting the product of the means of (5-2) and (5-8), (as given by (5-6)), from (5-11). The resulting formula may be written in terms of cumulants if desired and in this form a working form for the covariance not requiring the estimation of any polyspectra of order higher than k could be obtained by simply assuming that these spectra are zero. Estimation of the variability does seem to be more important to the design rather than the assessment after the experiment in any case.

Now turn to the estimation of the polyspectrum. Consider first the case in which the product moments of order less than k are known.

Consider the estimator,

$$(5\text{-}2) + \sum_{p=2}^{k} (-1)^{p-1}(p-1)!\int\cdots\int g_1(\tau_1)\cdots g_k(\tau_k)m_{\upsilon_1}(\tau_{\upsilon_1})\cdots m_{\upsilon_p}(\tau_{\upsilon_p})d\tau_1\ldots d\tau_p\ , \tag{5-12}$$

in which $(\upsilon_1, \ldots, \upsilon_p)$ denotes a grouping of the integers 1 through k into p groups.

The expected value of this expression is easily seen to be,

$$\int \cdots \int g_1(\tau_1) \cdots g_k(\tau_k)\ C_{1 \ldots k}(\tau_1, \ldots, \tau_k)\ d\tau_1 \ldots d\tau_k \quad , \tag{5-13}$$

$$= \int \cdots \int \delta(\chi_1 + \ldots + \chi_k)\ G_1(\chi_1) \cdots G_k(\chi_k)\ C_{1 \ldots k}(\chi_1, \ldots, \chi_k)\ d\chi_1 \ldots d\chi_k \tag{5-14}$$

and (5-12) is consequently providing a weighted estimate of the desired polyspectrum.

The covariance of two expressions of the form (5-12) is identical with the covariance of the added terms of the form (5-2). This covariance has been derived earlier.

Consider now the case in which the lower-order product moments are not known and must therefore be estimated from the data. The formulas for the means and variances of the natural estimate are complicated even in the second-order case, see [2] pp. 139-146.

For simplicity consider the second estimate proposed for this case in Section 4. It was,

$$\Sigma(-1)^{p-1}(p-1)!\ \hat{n}_{\upsilon_1}(t_{\upsilon_1}) \cdots \hat{n}_{\upsilon_p}(t_{\upsilon_p}) \quad , \tag{5-15}$$

where the $\hat{n}$'s denote the estimated product moments as derived from the filtered series $\{X_j(t, \omega_j) + iX_j(t, \omega_j)\}$. (5-15) may be written,

$$\Sigma(-1)^{p-1}(p-1)! \int\cdots\int g_1(\tau_1)\cdots g_k(\tau_k)\ \hat{m}_{\upsilon_1}(\tau_{\upsilon_1})\cdots m_{\upsilon_p}(\tau_{\upsilon_p})\ d\tau_1\cdots d\tau_k) \qquad (5\text{-}16)$$

$$= \int\cdots\int g_1(\tau_1)\cdots g_k(\tau_k)\{\Sigma(-1)^{p-1}(p-1)!\ \hat{m}_{\upsilon_1}(\tau_{\upsilon_1})\cdots\hat{m}_{\upsilon_p}(\tau_{\upsilon_p})\}d\tau_1\cdots d\tau_k. \qquad (5\text{-}17)$$

Under regularity conditions allowing the interchange of the operations of integration and of taking an expected value, the expected value of (5-17) is,

$$\int\cdots\int g_1(\tau_1)\cdots g_k(\tau_k)\{\Sigma(-1)^{p-1}(p-1)!\ E[\hat{m}_{\upsilon_1}(\tau_{\upsilon_1})\cdots\hat{m}_{\upsilon_p}(\tau_{\upsilon_p})]\ d\tau_1\cdots d\tau_k. \qquad (5\text{-}18)$$

Now,

$$\hat{m}_{\upsilon_1}(\tau_{\upsilon_1})\cdots\hat{m}_{\upsilon_p}(\tau_{\upsilon_p}) \ = \ \frac{1}{(2T)^p}\int_{-T}^{T}\cdots\int_{-T}^{T} Y_{\upsilon_1}(t_1)\cdots Y_{\upsilon_p}(t_p)\ dt_1\cdots dt_p\ , \qquad (5\text{-}19)$$

where,

$$Y_\upsilon(t_j) \ = \ X_{i_1}(t_j+\tau_{i_1})\ X_{i_2}(t_j+\tau_{i_2})\cdots X_{i_q}(t_j+\tau_{i_q})\ , \qquad (5\text{-}20)$$

if υ corresponds to the grouping $(i_1,\ldots,i_q)$.

