# VALUE and EXPENDITURE

PART I

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# VALUE and EXPENDITURE

An Investigation of Concepts

Relating to

The Consumer

By S. N. Afriat

## Part I

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# PREFACE

"Economics is an unfinished science"
F. Zeuthen.

It can be considered that economics relates to that part of the activities of a society which derives from the exchange of commodities. Thus the ultimate generator of economic activity is the need for possession, or, more essentially, for consumption. Accordingly there is, at least in principle, a dependence of theory of an economy on theory of the consumer; and this obtains a special importance for the consumer in economic theory.

In consumer theory there is in the first instance the scheme in which the consumer is pictured. Here this is embodied in the concept of an expenditure system, which is to express the idea of a consumer behaviour, and which is to become the object of mathematical elaboration and investigation.

The furnishing of this abstract concept of an expenditure system with various auxiliary concepts and structural features, and the definition of some special conditions on it which have an empirical source, builds up the matter of the subject, which is nothing more than an exhibition of the logical working of a theoretical image of the consumer. To put it in another way, it presents a mathematical subject, consisting of formal definitions, propositions and demonstrations. This is the content of Part I of this investigation, as contained in this first Report.

Such a theory can in itself tell us nothing about what we should of necessity expect of the consumer. The consumer can do what he likes; and no condition of behaviour can be mentioned but that he is at liberty to escape from it. However, it gives the basic language for treating the consumer. It should eventually show how to observe, and to formulate knowledge of the consumer, and find a basis for valid expectations; and also to give systematic concepts for often obscure, but nevertheless indispensable notions which are entertained.

Thus, Part II of this investigation, to be given in the next Report, approaches the old "index-number problem". It develops the very frequently used, but not sufficiently articulate ideas of standard and cost of living into a coherent scheme of analysis, by which is meant a scheme for measurement by observation, together with principles for the interpretation.

For something of the form of this analysis, some features of which are perhaps unexpected but nevertheless natural, a brief, very fragmentary and incomplete suggestion of it is to be found in Afriat [2], the second page, just as the first page there refers, though again quite incompletely, and in some details speculatively, to the matter elaborated in this Report.

The conceptual engineering needed to make something of the index-number problem, in a sense to define the cost of living, is made possible by some algebraical propositions, which show the way through the familiar and seemingly hopeless impasse in the subject (Samuelson [44], pp. 146 ff.) to arrive at a concept of measurement. An impediment to much of the usual thinking lies in the exclusive use of a graphical approach, which merely reveals apparent difficulties, and not the intrinsic possibilities. In any case, in this particular problem, the usual picture is in fact a serious illegitimacy, since it misrepresents the nature of the problem, and the crucial determinacies and indeterminacies involved; and dimension, which is inescapable in any geometrical figure, is a thing which is eventually to disappear, in the form of the algebraical analysis. For the generally elusive matter of the measurement of the "level" of prices, and individual and communal "welfare", there seems to be needed the attitude that there is a right way of doing a thing, even if that thing is impossible. For, in this matter, whether it is possible or impossible, it has to be done. It is at the centre of Trade Union negotiations, and all questions about the fitting adjustment of wage for a change of prices. It is required in the definition of real wage, and as a component in the Keynesian concept of national income. It is the source of judgement for questions about inflation. The vague but indispensible notions of standard and cost of living are much needing conception as systematic terms of measurement. In all this tangle of questions, there is called for that expert treatment which, according to a familiar definition, recognizes not just how much is enough, but also how little will do. It is such a treatment that has to be attempted in Part II, but it will be with the aid of some algebraical and other propositions that will make the whole task much more approachable.

The matter for Part II, on the analysis of expenditures, while related and complementary to the investigations of Parts I and II, is more of the nature of statistical method. The concern is with theory of models, the paramenters of which give a problem of estimation — the avoidance of which can be considered one of the central points of Part II. Beside predictors of distributions of expenditures, a special interest is in the technique of actually finding statistical maps for scales of consumer preferences, even a construction for what fits the idea of the much discussed "social welfare function", at any rate a function which is supposed to give the standard of consumption in terms of its composition.

Throughout all the work in Parts I-III, there is no occasion for recourse to the ideas of utility theory—as distinct from the mere matter of preferences—such as are found in the earliest thinking on the subject, and also in newer theories such as that of von Neumann and Morgenstern [52]. Of course, historically, utility theory has been very much part of consumer theory, but with an entanglement of distinct ideas which often belong together but on occasions ought to be separated. The familiar long drawn-out cardinalist-ordinalist controversy is the result of this tangle. The concern is so far only with preference in the sense of a relation, though there are auxiliary concepts, such as, most typically, a numerical function which is a gauge of preference. Therefore the word utility has been avoided, in order to prevent any linkage with the large and confusing literature which is taken up with concepts of marginal utility. For an historical account of this, reference is made to Stigler [48].

However, with the subject taken thus far, the pattern is not complete. There is not the means of representation for phenomenon such as that a penny to a rich man is not quite the same thing as a penny to a poor man, or for considerations about incentives, of saturation, and so forth, with many important related phenomena; or, to take another line, there is the currently much considered matter of behaviour in the face of risk. To give account of these and kindred patterns in behaviour (and with the work of Professor Savage [47] there must be included among these the practice of statistical analysis) it is necessary to introduce the ideas of utility theory, including that important instrument in the theory which is given by the von Neumann-Morgenstern utility index. This is the subject for Part IV. With cardinal utility, after its periodic expulsion and rehabilitation, at present suffering official expulsion from consumer theory (which it partially deserves for trepassing where not needed — and which expulsion has been presented by some writers as a kind of discovery!) it has to be established in its proper place. Then we shall be in the company of those whom Professor Samuelson ('A Comment', Survey of Contemporary Economics, Vol. II, p. 38) has described as "a few utilitarians, drunk on poorly understood post-Newtonian mathematical moonshine", by whom he must mean those investigators of that form and method of analysis which goes with the ideas of utility theory. Has not Professor Samuelson thrown out a baby with the bath-water?

Professor Hart, of Columbia University, on being questioned about who exactly "the consumer" was, said that in the literature the consumer and the individual were often synonymous, but that this was just because many of the workers on consumer theory happen to have been bachelors. It is convenient now to be abstract about the matter, and take the consumer to be any agent whose action is to consume different amounts of certain commodities on various occasions when they have given prices, thus including within the widest possibility the individual, the family and the population as possible objects of discussion.

Here there arises that troublesome aggregation pseudo-problem, concerned with derivation of the preferences of a society from those of the individuals comprising its population. Professor Arrow [6], for example, has studied certain abstract questions of social choice and individual values. However, a serious point that may not have been expressed plainly enough by welfare economists is that the one can be irrelevant to the other. A society cannot sensibly be considered as a mere sum of individuals, and this is not merely out of consideration for the celebrated Duesenberry Effect (the observation of Professor Duesenberry [14] that individuals influence each other, which caused a minor revolution in consumer theory.[1])

The concept of an expenditure system: $x = x(u)$ $(u'x=1)$, which is here presented is essentially the traditional concept: $x = x(p,e)$ $(p'x=e)$, homogeneous of degree zero, with built-in recognition of the absence of "monetary illusion", which requires that the actions of the price and expenditure variables $p$ and $e$ be merged in the action of the relative price variables $u = p/e$. With this done — and it is in any case natural to eliminate redundant variables — pitfalls which attend the thinking of prices and expenditure separately are automatically avoided.

1 Robert W. Clower, 'Professor Duesenberry and Traditional Theory'. Rev. Econ. Studies 19 (1952), 165-178.

The literature of consumer theory is full of faults and inconveniences which begin here. For an important example, Professor Hicks' arguments about Income and Substitution Effects need reformulation in this respect. In any case, his statement in Value and Capital [22], p. 309, § 7, is wrong. The change in expenditure (which there is called income) is not such "as would enable the consumer, if he chose, to buy the same quantities of all goods as before". It is a change which would leave the consumer on the same indifference surface; in which case, therefore, unless prices be parallel, the original point on the surface would not be attainable, except at greater expenditure, by virtue of the convexity of the surface. It is more appropriate to consider a general change, due to joint changes in prices and expenditure, and resolve it into a change which results in indifference, and a change which can be obtained by a change in expenditure at constant prices. In the account which is to be given here, such a general resolution of any small change is naturally effected by a pair of complementary oblique projections.

Contrary to what seems to have been common, there is considered just distribution of expenditure on any set whatever of commodities, rather than the spending of income, on all commodities. This is more general; and it is more appropriate, seeing that income is not always spent — it may even be overspent — and also that we are never sure what all the commodities really are. This is formally a small step, but from the point of view of method an important one. Also it makes systematic analysis in terms of several composite commodities possible, and convenient. It means that a self-contained analysis can be given for any assigned group of commodities, selected from among all those that are available, including savings. Of course the commodities and other factors outside the group, along with the usual sun-spots and other mysterious influences, may or may not have an important bearing on those within the group.

A scale applied to a set, which is to be defined as a binary relation between its elements with the properties of antisymmetry and complementary transitivity, is logically the same as the preference relation considered by Professor Arrow [6], in fact as the complement of a complete and transitive relation. However, preference is observable, through the relation of a selected object to a rejected one; and so, from the point of view of method, it is more logical and fitting to put the preference relation first, rather than its complement, which can only be formed when all preferences are decided. Professor Arrow gives a number of the properties of a preference relation according to his stated definition, but not the complete set of essential properties, which are gathered together in the most important, and apparently not altogether immediate theorem, that a preference scale is, according to the axiomatic definition, what we in any case want it to be: structurally the same thing as a complete order of the components of a partition of the set. The indifference relation defined for the scale — as its symmetric complement — is proved a relation of equivalence, the classes of which, defining the indifference classes of the scale, form the partition which is to be completely ordered. A scale appears as a special kind of order; and, in general, a consistent set of preferences form an order which is not a scale, but which can, in general in a variety of ways, be refined to a scale. Thus we seldom deal immediately with a relation satisfying the properties of a scale, as distinct from those of an ordinary order; and never immediately with a relation, such as Professor Arrow's, which is the complement of a scale. One of the remarkable properties of a differentiable expenditure system is that, if its preference relation is an order, then it turns out to be a scale.

The crucial feature of an expenditure system is the assemblage of its choices, and their revealed preferences; and the natural thing to consider about these preferences is the relation obtained by taking their transitive closure, which includes, with all preferences forming a chain, the preference between the extremities of the chain, here defining the preference relation of the system. If the preferences revealed by each of the choices separately can possibly belong to the same scale of value, which condition defines the coherence of the choices — and which possibility is the principal consideration — then the preference relation must be in order. Thus there arises the order condition applied to an expenditure system, expressing the consistency condition for the revealed preferences, — which is equivalent to the hypothesis that the preferences can all together belong to the same scale, and to the coherence condition for the choices. It is the same as the condition which Professor Houthakker [24] has called semi-transitivity, and is called by Professor Samuelson [46] the Strong Axiom of revealed preference, to distinguish it from his own Weak Axiom. This order condition, and the method it carries in its derivation, is inseparable from the real substance of the revealed preference idea, unfortunately missed by Professor Samuelson, who seems however to be much associated with that idea, but formulated, though again somewhat incompletely, in regard to the basic method carried in the principle of the derivation, by Professor Houthakker. The principle of the derivation is that all the different choices, say at different times, are results of the operation of the same scale of value; in other words, that the choices admit a common motivating scale, whose operation persists through and determines each of them. Contradiction of the axiom contradicts the original hypothesis and gives information on changes of preference. It is plain that Pareto [36] well understood the idea, though he did not say much about it. This has also been indicated by Professor Georgescu-Roegen [19].

The nature of the mathematical investigation which is here set out is, in an important aspect, to show the structure of expenditure systems with consistent preferences. In order to proceed to the wanted conclusions, it has been necessary to suppose some further conditions on an expenditure system. The first such condition is that of regularity, or that the mapping $u \to x(u'x=1)$ given by the system, of expenditure balances $u$ into commodity compositions $x$, subject to the balance condition $u'x=1$, is moreover a one-to-one correspondence $u \leftrightarrow x(u'x=1)$ between balances and compositions, defining a duality between them, by which balance and composition become interchangable terms in any definition or proposition. This condition is generally fitting in the subject, as will appear; and it gives the mathematically useful principle of duality, which is an instrument in many of the demonstrations. It yields a perfect analytical development to the analogy between producer and consumer analysis on which comment has been made by W. J. Baumol and H. Makower [8]. With it, and with no further assumption, not even continuity, there is obtained the theory of the critical surfaces and value frontiers, which mark out the relation of any given composition to all other compositions, according to the supposed consistent relation of preference. A substantial amount of analysis goes to proving that these surfaces are strictly convex — a tangent at a point of a surface cutting the surface just in that point — and smooth — there being just one tangent at any point — thus showing that such surfaces are just as economists almost always draw them. Also, various logical features in the nature of these surfaces, which have several different possible definitions, are shown as a result of the investigation. These features have, directly or indirectly and in different measures, been the concern of many writers, including Hicks and Allen [21], Samuelson [45], Houthakker [24], and Little [27].

The critical surfaces are the integral surfaces of the differential equation $u'dx = 0$, associated with the expenditure duality defined by a regular expenditure system. There appear to exist a pair of them at every given point, the inferior and superior value frontiers, every point to the one side of one being inferior to the given point in preference, every point to the other side of the other being superior, while every point on or between them is neither inferior nor superior, but indifferent.

The supposition of differentiability, or the weaker condition which is defined, and called uniformity, and which is implied by differentiability, implies that there can be at most one critical surface on every point. Therefore, the pair of value frontiers must coincide, and be identical with the indifference domain, which now appears as the unique critical surface through the point. There is thus obtained the classical picture of a continuous series of smooth, strictly convex indifference surfaces to represent the preferences of the consumer.

However, with this graphical map of a system of preferences, in which it is exhibited as a scale whose indifference domains are integral surfaces, there is still to be found the differentiable function which is to be a gauge for preferences, the often so-called utility function, which has been almost invariably the starting point for expositions of consumer theory. For this, an appeal has to be made to theory of differential equations, and it is here that that will-o-the-wisp of Professor Hicks (Value and Capital [22], p. 19, footnote), integrability, enters, and even dominates the picture.

The existence of the value frontiers as critical surfaces obtains the integrability of the differential equation $u'dx = 0$, in the sense of the existence of an integral surface through every point. Now, given differentiability, or the less stringent condition of uniformity, the integrability of the equation is equivalent to the integrability of the form $u'dx$, in the sense of the existence of a function $\lambda$, the integrating factor, and a differentiable function $\phi$, the integral, such that $\lambda\, u'dx = d\phi$. The level surfaces of any integral of the form, that is the surface on which it takes a constant value, are identical with the integral surfaces of the equation, and thus with the indifference surfaces of the system. By a continuity argument, any integral of the differential form appears as a gauge of preferences.

There is thus the result that, with differentiability, which is fundamental anyway in other parts of the theory, [1] or with the weaker uniformity condition, the hypothesis of consistent preferences leads to the representation of behaviour by a numerical objective function, on which consumer theory has always relied. It could be such a kind of result that Professor Houthakker [24] sets out to consider. However, he is not fully explicit; and, besides, he says at the end of his argument that he joins Professor Hicks in the conclusion that integrability is a 'will-o-the-wisp'. A strange conclusion, and a strange concept, especially seeing that it is around the condition of integrability that the general mathematical argument turns. It is through that condition that the theory in relational-geometrical terms links with the arithmetical theory, which gives an equivalent of the consistency condition in terms of the partial derivatives of the system — these entering through the formation of the substitution coefficients, or "residual variability" coefficients of Slutsky [48]. This equivalence is one of the main results of this theory. It is mainly that the local condition, given by the symmetry and the negativity

1 It is in the definition of differentiability to eliminate the difficulties suggested by J. Mosak [34].

conditions on the substitution matrix $s = x_u (1-ux')$ everywhere, is equivalent to the global condition, given by the non-reflexivity of the preference relation of the system.

In a recent statement, Professor Arrow [7] has informed us that Professor Georgescu-Roegen [18] "conclusively showed that the real issue behind the integrability problem was the question of transitivity". However we have to consider transitivity anyway, for a system of preferences. The important thing then is antisymmetry; and with this integrability is obtained. Many other writers seem to share the view that transitivity is the all-important criterion. But a transitive relation can always be obtained by suitably enlarging any given relation. The significant criterion is antisymmetry, which is the expression of consistency. Professor Arrow also tells us that "A good deal of effort has gone into finding assumptions on the demand function which would imply the existence of an ordering from which it could be derived". It belongs to the most fundamental understanding of demand functions and revealed preferences that this is the so-called Houthakker Strong Axiom. This is well substantiated by Professor Houthakker [24]. Professor Arrow then goes on to tell us that "Despite a common opinion, it has not yet been shown that the Weak Axiom is not sufficient to insure the desired result. The question is still open." Now, the so-called Samuelson Weak Axiom is, it so happens, weaker than the so-called Houthakker Strong Axiom. Therefore, with the Strong Axiom necessary, the Weak Axiom cannot be sufficient. But perhaps I have not understood Professor Arrow.

A special reference should be made now to the work of Eugen Slutzky, and other ideas which involve it, since he appears to be the first to have presented a significant new and distinct mathematical concept for consumer theory, beyond those ideas of Pareto and his predecessors, which have more closely to do with the form and the primitive terms for such a theory. The maximized numerical objective function, the so-called utility function, which for a long time had been about all there was to consumer theory, was supposed not directly "observable", but in some way to have an implicit existence in behaviours which satisfied appropriate "observable" conditions. To translate to the presently used terms, let an expenditure system which is thus associated with a function by the condition that for any expenditure balance $u$ the corresponding commodity composition $x$ give a proper maximum of the function under the condition $u'x = 1$, be said to have the function as objective. The condition that a function be an objective breaks into two parts, the equilibrium condition, that it be stationary, and then the stability condition, that the stationary value then be a proper maximum. The equilibrium condition, which is immediately seen as the same as the often considered integrability, was shown by Slutzky to imply the symmetry of a certain matrix, involving the partial derivatives of the system, but not involving the objective function at all. The symmetry condition would thus seem to be "observable", and necessary for the existence of the unobservable objective. Slutzky also gave a statement of the stability condition in terms of the derivatives of the objective function; but, so far ss I can see, he does not seem to have obtained its transformation into a condition on that same matrix of coefficients for which he had arrived at the symmetry. Such a condition appears in Hicks [22], and again in Samuelson [44]. The condition, to give it precisely, is that the matrix of Slutzky coefficients be negative definite for points different in direction from the balance point. This negativity condition is transformable into a set of inequalities in the Slutzky coefficients. While Slutzky [48] deduced the symmetry condition from the equilibrium structure of the system, by direct differentiation of the equilibrium equations, and thus showed the symmetry to be necessary for such a structure, he ignored the tougher question of the sufficiency. This is considered in Samuelson [46], where there is stated an identity which would show the equivalence of Slutzky's symmetry condition with the integrability conditions, given in the form of Antonelli [5]. In this form they are also expressed as a symmetry, but as applied to an unsymmetrical system of coefficients, in which one coordinate needs to be arbitrarily distinguished from the rest. Here I give another proof, in which the Slutzky symmetry condition is shown directly equivalent to the classical integrability conditions, here described as acyclicity. This is by deriving identities which exhibit the antisymmetry coefficients as linear functions of the cycle coefficients. Thus the Slutzky symmetry condition is necessary and sufficient for the integrability, which gives the equilibrium structure to the system.

The importance of the concept given by Slutzky, in the definition of his coefficients, is shown in the result in which they reveal their real power for the theory. This result is that the symmetry and the negativity condition on the matrix of these coefficients is necessary and sufficient for the preferences revealed by the system to be consistent. There is a substantial mathematical content in this result, especially when it is taken along with the supplementary results; and it is proved at some length in this report, though still with one or two features left only briefly sketched. I notice that the negativity condition is sometimes neglected, as part of the conditions for a consistent system of preferences to be defined. However it is necessary, along with the symmetry, and inseparable from the operation of consistent preferences in all the choices. Also there seems to be held the idea of the independence of the symmetry conditions. This is important in knowing the freedom left in a system after the requirement that it satisfy the symmetry conditions. That only a proper subset of these conditions can be independent follows even from their equivalence with the integrability conditions and the number of these that can be independent. Here there will be given results which moreover show exactly the system of dependencies between them.

It seems worth considering, as a conjecture, that uniformity and consistency globally may imply differentiability locally. Then it would be possible to say that, given uniformity, a necessary and sufficient condition for consistency is that the partial derivatives exist everywhere, and satisfy the substitutional symmetry and negativity conditions.

L. W. McKenzie [31] has presented an idea for the derivation of the Slutzky conditions, by twice differentiating the support function of the convex body formed by the set of compositions not inferior to a given composition. Though, as with Slutzky's own derivation, it does not yield the sufficiency, it is valuable for added insight it gives into the Slutzky conditions; and it seems to be almost alone as a clear and distinct idea contributed to the subject since Slutzky's paper of 1915.

The fittingness of the use of the word rational which has become customary, and which makes a fitful appearance here, may well be questioned. It is too sweeping to call a person rational merely because he apparently consumes according to a well-defined scale of values. It seems permissible to allow that a consumer could be a perfectly rational person, and yet reveal outrageously inconsistent preferences, within any prescribed framewrok for observing them. The realities of actual choices have depths which no formula for deciding them can reach. All that is meant by consumer rationality is a rigidity of revealed preferences — themselves an artificiality in the particular form by which they are decided — or a constancy of motive in consuming, which does not express the proper essence of rationality. The rational seems more to have to do with the way in which account is taken of objective ends and means, by the faculty of reason, this being specifically with a conscious, and logical process of thought. Rationality is, in any event, a form of regularity; and to this extent there is an appropriateness in the use of the word. It is a tempting word, and using it gives the impression of having cut through an old and knotty problem — though nothing could be further from the case. In game-theory however, when preferences, themselves neither rational nor irrational, are given in advance, a strategy which belongs to a solution of a game does represent the strategy, logically implicit in the ends and means, which it is well fitting to call rational.

Enquiries often relate to the question of the "realism" of certain central "assumptions" of consumer theory. The question may be asked of the condition of consistency of preferences. However, a consistent consumer is a concept, like a particle with uniform motion in a straight line under no forces, such as Newton introduced — though not because he expected ever to see such a thing anywhere, but in order that he be able to explain the motion of a particle through deviation from uniformity, and make an equivalence between dynamical acceleration and configurational force. Accordingly, consistency of preferences is just to be observed; whereas inconsistencies are also to be observed but may also be explained. Everybody knows that preferences change: the important thing is to have the analytical machinery which grasps when they change, and in what manner. In fact, the supposition that consumer theory has to contain universal empirical assumptions or propositions is not appropriate. Its purpose is purely for the elaboration of the forms in which it is fitting to observe, and to analyse.

We have, from a well-known philosopher, something to the effect that "There is nothing good but that desiring makes it so". This would seem to deny any meaning to ethical questions; or it turns them into the elaborate form of deciding what it is good to desire, and then defining the good through identity with that. Consumer theory has nothing to do with what it is good to desire; merely with what is desired. Thus it has nothing ethical about it, contrary to turns in the arguments of certain writers, such as Dr. Little [28], which make an interesting, but misleading involvement. There is, in a general sort of way, no better guide to the precise nature of Utility and All That than Professor Robertson [39, 40], in his well-known essay with that title, and its sequel Utility and All What? It is as well to remark that there should not be seen in this work a further inflating of that "vast parti-coloured mathematical balloon" towards which Professor Robertson says he feels such allergia; rather, a tying of that balloon safely where it belongs. There is also in it the purpose of trying to show consumer theory as a legitimate and interesting mathematical subject, instead of fantastically disordered sphere of discussion, strewn with mangled propositions — it is too much to say false, they are not always precise enough.

It is instructive of something in the state of economic science, not treated amply enough by Professor Koopmans [26], to follow Dr. Little [28], who says (p. 1) "Economists have used no methods of scientific research in arriving at their conclusions about economic welfare; and since there are no methods of scientific research involved there can be no methodology". What then can be the nature of welfare economics, if it is not science? Too obviously it is not pure art, though Dr. Little (p. 1) delicately pretends just to "the exposition, the criticism, and the appreciation" of his subject. In his excellently scientific book Economic Theory and Method, Professor Zeuthen opens with the remark "Economics is an unfinished science." This also gives the spirit and perspective of the present work. It seems that more elementary questions need to be considered before reaching many really problematic ones, which are already much discussed. So many economic concepts — national income is a good example — depend on others, the meaning of which is far from clear, and which ought to be settledfirst, if the subject is to be properly founded. Exploration of form is an instrument of discovery. It is, if anything, the universal method in science; and has to be entertained as the real authority, rather than a citation from the learned literature — which, in the present sphere, however 'authoritative', has been somewhat inconclusive. Any science carries an essential element of its recommendation in its own form. This is conspicious in the fundamental mathematical sciences, physical and otherwise. It seems very plausible that economics, as ordered formal knowledge, will eventually find a place among these, with its own peculiar characteristic conceptions, such as are only just beginning to emerge at all distinctly. Already the Theory of Games and Economic Behavior of von Neumann and Morgenstern has given an instrument of great universality which has still to be fully exploited as fitting, in the formal theories of economics. An important step in this direction is represented by Martin Shubik's recent book Strategy and Market Structure (New York, 1959).