The expected value of (5-19) may now be written as,

$$\frac{1}{(2T)^p}\int_{-T}^{T}\cdots\int_{-T}^{T} m_{\upsilon_1\cdots\upsilon_p}(t_1+\tau_{\upsilon_1},\ldots,t_p+\tau_{\upsilon_p})\ dt_1\cdots dt_p\ , \qquad (5\text{-}21)$$

where $t+\tau_\upsilon$ denotes $(t+\tau_{i_1},\ t+\tau_{i_2},\ldots,\ t+\tau_{i_q})$ if υ corresponds to the grouping $(i_1,\ldots,i_q)$.

This may be substituted into (5-18) to obtain an expression for the required expected values. A further simplification will be obtained later in this section. The reader may gain some confidence in this result if he notes that in Appendix 1 it is shown that for a wide class of processes,

$$(5\text{-}21) \rightarrow m_{\upsilon_1}(\tau_{\upsilon_1}) \ldots m_{\upsilon_p}(\tau_{\upsilon_p}) \quad , \qquad (5\text{-}22)$$

as $T \rightarrow \infty$.

Up to this point the formulas have been derived relative to the second estimation technique of Section 4. To derive the corresponding formulas for the first technique let,

$$G(\omega_1, \ldots, \omega_k) = \Sigma \lambda_{i_1 \ldots i_k} e^{i(\omega_i i_1 + \ldots + \omega_k i_k)} \quad , \qquad (5\text{-}23)$$

and substitute $G(\omega_1, \ldots, \omega_k)$ wherever the product $G_1(\omega_1) \ldots G_k(\omega_k)$ occurs in the results derived above.

For the third technique simply substitute,

$$G_i(\omega) = L_i(\omega - \omega_i) \quad , \qquad (5\text{-}24)$$

into the results where $L_i(\omega)$ denotes the transfer function of the low-pass filter employed in the complex demodulation of $X_i(t)$.

This section will be concluded by a brief investigation of the asymptotic behavior of the proposed estimates including a sketch of a proof of their asymptotic normality.

Consider the estimate (5-17). In terms of the spectral representation of the process, this estimate is equivalent to,

$$\int \cdots \int G_1(\omega_1)\cdots G_k(\omega_k)\ \Phi_T(\omega_1,\ldots,\omega_k)\ dZ_1(\omega_1)\cdots dZ_k(\omega_k) \quad , \tag{5-25}$$

where,

$$\Phi_T(\omega_1,\ldots,\omega_k) = \Sigma(-1)^{p-1}(p-1)!\ \frac{\sin \tilde{\omega}_{\pi_1} T \cdots \sin \tilde{\omega}_{\pi_p} T}{\tilde{\omega}_{\pi_1} T \cdots \tilde{\omega}_{\pi_p} T} \quad . \tag{5-26}$$

Since,

$$E\{dZ_1(\omega_1)\cdots dZ_k(\omega_k)\} = \delta(\omega_1 + \cdots \omega_k)\ M(\omega_1,\ldots,\omega_k)\ d\omega_1\cdots d\omega_k \quad , \tag{5-27}$$

$$= \Sigma(-1)^{p-1}(p-1)!\ \delta(\tilde{\omega}_{\pi_1})\cdots\delta(\tilde{\omega}_{\pi_p})\ C_{\pi_1}(\omega_{\pi_1})\cdots C_{\pi_p}(\omega_{\pi_p})\ d\omega_1\cdots d\omega_k \quad , \tag{5-28}$$

the expected value of (5-24) may be written,

$$\int \cdots \int G_1(\omega_1)\cdots G_k(\omega_k)\ \Phi_T(\omega_1,\ldots,\omega_k)\ \Sigma(-1)^{p-1}(p-1)!\ \delta(\tilde{\omega}_{\pi_1})\cdots\delta(\tilde{\omega}_{\pi_p})$$
$$C_{\pi_1}(\omega_{\pi_1})\cdots C_{\pi_p}(\omega_{\pi_p})\ d\omega_1\cdots d\omega_k \quad . \tag{5-29}$$

The function Φ_T defined by (5-26) may be seen to have the property that,

$$\delta(\tilde{\omega}_{\upsilon_1})\cdots\delta(\tilde{\omega}_{\upsilon_q})\ \Phi_T(\omega_1,\ldots,\omega_k) = 0, \qquad q > 1 \tag{5-30}$$

$$= \delta(\omega_1 + \cdots \omega_k)\ \Phi_T(\omega_1,\ldots,\omega_k)$$

$$q = 1 \tag{5-31}$$

for groupings $(\upsilon_1,\ldots,\upsilon_q)$. Using this property the expected value of (5-25) is seen to be,

$$\int\cdots\int G_1(\omega_1)\cdots G_k(\omega_k)\ \Phi_T(\omega_1+\cdots+\omega_k)\ C(\omega_1,\ldots,\omega_k)\ d\omega_1\cdots d\omega_k \quad , \tag{5-32}$$

which as $T \to \infty$, tends to,

$$\int\cdots\int G_1(\omega_1)\cdots G_k(\omega_k)\ \delta(\omega_1+\cdots+\omega_k)\ C(\omega_1,\ldots,\omega_k)\ d\omega_1\cdots d\omega_k \quad , \tag{5-33}$$

as desired.

In order to investigate the asymptotic distribution of the proposed estimate (5-25), consider its j-th-order cumulant. This cumulant may be written,

$$\int\cdots\int \Phi_T(\omega^1)\cdots\Phi_T(\omega^j)\ G^1(\omega^1)\cdots G^j(\omega^j)\ \delta(\tilde{\omega}^1+\cdots+\tilde{\omega}^j)$$
$$C_{(1\ldots k)\ldots(1\ldots k)}(\omega^1,\ldots,\omega^j)\ d\omega^1\cdots d\omega^j \quad , \tag{5-34}$$

where,

$$\omega^n = (\omega_1^n,\ldots,\omega_k^n) \quad ,$$

$$G^1(\omega^n) = G_1(\omega_1^n)\cdots G_k(\omega_k^n) \quad ,$$

and $C_{(1\ldots k)\ldots(1\ldots k)}(\omega^1,\ldots,\omega^j)$ is the j-th cumulant of $dZ_1(\omega_1)\cdots dZ_k(\omega_k)$. This cumulant may be expanded in terms of the cumulants of the basic process and is seen to involve delta functions. However, because of the relations (5-30), (5-31) these delta functions actually drop out leaving an expression of the form,

$$\int \cdots \int \Phi_T(\omega^1) \cdots \Phi_T(\omega^j)\; G^1(\omega^1) \cdots G^j(\omega^j)\; \delta(\tilde{\omega}^1 + \cdots + \tilde{\omega}^j)\; D(\omega^1, \ldots, \omega^j)\; d\omega^1 \cdots d\omega^j \;, \tag{5-35}$$

where D is bounded if the cumulants of the original process are bounded. Assuming that the filter functions G are bounded and integrable, (5-35) in absolute value is less than,

$$M \int \cdots \int \left| \frac{\sin \Theta_1 T \cdots \sin \Theta_j T}{\Theta_1 T \cdots \Theta_j T} \right| \delta(\Theta_1 + \cdots + \Theta_j)\; d\Theta_1 \cdots d\Theta_j + O\left(\frac{1}{T^{2j}} \right) \quad , \tag{5-36}$$

for some constant M.