Dr. Little, in his remark (Critique, p. 2) "I do not attempt to deduce any new theorems", signals another troublesome confusion. There do not seem to be any proper theorems at all in Dr. Little's book — at most a few somewhat tentative, and even incompletely understood definitions. Usually, a theorem has an hypothesis, and a conclusion. It is a mathematical proposition, and for such there may be consulted the well-known definition of Bertrand Russell [41]. In any case, the idea is well enough understood by anyone who has understood elementary Euclidean geometry.

However, many statements which are called theorems in the economic literature are nothing of the sort. What exactly is Professor Hicks' Index-Number Theorem? ([23], p. 181). It could be a hypothesis, or a conclusion; it could be true, and even false; and it has some, as it happens there unstated, sense; but the hardest thing at all to see is that it is a theorem. The same is true generally throughout Professor Hicks' Revision [23]: it is difficult to distinguish an intended hypothesis, or conclusion, and rather often even a perfectly definite sense. It is true — to be fair — that, in spite of their language, these writers may in fact be participating in that fine search for empirical propositions. But the muddle is more than one of mere speech: though Dr. Little would not like it, it is of method.

For further instruction, let us now contemplate Dr. Little's remark (Critique, p. 1) "I believe that any further extension of welfare theory is unlikely to be at all valuable, except as a mathematical exercise". Does this mean that the subject if hopeless within its present conception; or that it is complete, with the last word, presumably Dr. Little's, as having been said? Has a significant line now been drawn between "ordinary" and "mathematical" language and thought? These are interesting questions, but for the moment slight comment only can be made, mostly on the last, to a certain point. Familiarity with a mathematical concept makes thinking in terms of it "ordinary", and, moreover, such concepts can be seen operating everywhere in thought. With all this and generally, there is no profound purpose in distinguishing two kinds of economics, one of which is called mathematical. Thus the present work is mathematics, and also economics, and it cannot be parcelled into the one and the other. It has been a fashion with economists to lump all formulae and things that look like that together, and get them out of the way in a Mathematical Appendix.

In this work it would have seemed an inspiration to have had an Economical Appendix. But then I should surely have had to answer that question which economists will often ask "What is the economic significance of it?", and still I do not quite know what is meant.

Mathematics is more than a language. Indeed, if we are to believe L. E. J. Brouwer [12], who says "Mathematics is an essentially languageless activity of the mind .....", it is not even that. It is objective thought; and certain of such as is needed in economics is to be found nowhere else. It does not merely give a language with which, if one chooses, to express what may happen anyway to be in economic theory; but it can itself be indivisible with such theory. Professor J. A. Schouten, in the final speech at the International Congress of Mathematicians at Amsterdam in 1954 (<u>Proceedings</u>, Vol. I, p. 157) remarked that "..... it is much easier to avoid misunderstanding in the field of physics than in the field of economics or political science." Now mathematics is in one of its aspects the science of exact formulation and that means that better understanding may arise when problems can be formulated mathematically". This remark gives the key to changes which are taking place in economics, and which will, as seems to be the increasingly established opinion, continue to act towards the establishment of economics as a science whose rigour is equal to its apparent importance.[1]

It is to be taken as true — regarding a disdain for further mathematical effort such as has just been noted — that the careful statement of the obvious is not a negligible part of systematic exposition. But, if problems of the type such as arise in welfare economics are to proceed beyond such a stage, new forms and methods of precise statement, even new theorems, are just what is needed,

1 Some further discussions which are relevant appear in O. Morgenstern, "Professor Hicks on Value and Capital". <u>Journal of Political Economy</u> 49 (1941), 361-393.

allowing that, in the end, it may only be possible to look very critically at some of the questions, and to decide that, by their nature, only the simplest and most obvious things are worth doing and appropriate. That seems to be a kind of critique that is finally needed, and such as it would be well to develop. There always stands out the fact that all the general notions with which are built the many unavoidable questions in which value features as a critical term — it is difficult to exaggerate their number or their importance — call for every analysis that will make them better understood.

Preferences between objects are not absolute but conditional: priorities can be reversed when conditioning factors change, like, for a simple and familiar example, swords and ploughshares, in peace and war. Also, value is a concept in a form of analysis the application of which we are free to try, and to accept or reject: it is possible to argue that the thinking of behaviour in terms of preferences is a nice, but unprofitable artificiality, or alternatively, that preferences may be real enough, but very unstable. However, values are the only means we have of making sense of human behaviour. (And it could be for that matter of any living activity. What other abstract principle has Darwin given us, but that there are preferences in creation. Let us note also the idea working as a principle in the detection of crime — it is elementary to suspect anyone with a motive!)

The genesis of a particular mathematical subject is in essential forms, elicited from perceptions, in some sphere of experience. Here that sphere is the activity of consuming, together with the idea of analysis in terms of values. The data for any knowledge of the consumer is market data, giving prices of commodities, and quantities consumed at those prices, in a certain, necessarily finite number of occasions. Theory of the consumer should lead to statements of what questions can be asked of the data. It should give a general methodology about such questions, how they and their answers are formulated, and their conceptual meaning for different possible issues. As a subject in itself, at least in its fundamental terms, it seems to have fairly conspicuous natural limits, the tracing of which is one of the present interests.

There is not intended to be an approach from a particular point of view or subject, but rather, as far as possible, one just in accordance with the nature of the matter in itself. This makes for difficulties in the exposition, as regards ready communication. But is is impossible to do real justice to a reader without first doing justice as far as possible to the subject. It may of course be that in getting intrinsic difficulties out of a subject, which is by appearance comparatively simple, everything may become unexpectedly more difficult; but these new difficulties may well be mostly bound up with familiarity and training, which the subject itself does not, so to speak, care about. This exposition is mostly concerned with getting a subject stated, on terms which are entirely its own, with the hope that in due course it will get any further exposition it needs to make it accessible where wanted.

It is hardly possible or fitting to record every occasion which has been for the advancement of this work. But it can be recorded that some talks with Mr. Robin Marris in Cambridge, about the "spread" between the Paasche and Laspeyres indices, were the background for thinking which finally led to this general investigation of concepts. The scrutiny of principle is sometimes no doubt an impediment to action: probably if we thought about index-numbers too much, they would never get compiled. But index-numbers are supposed to stand for something. They seem almost in present days to have attained to the character of a sacred institution; but they are, in the nature of what they pretend to be, not a mere convention. To ask that the principle of their calculation be conceptually correct is a request which has to be respected

— even if we can only admit that we are then left at sea. The request itself can be made intelligible only by something of a theory of index-numbers, and it certainly calls for just such a theory to answer it. However, there does not seem to be a general show of such concerns among those who profess to the job of calculating the index-numbers. It could be of course that they are too busy measuring those all-influential "weights". Dr. Prais (before the Royal Statistical Society, April 16th, 1958; the Journal, Series A, 121) remarks, apparently with perfect confidence, "... practitioners have come to substantial agreement on what a cost-of-living index number should to-day measure in principle ...". What deep satisfaction for the practitioners! The fact is, they have no idea whatsoever about such a principle. [1] For a quite simple immediate discomfort, the Paasche and Laspeyres indices seem to compete on an equal footing to represent the same thing. Yet they are never the same: between them there is that unfortunate "spread". There are other discomforts; but the next Report has these for examination, together with exposition of a theory of index-number calculations for which the present material has been set out as a preparation. In this Report, as is altogether normal in the field, there is hardly any trace or suggestion of calculation. The view has been expressed (Morgenstern [33]) that "It may well be said that a modern economic theory will now no longer be considered finished until it is clearly shown how it can give numerical results". Consumer theory has hardly ever satisfied such a condition: it is a field ruled by authoritative expositions and disputes — in which the personal equation has been a large equation, in the balance between imponderables. Methods of index-number calculation and of demand analysis,

1 For an exact and informative statement relating to the situation, reference is made to W. Leontief "Composite commodities and the Problem of index number", <u>Econometrica</u> 4 (1956), 39-59.

which should most perfectly in principle have their source in such a theory, have with but few exceptions been without such foundation. Subsequent parts of this investigation are concerned with theories of measurement and calculation which have their origin precisely in the here described formal theory.

Concern with proofs of theorems is a very necessary requirement which, once discharged, can more be left aside, and the attention can then be given over entirely to problems of methodology. There is needed a further exposition in which there is not the sometimes intricate diversion of dealing with proofs, but only with concepts and principles; and it is hoped this may be supplied eventually.

This work in part had its beginnings in the Department of Applied Economics, Cambridge; and the interest even earlier, in a collaboration with Mr. J. R. Bellerby of the Agricultural Economics Research Institute, Oxford. It was continued, intermittantly, during a Fellowship in the Department of Mathematics of the Hebrew University, Jerusalem, in 1956-58. However, it is only since September 1958, after joining the Econometrics Research Program of Professor Morgenstern, that it has shown anything of its present shape, in which there can at least be a view towards a systematic and, within certain limits, complete statement. Some talks given here to a number of economists and mathematicians concerning the material, and the discussions which went with these, did a great deal to assist this development, to find important issues which had to be clarified, and advantageous formulations. I have an old debt of thanks to sincerely acknowledge to Mr. Bellerby; and also to Professor J. R. N. Stone in Cambridge. Now I have a very great one to Professor Morgenstern, to whom I am grateful for having made this writing fully possible, and very agreeable, and for having assisted it in very many ways. Without his interest and sympathy, it seems this task would have been a longer one, and much more difficult.

I wish also to thank very much Mrs. L. Diaforli who, with skill and patience, has done the typing. This work has had the partial support of the Office of Naval Research of the United States Government.

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## TABLE OF CONTENTS

### PART I

# PART I. THE CONSISTENT CONSUMER

## I. Value and Choice.

### 1. Logical notation.

If p is a proposition, then $\sim p$ denotes the proposition which is its *negation*, so that the assertion or denial of p is equivalent to the denial or assertion of $\sim p$.

The condition of *implication* from a proposition p to a proposition q is indicated by $p \Rightarrow q$, and is that the assertion of p carries with it the assertion of q. Mutual implication defines the *equivalence* $p \Leftrightarrow q$ of the propositions; and *equivalence by definition* is indicated by $p \equiv q$, with the term being defined appearing on the left.

Given two propositions p, q they may be composed to form the further propositions denoted by $p \wedge q$, $p \vee q$ which are called their *conjunction* and *disjunction*, the assertion of the one or the other of which is equivalent to the assertion of both of p and q or of either of p and q.

Now for a set of propositions $p(x)$ $(x \in C)$ indexed in a set C, the conjunction of all the members is denoted by $\bigwedge_{x \in C} p(x)$, the assertion of which is equivalent to the assertion of every proposition $p(x)$ $(x \in C)$, in other words the assertion of $p(x)$ for all $x \in C$; and the assertion of the disjunction, denoted by $\bigvee_{x \in C} p(x)$, is equivalent to the assertion of one or other of the propositions $p(x)$ $(x \in C)$, or that there exists an $x \in C$ such that $p(x)$.

Such an indexed set of propositions gives the concept of a *propositional function* $p(x)$ defined for $x \in C$, which gives a proposition with the values of truth or falsehood for any element

$x \in C$. The conjunction and disjunction operators $\bigwedge_{x \in C}$, $\bigvee_{x \in C}$ applied to a propositional function $p(x)$ define the *universal* and *existential quantifiers*, which obtain the assertion of $p(x)$ for all and for some $x \in C$.

A useful convention which is adopted is that by the assertion of merely $p(x)$ is to be understood the assertion of $\bigwedge_{x \in C} p(x)$.

The scope of the logical connectives, of conjunction, disjunction and implication, and the quantification operators, is indicated in the usual way by brackets and punctuation.[1]

## 2. Algebra of relations.[2]

Let R be a *binary relation* between the elements of a set C, that is a propositional function $xRy$ $(x,y \in C)$ of the ordered couples $(x,y)$ of elements of C. The conditions for R to be *reflexive*, and to be *non-reflexive* are defined by

$$x=y \Rightarrow xRy, \quad xRy \Rightarrow x \neq y,$$

*symmetric*, and *antisymmetric*, by

$$xRy \Rightarrow yRx, \quad xRy \Rightarrow \sim yRx,$$

to be *complete*, by

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[1] Reference is made to almost any of the standard works on symbolic logic, for example Quine [38] or Tarski [49], for general account of symbolic methods. What is needed here is extremely simple, but nevertheless indispensible. All the symbolic statements have the most direct translation into words; but these translations obscure essential form, which is immediately evident in the symbolic statement.

[2] Accounts are to be found in various works on logic, most especially in Russell and Whitehead [42]. However, an amount of this material (viz. Theorem III and definitions involved) does not appear to have had any consideration elsewhere.

$$x=y \,.\!\Rightarrow\!.\, xRy \vee yRx,$$

and to be *transitive*, by

$$xRy \wedge yRz \,.\!\Rightarrow\!.\, xRz.$$

The transitivity condition has the equivalent extended form

$$x_o R x_1 \wedge \ldots\ldots \wedge x_{m-1} R x_m \,.\!\Rightarrow\!.\, x_o R x_m.$$

Any relation R has a *conjugate* $R'$, a *complement* $\overline{R}$ and a *symmetric complement* $\tilde{R}$, defined by

$$xR'y \equiv yRx, \quad x\overline{R}y \equiv \sim xRy, \quad x\tilde{R}y, \equiv \sim xRy \wedge \sim yRx.$$

The complement of the conjugate of any relation is the same as the conjugate of the complement,

$$(\overline{R})' = \overline{(R')}$$

and so, without ambiguity, they both may be indicated by $\overline{R}'$.

A relation R is said to have the property of *complementary transitivity* if its complement $\overline{R}$ is transitive, equivalently, if

$$\sim xRy \wedge \sim yRz \,.\!\Rightarrow\!.\, \sim xRz.$$

There are two special complementary pairs of relations, the *universal* and the *null relations* $\nabla$ and $\Delta$, which always and which never hold, respectively,

$$x\nabla y, \quad \sim x\Delta y;$$

and also the relations of *identification* and *distinction* I and D defined by

$$xIy \equiv x=y, \quad xDy \equiv x=y.$$

Operations of *disjunction*, *conjunction* and *adjunction* of relations, to obtain the *sum* $Q \vee R$, *product*

$Q \wedge R$ and *resultant* $QR$ of any two relations $Q$, $R$ are defined by

$$x(Q \vee R)y \equiv xQy \vee xRy, \quad x(Q \wedge R)y \equiv xQy \wedge xRy,$$

$$x(QR)y \equiv \bigvee_{z} xQz \wedge zRy.$$

Adjuction is distributive over disjunction,

$$P(Q \vee R) = PQ \vee PR;$$

also it is associative,

$$P(QR) = (PQ)R,$$

so that any sequence of relations $R_1, \ldots, R_m$ has a well defined resultant $R_1 \ldots R_m$, and the mth power $R^m$ of any relation R can be defined as the resultant of a sequence of m relations identical with R,

$$R^m = \overbrace{R \ldots R}^{m}.$$

Adjunction of relations is not generally commutative,

$$QR = RQ;$$

however, adjunction of different powers of the same relation is commutative,

$$R^m R^n = R^{m+n} = R^{n+m} = R^n R^m.$$

The relation of *implication* $Q \Rightarrow R$ between relations $Q$, $R$ is defined by

$$Q \Rightarrow R \equiv xQy \Rightarrow xRy.$$

Equivalent statements for the conditions of reflexivity and non-reflexivity for a relation R are given by

$$I \Rightarrow R, \quad R \Rightarrow D,$$

of symmetry and antisymmetry, by

$$R \Rightarrow R', \quad R \Rightarrow \bar{R}',$$

of transitivity, by

$$R^2 \Rightarrow R,$$

and of completeness, by

$$D \Rightarrow R \vee R'.$$

Also, the symmetric complement has the definition

$$\tilde{R} = \bar{R} \wedge \bar{R}'.$$

THEOREM I. <u>A transitive relation is antisymmetric if and only if it is non-reflexive</u>:

$$R^2 \Rightarrow R : \Rightarrow : R \Rightarrow D . \Leftrightarrow . R \Rightarrow \bar{R}'.$$

It is plain that, in any case, antisymmetry implies non-reflexivity; for

$$xRy \Rightarrow \sim yRx . \Rightarrow . xRx \Rightarrow \sim xRx$$

$$\Rightarrow . xRy \Rightarrow x = y.$$

Now assume transitivity,

$$xRy \wedge yRz . . \Rightarrow . xRz.$$

Then

$$xRy \wedge yRx . . \Rightarrow . xRx,$$

so that

$$xRx : \Rightarrow : \sim . xRy \wedge yRx,$$

that is

$$xRy : \Rightarrow x = y . . \Rightarrow . xRy \Rightarrow \sim yRx.$$

THEOREM II. <u>The non-reflexivity of a relation is equivalent to the reflexivity of its symmetric complement</u>: $R \Rightarrow D . \Leftrightarrow . I \Rightarrow \tilde{R}.$

For

$$\sim xRx \Leftrightarrow x\tilde{R}x.$$

THEOREM III. <u>Given</u> $R \Rightarrow \bar{R}'$, <u>the conditions</u>

$$R\bar{R}' \Rightarrow R, \quad \bar{R}^2 \Rightarrow \bar{R}, \quad \bar{R}'R \Rightarrow R$$

<u>are equivalent, and imply</u>

$$R^2 \Rightarrow R, \quad R\tilde{R} \Rightarrow R, \quad \tilde{R}R \Rightarrow R, \quad \tilde{R}^2 \Rightarrow \tilde{R}.$$

Thus, assume

$$xRy \Rightarrow {\sim}yRx.$$

Then

$$xRy \wedge {\sim}zRy \,.\Rightarrow.\, xRz$$

gives

$${\sim}xRz \,.\Rightarrow.\, {\sim}xRy \vee zRy,$$

and then

$${\sim}xRz \wedge {\sim}zRy \,.\Rightarrow.\, {\sim}xRy,$$

and reversely; so $R\bar{R}' \Rightarrow R$ and $\bar{R}^2 \Rightarrow \bar{R}$ are equivalent together, and similarly with $\bar{R}'R \Rightarrow R$.

Assume again antisymmetry, and then complementary transitivity, with its now demonstrated equivalents. Then

$$\begin{aligned} xRy \wedge yRz \,.\Rightarrow.\, & xRy \wedge {\sim}zRy \\ .\Rightarrow.\, & xRz, \end{aligned}$$

so that $R^2 \Rightarrow R$. Also

$$\begin{aligned} xRy \wedge y\tilde{R}z \,.\Rightarrow.\, & xRy \wedge {\sim}zRy \\ .\Rightarrow.\, & xRz, \end{aligned}$$

so that $R\tilde{R} \Rightarrow R$; and similarly, $\tilde{R}R \Rightarrow R$. Finally,

$$\begin{aligned} x\tilde{R}y \wedge y\tilde{R}z \,.\Rightarrow.\, & (x\bar{R}y \wedge y\bar{R}z) \wedge (z\bar{R}y \wedge y\bar{R}x) \\ .\Rightarrow.\, & x\bar{R}z \wedge z\bar{R}x \\ .\Rightarrow.\, & x\tilde{R}z, \end{aligned}$$

so that $\tilde{R}^2 \Rightarrow \tilde{R}$.

3. Transitive closure.

A *link*, in a relation R, or an R-link, is defined as an ordered pair of elements (x,y) with the first member x in the relation R to the second member y, that is xRy. A relation is given when the set of ordered pairs formed by all its links is given. This set defines the *graph* of the relation R, and is also denoted by R, it being a subset of the set (C,C) of all ordered pairs of elements of C; thus,

$$R \subset (C,C), \qquad (x,y) \in R \equiv xRy.$$

The relation of implication between relations is, accordingly, the same as the relation of inclusion between their graphs.

Links in an ordered pair of the form (x,y), (y,z) are said to be *coupled*, in their given order, and to have the link (x,z) as *resultant*.

A sequence of links in which each link is coupled in order with its successor defines a *chain*, an R-chain being defined as a chain of R-links. Accordingly, a sequence $(x_o,\ldots,x_r)$ of r+1 elements defines a chain $((x_o,x_1),\ldots,(x_{r-1},x_r))$ with r links. Such a chain is said to *ascend* from the first element $x_o$ to the last $x_r$, or to *descend* from the last to the first. A chain ascending from x to y in R also gives a chain descending from x to y in the conjugate relation R'.

The *resultant link* of a chain, ascending from x to y, is defined as the link (x,y) which it determines between its extremities.

An ordered pair of chains is said to be *coupled* if the last link of the first chain is coupled with the first link of the last. A coupled pair of chains may be joined together to form a third chain. The resultant links of the chains in a coupled pair are coupled, and have resultant which is the resultant link of the chain obtained by joining them.

The property of transitivity of a relation R is that the resultant of a coupled pair of R-links is an R-link, or, in an equivalent, more extended form, the resultant link of any R-chain is an R-link.

Now, if R is any relation, it is possible to form the relation $\vec{R}$ whose links are the resultant links of all the chains of R, that is, $\vec{R}$ is defined by defining an $\vec{R}$-link as the resultant link of an R-chain.

The relation $\vec{R}$ thus constructed from a relation R is necessarily transitive, by the form of the construction. For, any coupled $\vec{R}$-links are the resultant links of a coupled pair of R-chains, and their resultant is the resultant link of the R-chain obtained by joining these R-chains, and is thus an $\vec{R}$-link; that is, the resultant of a coupled pair of $\vec{R}$-links is an $\vec{R}$-link; so $\vec{R}$ is transitive.

Moreover, for any transitive relation T, and any relation R, if the graph of T contains the graph of R then it must contain the graph of $\vec{R}$. For then any R-chain is a T-chain, and the resultant of any T-chain is a T-link, by the transitivity of T. Thus the resultants of the R-chains, which define the $\vec{R}$-links, are T-links; so the graph of T contains the graph of $\vec{R}$. Thus it appears that $\vec{R}$ is transitive, implied by R, and implies every transitive relation implied by R, for which property of $\vec{R}$ there is the following statement.

THEOREM I. $\vec{R}$ <u>is the minimal transitive relation implied by</u> R.

By this property, $\vec{R}$ is called the *transitive closure* of R.

Since

$$xR^m y \equiv \bigvee_{z_o,\ldots,z_m} (x=z_o \wedge y=z_m) \wedge (z_o R z_1 \wedge \ldots \wedge z_{m-1} R z_m),$$

an $R^m$-link is the same as the resultant of an R-chain of m links. Therefore, since the $\vec{R}$-links are the resultants of R-chains, an $\vec{R}$-link is an $R^m$-link for some m; so the graph of $\vec{R}$ is the union of the graphs of the powers $R^m (m=1,2,\ldots)$ of R; whence, equivalently,

$$\vec{R} \equiv \bigvee_m R^m.$$

So, more explicitly,

$$x\vec{R}y \equiv \bigvee_m \bigvee_{z_o,\ldots,z_m} (x = z_o \wedge y=z_m) \wedge (z_o R z_1 \wedge \cdots \wedge z_{m-1} R z_m)$$

The relation $\vec{R}$ as given by this formula can be verified in a formal algebraic way to have the essential defining properties for the transitive closure. Firstly, it is verified that $R \Rightarrow \vec{R}$, and secondly, that $\vec{R}^2 \Rightarrow \vec{R}$, since

$$\vec{R}^2 = (\bigvee_{m=1} R^m)^2 = \bigvee_{m,n=1} R^{m+n} = \bigvee_{m=2} R^m \Rightarrow \vec{R},$$

by the commutativity of the adjunction of different powers of the same relation, and by the distributivity of adjunction over disjunction. Finally, if

$$T^2 \Rightarrow T \quad \text{and} \quad R \Rightarrow T,$$

then

$$R^m \Rightarrow T^m \Rightarrow T, \text{ so that } \vec{R} \Rightarrow T.$$

Hence there is concluded the following, which is Theorem I with a somewhat different derivation.

(THEOREM I) <u>If</u>

$$\vec{R} = \bigvee_{m=1} R^m,$$

<u>then</u>

$$R \Rightarrow \vec{R}, \quad \vec{R}^2 \Rightarrow \vec{R}$$

<u>and</u>

$$R \Rightarrow T \wedge T^2 \Rightarrow T \;.\Rightarrow.\; \vec{R} \Rightarrow T.$$

This states merely that the relation $\vec{R}$ given by the given formula has the property of being the minimal transitive relation implied by R, that is, it is transitive and implied by R and implies every transitive relation which is implied by R.