Because,

$$\int \cdots \int \frac{\sin \Theta_1 \cdots \sin \Theta_m}{\Theta_1 \cdots \Theta_m} \; \frac{\sin(\Theta_1 + \cdots + \Theta_m)}{(\Theta_1 + \cdots + \Theta_m)}\; d\Theta_1 \cdots d\Theta_m \quad , \tag{5-37}$$

is absolutely integrable (as may be demonstrated by arguments simular to those in [23]).

$$\int \cdots \int \left| \frac{\sin \Theta_1 T}{\Theta_1 T} \cdots \frac{\sin \Theta_j T}{\Theta_j T} \right| \delta(\Theta_1 + \cdots + \Theta_j)\; d\Theta_1 \cdots d\Theta_j = O\left(\frac{1}{T^{j-1}} \right) \;. \tag{5-38}$$

Consequently under the above conditions the j-th cumulant of $T^{1/2}$ (5-25) approaches 0 for $j > 2$.

Consider now the 2nd cumulant or variance of $T^{1/2}$ (5-25). It is,

$$T \int \cdots \int \Phi_T(\omega^1)\; \Phi_T(\omega^2)\; G^1(\omega^1)\; G^2(\omega^2)\; \delta(\tilde{\omega}^1 + \tilde{\omega}^2)$$

$$\{M_{1\ldots k,1\ldots k}(\omega^1, \omega^2) - M_{1\ldots k}(\omega^1)\; M_{1\ldots k}(\omega^2)\}\; d\omega^1 d\omega^2 \quad . \tag{5-39}$$

As $T \to \infty$, this expression may be seen to approach,

$$\int\cdots\int G^1(\omega^1)\ G^2(\omega^2)\ \Sigma\delta(\tilde{\omega}_{\pi_1})\cdots\delta(\tilde{\omega}_{\pi_p})\ C_{\pi_1}(\omega_{\pi_1})\cdots C_{\pi_p}(\omega_{\pi_p})\ d\omega^1 d\omega^2\ , \tag{5-40}$$

under regularity conditions upon the G's and C's as a result of a theorem of [22]. The summation in (5-39) extends over all groupings of (ω^1, ω^2) such that each unit of a grouping involves terms from both ω^1 and ω^2.

In summary of these last calculations, it has been demonstrated that the proposed estimator, (5-25), under regularity conditions is asymptotically normal with mean (5-33) and variance $\frac{1}{T}$ (5-40).

That estimates of (3-19), (3-20) and (3-21), obtained by substituting estimates of the polyspectra as obtained above, will be asymptotically normal follows from the asymptotic normality of the polyspectral estimates and a Slutsky type theorem.

6. Moments or Cumulants?

The first possible definition of a polyspectrum likely to come to one's mind, is to define it as the Fourier transform of the product moment, that is by,

$$\delta(\omega_1+\cdots+\omega_k)\ M(\omega_1,\ldots,\omega_k) = \int\cdots\int e^{i(\omega_1 t_1+\cdots+\omega_k t_k)} m_{1\ldots k}(t_1,\ldots,t_k)dt_1\cdots dt_k . \tag{6-1}$$

(The reader is reminded that the delta function is introduced by the stationarity condition,

$$m_{1\ldots k}(t+t_1,\ldots,t+t_k) = m_{1\ldots k}(t_1,\ldots,t_k) \quad \text{for all } t.)$$

This definition has the disadvantage that the function M contains delta functions for a wide class of processes. To see this, consider M upon a surface of the form,

$$\omega_1 + \ldots + \omega_{k_1} = 0 \ ; \quad \omega_{k_1+1} + \ldots + \omega_{k_2} = 0 \ ; \ldots \ ; \ \omega_{k_j+1} + \ldots + \omega_k = 0 \ . \tag{6-2}$$

In this case,

$$\delta(\omega_1+\ldots+\omega_k)\, M(\omega_1,\ldots,\omega_k) = \int\ldots\int e^{i(\omega_1 t_1+\ldots+\omega_k t_k)}\, m(t_1,t_2,\ldots,t_k)dt_1\ldots dt_k, \tag{6-3}$$

$$= \int\ldots\int e^{i\Phi(\omega,t,\tau)}\, m(t_1,\ldots,t_k)dt_1\ldots dt_k \ , \tag{6-4}$$

for all $\tau_1,\ldots,\tau_j$ where,

$$\Phi(\omega,t,\tau) = \{\omega_1(t_1+\tau_1)+\ldots+\omega_{k_1}(t_{k_1}+\tau_1)\} +\ldots+\{\omega_{k_j+1}(t_{k_j+1}+\tau_j)+\ldots\omega_k(t_k+\tau_j)\}. \tag{6-5}$$