A *cycle* is defined as a chain with coincident extremities; and an R-cycle in then a cycle formed of R-links. A relation R is defined to be *acyclic* if no R-cycles exist.[3]

THEOREM II. <u>A relation R is acyclic if and only if its transitive closure $\vec{R}$ is antisymmetric, or equivalently non-reflexive</u>.

It is clear that an element x is on an R-cycle composed of some m links if and only if it gives a reflexivity $xR^m x$ of a power $R^m$ of R; and then, equivalently, of the transitive closure $\vec{R}$ which is the union of these powers. Thus the acyclcicity of R is equivalent to the non-reflexivity of $\vec{R}$; and, since $\vec{R}$ is transitive, this is equivalent to the antisymmetry of $\vec{R}$, by Theorem 2.I.

## 4. <u>Equivalence, order and scale.</u>[4]

A reflexive, symmetric and transitive relation defines an *equivalence*. Thus, the relation of identification I and the universal relation are the simplest possible examples of equivalences: in the one each element is equivalent just to itself, and in the other all elements are equivalent.

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3 von Neumann and Morgenstern [52].

4 The concept of equivalence and order are, of course, completely standard. However, the axiomatic concept of a scale appears to be newly introduced here, and is a much needed logical term in value theory, for the want of which some essential theory has lacked the means of statement. Reference is made to Von Neumann and Morgenstern [52] for the original modern approach to the logic of preferences, and also to Arrow [6], Samuelson [44], Hicks [22], Houthakker [24] and Little [28], for further discussions with special reference to the consumer. For more abstract treatment of relations generally, reference is made to Birkhoff [9], Birkhoff and MacLane [10], and also to works on symbolic logic, such as Quine [38] and Russell and Whitehead [42].

A complementary set of subsets of C is a set of subsets which are mutually disjoint and which have C for their union. Such a complementary set of subsets defines a partition $\Pi$ of C, with these subsets for components. Every element x of C belongs to one and only one component $\Pi_x$ of $\Pi$, which is said to have x as representative. The two extreme examples of partitions are those in which all the elements together form a single component, and those in which each single element itself is a component.

Any partition $\Pi$ determines an equivalence Q by the condition that elements are to be Q-equivalent if and only if they represent the same $\Pi$-components.

Thus,

$$xQy \equiv \Pi_x = \Pi_y$$

defines a relation Q determined by a partition $\Pi$, which is readily verified to be an equivalence.

A partition $\Pi$ and an equivalence Q thus related are said to represent each other. Thus, the condition for an equivalence Q and a partition $\Pi$ to represent each other is that, for all elements $x,y \in C$ and for all components $\alpha,\beta \in \Pi$,

$$x\in\alpha \wedge y\in\beta \;.\Rightarrow.\; xQy \Leftrightarrow \alpha = \beta.$$

It has been pointed out that <u>every partition represents an equivalence</u>. For example, the extreme partitions, which have been described, represent the universal relation and the relation of identification, respectively. Now it is to be seen that <u>every equivalence represents a partition</u>. For, let $Q_x$ denote the set of elements y which have Q-equivalence with x, that is such that yQx, this set defining the Q-equivalence c l a s s  of x; so

$$y \in Q_x \equiv yQx.$$

Then, by reflexivity,

$$x \in Q_x;$$

and, by symmetry and transitivity,

$$xQy \;.\Rightarrow.\; z \in Q_x \Leftrightarrow z \in Q_y,$$

that is,

$$xQy \Rightarrow Q_x = Q_y,$$

and also

$$z \in Q_x \cap Q_y \;.\Rightarrow.\; xQz \wedge zQy$$

$$.\Rightarrow.\; xQy,$$

so that

$$Q_x \cap Q_y \neq 0 \Rightarrow xQy.$$

With $x \in Q_x$, it now appears that

$$xQy \Leftrightarrow Q_x = Q_y, \quad x\bar{Q}y \Leftrightarrow Q_x \cap Q_y = 0.$$

Accordingly, the equivalence classes form a complementary set of subsets, which determine a partition $\Pi$ of C represented by the equivalence Q:

$$Q_x = \Pi_x.$$

A non-reflexive, transitive relation defines an  o r d e r.  Since, for a transitive relation, the conditions of non-reflexivity and antisymmetry for equivalent, an antisymmetric transitive relation also defines an order. An order relation is called a  c o m p l e t e  o r d e r  if it also satisfies the condition for being complete, by which every pair of elements are related in the order one way or the other. Otherwise it is called a  p a r t i a l  o r d e r.

If one order implies another, then it is said to *refine* the other, or the other is said to be a *refinement* of the one. The operation of refining an order is to construct an order which refines it. Every order can be refined to a complete order, given the axiom of choice;[5] however, the way is not unique. In other words, assuming the axiom of choice, *every partial order implies a complete order*.

Given any relation, in general not an order, it may or may not imply, or be contained in, an order relation. *A necessary and sufficient condition that a relation imply an order is that its transitive closure be non-reflexive*. For if $R \Rightarrow Q$ where Q is an order, and therefore transitive, it follows that $\vec{R} \Rightarrow Q$; and now the non-reflexivity of the order Q requires that of $\vec{R}$. Conversely, if $\vec{R}$ is non-reflexive, it is, since already transitive by construction, an order, implied by R, so the required construction is obtained. Moreover, *any order implied by R is determined to the extent of being a refinement of* $\vec{R}$.

A relation with the properties of antisymmetry and complementary transitivity defines a *scale*. Then the symmetric complement $\tilde{S} = \bar{S} \wedge \bar{S}'$

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[5] I am indebted to Dr. R.O. Davies for pointing out to me this proposition. The orders which properly refine any given partial order form the elements of a non-empty set, which is itself partially ordered by the refinement relation which is defined between its element. The orders which are the maximal elements in this partial order are the same as the complete orders which refine the originally given partial order (in general, there are several of them).

of a scale S defines the relation of *indifference* in that scale.[6] For any two elements x,y there are now the three possibilities, which are mutually exclusive and complementary, by the antisymmetry of S and the definition of $\tilde{S}$:

$$xSy, \quad x\tilde{S}y, \quad xS'y.$$

Thus, if they do not have one of the two relations of *preference* in S, that is xSy or xS'y, where xS'y stands for the conjugate relation ySx, then they have the symmetric relation of indifference $x\tilde{S}y$, by the definition of indifference as the negation of preference. In another terminology, any element is either *better* or *worse* than another, or they are *of the same value*, in the scale.[7]

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[6] Indifference, as determine by $\tilde{P} = \bar{P} \wedge \bar{P}'$, appears as derivative from preference, as determined by P, and not as more primitive than preference. Thus an indifference, as belonging to a set of revealed preferences, may be replaced by a preference, when that set of revealed preferences is enlarged. If some preferences are considered as revealed, other possible preferences may just as well be concealed; and no record of indifference can be allowed as a final record. However, an established manner of thought in value theory seems to take indifference as a primitive term, rather than as an auxilliary. It seems to take the construction of indifference classes as a fundamental task, rather than the establishment of preferences. This can be considered a fault, since, in action, it is only preference which is revealed, in the relation of the selected to the rejected; and no action or set of actions can point to an indifference. Indifference can only merely be a concept which applies to our knowledge of the state of an agent. It is a gap in our knowledge of preference, as is expressed in the definition $\tilde{P} = \bar{P} \wedge \bar{P}'$. Even if the agent may have a genuine indifference, whatever that may be, it can be of no use or consequence. In the nature of the logical situation, we can never know it, by any amount of observation. After all, without affecting anything, the agent may secretly become more discriminating - and capable of making distinctions which would be uncalled for on any possible occasion!

[7] In usages in economics, "same value" may only be in a monetary sense, and must be distinguished as only possibly a part in the more inclusive concept, which is the present concern, which may have nothing at all to do with money.

THEOREM I. <u>If S is a scale, then it is moreover an order, and its indifference relation $\tilde{S}$ is an equivalence</u>.

For, in the first place, it appears from Theorem 2.III that if S is a scale, then both S and $\tilde{S}$ are transitive. Now S is an order, by its antisymmetry together with transitivity. Also, by Theorem 2.II, the non-reflexivity of any relation is equivalent to the reflexivity of its symmetric complement. But S is non-reflexive, since it is an order; and therefore $\tilde{S}$ is reflexive. Moreover, the symmetry of $\tilde{S}$ is in the form of its definition. Thus $\tilde{S}$, being now reflexive, symmetric and transitive, is an equivalence.

Since the relation $\tilde{S}$ of indifference in a scale S is an equivalence, it determines a partition $\Sigma$ of the set C, whose components, which are the equivalence classes of $\tilde{S}$, may be taken to define the *indifference classes* of S. Thus, for elements $x,y \in C$ and indifference classes $\alpha, \beta \in \Sigma$,

$$x \in \alpha \wedge y \in \beta \ .\Rightarrow.\ x\tilde{S}y \Leftrightarrow \alpha = \beta.$$

It appears from this theorem that a scale is a special kind of order. Though every scale is an order, not any order is a scale. However, the concepts coincide under the condition of completeness.

THEOREM II. <u>The complete order and complete scale conditions are equivalent</u>.

It is only necessary now to observe that, with completeness and antisymmetry, transitivity implies complementary transitivity, thus,

$$\begin{aligned} {\sim}xSy \wedge {\sim}ySz \ &.\Rightarrow.\ ySx \wedge zSy \\ &.\Rightarrow.\ zSx \\ &.\Rightarrow.\ {\sim}xSz. \end{aligned}$$

Consider an equivalence Q, and the resulting partition $\Pi$ of the elements of C, together with a complete order $\mathcal{R}$ of the components of $\Pi$, these being the Q-equivalence classes. A relation S between the elements of C is defined by the relation $\mathcal{R}$ between the Q-equivalence classes to which they belong, thus,

$$xSy \equiv Q_x \mathcal{R} Q_y .$$

It is said that the relation $\mathcal{R}$ between the Q-equivalence classes of the elements *extends* to the relation S between the elements, and also that the relation S between the elements *reduces*, by identification under the equivalence Q, to the relation $\mathcal{R}$ between the Q-equivalence classes. Such a relation S which reduces by identification under an equivalence is said to be *reducible*.

The condition that a relation $\mathcal{R}$ between the classes of an equivalence Q between the elements of a set C extend to a relation S between the elements of C, or, what is the same thing, that the relation S between the elements of C reduce, by identification under the equivalence Q, to a relation $\mathcal{R}$ between the Q-classes, is that for elements $x, y \in C$ and components $\alpha, \beta \in \Pi$, of the partition $\Pi$ which represents Q,

$$x\in\alpha \wedge y\in\beta \ .\Rightarrow.\ xSy \Leftrightarrow \alpha\mathcal{R}\beta .$$

By this condition, the relation S between the elements of the set C is said to represent, and to be represented by, the relation $\mathcal{R}$ between the components of the partition $\Pi$ of C:

$$S \longleftrightarrow (\mathcal{R}, \Pi).$$

THEOREM III. <u>A complete order $\mathcal{R}$ of the classes in an equivalence</u> Q <u>between the elements of a set</u> C <u>extends to a scale on the elements of</u> C.

If

$$xSy \Leftrightarrow Q_x \mathcal{R} Q_y ,$$

where $\mathcal{R}$ is a complete order, then, by the antisymmetry of $\mathcal{R}$

$$xSy \Rightarrow Q_x \mathcal{R} Q_y \Rightarrow \sim Q_y \mathcal{R} Q_x \Rightarrow \sim ySx,$$

so S has the property of antisymmetry. Also, by the completeness, then by the transitivity, and then again by the antisymmetry of $\mathcal{R}$,

$$\begin{aligned} \sim xSy \wedge \sim ySz \;:\Rightarrow:\; & \sim Q_x \mathcal{R} Q_y \;.\wedge.\; \sim Q_y \mathcal{R} Q_z \\ :\Rightarrow:\; & Q_y \mathcal{R} Q_x \vee Q_y = Q_x \;.\wedge.\; Q_z \mathcal{R} Q_y \vee Q_z = Q_y \\ :\Rightarrow:\; & Q_z \mathcal{R} Q_x \vee Q_z = Q_x \\ :\Rightarrow:\; & \sim Q_x \mathcal{R} Q_z \\ :\Rightarrow:\; & \sim xSz \;, \end{aligned}$$

so S has the property of complementary transitivity, and is therefore a scale.

THEOREM IV. <u>Any scale S reduces by identification under the equivalence</u> $\tilde{S}$, <u>to a complete order $\mathcal{S}$ of the partition Σ formed by the classes of</u> S; <u>that is</u> , <u>for all elements</u> x,y∈C <u>and components</u> α,β∈Σ,

$$x\in\alpha \wedge y\in\beta \;:\Rightarrow:\; x\tilde{S}y \Leftrightarrow \alpha = \beta \;.\wedge.\; xSy \Leftrightarrow \alpha \mathcal{S} \beta,$$

when $\mathcal{S}$ is a complete order.

In Theorem 2·II it appears that, if S is a scale with indifference relation $\tilde{S}$, then

$$xSy \wedge y\tilde{S}z \;.\Rightarrow.\; xSz, \quad x\tilde{S}y \wedge ySz \;.\Rightarrow.\; xSz,$$

whence

$$x\tilde{S}y \wedge z\tilde{S}w \;.\Rightarrow.\; xSz \Leftrightarrow ySw.$$

It follows from here that the relation $\mathcal{S}$ between indifference classes α,β defined by

$$\alpha \mathcal{S} \beta \equiv \bigvee_{x\in\alpha,\, y\in\beta} xSy$$

has the property

$$\bigwedge_{x \in \alpha, y \in \beta} xSy \iff \alpha \mathcal{S} \beta.$$

So two elements have the relation S if and only if their indifference classes have the relation $\mathcal{S}$. Now it is immediate that

$$\alpha \neq \beta \;.\!\iff\!.\; \alpha \mathcal{S} \beta \vee \beta \mathcal{S} \alpha,$$

which shows $\mathcal{S}$ to be non-reflexive and complete. Also the transitivity of $\mathcal{S}$ is implied by that of S, and the manner in which $\mathcal{S}$ represents S. The relation $\mathcal{S}$ to which S has been shown to reduce by S, is now shown to exist, and to be non-reflexive, transitive and complete. It is therefore a complete order of the indifference classes, which extends to a scale applied to the elements:

$$S \longleftrightarrow (\mathcal{S}, \tilde{S}).$$

Thus, by Theorems III and IV, a scale appears as representing and as represented by a complete order of its indifference classes. In other words, the concept of a scale is essentially that of a completely ordered partition.[8]

For a scale S on a set of C, a numerical function $\phi$ defined on C is said to *measure* S if

$$\phi(x) < \phi(y) \Rightarrow xSy,$$

and to measure S *completely* if

$$\phi(x) < \phi(y) \iff xSy.$$

A function which measures a scale is called a *gauge*, of that scale. Thus, a preference gauge is to be a gauge which measures a preference

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8 The conclusion is that a scale as defined here is what everyone has in mind for a scale - the objects put into classes, and then these classes put into a complete order.

scale, being greater or less according as the object is better or worse.[9]

5. Choice and motive.[10]

An *action*, except when it is merely a constraint, consists in a *choice*, given by an object selected and others rejected. A significance is attributed to a choice, which lies in the relation of the selected to the rejected, by which it is taken that the selected is revealed as *preferred*. Choices are made under certain restrictions, and they reveal certain preferences: the preferences are considered as underlying the choice, and to be essential elements of behaviour, which can be a ground for deciding expectations of action, whereas the restrictions are accidental.

A *scale of value* is to be a scale deciding the better and the worse between potential alternatives of choice, to the end that, on an occasion of choice, with any possible restriction, the chosen object will be the best.

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9. Such a function might, in accordance with traditional usage, have been called a "utility function". However it would then, with the name, inherit the also traditional confusion of controversy, and mystery, which has surrounded the notion of utility (for the anatomy of the subject, see Majumdar [29]). Here the term utility is reserved for a greater structure of meaning than operates in the immediate matter. A utility function is indeed to be a preference gauge, but also it is to be more than that.

10 It seems needed to investigate further the usages, which are fundamental in the language and method of the present subject, and which the purpose is now to stabilize. But here the main adopted task is considered to be mathematical, and to consist merely in the formal statement of a body of definitions, porpositions and demonstrations. Evidently, many of the difficulties in the discussions of the consumer, which appear in the literature, stem directly from the incompleteness in the common understanding, about form and method, in the first ideas.

A scale of value which contains the revealed preferences is admitted as a *motive* for the choice, the selected appearing as better than the rejected in any such scale, and therefore determined by the scale as the best available on the occasion.

The *revealed preference relation* of a set of choices is defined as the transitive closure of the set of their revealed preferences. The choices can have a scale as a common motive, by which condition they are defined to be *coherent*, *in* the scale, if and only if their revealed preference relation is an order. Then any common motive scale, obtaining their coherence, can be considered *revealed* to the extent of being a refinement of their revealed preference order. For the supposition that any single choice has a motive, there can be no contradiction; but if it is that for a multiplicity of choices there is a single motive, in other words that the choices are coherent, then contradictions are possible, and are shown, in immediate analysis, by reflexivities in the revealed preference relation.

Even if constancy of motive may be a state never found in the living, it is required as a concept in order that any behaviour may be explained by its deviation from such a state, and also that value forces, which cause deformation of scales of value, may have the essential reference for their definition.

With an agent making repeated actions, *rationality* is manifested in constancy of motive, obtaining the coherence together of all choices which are made, having expression in the consistency of revealed preferences, given by the non-reflexivity of their revealed preference relation.[11] The contrary of the condition of rationality, caprice, is shown in the conflicting preferences revealed by incoherent choices, obtaining reflexivities in their joint revealed preference relation. Value theory is concerned with the formation of method which will represent activity as rational, and with an exhibited motive in some system of value. The essential questions for the representation seek a definition of (i) the valued potential alternatives, (ii) the scale applied to these alternatives and (iii) the restrictions which obtain the actual from the potential. The intention is to obtain a picture of the activity with the form: every acted alternative is such as to attain the extreme of the scale under a restriction. The condition for the individuality of an activity, by which it may be taken to proceed from a single undivided rational agent, is then that all actions in it can be motivated, or made to cohere in a single system of value.

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11 Rationality is thus to mean governed by preferences. However, the preferences themselves are irrational. They are merely given, as belonging to the character of the agent. This use of the word rational is quite usual in economics, and other sciences concerned with similar aspects of human behaviour. But, as pointed out to me by Professor Feis, it is not perfectly satisfactory, seeing how it would seem to reduce to a very trite matter the greater question which has to do with the conscious faculty of reason. Nevertheless a word is needed, and it is hard to find an available substitute. The notion of rationality is here a constancy, manifesting itself in a consistancy; and this is quite consistent with, though not identical with, the general understanding of the word.

6. Expenditure systems.

There is to be considered an agent whose action is to make expenditures on commodities, in order to obtain consumption. Various simple commodities are taken to form the elements of a single composite commodity, which is to be the matter for choice. An assemblage of amounts of the simple commodities determines an amount of the composite commodity, which is to be called a *commodity composition*. A composition is represented by a vector x, ranging in a region C of the *composition space*, the elements $x_1, \ldots, x_n$ of which give constituent amounts of the commodities, these being supposed all positive numbers.

On any occasion, the simple commodities have *prices*, that is amounts $p_1, \ldots, p_n$ of money required in exchange for unit amounts of the commodities, forming a vector p which determines the expenditure e for any composition x ; thus

$$e = p_1 x_1 + \ldots + p_n x_n = p'x,$$

represents the expenditure e as a bill, a total of elementary expenditures, each of which is the product of a price with an amount, and obtains it as the scalar product of the price and composition vectors. Given prices p, the condition $e = p'x$ admits those compositions x which obtain *balance* with an expenditure e.

The *relative prices*, in the event of a composition x at prices p, are to be the prices given relative to the expenditure $e = p'x$, or those fractions $u_1, \ldots, u_n$ of the expenditure required in exchange for unit amounts of the commodities, forming the elements of the vector

$$u = p/e.$$

In other words, they are the prices with the total expenditure taken as unit of money. By their definition, they satisfy the *balance condition*

$$1 = u_1x_1 + \dots + u_nx_n ,$$

which merely sums the proportions $u_1x_1, \dots, u_nx_n$ in which expenditure distributed over the commodity elements. Thus, to a certain distribution of expenditure among the commodities, there corresponds consumption with a certain composition, the correspondence being established by the relative prices. The relative prices are now to define an *expenditure balance*, represented by a vector u in a region B of the *balance space*, the elements of which are all positive numbers. A composition x is defined to be *within*, *on* or *over* a balance u according as $u'x \lesseqgtr 1$.

The condition $p'y \leqslant p'x$ states that composition y requires at most the same expenditure $e = p'x$ as x at prices p; and it is equivalent to $u'y \leqslant 1$, or that y should be within the balance $u = p/e$.

An hypothesis which can be applied to the consuming agent is that the composition x of consumption is given a determination by the prices p on the occasion taken together with the expenditure e,-thus under the condition $e = p'x$-, in such a way that composition remains unchanged when prices and composition are changed in the same ratio, such a change representing a change just in the unit of money; in other words, in a way which is "without monetary illusion". That is,

$$x(p,e) = x(p\lambda, e\lambda) \quad (\lambda > 0);$$

in particular, with $\lambda = 1/e$,

$$x = x(p/e, 1) = x(u).$$



The condition of absence of monetary illusion is required in order that the artificial unit of money should have no significance for choice. It thus leads to the determination of composition x by the balance u, or just by the range admitted by the choice; and this is expressed in the concept of an <u>expenditure system</u> E, which is to be a mapping of balances into compositions, subject to the balance condition, thus,

$$E: B \rightarrow C \ (u \rightarrow x; \ u'x = 1).^{12}$$

It is considered that a consumer must, in potentiality, obtain a composition x on every balance u; and the determination of x as a function $x = E(u)$ of u subject to the balance condition $u'x = 1$ now gives the concept of an expenditure system to express the idea of a *consumer behaviour*.

A pair of vectors (u,x) with $u'x = 1$ is to define a *choice*, with the understanding that x, on the balance u, is chosen from all those compositions y which are within this balance, having $u'x \leqslant 1$.

If E is an expenditure system, the choices (u,x) with $x = E(u)$ define the *choices of the system*. They form a set

[12] The step from writing $x = x(p,e)$, homogeneous of degree zero, to writing $x = x(u)$, $u = p/e$, is not great, but it is nevertheless important. To omit it is to retain inessential complication which is a bar to the most direct logical insight, and makes, as in apparent historically, some needed results hard to attain. The term "expenditure system" appears in Stone [49]. It seems useful to keep it as distinct from "demand system", which, in accordance with existing usage, may be appropriate for considering composition x just as a function $x = x(p)$ of prices p; in which case expenditure $e = p'x$ also appears as a function $e = e(p)$. In this way, a demand system determines an expenditure system, but not reversely. However, an expenditure system, together with a functional dependence of expenditure on prices, determines a demand system; for then $u = p/e(p) = u(p)$, so that $x = x(u) = x(p)$.

$\{(u,x)\}_{u\in C}$ , which constitutes the graph of the system, there being one choice for every balance $u\in C$ and the corresponding $x = E(u)$.

An expenditure system is defined to be r e g u l a r when the mapping of balances into compositions determines a one to one correspondence between balance and compositions. Thus,

$$E: B \longleftrightarrow C \; (u \longleftrightarrow x; \; u'x = 1)$$

indicates a regular expenditure system E.

An expenditure system is called c o n t i n o u s if it gives x as a continuous function of u: $y \to x \; (v \to u)$. Also it is called d i f f e r e n t i a b l e if the partial derivatives

$$x_{ij} = \partial x_i / \partial u_j$$

of the elements $x_i$ of $x$ with respect to the elements $u_j$ of u all exist, for the transformation of u into x, forming a matrix $x_u$, and the infinitesimals have the transformation

$$dx = x_u du \; ,$$

for the differential $dx$ of $x$ corresponding to a differential $du$ of $u$.

A necessary and sufficient condition for a differentiable expenditure system to be regular is that the mapping between differentials be invertible, and, equivalently, this is that the partial derivative matrix $x_u$ be regular and invertible.[13] In this case the system can be inverted to give u as a function of x with partial derivative matrix $u_x$ which is the inverse of $x_u$; thus,

$$du = u_x dx,$$

from whence it appears that

$$u_x x_u = x_u u_x = 1.$$

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[13] It is true that the regularity of $x_u$ makes the mapping only locally one-to-one; but it will appear that, with the rationality condition, it obtains this property also globally.