Making the change of variables $t + \tau \rightarrow t$ in the integral (6-4) leads to,

$$\int\ldots\int e^{i(\omega_1 t_1+\ldots+\omega_k t_k)}\, m(t,\tau)\, dt_1\ldots dt_k \quad , \tag{6-6}$$

where,

$$m(t,\tau) = m(t_1-\tau_1,\ldots,t_{k_1}-\tau_1;\ldots;\ t_{k_j+1}-\tau_j,\ldots,t_k-\tau_j) \ . \tag{6-7}$$

The τ's in the integral may now be averaged out without affecting the value of the integral giving,

$$\int \cdots \int e^{i(\omega_1 t_1 + \dots + \omega_k t_k)} \{ \frac{1}{(2T)^j} \int_{-T}^{T} \cdots \int_{-T}^{T} m(t,\tau) d\tau_1 \dots d\tau_j \} dt_1 \dots dt_k \quad . \qquad (6\text{-}8)$$

Letting $T \to \infty$, under conditions presented in Appendix 1, the inner term in (6-8) approaches $m_{\pi_1}(t_{\pi_1}) \dots m_{\pi_j}(t_{\pi_j})$ where $(\pi_1; \dots ; \pi_j)$ is the grouping of subscripts $(1,2,\dots,k_1;\dots;k_j+1,\dots,k)$. The limit of (6-8) as $T \to \infty$ is consequently seen to be,

$$\delta(\tilde{\omega}_{\pi_1}) \dots \delta(\tilde{\omega}_{\pi_j}) \, M_{\pi_1}(\omega_{\pi_1}) \dots M_{\pi_j}(\omega_{\pi_j}) \quad . \qquad (6\text{-}8)$$

(The reader is reminded that $\tilde{\omega}_{\pi_1} = \omega_1 + \dots + \omega_{k_1}$ for example.)

By inspection one notes that the additional delta functions in (6-8) might have been avoided by taking the Fourier transform of,

$$m_{1\dots k}(t_1,\dots,t_k) - m_{\pi_1}(t_{\pi_1}) \dots m_{\pi_j}(t_{\pi_j}) \quad , \qquad 6\text{-}9)$$

rather than simply $m_{1\dots k}(t_1,\dots,t_k)$.

Consider the problem of finding an expression involving the product moments that does not have awkward delta functions in its Fourier transform. If one subtracts the terms causing delta functions on surfaces of the form,

$$\tilde{\omega}_{\upsilon_1} = 0 \; , \; \tilde{\omega}_{\upsilon_2} = 0 \quad , \qquad (6\text{-}10)$$

one is led to,

$$m_{1\dots k}(t_1,\dots,t_k) - \Sigma m_{\upsilon_1}(t_{\upsilon_1}) m_{\upsilon_2}(t_{\upsilon_2}) \quad , \qquad (6\text{-}11)$$

where the summation extends over all possible groupings of $1,\dots,k$ into two groups.

Unfortunately this expression still possesses spikes on surfaces of the form,

$$\tilde{\omega}_{\pi_1} = 0, \quad \tilde{\omega}_{\pi_2} = 0, \quad \tilde{\omega}_{\pi_3} = 0 \,, \tag{6-12}$$

for consider the Fourier transform of (6-11) on this surface. It is

$$\delta(\tilde{\omega}_{\pi_1})\, \delta(\tilde{\omega}_{\pi_2})\, \delta(\tilde{\omega}_{\pi_3})\, M_{\pi_1}(\omega_{\pi_1})\, M_{\pi_2}(\omega_{\pi_2})\, M_{\pi_3}(\omega_{\pi_3})$$

$$- S(3,2)\, \delta(\tilde{\omega}_{\pi_1})\, \delta(\tilde{\omega}_{\pi_2})\, \delta(\tilde{\omega}_{\pi_3})\, M_{\pi_1}(\omega_{\pi_1})\, M_{\pi_2}(\omega_{\pi_2})\, M_{\pi_3}(\omega_{\pi_3}) \quad , \tag{6-12}$$