## 7. Revealed preference.

The consumer is considered as an individual capable of existing in certain *states*, and for whom an action is a process of transition from one state to another. If an action has motive, it is a transition from one state to a higher one in the motivating scale of value; and, in ultimate equilibrium, it is to one which is the best within the existing limitations of possibility.

Each state of the individual is now to be viewed as determined by a consumption, with a certain composition x, together with a *reserve*, given by an amount of money $\rho$. The reserve can be considered as the money equivalent of all expendible assets, exterior to the considered consumption. Thus, there is made the identification $s = (x,\rho)$ of any state s with a composition x of consumption and a reserve $\rho$ of money. The restriction which defines the alternative states attainable by the individual in an event is given by the ratios of amounts in which different goods may be exchanged in the market in the event, these ratios being given when the prices, the exchange ratios with money, are given; and they are the ratios of respective prices. The *amplitude* of a state, at given prices, is defined as the amount of money which has exchange equivalence with all the goods of the state together; thus,

$$\alpha = p'x+\rho$$

is the amplitude of the state $s = (x,\rho)$, with consumption x and reserve $\rho$, at prices p. The states which are exchangable, being the alternatives of choice admitted by a given exchange restriction, are all those with a given fixed amplitude. This common amplitude of the alternative states defines the amplitude of the choice.

The composition obtained in any event is to be contained in the result of a restricted choice between valued alternative states. Thus an amplitude $\alpha$ and prices p are given in an event, and the composition x found in that event is to be that which obtains a state $(x,\rho)$ which is higher, in the motivating scale M which is operative, than any other state $(y,\rho)$ attainable under restriction to the given amplitude; thus,

$$y \neq x \Rightarrow (y,\sigma)M(x.\rho) \qquad (\alpha = p'x+\rho = p'y+\sigma).$$

A good is defined as a coordinate of state for which it can always be assumed that there holds the law of increase, that "more" is "better". In this way, the possession of a commodity is to be considered a good.

For any states s, t let

$$s \subset t$$

mean that all the elementary goods in s are in amount at least the corresponding amounts in t, and not all equal.

It is to be assumed, for a scale of value M applied to states, that it fulfills the law of increase

$$s \subset t \Rightarrow sMt,$$

that is, states are always preferred in which some goods are greater and no goods are less, in which case M will be called an increasing scale $S_\rho$ applied to compositions

$$xS_\rho y \equiv (x,\rho)M(y,\sigma)$$

It is possible that for some different $\rho,\sigma$ and for some x,y there may hold the relations

$$xS_\rho y, \quad yS_\sigma x \ ;$$

that is, at the reserve level $\sigma$, x is preferred to y, while at the reserve level $\rho$, y is preferred to x. In this case, movement in the level of reserve appears as a *force*, affecting the value of compositions. In the contrary case, that is

$$xS_\rho y \iff xS_\sigma y,$$

when the value of compositions remains *rigid* when subject to changes in reserve, or is independent of the reserve, in which case there is said to be *rigid reduction* of the scale from states to compositions, there is defined a scale of compositions S such that

$$xSy \iff (x,\rho)M(y,\rho)$$

for every $\rho$.

With the condition of rigid reduction, of M to S, together with the law of increase, and the transitivity, which is axiomatic for a scale, there follows

$$\begin{aligned}(x,\rho)M(y,\sigma)\wedge\rho \geqslant \sigma . &\Rightarrow . (x,\rho)M(y,\sigma)\wedge(y,\sigma)C(y,\rho)\\ &.\Rightarrow . (x,\rho)M(y,\sigma)\wedge(y,\sigma)M(y,\rho)\\ &.\Rightarrow . (x,\rho)M(y,\rho)\\ &.\Rightarrow . xSy.\end{aligned}$$

Now suppose, in an event with prices p, the composition chosen is x. Whatever it is, let $\rho$ denote the reserve, so the amplitude of choice is $\alpha = p'x+\rho$. The chosen state is $(x,\rho)$, and any alternative state is $(y,\sigma)$ where $\alpha = p'x+\sigma$, so that $\sigma \geqslant \rho$ if $p'y \leqslant p'x$. The principle of revealed preference gives

$$\alpha = p'y+\sigma\wedge y\neq x \Rightarrow (y,\sigma)M(x,\rho);$$

and it now appears accordingly that

$$y \neq x \wedge p'y \leqslant p'x \Rightarrow (y,\sigma)M(x,\rho) \wedge \sigma \geqslant \rho$$

$$\Rightarrow ySx.$$

This gives the special form which the primitive principle of revealed preference takes in application to the consumer, in which it is joined with the law of increase, that "more" is "better", and the condition of rigid reduction, that the supposed generally motivating scale of value, applying more comprehensively to states, which have consumption just as a component, induces a well-defined scale of value on compositions, undisturbed by changes in the residual factors of state, here exemplified in the concept of reserve. The principle of consumer's revealed preferences is thus broken down to the primitive principle, together with notions peculiar to the consumer.[14]

8. Rationality and coherence.

Let there be given an expenditure system E. Then to any balance $u \in B$ there corresponds a composition $x \in C$ on u, and therefore a choice (u,x); thus

$$E: u \rightarrow (u,x) \quad (u'x = 1).$$

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[14] Preferences are revealed in any action, assuming motive to operate, or merely by definition of the analysis of behaviour in terms or preferences. "Revealed preference", which originates with Samuelson [45] as a principle in consumer theory, and is taken further, most especially, by Houthakker [24], belongs to a primitive and universal principle, which has immediate use by everyone. It is an important element at the basis of the knowledge which embodies out expectations of the behaviour of more or less stable individuals. It carries the notion that preference, as revealed by, and as, in the principle of the concept, controlling choice-in the sense of choice which is most general, and often somewhat complex and difficult in application, to the confusion of philosophers, but which bears on every kind of activity-are an amportant part of the definition of character. Their stability is correlated with stability of character, which is inseparable from the perpetuated identity of the individual.

The rationale of the principle of revealed preference for the consumer seems often to have been somehwat inadequately given, and this has interfered with the correctness and completeness of its application. For example, cf. Morgenstern [32].

For any $u \in B$, let a relation $P_u$, to be called the *revealed preference relation* of the choice $(u,x)$ determined by $u$, be defined by

$$yP_u x \equiv y \neq x \wedge u'y \leqslant 1.$$

It is a relation which holds just between the composition $x$ on the balance $u$, and every other composition $y$ which is within that balance; and it may be stated by saying that $x$ is *revealed preferred* to all those other compositions $y$, *by* the choice $(u,x)$.

The principle of revealed preference applied to the consumer gives that if composition $x$ is obtained at prices $p$, with a composition value scale $S$ as motive, then

$$y \neq x \wedge p'y \leqslant p'x \Rightarrow ySx,$$

that is, $x$ must be preferred, in any scale $S$ admitted as motive, to every other composition $y$ which requires at most the same expenditure as $x$ at the prices $p$; equivalently,

$$P_u \Rightarrow S,$$

which may be called the condition for a scale $S$ to be admitted as *motive* for a choice $(u,x)$.

Any choices are together said to be *coherent* if they admit a scale $S$ as a common motive; and then they are said to *cohere in* that scale.

The expenditure system $E$ will be said to admit a scale $S$ as motive if $S$ is admitted as a motive of all its choices, thus,

$$\bigwedge_{u \in B} P_u \Rightarrow S\ ;$$

and if $E$ admits a motive, then it will be called *rational*. Thus an expenditure system $E$ admits a scale $S$ as motive if all its choices together cohere in $S$. The rationality of an expenditure system is thus equivalent to the coherence of all its choices.

In view of the axiomatic transitivity of S,

$$\bigwedge_{u\epsilon B} P_u \Rightarrow S \;.\!\Leftrightarrow\!.\; P \Rightarrow S,$$

where

$$P = \overrightarrow{\bigvee_{u\epsilon B}} P_u = \bigvee_{m=1}^{\infty} (\bigvee_{u\epsilon B} P_u)^m$$

is the transitive closure of the sum $\bigvee_{u\epsilon B} P_u$ of the revealed preference relations $P_u$ of all the choices (u,x) of the system E, and defines the r e v e a l e d p r e f e r e n c e r e l a t i o n P o f a n e x p e n d i t u r e s y s t e m E.[15] Accordingly, a necessary and sufficient condition for an expenditure system E to admit a scale S as motive is given by

$$P \Rightarrow S;$$

whence, any scale which motivates an expenditure system must be a refinement of its revealed preference relation, and can be considered r e v e a l e d to this extent. But a necessary and sufficient condition that a relation imply an order is that its transitive closure be non-reflexive; wherefore, an expenditure system is rational if and only if its revealed preference relation, being now non-reflexive by hypothesis, and transitive by construction, is an order; and any scale which can be a motive of the rational expenditure system is a refinement of this order.

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[15] If the distinction would be an advantage, just the preferences determined by the relations $P_u$ could be called revealed; and the further preferences determined by P could then be called inferred (under the hypothesis of rationality). Then any of the preferences, revealed or inferred, define the preferences of E.

## 9. Consistency.[16]

A composition x is said to *occur on* a balance u, in an expenditure system E, if $x = E(u)$; also x is said merely to *occur* in E if it occurs on some balance.

A series of compositions $z_0, \ldots, z_k$ is said to form a *preference chain* of E, *descending* from $z_0$ to $z_k$, and *ascending* from $z_k$ to $z_0$, if $z_0, \ldots, z_{k-1}$ occur in E on balances $w_0, \ldots, w_{k-1}$ and

$$w_0' z_1 \leqslant 1 \wedge \ldots \wedge w_{k-1}' z_k \leqslant 1.$$

The condition yPx, that y be revealed inferior in preference to x in E, is now that there exists a preference chain of E descending from x to y; thus,

$$yPx \Longleftrightarrow \bigvee_k \bigvee_{w_0, \ldots, w_{k-1}} (x = z_0 \wedge y = z_k) \wedge (w_0' z_1 \leqslant 1 \wedge \ldots \wedge w_{k-1}' z_k \leqslant 1) ,$$

where it is understood that $w_i$ determines $z_i$ $(i = 0, 2, \ldots)$

A preference chain of E which does not have any distinct extremities, or which, equivalently, has each of its terms at the same time as beginning and end, defines a *preference cycle* of E. Thus, a cycle of compositions $z_0, z_1, \ldots, z_k, z_0 \ldots$ determines a preference cycle of E if they occur on balances $w_0, w_1, \ldots, w_k, w_0, \ldots,$

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16 It could seem an unnecessary complication to have three names for what here amounts to the same condition, given by the rationality of a behaviour, the coherence of the choices which belong to the behaviour, and the consistancy of the preferences which belong to the choices. However, the need for the distinction becomes apparent when a behaviour, a set of choices, and a set of preferences have to be considered separately.

and

$$w'_o z_1 \leqslant 1 \wedge \ldots\ldots \wedge w'_{k-1} z_k \leqslant 1 \wedge w'_k z_o \leqslant 1.$$

Given an expenditure system E, with revealed preference relation P, a pair of preferences xPy, yPx, which are to be called *opposites*, defines a *contradiction* in the revealed preferences of E. A preference contradiction is thus a symmetry

$$xPy \wedge yPx$$

in the relation P. The revealed preferences of E are called *consistent* when there are no contradictions, that is to say when there is the condition

$$\sim . xPy \wedge yPx,$$

which is equivalent to the condition

$$xPy \Rightarrow \sim yPx$$

for the antisymmetry of P. The consistency of the revealed preferences of E is thus represented by the antisymmetry of the relation P.

An *absurd* preference is one of the form xPx, the absurdity being in the real impossibility of an object being at the same time selected and rejected in the same act. Evidently, a pair of opposite preferences by transitivity imply an absurd preference; moreover, an absurd preference, duplicated, represents a trivial pair of opposite preferences. Therefore the absence of absurd preferences, that is the condition

$$\sim xPx ,$$

which is equivalent to the condition

$$xPy \Rightarrow x \neq y$$

for the non-reflexivity of P, is equivalent to the consistency of the revealed preferences of E, expressed in the antisymmetry of P, and, moreover, to rationality, which has been seen to require just this condition.

Any reflexivities xPx of P are for compositions x which appear on preference cycles; and conversely, each composition x which appears on some preference cycle gives a reflexivity of P. Therefore, a necessary and sufficient condition for consistency is the absence of preference cycles, which defines the condition of *acyclicity* applied to P.[17]

Consistency, as acyclicity, thus has statement in the negation of all preference cycles:

$$\bigwedge_k \bigwedge_{w_o,\ldots,w_k} :\sim .w_o'z_1 \leqslant 1 \wedge \ldots \wedge w_{k-1}'z_k \leqslant 1 \wedge w_k'z_o \leqslant 1,$$

and, equivalently, in the form of a condition for the breaking of every preference cycle, as

$$\bigwedge_k \bigwedge_{w_o,\ldots,w_k} :: w_o'z_1 \leqslant 1 \wedge \ldots \wedge w_{k-1}'z_k \leqslant 1 \;.\Rightarrow.\; w_k'z_o > 1. \quad [18]$$

Let $z_o,\ldots,z_k$ be a series of compositions forming an ascending preference chain of a consistent expenditure system E, in which they occur with balances $w_o,\ldots,w_k$. Then every partial series $z_r,\ldots,z_s$ $(r<s)$ of consecutive compositions must also form a chain, and not a cycle,

---

[17] cf. von Neumann and Morgenstern [52], where there is a treatment along similar lines.

[18] This is the form of the condition which Houtakker [24] has called "semi-transitivity". In fact, here it appears as the condition for the non-reflexivity, or equivalently the antisymmetry, of the in any case transitive relation P. If there is an error in terminology, it reflects an error of method.

so that, with

$$w_r'z_{r+1} \leqslant 1 \wedge \ldots \wedge w_{s-1}'z_s \leqslant 1 \;.\Rightarrow.\; w_s'z_r > 1,$$

it appears that

$$w_s'z_r > 1 \;(r < s).$$

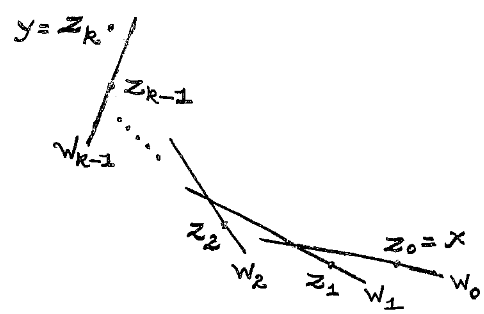
Thus there is obtained the following consistency scheme, for a preference chain in a consistent system:

$$\begin{array}{llllll}
\cdot & w_0'z_1 \leqslant 1 & & & & (>0) \\
w_1'z_o > 1 & \cdot & w_1'z_2 \leqslant 1 & & & \\
w_2'z_o > 1 & w_2'z_1 > 1 & \cdot & \ddots & & \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & & \ddots & \\
w_{k-1}'z_o > 1 & w_{k-1}'z_1 > 1 & w_{k-1}'z_2 > 1 & \ldots\ldots & & w_{k-1}'z_k \leqslant 1 \\
w_k'z_o > 1 & w_k'z_o > 1 & w_k'z_2 > 1 & \ldots & w_k'z_{k-1} > 1 & \cdot
\end{array}$$

If $w_r'z_s \leqslant 1$ for some $r<s-1$, then the reduced series

$$z_o, \ldots, z_r, z_s, \ldots, z_k$$

of compositions obtained by omitting all those in the chain between $z_r$ and $z_s$ is also a preference chain descending from $z_o$ to $z_k$, defining a r e d u c t i o n of the original chain; and a chain is to be called i r r e d u c i b l e if it has no reductions. Thus, for an irreducible preference chain in a consistent system, the consistency scheme may be argumented by the further conditions $w_r'z_s > 1$ $(r<s-1)$ indicated in the remaining positions above the diagonal.



10. Cross-deviations.[19]

On two occasions, to be indicated by o and 1, let $p_o$, $p_1$ be the vectors of prices and $x_o$, $x_1$ the commodity compositions. The direct expenditure on occasion o is simply the expenditure

$$e_o = p_o' x_o$$

on that occasion; and the cross-expenditure from occasion o to occasion 1 is defined to be the expenditure

$$e_{o1} = p_o' x_1$$

required for the composition $x_1$ of commodities on occasion 1 at the prices $p_o$ on occasion o. So, by definition, $e_{oo} = e_o$, or the cross-expenditure from one occasion to the same occasion is the same as the direct expenditure in that occasion. The relative cross-expenditure from o to 1 is now defined as the ratio of the cross-expenditure from o to 1 with the direct expenditure in o, thus

$$f_{o1} = \frac{e_{o1}}{e_{oo}} = \frac{p_o' x_1}{p_o' x_o} = u_o' x_1 ,$$

where $u_o = p_o/e_o$ gives the relative prices in o. Accordingly, $f_{oo} = u_o' x_o = 1$, by definition. The ratio of the pair of relative cross-expenditures between two occasions defines the principal ratio, for the one occasion on the other, thus

$$R_{o1} = \frac{f_{o1}}{f_{1o}} = \frac{e_{o1} e_{11}}{e_{oo} e_{1o}} = \frac{p_o' x_1 p_1' x_1}{p_o' x_o p_1' x_o} = \frac{u_o' x_1}{u_1' x_o} ,$$

[19] These are the "observables" which enter into the index-number calculations, for which the account is given in Part II, which give measurement of standard and cost of living. There are two cross-expenditure and then two cross-deviations measures between, say, every pair of years, ordered one way and the other.

so

$$R_{oo} = 1, \quad R_{o1}R_{1o} = 1,$$

by definition. Now the relative cross-expenditure deviation, or merely the *cross-deviation*, from o to 1, is defined as the deviation of the relative cross-expenditure from unity, thus

$$D_{o1} = f_{o1} - 1 = u_o' x_1 - 1 \;;$$

and this definition gives

$$D_{oo} = 0.$$

Revealed preference now has the formulation that if $S_o$ is the composition value scale operative on occasion o, then

$$D_{o1} \leqslant 0 \Rightarrow x_1 S_o x_o \;;$$

that is, composition $x_1$ is revealed inferior to $x_o$, in the scale of value operative on occasion o, if the cross-deviation from o to 1 is non-positive.

# II. Rational Expendiute Dualities.

## 1. Duality

An *expenditure duality* is defined as a regular expenditure system, that is a system which obtains a one to one correspondence between expenditure balances and commodity compositions.[1] Thus

$$E: B \longleftrightarrow C \ (u \longleftrightarrow x;\ u'x = 1)$$

indicates an expenditure duality E. A balance and composition u and x which are thus uniquely paired with each other in the duality E are said to be the *duals* of each other, in E, and, by this condition, to form a *choice* (u,x) of E.

There is a perfect symmetry between balance and composition in the concept of a duality; therefore, in any definition or proposition relating to an expenditure duality, the roles of composition and balance may be interchanged, to obtain a *dual* definition or proposition. This reciprocation of concepts, besides being an instrument for demonstrations, gives an expression of the duality which exists in the analysis of supply and demand, there being, in essential form, one analysis, which can be interpreted for one side or for the other.

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[1] In case this concept of expenditure duality should seem unfounded, even from the point of view of the orthodoxy of the subject, so far as that exists, it should be noted that, along with the dependence $x=x(u)$, which is at the foundation of consumer theory, the dependence $u=u(x)$ has an already existing acceptance, though it is more concealed. For example, the altogether standard idea that x is determined from u as obtaining the maximum of a function $\phi(x)$ under the constraint $u'x=1$ gives the condition $u\lambda=\phi_x$, $\lambda=x'\phi_x$, which obtains $u=\phi_x/\lambda=u(x)$ as a function of x. Also, a question of "integrability" is a very standard topic in discussions. It has been much considered, but not everywhere is it plainly stated what exactly it is that has to be integrable, and in what sense (apparently the matter was first raised in Volterra's review of Pareto's Manuel d'Æconomie Politique). If it is the differential form $u'dx$, then, presumably, again u is a function of x. Thus the concept of a duality, giving a pairing of balances and compositions u and x, seems implicity granted in the most usual forms of discussion of the consumer, even if not altogether directly. Even if the phenomenon of "saturation", which is against the concept, is to be admitted, it is always possible to replace strict saturation by near saturation — as near as no matter.

## 2. Dual and induced preferences.

Let there be given an expenditure duality E, the choices (u,x) of which, obtaining the pairing between balances and compositions, may be indicated, from either side, either by the balance u or by the composition x, thus,

$$x \longleftrightarrow (u,x) \longleftrightarrow u.$$

The c r o s s - d e v i a t i o n from a choice (u,x) to a choice (v,y) of E, with indication of choice from the side of composition, is defined by

$$D_{xy} = u'y - 1.$$

In a dual fashion,

$$D_{vu} = y'u - 1.$$

But

$$u'y-1 = y'u-1,$$

and therefore

$$D_{xy} = D_{vu}.$$

The composition y is w i t h i n, o n or o v e r the expenditure balance u according as $D_{xy} \lesseqgtr 0$; and dually, and equivalently, the balance u is within, on or over the composition x, now according as $D_{vu} \lesseqgtr 0$.

The revealed preference relation $P_u$ of a choice (u,x) may now be indicated equivalently by $P_x$, since, in a duality, to every x there corresponds a u; thus,

$$P_u = P_x,$$

where, with indication from the side of composition, there is the definition

$$yP_x x \equiv D_{xy} \leqslant 0.$$

In a dual fashion, a choice $(v,y)$ has a *dual revealed preference relation*, which is the relation $Q_v = Q_y$ between balances, defined by

$$uQ_y v \equiv D_{vu} \leqslant 0.$$

Thus

$$yP_x x \Leftrightarrow uQ_y v.$$

Associated with any expenditure duality E, there is thus defined a pair of relations P, Q the one between compositions and the other between balances. They are given by the formulae

$$P = \overrightarrow{\bigvee_{x\in C}} P_x, \quad Q = \overrightarrow{\bigvee_{v\in B}} Q_v$$

that is, by the transitive closures of the sums of the preferences and the dual preferences revealed by the choices; and they are to be called the *preference relation* P and *dual preference relation* Q of E, respectively. They are defined equivalently as the minimal transitive relations under the conditions

$$D_{xy} \leqslant 0 \Rightarrow yPx, \quad D_{vu} \leqslant P \Rightarrow uQv,$$

where, since $D_{xy} = D_{vu}$, it appears that

$$yPx \Leftrightarrow uQv.$$

Equivalently, for any pair of series $z_o, \ldots, z_k$ and $w_o, \ldots, w_k$ which are duals in E, the one is a chain of compositions descending in P if and only if the other is a chain of balances ascending in Q.

A preference between compositions $x,y$ *induces* the corresponding preference between the dual balances $u, v$; and similarly from the dual side. Thus the relations P, Q between compositions and balances induce relations $P^*$, $Q^*$ between balances and compositions according to the definitions

$$vP^*u \equiv yPx, \quad xQ^*y \equiv uQv.$$

Now it appears that

$$vP^*u \Longleftrightarrow uQv, \quad xQ^*y \Longleftrightarrow yPx \;;$$

that is

$$P^* = Q', \quad Q^* = P',$$

where $P'$, $Q'$ are the conjugates of the relations $P, Q$. Thus:

THEOREM. _The induced preference relation of an expenditure duality is the conjugate of the dual preference relation._

Together with this theorem there is its dual equivalent.

## 3. Value domains.

Let E be a rational expenditure duality, with preference order P. The *inferior* and *superior domains* of any composition x are defined as the sets Px and xP of compositions y such that yPx and xPy; and the complements $\bar{P}x$, $x\bar{P}$ of these define the *non-inferior* and *non-superior domains*, formed of those compositions y such that $y\bar{P}x$ and $x\bar{P}y$, where $\bar{P}$ is the relation which is the complement of P.

The complement of the union of the inferior and superior domains of x, or equivalently the intersection of the non-inferior and non-superior domains, defines the *indifference domain* $\tilde{P}_x$ of x, of compositions y such that $y\tilde{P}x$, where $\tilde{P} = \bar{P} \wedge \bar{P}'$ is the symmetric complement of P; thus,

$$\tilde{P}_x = x\bar{P} \cap \bar{P}x.$$

Since $\tilde{P}$ is symmetric, it follows that

$$y \in \tilde{P}_x \Leftrightarrow x \in \tilde{P}_y \ .$$

The transitivity of P gives

$$xPy \ .\Rightarrow.\ yP \subset xP \wedge Px \subset Py;$$

and since,

$$y \in xP \cap Px \ .\Leftrightarrow.\ xPy \wedge yPx,$$

the antisymmetry of P gives

$$xP \cap Px = 0.$$

All these statements have their duals, in which balances and compositions are interchanged, and the preference relation is replaced by the dual preference relation.

## 4. Convexity.[2]

The *segment* joining a finite set of elements $x, y, \ldots \in C$ is defined as the set $[x, y, \ldots]$ of elements of the form $x\lambda + y\mu + \ldots (\lambda, \mu, \ldots \geq 0;$ $\lambda + \mu + \ldots = 1)$.

A set K C is called *convex* if it contains, with any pair, or equivalently with any finite set of its elements, the segment which joins them, thus,

$$x, y, \ldots \in K \Rightarrow [x, y, \ldots] \subset K.$$

The intersection of any convex sets in a convex set. The intersection of all the convex sets containing a given set S is a convex set [S], containing S, and contained in every convex set K containing S, thus,

$$S \subset [S], \quad S \subset K \Rightarrow [S] \subset K.$$

---

[2] For this subject, general reference is made to Fenchel [16], Eggleston [15].

It thus appears with the property of being the minimal convex set containing S, by which property it is called the *convex cover* of S. The convex cover of any finite set is merely the segment joining its elements, this identity being already expressed in the notation.

A convex set if necessarily connected. A closed convex set with non-empty interior is to be called a *convex body*. A *concave set* is defined as the complement of a convex set.

5. Px, xP are concave, convex and open.

Any preference chain descending from x to a composition y in Px can be continued to any composition z within the balance v dual to y, so that z is also in Px, by the definition of P; that is

$$y \in Px \wedge v'z \leqslant 1 \Rightarrow z \in Px.$$

It follows that Px is identical with the union of the balance regions $\{z; v'z \leqslant 1\}$ which belong to the compositions $y \in Px$. But

$$y \in Px \Leftrightarrow v \in uQ.$$

Accordingly,

$$Px = \bigcup_{v \in uQ} \{z; v'z \leqslant 1\},$$

and equivalently,

$$\text{(1)} \qquad \overline{P}x = \bigcap_{v \in uQ} \{z; v'z > 1\},$$

whence:

PROPOSITION (i). Px is concave and xP convex.

For it follows from the previous formula, (1), that $\overline{P}x$ is convex, being the intersection of convex half-spaces given by a set of balances; and, equivalently, its complement Px is concave. Then also

$$(2) \qquad v \in uQ \Leftrightarrow \{z; v'z > 1\} \supset \bar{P}x,$$

whence, since $\bar{P}x$ is convex, $uQ$ appears as convex; and, dually, $xP$ is convex.

PROPOSITION (ii). $\qquad y \subset x \Rightarrow yPx.$

That is, $P$ is an increasing scale. For, if $y \subset x$, then $u'y \leqslant u'x$, since the elements of $u$ are positive, whence, since $u'x = 1$, and by the definition of $P$,

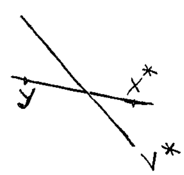
$$y \subset x \Rightarrow y{=}x \wedge u'y \leqslant 1$$

$$\Rightarrow yPx.$$

PROPOSITION (iii). <u>For every $y \in Px$ there exists a $z \in Px$ such that</u> $y \subset z$.

For if $y \in Px$ there exists a preference chain descending from $x$ within $Px$, with $y$ as ultimate composition, and a penultimate composition $x^*$ with $y$ within its dual balance $u^*$, that is $u^{*\prime}y \leqslant 1$. If $u^{*\prime}y < 1$, then, since the elements of $u^*$ are positive, there exists a $z \supset y$ such that $u^{*\prime}z = 1$, so the chain can be continued from $x^*$ to $z$, giving $z \in Px$. Otherwise, if $u^{*\prime}y = 1$, take a composition $y^*$ in the interior of the segment $[x^*, y]$, with dual balance $v^*$. Now, since $u^{*\prime}x^* = 1$ and $u^{*\prime}y = 1$ it follows that $u^{*\prime}y^* = 1$. But the supposed antisymmetry of preferences requires

$$u^{*\prime}y^* = 1 \Rightarrow u^{*\prime}x^* > 1.$$




However,

$$v^{*\prime}x^* > 1 \Rightarrow v^{*\prime}y < 1\ ,$$

since the segment cuts the balance, with its extremities on either side. Therefore $v^{*\prime}y < 1$, and, since the elements of $v^*$ are positive, there exists a $z \supset y$ such that $v^*z = 1$. The chain descending from $x$ to $x^*$ may now be continued to $y^*$ and then to $z$, so that $z \in Px$.

PROPOSITION (iv). Px __and__ xP __are__ __open__.

If y is on the frontier $(Px)^*$ of Px, and $z \supset y$, then $z \in \overline{P}x$, so, by Proposition (iii), y cannot belong to Px; whence Px is open, and therefore $\overline{P}x$ is closed. Now (1) and (2) imply that uQ is open, and so, dually, xP is open.

## 6. Contensional dualities.[3]

The domains B, C are now to be considered as contained in a conjugate pair of real linear spaces of finite dimension, which are to be distinguished as the *intensional* and *extensional spaces*; so $u'x (u \in B,\ x \in C)$ is a real bilinear function on B, C.

Intensional and extensional elements u and x are said to *include* or *exclude* each other according as $u'x > 1$ or $u'x < 1$; and they are said to be *on* each other, or to be *conjugate*, if $u'x = 1$.

By the symmetry of these relation of inclusion, exclusion and conjugation between intensional and extensional elements their roles are exchangeable in any definition or proposition constructed on these relations.

An intensional element u is said to *divide* the extensional domain C if there are elements of C in each of the relations of inclusion and exclusion with u, or what is the same thing, if u is conjugate with some element of C: and similarly for an extensional element x dividing the intensional domain B. The intensional and extensional domains B, C are said to be *conjugate* if each includes exactly all the elements which divide the other, or, equivalently, each includes all the elements which are conjugate to some element of the other.

---

[3] The concept here really amounts to that of a convex body, with a formulation which does justice to the duality, existing perfectly between points and half-spaces, and which specially suits the present subject.

Now let B, C denote such a pair of conjugate domains in the intensional and extensional spaces. In correspondence with any extensional subset $X \subset C$, there is defined an intensional subset $\{X\} \subset B$, formed of just those elements $u \in B$ which include all the elements x of X, and which may be called the inclusion domain or merely the *inclusion* of X; thus

$$u \in \{X\} \equiv \bigwedge_{x \in X} u'x > 1.$$

The inclusion of the inclusion of X is a set containing X, which is to define its *completion* [X], thus

$$[X] = \{\{X\}\} ;$$

and, as implied in the notation, this turns out identical with the convex cover of X in C. Now X is said to be *complete* if it is identical with its completion, thus,

$$X = [X];$$

and so completeness is equivalent to convexity. The completion of any set is complete, so that any iteration of the operation of taking the completion is equivalent to the simple operation:

$$[[X]] = [X].$$

Again, all these definitions and propositions may be replaced by their dual forms, with intensional and extensional elements exchanged.

A pair of intensional and extensional subsets $U \subset B$, $X \subset C$ is said to define a *contensional duality* in the conjugate domains B, C if each is the domain of inclusion of the other, in these domains. Thus a contensional duality (U,X) satisfies the conditions

$$U = \{X\}, \quad X = \{U\},$$

where U and X define the intensional and extensional *sides* of the duality. It appears that

$$[U] = \{\{U\}\} = U, \; [X] = \{\{X\}\} = X,$$

so that each side of a contensional duality is complete. Any complete intensional or extensional set is a side of a contensional duality. Any set in either B or C, is said to *generate* the duality for which its completion gives one of the sides. A contensional duality is defined to be *regular* if each of its sides has non-empty interior, and otherwise to be *singular*.

With any intensional element u there are associated the propositional functions of the extensional elements defined by

$$u(x) \equiv u'x > 1, \quad u^*(x) \equiv u'x < 1$$

asserting that x is included, and excluded by u. Their negations, which assert that x is not included, that is excluded or conjugate, and not excluded, that is included or conjugate, are given by

$$\bar{u}(x) \equiv u'x \leqslant 1, \quad \bar{u}^*(x) \equiv u'x \geqslant 1.$$

Thus, associated with any intensional element u, there are four propositional functions, indicated by $u$, $u^*$, $\bar{u}$ and $\bar{u}^*$, and which are to define linear *barriers*, the first pair of which having open domains, and the second pair closed. A complemented lattice is generated by such propositional functions, or the barriers represented by them, with conjugation and disjunction giving upper and lower bounds in the relation of implication, and with complementation given by negation. The *consistency* and *independence* of a set of barriers, or more particularly of a set of intensional elements, is defined in terms of the consistency and independence of the associated propositional functions. For a set of intensional elements $U \subset B$, an *implication*

$$U \Rightarrow v$$

means that, for all x, v(x) is implied by the conjunction of the u(x) for all u∈U, that is

$$\bigwedge_x : \bigwedge_{u\in U} u(x) \;.\Rightarrow.\; v(x),$$

and gives the condept of the d e p e n d e n c e of v on all the u∈U. It appears that an equivalent formulation of this condition is that v belongs to the completion of U, which is also the convex cover of U; thus,

$$v\in[U].$$

A set of barriers is consistent if and only if their completion does not contain the null barrier, and independent if none lies in the completion of the remainder.

7. Support and contact conjugacy.

Let (U,X) be a regular contension in the conjugate domains (B,C). Let $U^o$ and $U^*$ denote the interior and frontier of the intensional side U; and similarly for the extensional side. The frontiers of the sides are a pair $(U^*;X^*)$ of surfaces, the elements of which define the s u p p o r t s and c o n t a c t s of the contension. Any supporting element $u^*\in U^*$ is conjugate to some contact element $x^*\in X^*$ but to no interior element $x^o\in X^o$, all of which it includes:

$$\bigwedge_{u^*}\bigvee_{x^*} u^{*\prime}x^* = 1, \qquad \bigwedge_{u^*,\, x^o} u^{*\prime}x^o > 1.$$

The set of contacts conjugate to a given support, which is thus non-empty, defines its z o n e o f c o n t a c t. Similarly, every contact has a z o n e o f s u p p o r t, given by its conjugate supports, while it includes every interior intensional element:

$$\bigwedge_{x^*}\bigvee_{u^*} u^{*\prime}x^* = 1, \qquad \bigwedge_{x^*,\, u^o} u^{o\prime}x^* > 1 .$$

The zones of support or contact of a given contact or support are singular complete closed sets, equivalently they are convex bodies within the space of lower dimension, of elements conjugate to that contact or support.

A strict contension is defined to be one in which the support and contact conjugacy is a reciprocity, pairing supports and contacts in a one-to-one correspondence by the condition that they be conjugates. The zone of contact or support of a given support or contact in this case consists of a single element.

A support is said to have *zero contact* if its zone of contact has surface measure zero, and *simple contact* if this zone consists of a single element. Similary, a given contact can have *zero* and *simple support*.

In a regular contension, almost every contact has simple support, and almost every support simple contact, while in any contension, almost every contact has zero support and almost every support zero contact. A strict contension is one in which every contact or support has simple support or contact.

The following proposition is important later: the interior supports of the zone of support of a given contact have the same zone of contact, which is the common contact of all the supports of the zone.[4]

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[4] A proof of this proposition has been kindly obtained for me by R. Phelps. It is to be found in a note at the end of this report.

8. <u>The contacts and supports of $\bar{P}$x.</u>

It appears from 5(2) that uQ is the domain of $\bar{P}$x,

$$uQ = \{\bar{P}x\} .$$

Thus they form a contension (uQ, $\bar{P}$x), the supports and contacts of which are given by the frontiers of its sides. The frontier of $\bar{P}$x is the same as the frontier (Px)* of its complement Px. The support and contact surfaces of the contension are thus the frontiers (uQ)*, (Px)* of uQ and Px. Thus, in view of 5(2) and Propositions 5, (i) and (iv), there appears;

PROPOSITION (i). <u>The frontiers</u> (uQ)*, (Px)* <u>of</u> uQ, Px <u>give the supports and contacts in the contension</u> (uQ, $\bar{P}$x) <u>determined by</u> $\bar{P}$x.

The supports and contacts in the contension determined by $\bar{P}$x may also be called the supports and contacts of the convex body $\bar{P}$x.

PROPOSITION (ii). <u>The dual of any support of</u> $\bar{P}$x <u>lies in its zone of contact.</u>

For let v be any supporting balance, with dual composition y. Then if y is not a contact, lying in (Px)*, it must lie in Px. Now if z is a contact of v, therefore on (Px)* and distinct from y, then z is revealed inferior to y in Px, and therefore must lie in Px as well as (Px)*, which is impossible, since, by Proposition 5(iv), Px and (Px)* are disjoint.

PROPOSITION (iii). <u>There is a unique support of</u> $\bar{P}$x <u>with a given zone of contact.</u>

For were there a distinct two, by Proposition (ii) their dual compositions would lie in the common zone of contact, and therefore each composition would lie in the dual balance of the other, which is impossible, by the antisymmetry of preference.

PROPOSITION (iv). <u>On every contact of $\bar{P}x$ there is a unique support.</u>

By the general proposition stated at the end of paragraph 7, that the interior supports of the zone of support of a given contact have the same zone of contact, the common contact of all the supports of the zone, it appears that if the zone of support of a contact of $\bar{P}x$ were to consist of more than one support, it would be a convex set with at least two distinct interior supports; and these would have the same zone of contact, which is impossible, by Proposition (iii).

PROPOSITION (v). $\quad v \to u \; (yPx;\; y \to x)$.

With any composition y in Px, the *perspective* $V_y$ of $\bar{P}x$ from y is defined as the complement in Px of the convex closure of $\bar{P}x$ with y. Its common frontier with $\bar{P}x$ defines the *face* $F_y$ of $\bar{P}x$ from y, the frontier of which defines the *horizon* of $\bar{P}x$ from y. The set $K_y$ of directions of the supports of $\bar{P}x$ with contact on the face $F_y$ defines the *curvature* of $\bar{P}x$ facing y. Its frontier is the set of directions of the supports of $\bar{P}x$ which are on y, these being the supports with contact on the horizon of $\bar{P}x$ from y.

If $z \in C_y$, its dual balance w is parallel to a support on the face $F_y$ of $\bar{P}x$ from y. So if $|w|$ denotes the direction of w then

$$z \in V_y \Rightarrow |w| \in K_y \; .$$

In particular, $y \in V_y$, and therefore $|v| \in K_y$ , where v is the balance dual to y. Also, if $z \in V_y$, the convex closure of $\bar{P}x$ with y contains its convex closure with z, and the same for the complements in Px, which are the perspectives of $\bar{P}x$ from y and z. Therefore

$$z \in V_y \Rightarrow V_z \subset V_y .$$

But

$$V_z \subset V_y \Rightarrow F_z \subset F_y \Rightarrow K_z \subset K_y.$$

Therefore

$$z \in V_y \Rightarrow K_z \subset K_y.$$

Now let $(N_x, N_{|u|})$ denote the set of balances on compositions in a neighbourhood $N_x$ of x with directions in a neighbourhood $N_{|u|}$ of the direction $|u|$ of u. Then, if $N_u$ denotes any neighbourhood of the balance u, there exist neighbourhoods $N_x$, $N_{|u|}$ such that

$$(N_x, N_{|u|}) \subset N_u.$$

For any neighbourhood $N_x$ of x there exists a composition $y_0$ whose perspective $V_{y_0}$ lies within $N_x$. Also for any neighbourhood $N_{|u|}$ of $|u|$ there exists a composition $y_1$ such that the curvature $K_{y_1}$ of $\bar{P}x$ facing $y_1$ is contained in $N_{|u|}$. Now take any composition y in $V_{y_0} \cap V_{y_1}$, so that $V_y \subset V_{y_0} \cap V_{y_1}$. Then

$$\begin{aligned} z \in V_y &\Rightarrow z \in N_x \wedge |w| \in N_{|u|} \\ &\Rightarrow w \in (N_x, N_{|y|}) \\ &\Rightarrow w \in N_u. \end{aligned}$$

Therefore, for any neighbourhood $N_u$ of u there has been found a neighbourhood $V_y$ of x in Px such that $w \in N_u$ for all $z \in V_y$, which proves the proposition.

PROPOSITION (vi). <u>On every support of $\bar{P}x$ there is a unique contact.</u>

Take any support v of $\bar{P}x$; then, by Proposition (ii), its dual composition y is in the zone of contact. If this zone of contact is not just y, let z be another contact in the zone, distinct from y. Then, by the conditions for a duality, its dual balance w must be different from v; and so, by Proposition (iv), it cannot be a support of $\bar{P}x$. Now, since $v \neq w$, there exist disjoint neighbourhoods $N_v$, $N_w$ of v, w. By the argument in the proof of Proposition (v), there exists a neighbourhood $N_z$ of z such that the balances dual to any composition in this neighbourhood, below v and w, lie in $N_v$, and then, by Proposition (v), there exists another neighbourhood $M_z$ of z such that the balance dual to any composition in this neighbourhood, below u and v, lies in $N_w$. Let $L_z = M_z \cap N_z$. Then $L_z$ is a neighbourhood of z containing compositions below v and w whose dual compositions must lie in both the neighbourhoods $N_v$ and $N_w$, which is impossible, since these neighbourhoods are disjoint.

PROPOSITION (vii). _The dual of a support or contact of $\bar{P}x$ is a conjugate contact or support._

By Propositions (ii) and (vi), the dual y of any support v is its unique contact. Moreover, by Proposition (iv), on any contact y there is a unique support v which has y as its unique contact and dual.

It appears from Propositions (iv) and (vi) that (uQ, $\bar{P}$x) is a strict contension, uniquely pairing supporting balances and contact compositions by their conjugacy. The support and contact conjugacy thus determines a reciprocity between balances and compositions, which, according to Proposition (vii), coincides with the original duality between them.

THEOREM. (uQ, $\bar{P}$x) _is a strict contension, for which the reciprocity between the supports and contacts forming PuQ)* and (Px)*, determined by conjugacy, coincides with the balance and composition duality._[5]

[5] This theorem embodies most of the propositions of paragraphs 5 and 8.

9. Critical surfaces and value frontiers.

The support and contact surfaces in the intensional and extensional spaces of a contension can be interpreted as reciprocal locus and envelope. Thus, according to the Theorem of paragraph 8, $(uQ)^*$ and $(Px)^*$ are surfaces in the balance and composition spaces which are related by their representing an evelope and locus which are reciprocals, such that the pairing which goes with the reciprocity between points and tangents coincides with that between balances and compositions in the duality. The dual arguments now give $\overline{Q}u$ a convex body with contacts and supports $(Qu)^*$ and $(xP)^*$, for which the same proposition applies. Thus it appears that $(xP)^*$, $(Px)^*$ are surfaces in the composition space with the property that the reciprocity between points and tangents is a pairing between compositions and balances coinciding with that of the duality. A surface with this property will be called a *critical surface* of the duality. For another, equivalent definition, the critical surfaces are the *integral surfaces* of the differential equation $u'dx = 0$, that is surfaces on which every differential element $dx$ satisfies this equation. Since a differential element of the surface is also a differential element of the tangent, the tangent consists in the locus of points $y$ satisfying $u'y = 1$, coordinated with the balance $u$. Thus, to every point $x$, the dual $u$ gives the tangent; whence the definitions are equivalent.

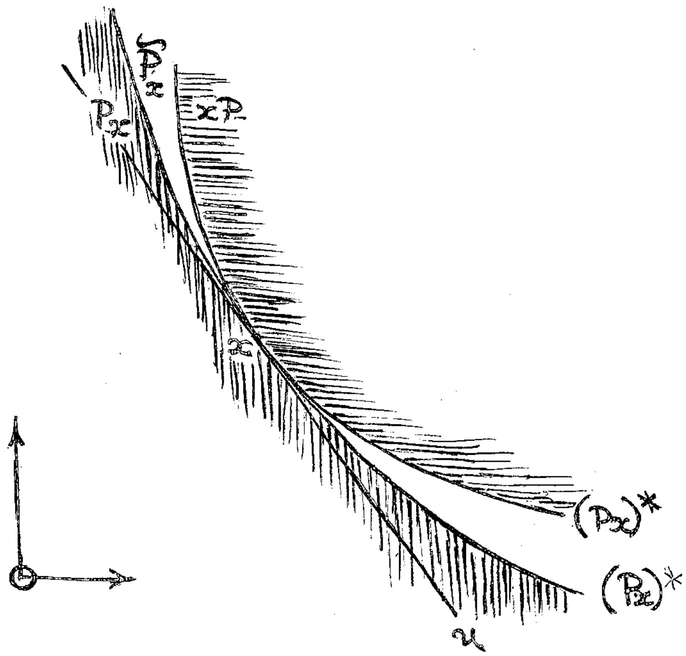
It has appeared thus that through every composition $x$ there exists a pair of critical surfaces of the duality, given by the frontiers $(Px)^*$, $(xP)^*$ of the value domains $Px$, $xP$ of $x$; and these are to be called the *value frontiers*, and distinguished as the *inferior* and *superior frontiers*, of $x$.

Since, by Proposition 5(iv), Px, xP are open, they are disjoint with their frontiers. Also, in view of the antisymmetry of P, they are disjoint with each other,

$$xP \cap Px = 0,$$

as appears in paragraph 3. Their frontiers therefore bound a closed region which is disjoint from both, and thus identical with the complement of their union, that is the indifference domain $\tilde{P}_x$ os x. Accordingly, the value frontiers (Px)* , (xP)* of a composition x appear with the following property:

THEOREM. The value frontiers, belonging to any composition, are a pair of critical surfaces through that composition, such that every composition to one side of one surface is inferior to that composition, and every composition to the other side of the other is superior, while every composition on or between them is indifferent.



10. Integrability, uniformity and regularity.

The value frontiers of any composition in a rational expenditure duality have been seen to be critical surfaces of the duality through that composition; and these are integral surfaces of the differential equation $u'dx = 0$. It follows that there is at least one integral surface of the equation through every point, or that the equation is *integrable*. Therefore, following from paragraph 9:

THEOREM I. *The rationality of an expenditure duality implies the integrability of its associated differential equation.*

A point on which there is a unique integral surface is to be called *regular*. Regularity at a point now requires the coincidence of its value frontiers, since these are integral surfaces, by paragraph 9.

A duality will be called *uniform*, in any region of balances, if there exist positive constants $\lambda, \Lambda$ such that

$$\lambda|v-u| < |y-x| < \Lambda|v-u|$$

for balances u, v ranging throughout that region. It is easy to show the following:

THEOREM II. *Uniformity in a closed region implies continuity and is implied by differentiability.*

Uniformity is thus a condition intermediate between continuity and differentiability. Moreover, the theory of differential equations leads to the following:

THEOREM III. *Uniformity implies the equivalence of integrability and regular integrability.*

For, given uniformity and integrability, in any two-dimensional coordinate section an integral surface satisfies an ordinary differential equation of the first order with the Lipschitz condition, which obtains its uniqueness in that section.[6] Thus, integral surfaces through a point cannot depart from each other in any two-dimensional coordinate section, and therefore they cannot depart from each other at all. Therefore:

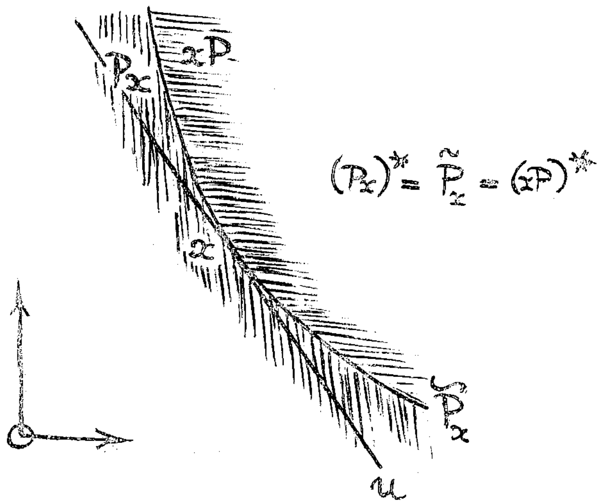
THEOREM IV. In a rational, uniform expenditure duality, the pair of value frontiers of any composition coincide with each other.

11. The scale theorem.

Consider a rational expenditure duality, which is regular at every composition, which condition is obtained, for example, under the hypothesis of differentiability or, more generally, of uniformity. The value frontiers $(Px)^*$, $(xP)^*$ of any composition x in this case coincide in the indifference domain $\tilde{P}_x$ of x:

$$(Px)^* = \tilde{P}_x = (xP)^*.$$

So, through every composition x, there is a unique critical surface, which is at the same time the inferior frontier, the superior frontier, and also the entire indifference domain of x.




[6] cf. Birkill [9], Forsyth [17] and Piaggio [37].