where $S(3,2)$ denotes the number of ways of putting three different things into two like cells. The contribution to (6-12) from $m_{1\ldots k}(t_1,\ldots,t_k)$ has been derived at (6-8). The contribution from $\Sigma m_{\upsilon_1}(t_{\upsilon_1})\, m_{\upsilon_2}(t_{\upsilon_2})$ results from the fact that any individual term in this sum which is not such that thr grouping (π_1, π_2, π_3) is a refinement of its grouping leads to a Fourier transform of zero, while if it is a refinement a contribution of the form,

$$\delta(\tilde{\omega}_{\pi_1})\, \delta(\tilde{\omega}_{\pi_2})\, \delta(\tilde{\omega}_{\pi_3})\, M_{\pi_1}(\omega_{\pi_1})\, M_{\pi_2}(\omega_{\pi_2})\, M_{\pi_3}(\omega_{\pi_3}) \quad , \tag{6-13}$$

is obtained (use the device of inserting dummy parameters and Appendix 1). The number of refinements is $S(3,2)$.

The delta functions in (6-12) may be removed by subtracting an appropriate expression. This leads one to the consideration of,

$$m_{1\ldots k}(t_1,\ldots,t_k) - \Sigma m_{\upsilon_1}(t_{\upsilon_1})\, m_{\upsilon_2}(t_{\upsilon_2}) + 2\Sigma m_{\upsilon_1}(t_{\upsilon_1})\, m_{\upsilon_2}(t_{\upsilon_2})\, m_{\upsilon_3}(t_{\upsilon_3}) \,. \tag{6-14}$$

Suppose that one continues in this way until he is considering the expression,

$$T_1 m_{1\ldots k}(t_1,\ldots,t_k) + T_2 \Sigma m_{\upsilon_1}(t_{\upsilon_1})\, m_{\upsilon_2}(t_{\upsilon_2}) + \ldots + T_{n-1}\Sigma m_{\upsilon_1}(t_{\upsilon_1})\ldots m_{\upsilon_{n-1}}(t_{\upsilon_{n-1}}) \ , \tag{6-15}$$

where $T_1 = 1$, $T_2 = -1$, $T_3 = 2$, this expression being such that it has no delta functions on surfaces corresponding to grouping the integers into n-1 or fewer groups. Consider its Fourier transform on the surface,

$$\tilde{\omega}_{\pi_1} = 0,\ldots,\tilde{\omega}_{\pi_n} = 0 \ . \tag{6-16}$$

It has the form,

$$\delta(\tilde{\omega}_{\pi_1})\ldots\delta(\tilde{\omega}_{\pi_n})\, M_{\pi_1}(\omega_{\pi_1})\ldots M_{\pi_n}(\omega_{\pi_n})\{\sum_{p=1}^{n-1} T_p\, S(n,p)\} \tag{6-17}$$

where $S(n,p)$ denotes the number of ways of putting n different things into p like cells with no cells empty. That $S(n,p)$ is the appropriate coefficient of T_p follows from the fact that the terms of the form,

$$m_{\upsilon_1}(t_{\upsilon_1})\ldots m_{\upsilon_p}(t_{\upsilon_p}) \ , \tag{6-18}$$

transform to a term of the form,

$$\delta(\tilde{\omega}_{\pi_1})\ldots\delta(\tilde{\omega}_{\pi_n})\, M_{\pi_1}(\omega_{\pi_1})\ldots M_{\pi_n}(\omega_{\pi_n}) \ , \tag{6-19}$$

whenever $(\pi_1,\ldots,\pi_n)$ is a refinement of $(\upsilon_1,\ldots,\upsilon_p)$ and to zero otherwise as a result of the theorem of Appendix 1. The number of such non-zero terms is $S(n,p)$.

To avoid the delta functions a term of the form,

$$\{\sum_{p=1}^{n-1} T_p S(n,p)\}\ \{\Sigma m_{\upsilon_1}(\omega_{\upsilon_1})\ldots m_{\upsilon_n}(\omega_{\upsilon_n})\}\quad , \tag{6-20}$$

must be subtracted. This means that,

$$T_n = -\sum_{p=1}^{n-1} T_p S(n,p)\quad . \tag{6-21}$$

In Appendix 2 it is shown that (6-21) implies

$$T_n = (-1)^{n-1}(n-1)!\quad .$$

One is consequently led to the consideration of the expression,

$$\Sigma(-1)^{p-1}(p-1)!\ m_{\upsilon_1}(t_{\upsilon_1})\ldots m_{\upsilon_p}(t_{\upsilon_p})\quad , \tag{6-22}$$

which is the expansion of the cumulant

$$c_{1\ldots k}(t_1,\ldots,t_k)\quad .$$

7. Acknowledgement

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Appendix 1

The following lemma will be required in the proof of the theorem of this appendix.

Lemma. Let $\{X_n\}$ be a sequence of random variables approaching μ in probability. Let Y be a random variable such that for some $\delta > 0$, (i) $E|Y|^{1+\delta}$ exists, (ii) for n sufficiently large $E|(X_n-\mu)Y|^{1+\delta}$ exists, then, $EX_nY \to \mu EY$.

Proof:

$$|EX_nY-\mu EY| = |E(X_n-\mu)Y| , \qquad (A1-1)$$

$$\leq E|(X_n-\mu)Y| , \qquad (A1-2)$$

$$= \int |X_n-\mu| \; |Y| \; dP_n(x,y) , \qquad (A1-3)$$

$$= \int_{|X_n-\mu|\leq\epsilon} |X_n-\mu| \; |Y| \; dP_n(x,y) + \int_{|X_n-\mu|>\epsilon} |X_n-\mu| \; |Y| \; dP_n(x,y) \qquad (A1-4)$$

where $P_n(x,y)$ denotes the joint probability function of X_n and Y. The first term in (A1-4) is $\leq \epsilon E|Y|$ and consequently may be made arbitrarily small by a choice of ϵ. The second term is,

$$\leq \{\int_{|X_n-\mu|>\epsilon} dP_n(x,y)\}^{\delta/1+\delta} \{\int |X_n-\mu|^{1+\delta} \; |Y|^{1+\delta} \; dP_n(x,y)\}^{1/1+\delta} . \qquad (A1-5)$$

The first term in (A1-5) may be made arbitrarily small as a result of the convergence in probability of $\{X_n\}$ to μ, while the second term is bounded; consequently (A1-4) may be made arbitrarily small and the lemma follows.

<u>Theorem</u>: Let $\{Y(t), Z(t)\}$ be a stochastic process such that

(i) $EY(t)$ and $EZ(t)$ exist and equal μ_y, μ_z respectively,

(ii) $E\{Y(t_1)\ Z(t_2)\}$ exists and equals $m_{yz}(t_1,t_2)$,

(iii) $\{Y(t)\}$ is ergodic in the sense that,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Y(t+\tau)\,d\tau \overset{p}{=} \mu_y \quad , \tag{A1-6}$$

(iv) $$E\{Z(t_2)\ \frac{1}{2T}\int_{-T}^{T} Y(t_1+\tau)\} = \frac{1}{2T}\int_{-T}^{T} E\{Z(t_2)Y(t_1+\tau)\}d\tau \quad , \tag{A1-7}$$

(v) $E|Z(t_2)|^{1+\delta}$ exists for some $\delta > 0$, and

(vi) $E|Z(t_2)(\frac{1}{2T}\int_{-T}^{T} Y(t_1+\tau)\ d\tau-\mu_y)|^{1+\delta}$ exists for T sufficiently large.

Under these conditions,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m_{yz}(t_1+\tau,\ t_2)d\tau = \mu_y\mu_z \quad . \tag{A1-8}$$

<u>Proof</u>:

$$\frac{1}{2T}\int_{-T}^{T} m_{yz}(t_1+\tau,\ t_2)d\tau = \frac{1}{2T}\int_{-T}^{T} E\{Z(t_2)\ Y(t_1+\tau)\}d\tau \quad , \tag{A1-9}$$

$$= E\{Z(t_2)\ \frac{1}{2T}\int_{-T}^{T} Y(t_1+\tau)d\tau\} \quad , \tag{A1-10}$$

from (A1-7). (A1-8) now follows immediately from the lemma.