The critical surfaces of the duality now appear with the property that each disconnects the composition domain C in to two domains, which may be distinguished as its inferior and superior sides, such that any composition on the inferior or superior side of the surface is inferior or superior in P to any composition on the surface, while all compositions on the surface are neither inferior nor inferior to each other in P, but have to each other the indifference relation $\tilde{P}$. Now $\tilde{P}$ appears as the equivalence on C represented by the partition $\Pi$ of C formed by the set of critical surfaces. Hence

$$x\tilde{P}y \Leftrightarrow \tilde{P}_x = \tilde{P}_y,$$

each point being regular, it being contained in just one critical surface; whence follows the transitivity

$$x\tilde{P}y \wedge y\tilde{P}z \Rightarrow x\tilde{P}z.$$

So, with the reflixivity of $\tilde{P}$ equivalent ot the non-reflexivity of P, by Theorem I.2.II, and the symmetry of $\tilde{P}$ following immediately from the form of its definition as a symmetric complement, $\tilde{P}$ appears as reflexive, symmetric and transitive, and therefore as an equivalence; and the components in the partition $\Pi$ which represents it have been identified with the critical surfaces.

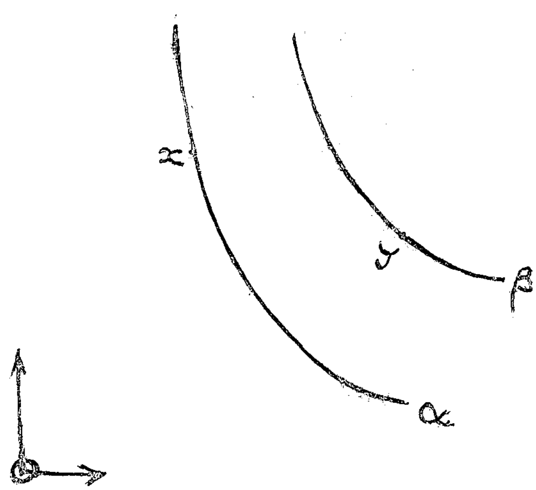
Any pair of critical surfaces $\alpha, \beta \in \Pi$, being, under the regularity hypothesis, either identical or disjoint, by their topological connectivity, must lie each entirely to one of the two sides of the other; and they will, evidently, lie on opposite sides of each other. Accordingly, if $\alpha \mathcal{P} \beta$ means that $\alpha$ lies to the inferior side of $\beta$, thus defining a relation $\mathcal{P}$ between critical surfaces, then this also means that $\beta$ lies to the superior side of $\alpha$. Moreover, it is again evident that every pair of surfaces, just provided they are distinct, have the relation $\mathcal{P}$ one way or the other, that is

$$\alpha \neq \beta \;.\Leftrightarrow.\; \alpha\mathscr{P}\beta \vee \beta\mathscr{P}\alpha \,,$$

which states $\mathscr{P}$ to be a complete, non-reflexive relation. Now, by the property of the sides of the critical surfaces, together with the definition of $\mathscr{P}$,

$$x\epsilon\alpha \wedge y\epsilon\beta \;.\Rightarrow.\; xPy \Leftrightarrow \alpha\mathscr{P}\beta \,,$$

so that the relation P between the elements of C is represented by the relation $\mathscr{P}$ between the components in the partition $\Pi$ of C. Since the symmetric complement $\tilde{P}$ of P has been identified with the equivalence represented by the partition $\Pi$, it now appears that P is a scale if and only if $\mathscr{P}$ is a complete order. Since has already been observed complete and non-reflexive, to obtain it as a scale it remains to note its transitivity, which follows directly from the transitivity of P and the way in which P is represented by $\mathscr{P}$.




$$xPy \Leftrightarrow \alpha\mathscr{P}\beta$$

It has been shown that under the condition of regularity applied to an expenditure duality E, implied for example by differentiability or more generally by uniformity, the rationality of E, which requires the preference relation P of E is be an order, obtains P as a scale. The rationality condition is merely that P be non-reflexive, and, since transitive by construction, therefore an order. It is proved that under this condition P is not just an order, but that more special kind of order which is termed a scale, for which there is representation by a completely ordered partition. Equivalently, P then has the properties of anti-symmetry and complementary transitivity, implying but not implied by the properties of non-reflexivity and transitivity which characterize an order. Thus, if the conditions that P be an order and a scale be called the *order* and *scale conditions*, respectively, as applied to E, the following theorem has been demonstrated.

THEOREM. For any expenditure duality, uniformity implies the equivalence of the order and scale conditions.

Thus the revealed preference relation of a rational, uniform expenditure duality is not just an order but, more particularly, a scale. This scale has, moreover, the particular structure which has been indicated, in which the indifference classes are the smooth, strictly convex critical surfaces.

## 12. Preference gauges and integrals.

With any expenditure duality E, there is associated the *differential form* u'dx, and the corresponding *differential equation* u'dx = 0. The critical surfaces of the duality are the integral surfaces of the differential equation. The equation, or the duality, is said to be *integrable* if there always exists at least one integral surface of the equation, or critical surface of the duality, on every composition; and the condition of *regular integrability* is that moreover there exists at most one.

Under the condition of uniformity, it appears from the theory of differential equations that integrability, of the duality, or the equation, whenever it holds, is always regular, and is equivalent to the integrability of the form, in the sense of its being proportional, by some factor $\lambda = \lambda(x)$, to the total differential $d\phi = \phi_x' dx$ of some function $\phi = \phi(x)$. That is,

$$\lambda u' dx = d\phi$$

or equivalently, with $\phi_x$ the vector of partial derivatives of $\phi$ with respect to the elements of x,

$$u\lambda = \phi_x,$$

where the factor of proportionality $\lambda$ defines an *integrating factor*, and $\phi$ the corresponding *integral* of the form, there being just one functionally independent integral of a given integrable form. The *level surfaces* of any function are defined as the surfaces on which the function takes a constant value. Subject to uniformity, the conditions of integrability for the form and the equation are equivalent, and the level surfaces of an integral of the form are the integral surfaces of the equation. Thus, for a rational uniform expenditure duality, there exists a differential function $\phi$ the level surfaces of which give the critical surfaces.

THEOREM I. The rationality of any expenditure duality implies its integrability, and, provided uniformity, which then obtains regular integrability, also that there exists a differentiable function $\phi$, given as any integral of the associated differential form, whose level surfaces are the critical surfaces of the duality.

The critical surfaces of a rational uniform expenditure duality now have a total order $\mathcal{M}$ defined by the order of magnitude of the values which any integral of the associated form takes on each. Thus, the scale M defined by

$$xMy \equiv \phi(x) < \phi(y)$$

is represented by this total order $\mathcal{M}$ of the critical surfaces, which appear as its indifference classes. But topological considerations show that the critical surfaces can have only one continuous order, in which surfaces neighbouring in position are neighbouring in order. The order of the critical surfaces, determined by the preference relation P, and by the scale M obtained from any integral $\phi$, are both continuous; and therefore they must be identical, that is $\mathcal{M} = \mathcal{P}$. Thus M = P; and hence the function $\phi$ is a gauge which completely measures of the preference scale P,

$$\phi(x) < \phi(y) \Longleftrightarrow xPy.$$

THEOREM II. The preference relation P of a rational uniform expenditure duality E is a scale, for which the gauges, completely measuring it, are given by the integrals $\phi$ of the associated differential form $u'dx$.

## 13. Objective maxima.

It follows from the definition of P that the composition x dual to any balance u is superior in P to all other compositions y under the condition $u'y \leqslant 1$, of being within the balance u, and, since $u'x = 1$, therefore to all other compositions y subject to the balance condition $u'y = 1$. Since, by Theorem 12.II, P is completely measured by a differentiable gauge function, there follows immediately:

THEOREM III. _If E is a rational uniform expenditure duality then there exists a differentiable function $\phi = \phi(x)$ such that the composition x dual to a given balance u is determined by the condition that it attains the absolute maximum of $\phi$ under the balance condition $u'x = 1$._[7]

A function for which any choice is such as to attain its absolute maximum is called an o b j e c t i v e  f u n c t i o n. Now all the choices of a rational uniform expenditure duality have any gauge function of the revealed preference scale as objective function.

According to Proposition 5(ii), P is an increasing scale, satisfying the condition

$$x \subset y \Rightarrow xPy;$$

and if $\phi$ is a gauge completely measuring P, it follows that

$$x \subset y \Rightarrow \phi(x) < \phi(y).$$

It appears now, with $\phi$ increasing, and all the elements of u positive, that an absolute maximum of $\phi$ subject to $u'x = 1$ is, equivalently, an absolute maximum of $\phi$ subject to $u'x \leqslant 1$.

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7 The converse is trivial. Evidently Houthakker [24] has such a type of theorem in mind. Its conclusion is the usual hypothesis which forms the starting point of consumer theory and is the hypothesis from which, by direct differentiation, Slutzky deduced his famous symmetry conditions.

THEOREM IV. An objective function of an expenditure duality under the condition $u'x = 1$ is, equivalently, an objective function under the condition $u'x \leqslant 1$.

If $\phi(x)$ is maximum subject to $u'x = 1$, it is stationary; and, with $\phi$ differentiable, the condition for this is

$$u\lambda = \phi_x,$$

where $\lambda$ now appears as the Lagrangian multiplier corresponding to the constraint $u'x = 1$, which gives it the value $\lambda = x'\phi_x$. From here it is again evident that $\phi$ is an increasing function; since, with u having all its elements positive, $\phi_x$ has all its elements of the same sign, and is therefore increasing or decreasing; and it can be taken to be increasing, if necessary by replacing it by its negative. Also $\phi$ is a convex function, since, moreover, it has convex level surfaces.

# III. Standard and Cost.

## 1. Differentiable dualities.

An expenditure duality

$$E: B \longleftrightarrow C(u \longleftrightarrow x,\ u'x = 1)$$

is d i f f e r e n t i a b l e if there exist the partial derivatives

$$u_{ij} = \partial u_i / \partial x_j, \quad x_{ij} = \partial x_j / \partial u_i$$

of the elements $u_i$ and $x_j$ of u and x with respect to each other forming matrices $u_x$ and $x_u$ which obtain the differential transformations

$$dx = x_u du, \quad du = u_x dx.$$

Then the partial derivative matrices, both the one way and the other, are regular, and are the inverses of each other:

$$u_x x_u = x_u u_x = 1.$$

## 2. The differential balance conditions.

By differentiating the balance condition

$$u'x = 1,$$

holding in the differentiable duality E, with x independent and u dependent, there is obtained the condition

$$u_x' x + u = 0,$$

involving the partial derivatives, which is necessary and sufficient for the constancy of $u'x$. Now the condition that, moreover, the constant value of $u'x$ should be unity is

$$x' u_x' x + 1 = 0.$$

Hence, a necessary and sufficient condition for the balance condition in terms of the partial derivatives is given by the conjuction of these conditions, which are to be called the d i f f e r e n t i a l balance conditions.

THEOREM. The balance condition $u'x = 1$ is equivalent to the conditions

$$u_x'x + u = 0, \quad x'u_x x = -1$$

and to the conditions

$$x_u'u + x = 0, \quad u'x_u u = -1.$$

The second pair of conditions is merely the equivalent dual form of the first.

## 3. Dual gauges and multipliers.

If the considered differentiable expenditure duality E is rational, its preference relation P is a scale, its associated differential form $u'dx$ is integrable, and any integral $\phi$ of $u'dx$ is a differentiable gauge, completely measuring P; thus,

$$\phi(x) < \phi(y) \Longleftrightarrow xPy,$$

where

$$u\lambda = \phi_x,$$

where $\lambda = x'\phi_x$ since $u'x = 1$, and $\phi_x$ is the vector of the partial derivatives

$$\phi_i = \partial\phi/\partial x_i$$

of $\phi$ with respect to the elements $x_i$ of x; equivalently,

$$(1 - ux')\phi_x = 0.$$

For the equivalent dual proposition, under the same conditions the dual preference relation Q, for which the induced conjugate is identical with P, is a scale, with any integral $\psi$ of $x'du$ as a differentiable gauge; thus

$$\psi(u) > \psi(v) \Longleftrightarrow uQv,$$

and

$$x\mu = \psi_u,$$

where $\mu = u'\psi_u$, so that

$$(1 - xu')\psi_u = 0,$$

where $\psi_u$ is the vector of the partial derivatives

$$\psi_j = \partial\psi/\partial u_j$$

of $\psi$ with respect to the elements $u_j$ of u. Here, whereas $\phi$ is a gauge of increasing value of x in P, $\psi$ is a gauge of decreasing value of u in Q, with equivalence, in the induced relation, to increasing value of x in P. Such a pair of functions $\phi$, $\psi$ are to define *dual gauges* of the expenditure duality E; and $\lambda$, $\mu$ define the associated *multipliers*.

If $\phi$, $\psi$ are any functions of x, u such that

$$\phi(x) = \psi(u)$$

is an identity, for the correspondence between u and x in E, they will be called *dual functions* of E. In this case $\phi$ is defferentiable if and only if $\psi$ is differentiable, and then the vectors $\phi_x$, $\psi_u$ of partial derivatives have the relations

$$\phi_x = u_x'\psi_u, \quad \psi_u = x_u'\phi_x,$$

as follows immediately from the rules for the partial differentiation of functions of functions.

THEOREM. *If $\phi$, $\psi$ are dual functions in a differentiable expenditure duality E, then the conditions*

$$(1 - ux')\phi_x = 0, \quad (1 - xu')\psi_u = 0$$

*are equivalent, and imply*

$$x'\phi_x + u'\psi_u = 0.$$

Multiply the first condition $u\lambda = \phi_x$, where $\lambda = x'\phi_x$, by $x_u'$ thus,

$$x_u' u\lambda = x_u'\phi_x,$$

and apply the differential balance condition and the partial derivative relation, to the respective sides, to obtain

$$-x\lambda = \psi_u,$$

where

$$-\lambda = u'\psi_u = \mu;$$

whence the theorem follows.

Accordingly, if a function is a gauge, then the dual function is a dual gauge, and the corresponding multipliers are the negatives of each other.

## 4. Value and relative marginal price.

Let $\phi(x)$ be a gauge of the preference scale P of a rational differentiable duality E, with dual $\psi(u)$. Let prices p and expenditure e obtain the relative prices $u = p/e$, which, in E, determine composition x with $u'x = 1$, or equivalently with $p'x = e$. The partial derivative of u with respect to e with p constant defines the *expenditure derivative*

$$u_e = -p/e^2 = -u/e,$$

of u; and the *relative expenditure derivative* is defined by

$$\dot{u} = eu_e = -u,$$

giving the change

$$du = \dot{u}de/e$$

in u corresponding to a relative change de e in e, with p constant. Correspondingly, the relative expenditure derivative of $\psi = \psi(p/e)$ is

$$\dot{\psi} = e\psi_e = e\psi_u' u_e = \psi_u' \dot{u} = -\psi_u' u = -\mu = \lambda,$$

where $\lambda$, $\mu$ are the multipliers belonging to the dual gauges $\phi$, $\psi$; and now

$$d\psi = \dot{\psi}\, de/e,$$

with p constant.

The number

$$X = \phi(x)$$

defines the v a l u e  i n d e x of the composition x, relative to the gauge $\phi$. The value indices of compositions, measured in any gauge, are the greater or the less according to the preferences between them; thus, if compositions x, y have value indices X, Y in the same gauge, then

$$X < Y \Longleftrightarrow xPy.$$

Now, through the dual gauge, the value index appears as a function

$$X = \psi(u)$$

of u, and hence with relative expenditure derivative

$$\dot{X} = \lambda,$$

$\lambda$ being the multiplier corresponding to the gauge $\phi$.

There is now made the definition

$$\mathring{X} = 1/U$$

for the r e l a t i v e  m a r g i n a l  p r i c e  i n d e x U conjugate to X, or merely the p r i c e  i n d e x U c o n j u g a t e t o  t h e  v a l u e  i n d e x X. Thus any gauge $\phi$ assigns to any choice (u,x) of E a pair of conjugate i n d e x  n u m b e r s (U,X), given by

$$X = \phi, \quad 1/U = \lambda,$$

where $\lambda$ is the multiplier corresponding to the gauge $\phi$.

If now, for expenditure e and prices p, giving relative prices

$$u = p/e,$$

the *marginal price index* P is derived with the definition

$$U = P/e,$$

then P determines the expenditure derivative $X_e = \partial X/\partial e$ of X, in which prices p are constantly by the relation

$$X_e = 1/P,$$

in conformity with the notion of price $\pi$ for a simple commodity, relative to the value index given by the simple physical amount $\xi$, the reciprocal of which gives the rate of change with expenditure $\epsilon$ of amount purchased; thus, $\pi\xi = \epsilon$ in the bill for a quantity $\xi$ purchased at price $\pi$, so that, with $\pi$ constant, $\xi_\epsilon = \partial\xi/\partial\epsilon = 1/\pi$, there being in this case the identity of the total and marginal prices.

The price index U, conjugate to the value index X, gives the marginal change dX, on X, for a relative change de/e in expenditure, by the relation

$$dX = de/eU;$$

or, in terms of the marginal price index P, and an absolute change de in expenditure e,

$$dX = de/P.$$

The following has been shown.

THEOREM. In a rational differentiable expenditure duality E, for any choice (u,x), conjugate value and price indices (U,X), with

$$e\partial X/\partial e = 1/U,$$

*are given by*

$$X = \phi, \qquad 1/U = \lambda,$$

*where* $\phi$ *is a gauge and* $\lambda$ *the corresponding multiplier.*

## 5. Contact and support functions.

Let $(B,C)$ represent a pair of conjugate intensional and extensional regions. An intensional element $v$ is said to be *dependent* on an intensional element $u$ if $v = u\lambda$ $(\lambda > 0)$; and similarly for extensional elements.

Let $(U,X)$ be a contension having $U$ $B$, $X$ $C$. To every intensional element $u \in B$ there exists a *dependent support* $u\sigma(u) = u^* \in U^*$ of the contension. Let $x^*$ denote any contact conjugate to the dependent support $u^*$ of any intensional element $u$. The function $\sigma(u)$, thus defined over the intensional elements $u \in B$, defines the *support function* of the contension; and the *contact function* $\gamma(x)$ has precisely the dual form of definition. The support function $\sigma(u)$ determines the mapping

$$B \to U^* \ (u \to u^* = u\sigma(u))$$

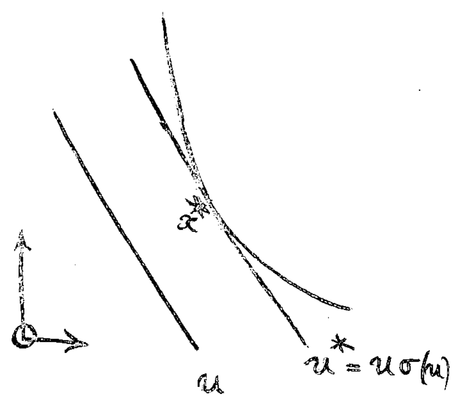
of the intensional region $B$ onto the supporting surface $*$ of the contension; and it has an equivalent definition by the property

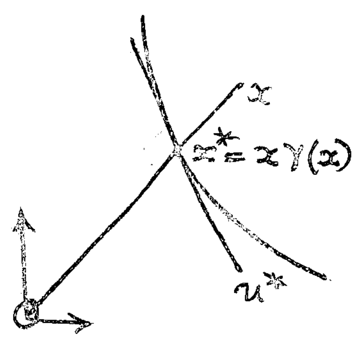
$$\sigma(u) = \inf_{x \in X} u'x = u'x^* \ (u \in B),$$

it being the case that, for any $u \in B$, the function $u'x$ is continuous in the region $X$, and therefore its inferior limit is its minimum in the closure $\overline{X}$ of $X$, attained at a contact element $x^*$, on the frontier $X^*$, conjugate to the dependent support $u^*$ of $u$; and similarly for the dually defined contact function $\gamma(x)$.

The gradient $\sigma_u$ of the support function $\sigma = \sigma(u)$ exists at every point u for which the dependent support u* has simple contact, in a unique conjugate contact x*, and is given by the unique conjugate contact of the dependent support; thus, $\sigma_u = x^*$. In the case of a regular contension, one in which both the sides have non-empty interiors, almost every support has simple contact. Therefore, the gradient $\sigma_u$ of the support function $\sigma = \sigma(u)$ of a regular contension exists almost everywhere, and, where it exists, it equals the there unique contact x* conjugate to the dependent support u* of u: $\sigma_u = x^*$. Again there is a similar proposition for the contact function.

In the case of a strict contension, every support has simple contact, and every contact simple support. Hence, the support function of a strict contension is differentiable everywhere; and the gradient $\sigma_u$ is now the always unique contact x* conjugate to the dependent support u* of u: $u \to u^* \longleftrightarrow x^*$. Thus the gradient $\sigma_u = x^*$ always lies on the contact surface X* of the contension, for all $u \in B$.




$u \to u^* \longleftrightarrow x^*$

$x \to x^* \longleftrightarrow u^*$

## 6. Parallel price loci.

An element u determines a direction $\langle u \rangle$, with the definition

$$v \in \langle u \rangle \equiv v = u\lambda \quad (\lambda > 0).$$

Accordingly,

$$u \in \langle u \rangle,$$

or u belongs to the direction which it determines.

A direction $\Theta$ is determined by any one of its elements, thus,

$$u \in \Theta \Longleftrightarrow \Theta = \langle u \rangle;$$

and any two directions are either disjoint, or identical if they have a common element. Two elements u, v are said to be *parallel* if they have the same direction, that is

$$\langle u \rangle = \langle v \rangle ,$$

for which an equivalent condition is that they be dependent, $v = u\lambda$ $(\lambda > 0)$.

Any given elements $u_1, \ldots, u_r$ determine the *sector* $\langle u_1, \ldots, u_r \rangle$, with the definition

$$v \in \langle u_1, \ldots, u_r \rangle = v = u_1\lambda_1 + \ldots + u_r\lambda_r \quad (\lambda_1, \ldots, \lambda_r \geqslant 0;\ \lambda_1 + \ldots + \lambda_r > 0) .$$

It is identical with the convex cover of the directions of $u_1, \ldots, u_r$,

$$\langle u_1, \ldots, u_r \rangle = [\langle u_1 \rangle \cup \ldots \cup \langle u_r \rangle] ,$$

defines the sector determined by these given directions. Generally, the sector determined by a set of elements is defined as the convex cover of the set formed by their directions.

The interscept $\Omega$ of a sector S on the unit sphere defines the *spherical map* of the sector; and the area $\omega$ of the map defines the *spherical angle* of the sector. A *regular* sector is defined to be one with positive angle.

To every point $\Theta$ on the unit sphere in the balance space, there corresponds a balance direction; and which may also be denoted by $\Theta$.

Given an expenditure system E, let $E(\Theta)$ denote the set of compositions obtained from balances belonging to a direction $\Theta$, that is

$$x \in E(\Theta) \equiv u \in \Theta .$$

Then $E(\Theta)$ defines a *parallel price locus* of the system E, giving all compositions obtained at prices parallel to a fixed direction $\Theta$. Thus, for given prices p, and any expenditure e, obtaining relative prices $u = p/e$, determining a composition x, the locus of compositions $x = x(u) = x(p/e)$, with p fixed and e varying, is the parallel price locus $E(\Theta)$ with $\Theta = \langle p \rangle$.

For any choice (u,x) of a uniform duality E, there corresponds a unique parallel price locus $E\langle u \rangle$, passing through x, or, equivalently, belonging to the direction $\langle u \rangle$ of balances parallel to u.

Let E be a rational differentiable expenditure duality, with preference scale P, determining a partition $\Pi$ of C into indifference classes, given by the critical surfaces, with complete order $\wp$. A preference gauge $\phi$ will be twice differentiable; and the parallel price locus $E\langle p \rangle$ then satisfies $\dot{x} = \phi_{xx}^{-1} p$.

Every composition x lies on a unique critical surface $\pi = \pi_x$, and a unique parallel price locus $E\langle u \rangle$. Again, any critical surface $\pi$ is cut by any parallel price locus $E(\Theta)$, for any balance direction $\Theta$, in a unique composition $x(\Theta, \pi)$. Thus there is a coordination

$$x \longleftrightarrow (\Theta, \pi) \qquad (\Theta = \langle u \rangle, \ \pi = \pi_x)$$

between compositions, and conjunctions of balance direction and indifference class. The mapping

$$S \longleftrightarrow \pi \ (\Theta \longleftrightarrow x \ (\Theta, \pi)) ,$$

between the unit sphere in the balance space, and a critical surface $\pi$, is one-to-one and continuous both ways, and is thus a homeomorphism. Any sector of balance directions, with spherical map $\Omega$ is associated with a region $x(\Omega,\pi)$ on any critical surface $\pi$, which is a topological image of $\Omega$. These regions together form a *tube* $E(\Omega)$, with the parallel price loci $E(\Theta)$ $(\Theta\in\Omega)$ as *filaments*. The filaments are continua on which the elements have a natural complete order which coincides with the complete order $\mathcal{P}$, of the critical surfaces on which they appear, and which is coordinated with a magnitude of expenditure; and the filaments are distinguished from each other by price direction. Any set of price vectors determine a sector of directions, and then a tube containing all the compositions which can arise at those prices, over the complete range of expenditure.

## 7. Trend and amplification.

Let E be a differentiable expenditure duality. Then, with $u = p/e$, there exist the partial derivatives of the elements of $x = x(u)$ with respect to e with p constant, forming the vector

$$x_e = x_u u_e$$

$$= -x_u u/e,$$

defining the *trend* of x, for changes from an absolute money expenditure e. It gives the direction of change of x for changes in expenditure at a absolute level e, with prices fixed at $p = eu$; and

$$dx = x_e de$$

is the change in x for an absolute change de in e. Now the *relative trend* is defined by

$$\dot{x} = ex_e$$

$$= -x_u u.$$

It also gives the direction of change of x, that is to say it is the tangent at x to the parallel price locus of E through x. It gives a change

$$dx = \dot{x}de/e$$

in composition x for a relative change de/e of expenditure at constant prices.

Since, by the balance condition, $u'x_u u = -1$, the relative trend vector satisfies the condition

$$u'\dot{x} = 1.$$

A change in expenditure at constant prices is to be called an *amplification*, and is given by the ratio of the expenditure change to the original expenditure.[1] An infinitesimal amplification, de e; and the corresponding differential of composition is given by dx = xde e.

## 8. Standard and relative cost.

Any composition x lies in a unique indifference class $\pi_x$ of the preference scale P, of any rational differentiable expenditure duality E; and this indifference class defines the *standard* of x, in regard to E. Standards are defined *higher* or *lower* according to the complete order .

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1 The expression is related to the concept of amplitude of choice used in § I.7.

Let prices p be fixed. Then, for every expenditure e, there is obtained on the parallel price locus E<p>, a composition x, determined in E by the balance $u = p/e$; and reversely, to every composition x on the parallel price locus E<p>, there corresponds an expenditure e, which obtains composition x at prices p.

The standard $\pi$ attained by an expenditure e at prices p is defined as the standard $\pi = \pi_x$ of the composition x which it obtains at those prices. Thus, in regard to given prices, there is a one-to-one correspondence between expenditure, and attained standard; thus

$$p: e \longleftrightarrow \pi,$$

giving

$$e = e(p,\pi), \text{ and reversely } \pi = \pi(p,e).$$

Two compositions x,y of different standards $\pi_x$, $\pi_y$ require different expenditures $e(p,\pi_x)$, $e(p,\pi_y)$ at the same prices p. Define

$$\rho_{xy}(p) = e(p,\pi_x)/e(p,\pi_y)$$

to be the r e l a t i v e c o s t of composition y on composition x a t p r i c e s p. It is the proportion in which expenditure must be adjusted at prices p, for composition obtained to move from the standard represented by x to the standard represented by y.

Now define

$$\rho_{xy} = \rho_{xy}(u),$$

and call this merely the r e l a t i v e c o s t of y on x.[2] It gives the ratio in which expenditure must be adjusted to move from the choice of composition x to a composition of the standard represented by a composition y, in other words, it gives the amplification necessary to move from x to

---

[2] This concept will later, in Part II, be involved in giving a sense, as a concept of measurement applied to market data, for the at present idefinite, but also idispensible idea of "cost of living".

the standard of y on the form $\rho_{xy}$-1.

Thus, $u^* = u/\rho_{xy}$ is the one balance parallel to u which touches the critical surface $\pi_y$ through y. Contact is at a composition x* which is the one composition on the parallel price locus through x which is indifferent to y in P, and which can be attained from x merely by a readjustment of expenditure in the ratio $\rho_{xy}$. With

$$\rho_{xy} = u'x^*$$

it appears that

$$\sigma(u) = \rho_{xy}$$

is the support function of the contension determined by y, with contact and support surfaces given by the critical surface through y and its dual, through v. Accordingly,

$$\rho_{xy} = \inf_{yPz} u'z = \min_{y\tilde{P}z} u'z = u'x^*,$$

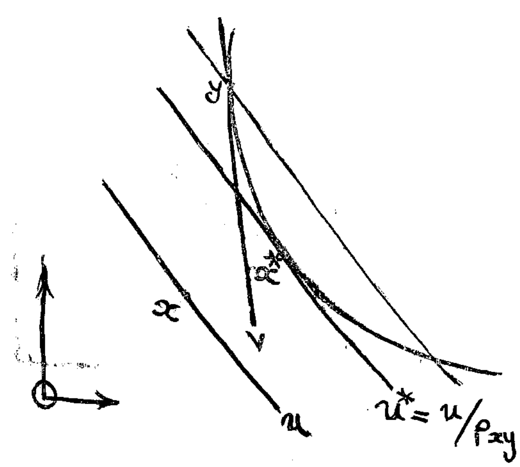
and so:

$$\rho_{xy} \leqslant u'y,$$

where the equality holds if and only if the balances u, v dual to x and y are parallel ($u \parallel v$), that is

$$u \parallel v \Longleftrightarrow \rho_{xy} = u'y;$$

and in this case $u^* = v$, that is the dependent support of u coincides with v.



## 9. The relative cost gauge.

The relative cost function $\rho_{xy}$, defined for a rational, uniform expenditure duality E, has the properties

$$\rho_{xy} = 1 \Leftrightarrow x\tilde{P}y,$$

$$\rho_{xy} < \rho_{xy} \Leftrightarrow yPz,$$

including the properties

$$\rho_{xy} < 1 \Rightarrow yPx, \quad \rho_{xy} > 1 \Rightarrow xPy.$$

For any composition x, the function

$$\phi(y) = \rho_{xy}$$

thus appears as a complete preference gauge of E,

$$\phi(y) < \phi(z) \Leftrightarrow yPz.$$

It will be called the *relative cost gauge*, associated with the composition x, which will then be called the *base point*.

Thus, among all the preference gauges of a rational, uniform expenditure duality, these having been identified with all the integrals of the differential form u'dx, of which only one is functionally independent, there is a special class defined, these being the relative cost gauges, of which there is just one associated with every composition.

10. Substitution, change and compensating amplification.

Let E be a rational, differentiable expenditure duality. Any displacement from one choice to another defines a *change*, but a displacement from one choice to an indifferent one, according to the revealed preferences, is to be called a *substitution* of E. Thus, a change

$$x \to y, \text{ or equivalently } u \to v,$$

in a substitution of E just if

$$x\tilde{P}y, \text{ or equivalently } v\tilde{Q}u \ .$$

Accordingly, among possible changes, there are certain ones which are distinguished as substitutions.

Any given change may be *compensated*, by a suitable adjustment of expenditure at fixed prices, to obtain a substitution. Thus a change $x \to y$ may be compensated by adjusting the expenditure y, in the ratio $\rho_{xy}$, so as to obtain instead of y a new composition y* which is a substitute for x by indifference in the revealed preferences, but which is in general different from x:

$$x \xrightarrow{\text{change}} y \xrightarrow{\text{compensation}} y^* \qquad (x\tilde{P}y^*)$$

$$x \xrightarrow{\text{substitution}} y^* \ .$$

By the balance condition $u'x = 1$ it follows that the matrix

$$I_u = ux'$$

is a projector, since it satisfies the idempotence condition

$$I_u^2 = (ux')(ux') = u(x'u)x' = ux' = I_u \ .$$

The complementary projector is

$$J_u = 1 - ux' = 1 - I_u$$

this also being idempotent:

$$J_u^2 = J_u.$$

Any idempotent is identical with the projector on its range and parallel to its null-space.[3] Considering $J_u$ as obtaining a projection

$$du^* = J_u du$$

of infinitesimal changes du of u in the balance space, the condition for an element du to belong to the range of $J_u$ is $J_u du = du$, and this is equivalent to the condition $x'du = 0$, which, since $\mu x'du = d\psi$, is the condition $d\psi = o$ for du to be on the dual critical surface through u, in other words, to be a substitution. Thus the infinitesimal changes in the range of $J_u$ are substitutions. Further, the condition $J_u du = 0$ for du to belong to the null-space of $J_u$ is equivalent to du being parallel to u, which is to say a parallel price displacement, equivalent to an expenditure adjustment at constant prices, defining an expenditure amplification. So the infinitesimal changes in the null-space of $J_u$ are expenditure amplifications. More specifically, the condition $J_u du = 0$ is the same as

$$du = u(x'du).$$

But

$$x'du = d\psi/\mu = de/e,$$

where $\psi$ is a dual gauge and $\mu = \dot{\psi}$ the corresponding multiplier, and $de/e$ defines an infinitesimal expenditure amplification. Thus an element du in the null-space of $J_u$ is an amplification

$$du = u\,de/e = -\dot{u}\,de/e.$$

3 Halmos [20] and Afriat [1] contain the relevant material on projectors.

THEOREM I. *The complementary projectors* $I_u = ux'$ *and* $J_u = 1 - ux'$ *give the resolution*

$$du = J_u du + I_u du$$
$$= du^* - \dot{u}de/e$$

*of any change* du *into the sum of a substitution and an amplification, when*

$$du^* = J_u du,\ \dot{u} = -u \text{ and } x'du = de/e.$$

An equivalent proposition is that any change du is compensated by the amplification $\dot{u}de/e = -I_u du$ to give the substitution $du^* = J_u du$. Accordingly, $I_u$ will be called the c o m p e n s a t i n g a m p l i c a t i o n p r o j e c t o r, and $J_u$ the c o m p e n s a t e d s u b s t i t u t i o n p r o j e c t o r.

Thus, projection of a change du by the projector $J_u$ determines the substitution

$$du^* = du + \dot{u}de/e$$

which derives from it when it is compensated by an adjustment of expenditure, with the prices remaining fixed. In the dual fashion, there is defined the compensating projector $J_x = J_u'$ applied to changes in composition.

To any change du in balance, there corresponds the dual change $dx = x_u du$ in composition; and to a substitution $du^*$ there corresponds the dual substitution $dx^* = x_u du^*$. Now, for a general change du of balance, resolved into substitution and amplification, there corresponds, for the dual change, the resolution

$$dx = x_u du$$
$$= x_u J_u du + x_u I_u du$$
$$= (x_u J_u)(J_u du) + (x_u I_u)(I_u du)$$
$$= s du^* + \dot{x}x'\dot{u}de/e$$
$$= dx^* - \dot{x}de/e$$

where

$$\dot{x} = -x_u u, \quad s = x_u(1-ux') = x_u + \dot{x}x',$$

and

$$dx^* = sdu^*, \quad \dot{u}de/e = ux'du.$$

Thus the dual change dx is correspondingly resolved into a substitution $dx^*$ and an amplification $-\dot{x}de/e$.[4] The matrix s which transforms substitution $du^*$ into dual substitution $dx^*$ defines the *substitution matrix* of the system. It not only gives this transformation between substitutions, but, in case du is not restricted to be a substitution, but is admitted as any change, the transformation of du by the substitution matrix s has the effect of obtaining the dual substitution which results from compensation of the change by amplification; thus,

$$sdu = sdu^*, \quad \text{where } du^* = J_u du,$$

because

$$sJ_u = x_u J_u J_u = x_u J_u = s.$$

THEOREM II. <u>If</u>

$$\dot{x} = -x_u u \text{ and } s = x_u J_u$$

<u>then</u>

$$dx^* = dx + \dot{x}de/e$$

<u>where</u>

$$dx^* = sdu, \quad \dot{u}de/e = ux'du,$$

<u>gives the substitution</u> $dx^*$ <u>which is the residual variation for when a general change</u> dx <u>receives its compensating expenditure amplification</u> $de/e$.[5]

[4] The substitutional change $dx^*$ left after the compensating amplification of a general change dx has been called by Slutzky [41] the residual variation.

[5] Substitution is in the sense of variation from one point to another in the same indifference class.

Since the projector $I_u$ is of rank one, the complementary projector $J_u$ is of rank n-1. Accordingly $s = x_u J_u$ is singular, since $J_u$ is singular; and moreover since $x_u$ is regular, and $J_u$ is of rank n-1, it follows that s is of rank n-1, or of nullity 1.

THEOREM III. <u>The nullity of the substitution matrix of a differentiable expenditure duality is unity</u>.

With $du = u_x dx$, and $dx^* = sdu = su_x dx$, it follows that the substitution $dx^*$ derived from a change dx as a result of compensation by amplification is obtained by the transformation

$$dx^* = K_u dx,$$

where

$$\begin{aligned} K_x &= x_u J_u x_u^{-1} \\ &= x_u (1-ux')x_u^{-1} \\ &= 1-\dot{x}u' , \end{aligned}$$

using the definition of $\dot{x}$ directly, and the differential form of the balance condition. It appears from here that $K_u$ is a projector, satisfying

$$K_u^2 = K_u ,$$

since this is the case for $J_u$. Alternatively it follows from the identity $u'\dot{x} = 1$ which is derived from the differential balance conditions.

The substitution matrix, in first place defined by $s = x_u J_u$, could have been defined equivalently by the formula

$$s = K_x x_u .$$

For the operations of transformation of the change, from balance du to composition dx, and expenditure compensation, are interchangeable in order, thus,

$$K_x x_u = x_u J_u ,$$

$J_u$ being the transform of $K_x$ by $x_u$. For another statement, the transitions

$$du \xrightarrow{J_u} du^* \xrightarrow{x_u} dx^*$$

and

$$du \xrightarrow{x_u} dx \xrightarrow{K_x} dx^*$$

are equivalent, and effected by s

$$du \xrightarrow{s} dx^*.$$

11. Substitution symmetry and negativity.

Let E be a rational, continuously differentiable expenditure duality, thus with trend vector and substitution matrix

$$\dot{x} = -x_u u \quad \text{and} \quad s = x_u(1-ux') = x_u + \dot{x}x'$$

existing and continuous.

For any compositions x, y let x* be the substitute for y which is the result of compensating the change from y to x by amplification, that is adjustment of expenditure at constant prices. So x* is the intersection of the critical surface through y and the parallel price locus through x, and is attained from x by expenditure amplification in the ratio given by the relative cost function $\rho_{xy}$.

The relative cost function is given by $\rho_{xy} = u'x^*$ and has been identified with the support function $\sigma^*(u)$ of the strict contension determined by y, with support and contact surfaces given by the dual critical surfaces through v, y; and it has appeared that $\sigma^* = \sigma^*(u)$, which is such that the support and contact surfaces are the envelope and locus determined by $\sigma^*(u) = 1$, is differentiable, with vector $\sigma^*_u$ of partial derivatives satisfying

$$\sigma^*_u = x^*$$

where $x^* = x^*(u) = x^*(u^*)$, the dual contact of the dependent support $u^*$ of u, is a function of u, and more immediately of $u^*$. But

$$du^* = J_{u^*}du$$

and, by the differentiability of E,

$$dx^* = x_{u^*}du^*$$

$$= x_{u^*}J_{u^*}du = s^*du,$$

where $s^*$ is the substitution matrix s of E evaluated at $x^*$. Accordingly, $\sigma^*_u = x^*$ is differentiable, with partial derivative matrix

$$x^*_u = s^*.$$

Equivalently, $\sigma^*$ is twice-differentiable, with second partial derivative matrix $\sigma^*_{uu}$ satisfying

$$\sigma^*_{uu} = s^* .$$

The continuous differentiability of E now obtains the symmetry of $\sigma^*_{uu}$, and therefore of $s^*$, since

$$\partial^2\sigma/\partial u_i\partial u_j = \partial^2\sigma/\partial u_j\partial u_i$$

if both sides exist and are continuous.

A differentiable expenditure system is defined to have the condition of *substitutional symmetry* if its substitution matrix is everywhere symmetric. It has appeared that a rational, continuously differentiable expenditure system is substitutionally symmetric,

$$s = s'.$$ [6]

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[6] The idea for this derivation of the Slutzky symmetry condition is due to McKenzie [31].

Now, in a rational, continuously differentiable expenditure duality, with symmetric substitution matrix s, consider any change from a choice (u,x) to a choice (u + du, x + dx), compensated, by expenditure adjustment, to a choice (u + du*, x + dx*) which is a substitute for (u,x); in which case

$$du^* = J_u du^*,$$

and

$$dx^* = x_u du^*$$
$$= x_u J_u du = s du .$$

Since the choices (u,x), (u + du*, s + dx*) are indifferent to each other by revealed preference, or, alternatively, by the convexity of the indifference surfaces, they must satisfy

$$u'(x + dx^*) > 1, \quad (u + du^*)'x > 1 ;$$

equivalently,

$$u'dx^* > 0, \quad du^{*\prime}x > 0 ,$$

since $u'x = 1$; from which, since

$$(u + du^*)'(x + dx^*) = 1 ,$$

it follows that

$$du^{*\prime}dx^* < 0$$

for all $du^* \neq 0$; [7] that is

$$du'J_u's du < 0;$$

for all $du \nparallel u$. But

$$J_u'x_u' = s' = S = x_u J_u, \quad \text{and} \quad J_u^2 = J_u,$$

so that

$$J_u's = s .$$

---

7 cf. Samuelson [44].



Hence the condition becomes

$$du'sdu < 0$$

for all $du \nparallel u$; otherwise $du'sdu = 0$; and this defines the condition of substitutional negativity.

The following theorem has now been proved.

THEOREM. For a continuously differentiable expenditure duality, rationality implies substitutional symmetry and negativity.

The substitutional negativity condition is that the substitution matrix s be negative in every direction except that of u; and a necessary and sufficient condition for this is that the principal minors, or just the leading principal minors, of s of order r have the sign $(-1)^r$ for $r = 1,\ldots,n-1$.

The sufficiency of the substitutional symmetry and negativity conditions for rationality will appear later.

12. Compensated change reciprocity.

Let (u,x) be any choice; then (du,dx) defines a compensated change if (u + du, x + dx) is a substitute. In this case

$$dx = sdu,$$

where, with rationality, s is symmetric, and negative definite in every cirection away from u. It follows that for two compensated changes $(du_o,dx_o)$, $(du_1,dx_1)$, necessarily having $du_o$, $du_1$ = o away from the direction of u,

$$du_o'dx_o = du_o'sdu_o < 0,$$
$$du_o'dx_1 = du_o'sdu_1 = du_1'sdu_o = du_1'dx_o ;$$

whence:

THEOREM: <u>In the case of rationality compensated changes</u> $(du_i, dx_i)$ $(i = o,1)$ <u>together satisfy the conditions</u>

$$du_o' dx_o < 0, \quad du_o' dx_1 = du_1' dx_o' \text{ .}$$ [8]

[8] cf. Hicks [21], where importance is given to questions and propositions which have at least something of the form here. However, in spite of its elegance, this theorem seems rather useless, seeing that it presupposes the main issues, these being the criteria for rationality and for a change to be a substitution. In Samuelson [44] , pp. 107 ff there are again some related discussions, and also the suggestion that a certain formula "contains almost all the meaningful empirical implications of the whole theory of consumer's choice". cf Little [27] , p. 96, where similar ideas are entertained. However, this formula is really Samuelson's own so-called Weak Axiom, in a sophisticated disguise, and so it is too weak for such a content.

## IV. Infinitesimal Structure.

### 1. Dual differential forms.

With an expenditure duality E there are associated, in a symmetrical fashion, the linear differential forms $u'dx$ and $x'du$, which are said thus to constitute a *dual pair* of forms. Differentiation of the balance condition $u'x = 1$ gives

$$u'dx + x'du = 0,$$

so they are the negatives of each other.

In the case of a differentiable duality, for which $dx = x_u du$, substitution for dx in the form $u'dx$, and use of the differential form $x_u'u + x = 0$ of the balance condition, obtains transformation of the form thus:

$$u'dx = u'(x_u du) = (u'x_u)du = -x'du.$$

Thus the transformation induced on the forms associated with the duality, by the infinitesimal transformation, converts each form into the negative of the other.

The conditions

$$\lambda u'dx = d\phi, \quad \mu x'du = d\psi$$

for the differential forms to be integrable, that is proportional, by some factors $\lambda$ and $\mu$, to perfect differential $d\phi$ and $d\psi$, are equivalent to the conditions

$$(1-ux')\phi_x = 0, \; (1-xu')\psi_u = 0\,;$$

and as has been shown, these conditions are equivalent for functions $\phi$, $\psi$ which are duals, that is satisfying

$$\phi(x) = \psi(u).$$

Therefore the dual integrability conditions, applying to the dual pair of forms, are equivalent.

## 2. Cycle coefficients and dual transformation.

The differential form $u'dx = u_i dx_i$ has coefficients $u_i$ with partial derivatives $u_{ij} = \partial u_i/\partial u_j$; and these determine the *cycle coefficients* $U_{ijk}$ of the form according to the formula

$$U_{ijk} = \sum_{\alpha,\beta,\gamma} u_\alpha u_{\beta\gamma} \delta^{\alpha\beta\gamma}_{ijk} ,$$

where $\delta^{\alpha\beta\gamma}_{ijk} = 1$, $-1$ or $0$ according as $\alpha,\beta,\gamma$ is a positive, negative or non-permutation of $i,j,k$. Similarly, the cycle coefficients of the dual form determine the dual cycle coefficients $X_{ijk}$ of the expenditure duality, according to the dual formula.

The dual sets of cycle coefficients are converted into each other by the transformation given by

$$\sum_{\alpha,\beta,\gamma} U_{\alpha\beta\gamma} X^{\alpha\beta\gamma}_{ijk} = X_{ijk},$$

and the dually stated inverse equations, where

$$U^{rst}_{ijk} = \sum_{\alpha,\beta,\gamma} x_{r\alpha} x_{s\beta} x_{t\gamma} \delta^{\alpha\beta\gamma}_{ijk} ,$$

and $X^{rst}_{ijk}$ is defined similarly. The functions $U^{rst}_{ijk}$, $X^{rst}_{ijk}$ constitute the dual sets of *cycle transformation coefficients* of the expenditure duality. Together they satisfy the identities

$$\sum_{\alpha,\beta,\gamma} X^{rst}_{\alpha\beta\gamma} U^{\alpha\beta\gamma}_{ijk} = \delta^{rst}_{ijk} .$$

THEOREM. The dual transformation of the form and cycle coefficients $u_i$ and $U_{ijk}$ of the duality into their duals $x_i$ and $X_{ijk}$ is given by the equations

$$\sum_{\alpha} u_{\alpha} x_{\alpha i} = -x_i, \qquad \sum_{\alpha,\beta,\gamma} U_{\alpha\beta\gamma} X^{\alpha\beta\gamma}_{ijk} = X_{ijk}$$

and the similar dual equations, where $x_{ij}$ and $X^{rst}_{ijk}$ are the differential and the cycle transformation coefficients.

3. Cycle identities.

Since $U_{ijk}$ is antisymmetric for permutation of the indices, there are $n(n-1)(n-2)/6$ cycle coefficients, in either of the dual sets, which are distinct without regard for distinction of sign, one for each selection of three distinct indices $i,j,k$ from the $n$ possible values. Further, the four cycle coefficients for the four sets of three indices taken from any four $\alpha,\beta,\gamma,\delta$ have a dependence given by the *cycle identity*

$$U_{\beta\gamma\delta} + U_{\alpha\delta\gamma} + U_{\delta\alpha\beta} + U_{\gamma\beta\alpha} \equiv 0.$$

So let $N(m)$ denote the number of cycle coefficients with indices ranging in $1,\ldots,m(m < n)$ which are not thus dependent. Then it is possible to see that

$$N(m+1) = N(m) + m - 1;$$

whence it follows by induction that

$$N(n) = (n-1)(n-2)/2.$$ [1]

[1] cf. Forsyth [7]

4. Acyclicity.

The differential form $u'dx$ is said to be *acyclic* if its cycle coefficients all vanish, that is

$$U_{ijk} = 0 (i,j,k = 1,\ldots,n).$$

It appears from the linear form of the dual transformation, between the dual sets of cycle coefficients, that the dual acyclicity conditions, for the dual pair of differential forms, are equivalent.

It follows from the cycle identities that there are at most $(n-1)(n-2)/2$ independent acyclicity conditions.

The condition of *differential symmetry* for the expenditure duality is defined by

$$u_{ij} = u_{ji} \quad (i,j = 1,\ldots,n).$$

or equivalently by the dual form of this condition; and it evidently implies acyclicity. There are exactly $(n-1)(n-2)/2$ independent differential symmetry conditions. So there must be at least that number of independent acyclicity conditions. But, as has already been seen, there is also at most that number. Accordingly, there are exactly $(n-1)(n-2)/2$ independent acyclicity conditions.

THEOREM. The number of independent conditions among the $n(n-1)(n-2)/6$ distinct acyclicity conditions

$$U_{ijk} = 0(i,j,k = 1,\ldots,n)$$

is $(n-1)(n-2)/2$.[2]

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[2] The formula is stated in Forsyth[17].

5. Acyclicity and integrability.

THEOREM. Acyclicity is equivalent to integrability.