Corollary 1. Let $\{X_1(t),\ldots,X_k(t)\}$ be a stochastic process with k-th-order product moment,

$$m_{1\ldots k}(t_1,\ldots,t_k) = E\{X_1(t_1)\ldots X_k(t_k)\} \quad .$$

Under conditions immediately deducible from the preceding conditions (i) to (v) by setting $Y(t) = X_1(t_1+t)\ldots X_j(t_j+t)$ and $Z(t) = X_{j+1}(t_{j+1}+t)\ldots X_k(t_k+t)$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m_{1\ldots k}(t_1+\tau,\ldots,t_j+\tau;t_{j+1},\ldots,t_k)d\tau$$

$$= m_{1\ldots j}(t_1,\ldots,t_j)\, m_{j+1\ldots k}(t_{j+1},\ldots,t_k) \quad . \tag{A1-11}$$

Corollary 2. In the same notation as Corollary 1, and under assumptions immediately deducible from the theorem,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m_{1\ldots j}(t_1+\tau,\ldots,t_\ell+\tau;t_{\ell+1},\ldots,t_j)\, m(t_{j+1}+\tau,\ldots,t_n+\tau;t_{n+1},\ldots,t_k)d\tau$$

$$= m_{1\ldots \ell}(t_1,\ldots,t_\ell)\, m_{\ell+1\ldots j}(t_{\ell+1},\ldots,t_j)\, m_{j+1\ldots n}(t_{j+1},\ldots,t_n)\, m_{n+1\ldots k}(t_{n+1},\ldots,t_k) \quad . \tag{A1-12}$$

There exist similar relations to (A1-12) if the integrand is the product of more than two terms.

Corollaries 1 and 2 provide another indication that the product moments are perhaps not the relevant quantities to be analyzing in many situations. The corollaries indicate that for a variety of stochastic processes, expressions involving lower-order product moments are immediately deducible from a product moment. This presence of lower-order information may prove distracting to one who has been following a step-by-step inquiry into the process.

It seems appropriate to end this paper on the folling not of pessimism. Experience with real random variables indicates that higher order moments are typically not efficient estimates of scientifically relevant parameters; consequently as the specifications of stochastic processes become tighter polyspectra are likely to prove less pertinent in a similar manner.

Appendix 2

Theorem: Let $S(n,m)$ denote the number of ways of putting n different things into m like cells with no cells empty. If,

$$T_n = - \sum_{p=1}^{n-1} T_p S(n,p) \quad , \qquad (A2\text{-}1)$$

for $n > 1$ and $T_1 = 1$, then,

$$T_n = (-1)^{n-1}(n-1)! \quad .$$

Proof: Assume that this result is true for $n \leq N-1$ and seek to prove it for $n = N$. That is, seek to prove,

$$T_N = - \sum_{p=1}^{N-1} (-1)^{p-1}(p-1)! \; S(N,p) \quad . \qquad (A2\text{-}2)$$

Now $S(n,p) = \Delta^p 0^n/p!$, see [15]. The right-hand side of (A2-2) is consequently,

$$= - \sum_{p=1}^{N-1} (-1)^{p-1} \Delta^p 0^N/p \quad , \qquad (A2\text{-}3)$$

$$= - \sum_{p=1}^{N-1} (-1)^{p-1} \{\Delta^p 0^{N-1} + \Delta^{p-1} 0^{N-1}\} \quad , \qquad (A2\text{-}4)$$

$$= - \sum_{p=1}^{N-1} (-1)^{p-1} \Delta^p 0^{N-1} - \sum_{p=0}^{N-2} (-1)^p \Delta^p 0^{N-1} \quad , \qquad (A2\text{-}5)$$

$$= (-1)^{N-1} \Delta^{N-1} 0^{N-1} \quad , \qquad (A2\text{-}6)$$

$$= (N-1)! \; (-1)^{N-1} \quad . \qquad (A2\text{-}7)$$

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