Suppose $\sum_{i=1}^{n} u_i dx_i$ is integrable, say with integrating factor $\lambda$ and integral $\phi$, so that $\lambda u_i = \phi_i$ where $\phi_i = \partial\phi/\partial x_i$. Then, by differentiation with respect to $x_i$,

$$\lambda_j u_i + \lambda u_{ij} = \phi_{ij} \quad (i,j = 1,\ldots,n),$$

where $\lambda_j = \partial\lambda/\partial x_j$ and $\phi_{ij} = \partial^2\phi/\partial x_i \partial x_j$; and it is assumed that $\phi_{ij} = \phi_{ji}$, this necessarily being so in the case of continuous differentiability. Hence

$$\lambda(u_{ij} - u_{ji}) = \lambda_i u_j - \lambda_j u_i,$$

so that

$$U_{ijk} = u_i(u_{jk} - u_{kj}) + u_j(u_{ki} - u_{ik}) + u_k(u_{ij} - u_{ji})$$

$$= \frac{1}{\lambda} \begin{vmatrix} u_i & u_j & u_k \\ \lambda_i & \lambda_j & \lambda_k \\ u_i & u_j & u_k \end{vmatrix}$$

$$= 0.$$

Thus integrability implies acyclicity. To prove the converse, consider a sequence of n steps in the integration of $\sum_{i=1}^{n} u_i dx_i$ under the hypothesis of acyclicity, where, at the mth step $(m < n)$, the form $\sum_{i=1}^{m} u_i dx_i$ is integrated to obtain an integral $\phi$ carrying $x_{m+1}, \ldots, x_n$ as parameters. It is to be shown that the mth step is attainable from the (m-1)th. Since the first step is an elementary integration, and thus attainable, it will then follow, by induction, that the nth, and final step is attainable, showing $\sum_{i=1}^{n} u_i dx_i$ to be integrable. Thus,

$$\lambda u_i = \phi_i \quad (i = 1,\ldots,m-1),$$

where the integrating factor $\lambda$ and integral $\phi$ have reference just to the (m-1)th step. Then

$$\lambda \sum_{i=1}^{m} u_i dx_i = \sum_{i=1}^{m-1} \phi_i dx_i + (\phi_m + \chi) dx_m,$$

where $\chi = \lambda u_m - \phi_m$. The acyclicity hypothesis implies the acyclicity of the form $\lambda \sum_{i=1}^{m} u_i dx_i$, with coefficients $\phi_i$ $(i = 1,\ldots,m-1)$ and $\phi_m + \chi$; and included among these acylicity conditions are the conditions

$$(\phi_m + \chi)(\phi_{ij} - \phi_{ji}) + \phi_i(\phi_{jm} - \phi_{mj} - \chi_j) + \phi_j(\phi_{mi} + \chi_i - \phi_{im}) = 0 (i,j=1,\ldots,m-1)$$

which reduce to the conditions

$$\phi_j \chi_i - \phi_i \chi_j = 0 \quad (i,j = 1,\ldots,m-1),$$

which imply that $\phi, \chi$ have a functional dependence, in which $x_m$ is carried as a parameter:

$$\chi = \chi(\phi, x_m).$$

Therefore

$$\lambda \sum_{i=1}^{m} u_i dx_i = d\phi + \chi(\phi, x_m) dx_m.$$

But

$$\frac{d\phi}{dx_m} + \chi(\phi, x_m) = 0$$

is an ordinary differential equation, the general theory of which gives the existence of an integrating factor $\mu$ and an integral $\psi$ such that

$$\mu(d\phi + \chi(\phi, x_m)dx_m) = d\psi .$$

Now

$$\lambda\mu \sum_{i=1}^{m} u_i dx_i = d\psi ;$$

so the mth step of integration is attainable from the (m-1)th; which completes the proof.[3]

## 6. Substitution coefficients.[4]

With a differentiable expenditure duality E there is associated the dual pair of substitution matrixes

$$s = x_u(1-ux'), \quad r = u_x(1-xu'),$$

the elements of which define the dual sets of *substitution coefficients* of E. They are transformed into each other according to the formulae

$$x_u r x_u' = s', \quad u_x s u_x' = r' ,$$

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[3] This theorem is very well known; but the present proof is given for the sake of completeness, especially in view of the importance of the result in this subject, and also because a general treatment appears to be omitted in standard works.

[4] The coefficients are Slutzky's in disguise, but a disguise which happens more to reveal their nature. The term substitution may be more misleading than appropriate. It is retained for want of a definitely better one, and out of regard for existing usage. cf. Hicks [22]. Slutzky's original term residual variability seems in some ways more fitting to the nature of the idea.

from which it follows that the conditions for the symmetry or the negativity of each of r and s are equivalent.

The antisymmetric matrices

$$k = \frac{1}{2}(s-s'), \quad h = \frac{1}{2}(r-r'),$$

which are the antisymmetric parts of s and r, define the dual pair of *antisymmetry matrices* of E. They are converted into each other by the transformations

$$x_u h x_u' = -k, \quad u_x k u_x' = -h,$$

from which it appears that $k = 0$ if and only if $h = 0$, and thus that the dual forms of the substitutional symmetry condition are equivalent. These symmetry conditions are expressed by the vanishing of the antisymmetry coefficients $k_{ij}$, for which $k_{ij} = -k_{ji}$ $(i,j = 1,\ldots,n)$; thus,

$$k_{ij} = 0 \quad (i,j = 1,\ldots,n).$$

THEOREM. Differential symmetry implies substitutional symmetry.

That is,

$$x_u - x_u u x' = x_u' - x u' x_u'$$

is implied by $x_u = x_u'$. For there is reduction to the condition

$$x_u u x' = x u' x_u',$$

and then to the condition

$$x_u u = -x,$$

which, with $x_u = x_u'$, is the same as the differential balance condition

$x_u' u = -x.$

7. Substitutional identities.

It has been shown that s is of rank n-1, or nullity 1; and therefore s and s' have null-spaces of dimension 1. Apparently su = 0, since

$$su = x_u(1-ux')u = x_u(u-u) = 0,$$

because u'x = 1. Therefore u spans the null-space of s. Moreover, s'u = 0. For

$$u's = u'x_u(1-ux') = -x'(1-ux') = -x' + x' = 0,$$

using both the immediate and the differential form of the balance condition. Accordingly, u also spans the null-space of s'.

THEOREM I. su = 0, s'u = 0 and the nullity of s is unity, so any vector annihilated by s or s' is dependent on u.

As a corollary of this theorem, which gives identities satisfied between the substitution coefficients, there follows

$$ku = 0,$$

that is, the antisymmetry coefficients satisfy the identities

$$\sum_j k_{ij}u_j = 0.$$

Accordingly, among the $\frac{1}{2}n(n-1)$ distinct symmetry conditions

$$k_{ij} = 0 \quad (i,j = 1,\ldots,n),$$

and most $\frac{1}{2}(n-1)(n-2)$ are independent, and imply the rest.

Since differential symmetry implies substitutional symmetry, and there are $\frac{1}{2}(n-1)(m-2)$ independent differential symmetry conditions, it follows that there is at least the same number of substitutional symmetry conditions, and now exactly that number. Therefore these given identities between the symmetry conditions give all the dependencies between them, leaving exactly the stated number independent.

THEOREM II. The substitutional symmetry conditions

$$k_{ij} = 0 \quad (i,j = 1,\ldots,n),$$

where $k_{ij} = \frac{1}{2}(s_{ij} - s_{ji})$, of which $\frac{1}{2}n(n-1)$ are distinct not regarding distinction of sign, have the dependencies determined by the identities

$$\sum_{j=1}^{n} k_{ij} u_j = 0, \quad (i = 1,\ldots,n)$$

which leave exactly $\frac{1}{2}(n-1)(n-2)$ of them independent and implying the rest.[5]

## 8. Substitutional symmetry and acyclicity.

THEOREM I. Substitutional symmetry is equivalent to acyclicity.

The cycle coefficients are defined by

$$X_{ijk} = x_i(x_{jk} - x_{kj}) + x_j(x_{ki} - x_{ik}) + x_k(x_{ij} - x_{ji}),$$

the trend coefficients by

$$\dot{x}_i = -\sum_j x_{ij} u_j,$$

and the substitution coefficients by

$$s_{ij} = x_{ij} + \dot{x}_i x_j,$$

where $x_{ij} = \partial x_i / \partial u_j$. Multiply the cycle coefficient $X_{ijk}$ by $u_k$ and sum over k, using the further formulae

$$\sum_i u_i x_i = 1, \quad x_j + \sum_i u_i x_{ij} = 0,$$

[5] There is a contrary belief, expressed, for example, in Samuelson [44], p. 107, that the symmetry conditions are all independent. The correct statement, though not the correct proof, is found in Samuelson [46].

thus,

$$\sum_k X_{ijk}u_k = x_i\left(\sum_k x_{jk}u_k - \sum_k x_{kj}u_k\right) + x_j\left(\sum_k x_{ki}u_k - \sum_k x_{ik}u_k\right) + \sum_k x_k u_k (x_{ij} - x_{ji})$$

$$= x_i\dot{x}_j + x_i x_j - x_j x_i + x_j\dot{x}_i + x_{ij} - x_{ji}$$

$$= (x_{ij} + \dot{x}_i x_j) - (x_{ji} + \dot{x}_j x_i)$$

$$= s_{ij} - s_{ji}.$$

Then $s_{ij} = s_{ji}$ if $X_{ijk} = 0$ $(i,j,k = 1,\ldots,n)$; so that acyclicity implies substitutional symmetry. For the converse, suppose $s_{ij} = s_{ji}$ $(i,j = 1,\ldots,n)$, that is

$$x_{ij} - x_{ji} = \dot{x}_j x_i - \dot{x}_i x_j.$$

Then

$$X_{ijk} = \begin{vmatrix} x_i & x_j & x_k \\ x_i & x_j & x_k \\ \dot{x}_i & \dot{x}_j & \dot{x}_k \end{vmatrix} = 0$$

Therefore substituional symmetry implies acyclicity; so the theorem is now proved. This theorem and the Theorem of paragraph 5 now give:

THEOREM II. <u>Substitutional symmetry is equivalent to integrability</u>.[6]

See Notes 2 and 3.

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[6] By differentiation of the equilibrium conditions, which give the integrability, Slutzky [47] showed that integrability implied substitutional symmetry; but the converse, though suspected, and even taken for granted, has never, to my knowledge, been proved. In Samuelson [44], p. 116, footnote, a sketch of a proof is offered. However, in carrying out the steps there implicity suggested we are required to invert a transformation which can be seen to be singular. In the same footnote there is suggestion of the idea, which seems to have some acceptance, that the symmetry conditions are sufficient for the existence of a "preference field". This, however, is

9. Structural identities.

Let E be a differentiable expenditure duality, thus with trend vector $\dot{x}$ and substitution matrix s. Then there is defined the matrix

$$X(\sigma) = \begin{pmatrix} \sigma & \dot{x}' \\ \dot{x} & s \end{pmatrix}$$

with the element σ arbitrary. Since s is singular, the value of the determinant of X(σ) is independent of σ; and this value is given as follows.

THEOREM I. $|X(\sigma)| = |x_u|$ .

Thus, by using elementary operations on the determinant,

$$|X(\sigma)| = \begin{vmatrix} \sigma & \dot{x}' \\ -x_u u & x_u(1-ux') \end{vmatrix}$$

$$= |x_u| \begin{vmatrix} \sigma & \dot{x}' \\ -u & 1-ux' \end{vmatrix}$$

$$= |x_u| \begin{vmatrix} \sigma & x'-\sigma x' \\ -u & 1 \end{vmatrix}$$

$$= x_u(\sigma + (\dot{x}'-\sigma x')u) = |x_u|,$$

since $u'x = 1$ and $u'\dot{x} = 1$.

Provided $\sigma \neq 0$, there is the expansion

$$|X(\sigma)| = \sigma|s-\dot{x}\sigma^{-1}\dot{x}'|;$$

and since, by Theorem I, this determinant is not zero, there may be introduced the matrix

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not at all the case. Given the substitutional symmetry conditions, the substitutional negativity conditions are then necessary and sufficient for preferences to be consistently defined. Questions relating to the existence of a "total utility function" have also had some consideration by Ville [51].

$$K(\sigma) = (s - \dot{x}\sigma^{-1}\dot{x}')^{-1} \quad (\sigma \neq 0),$$

for which the following properties can be verified.

THEOREM II. $X^{-1}(\sigma) = \begin{pmatrix} 0 & u' \\ u & K(\sigma) \end{pmatrix}$, $K(\sigma) - K(\rho) = (\sigma-\rho)uu'$.

These identities may be proved directly, by multiplying through by denominators, and using the relations

$$u'x = 1, \quad u'\dot{x} = 1, \quad su = 0, \quad s'u = 0.$$

See Note 2.

10. Inversion theorem.

Let E be integrable, say with integrating factor $\lambda$ and integral $\phi$, so that $u\lambda = \phi_x$. If E is continuously differentiable, then the matrix $\phi_{xx}$ of second partial derivatives of $\phi$ exists, is continuous, and must, moreover, be symmetric; and thus there is defined the symmetric matrix

$$\Phi = \begin{pmatrix} 0 & \phi_x'/\lambda \\ \phi_x/\lambda & \phi_{xx}/\lambda \end{pmatrix}, \quad \lambda = x'\phi_x ,$$

in conjunction with the integral $\phi$.

THEOREM. $\Phi^{-1} = X(\sigma)$, where $\sigma = -\lambda(\phi_x'\phi_{xx}^{-1}\phi_x)^{-1}$.

It is required to prove the relations

$$\dot{x}'\phi_x/\lambda = 1, \quad \sigma\phi_x'/\lambda + \dot{x}'\phi_{xx}/\lambda = 0.$$

$$s\phi_x/\lambda = 0, \quad \dot{x}\phi_x'/\lambda + s\phi_{xx}/\lambda = 1.$$

The two relations in the first column are readily verified, since $\phi_x/\lambda = u$, so they become $\dot{x}'u = 1$, $su = 0$. Now differentiation os

$$\phi_x = u\lambda, \quad \lambda = x'\phi_x$$

gives

$$\phi_{xx} = u_x\lambda + u\lambda_x', \quad \lambda_x = \phi_x + \phi_{xx}x;$$

whence

$$(1-ux')\phi_{xx} = (u_x + uu')\lambda.$$

Now there is obtained

$$s\phi_{xx} = x_u(1-ux')\phi_{xx} = x_u(u_x + uu')\lambda = (1-\dot{x}u')\lambda,$$

which is the last of the relations required. For the remaining relation,

$$(1-ux')\phi_{xx}x_u u = (u_x+uu')\lambda x_u u = (u-u)\lambda = 0,$$

so that

$$\phi_{xx}x = u(x'\phi_{xx}\dot{x}) = -u\sigma\lambda,$$

as required. Finally, since now

$$\sigma^{-1}\dot{x}\dot{x}' = s-\lambda\phi_{xx}^{-1}, \quad u'\dot{x} = 1, \quad u\lambda = \phi_x, \quad su = o,$$

by combination of these relations the expression for $\sigma$ is readily obtained.

COROLLARY. $K(\sigma) = \phi_{xx}/\lambda$.

COROLLARY. $\dot{x} = -\phi_{xx}^{-1}\phi_x\sigma$ , $s = \lambda\phi_{xx}^{-1} + \phi_{xx}^{-1}\phi_x\sigma\phi_x'\phi_{xx}^{-1}$ . [7]

COROLLARY. If E is integrable then s is symmetric.

## 11. Equilibrium and stability conditions.

It appeared in II.13 that for a uniform expenditure duality E, rationality, which is the non-reflexivity of the preference relation P, is equivalent to the condition that every composition x obtains the maximum of an objective function $\phi(x)$ under the constraint $u'x = 1$, that is with x on its dual balance u.

This condition for an absolute maximum resolves into an e q u i l i b r i u m c o n d i t i o n, that the function be stationary, and then a s t a b i l i t y c o n d i t i o n for this

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[7] cf. Hicks [22], appendix, and Samuelson [44].

equilibrium, which is that the stationary value be a maximum - in the case of local stability then a proper local maximum, and in the case of global stability then an absolute global maximum. With continuous differentiability for E, and local stability everywhere, the then obtained local convexity obtains the global convexity of the critical surfaces, which gives global stability; so local stability everywhere is equivalent to global stability.

The equilibrium condition is

$$u\lambda = \phi_x, \quad \lambda = x'\phi_x,$$

where $\lambda$ appears as the Lagrangian multiplier corresponding to the constraint $u'x = 1$. The equilibrium condition is thus identical with the integrability condition, which is proved equivalent to substitutional symmetry. Assuming the expenditure duality E continuously differentiable, so the integral $\phi$ is continuously twice-differentiable, and $\phi_{xx}$ exists, is continuous and symmetric, the stability condition assumes the form

$$dx \neq 0 \wedge \phi_x' dx = 0 \;.\Rightarrow.\; dx'\phi_{xx} dx < 0,$$

in other words, that the quadratic form $dx'\phi_{xx}dx$ be negative definite under the constraint $\phi_x' dx = 0$; and a necessary and sufficient condition for this is that, for $r = 3,\dots,n+1$, the leading principal minor of

$$\Phi = \begin{pmatrix} 0 & \phi_x'/\lambda \\ \phi_x/\lambda & \phi_{xx}/\lambda \end{pmatrix}$$

of order r have the sign $(-1)^{r-1}$.[8] By application of Jacobi's theorem on the minors of the adjugate,[9] an equivalent of this condition is that, for $r = n-1,\dots,1$ the final principal minors of $\Phi^{-1}$ have the sign $(-1)^r$.

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[8] Mann [30].

[9] Aitken [3].

But, by the inversion theorem of paragraph 10, this inverse matrix is identical with $X(\sigma)$, and so these minors are identical with minors of s, and the condition obtained, which is that, for $r = 1,\ldots,n-1$, the principal minors of order r of s have the sign $(-1)^r$, is the condition for s the be negative definite in every direction away from that of u, on which direction s is singular, since $su = 0$. The conclusions which have been obtained are contained in the following theorem.

THEOREM. _A necessary and sufficient condition for equilibrium is the symmetry of s, and a necessary and sufficient condition for the stability of equilibrium is the negativity of s in every direction different from that of u._

## 12. The infinitesimal conditions of consistency.

An expenditure duality E has been defined to be rational if its choices are coherent, admitting a common motive, or equivalently, if its preferences are consistent, no one being the contrary of another. This rationality condition is expressed by the non-reflexivity of the preference relation P of E; or, since P is defined transitive, also by the antisymmetry of P. Thus, for a rational expenditure duality E, the preference relation P is an order relation, being non-reflexive and transitive. This is a global condition, having reference to every choice of the duality, or to every point in the considered conjugate regions in the balance and composition spaces.

If E is differentiable, the local behaviour, the infinitesimal structure of E, is represented by the linear transformation between balance and composition differentials, the coefficients of which are the partial derivatives of the balance and composition elements with respect to each other. This transformation gives the response $dx = x_u du$ in $x$ to an infinitesimal change $du$ in the balance $u$. It may be enquired whether a local condition, that is a condition on local behaviour as determined by the partial derivative matrix $x_u$, can be decisive for the global rationality condition, as represented by the comprehensively determined relation $P$; that is, whether there is an equivalence between local conditions, applied to the partial derivative matrix $x_u$ at the different points of $u \in B$, and the global condition represented by the non-reflexivity of $P = \overrightarrow{\bigvee_{u \in B}} P_u$.

The answer is in the following theorem, which follows by collecting together various conclusions obtained under differentiability or a weaker condition, which give the equivalence of the rationality condition with the condition of an absolute maximum, with equilibrium with stability, and also with substitutional symmetry and negativity.

THEOREM. <u>Substitutional symmetry and negativity everywhere is necessary and sufficient for the rationality of a differentiable expenditure duality</u>.

## NOTE 1

The following is Dr. Phelps' proof of the proposition stated on p. 50. Let $\overline{U}$, $\overline{X}$ be dual convex bodies, so that

$$u \in \overline{U} \wedge x \in \overline{X} \Rightarrow u'x \geqslant 1.$$

Support and contact elements belong to the frontiers $U^*$, $X^*$ of U, X and are conjugate if $u'x = 1$. Let $C_u$, $S_x$ denote the zones of contact and support of any support and contact u, x. They are formed of the contacts and supports conjugate to u, x respectively. Evidently

$$x \in C_u \Leftrightarrow u \in S_x.$$

It is required to prove that, for any contact z,

$$u, v \in \text{Int } S_z \Rightarrow C_u = C_v.$$

It is enough to show that

$$u \in \text{Int } S_z \wedge v \in S_z \Rightarrow C_u \subset C_v.$$

The hypothesis here gives the existence of a $\lambda > 1$ such that

$$w = \lambda u + (1-\lambda) v \in S_z.$$

Take $y \in C_u$, so that $u'y = 1$. It is required to prove now that $v'y = 1$. Thus, since v, w are supports, and y a contact,

$$1 \leqslant v'y, \quad 1 \leqslant w'y = \lambda u'y + (1-\lambda)v'y = \lambda + (1-\lambda)v'y,$$

whence also $1 \geqslant v'y$, and hence $v'y = 1$.

NOTE 2

Some significance for the results in III.9, pp. 102 ff. is shown in the following remarks, which are also to indicate an inconclusive attempt at a construction of integrals, which would at the same time give an independent proof of the sufficiency of the substitutional symmetry condition for integrability, without first showing it equivalent to the acyclicity condition.

If $\phi$ is any integral, then any other integral is $\psi = \omega(\phi)$, where $\omega$ is an arbitrary function with derivatives $\omega'$, $\omega''$; and if $\lambda$, $\mu$ are the integrating factors, then

$$\psi_{xx}/\mu = uu'\lambda\omega''/\omega' + \phi_{xx}/\lambda .$$

It has appeared that to every integral $\phi$ there corresponds a function $\sigma$ such that $K_\sigma = \phi_{xx}/\lambda$ ; and for a further integral there is a function $\rho$ such that $K_\rho = \psi_{xx}/\mu$. Hence,

$$K_\rho - K_\sigma = uu'\lambda\omega''/\omega' ;$$

and now a further result of III.9 gives

$$\rho - \sigma = \lambda\omega''/\omega' .$$

If $\tau$ is an arbitrary function, and $\phi$ an existing integral which is to be constructed, then by the considered theorems,

$$K_\tau = \phi_{xx}/\lambda + (\tau - o)uu'$$

where $\zeta = \tau - \sigma$ is an unknown function, since $\sigma$ is unknown. Division of the i, jth element of K by $u_i$ now gives

$$\Omega_{ij} = \phi_{ij}/\phi_i + \zeta u_j .$$

There is now obtained a set of vectors $\Omega_i$ with elements $\Omega_{ij}$,

$$\Omega_i = [\log \phi_i]_x + \zeta u,$$

such that, for some function $\zeta$, the vectors $\Omega_i - \zeta u$ are gradiants of the form $[\log \phi_i]_x$.

## NOTE 3

A balanced system, equivalently a linear differential form where coefficients satisfy the balance condition may be said to be homogeneous if all the elements of x are homogeneous functions of u of the same degree. If the degree is to be integral the balance condition gives the degree to be -1; and the Euler identities for homogeneous functions now give the equivalent pair of relations

$$x_u u = -x, \quad u_u x = -u.$$

A system which satisfies the more general relations

$$x_u u = xp, \quad u_x x = uq,$$

which are equivalent, with pq = 1, may be called virtually homogeneous. If p, q are constant, the ordinary homogeneity condition is obtained. Immediate integrability is defined as the condition that unity being an integrating factor.

THEOREM. <u>The conditions for the substitutional symmetry and the differential symmetry of a homogeneous balanced system are equivalent; so an homogeneous balanced system is integrable if and only if it is immediately integrable</u>.

Thus, the condition for homogenity is $\acute{x} = -xp$, so $s = x_u - xx'p$; and hence $s = s'$ if and only if $x_u = x_u'$. The final conclusion now follows from the equivalence of substitutional symmetry and acyclicity together with the classical propositions that acyclicity is equivalent to integrability and differential symmetry to immediate integrability.

Let $f_i dx_i$ be a linear differential form whose coefficients are homogeneous functions, all of the same degree, such that $\mu = f_i x_i \neq 0$. Then its balanced equivalent $u_i dx_i$ has coefficients $u_i = f_i/\mu$ which are homogeneous functions of degree -1; and so, in accordance with Euler's theorem for homogeneous functions, there is obtained the homogeneity condition $u_x x = -u$. Now, if the given form is acyclic, so also is its balanced equivalent, which must therefore be substitutionally symmetric, and therefore differentially symmetric and immediately integrable. There is now obtained the following theorem. (Piaggio [37]), which is readily proved for $n = 3$ using vector methods, but not so readily in the general case.

<u>THEOREM</u>. <u>If the coefficients $f_i$ of an acyclic differential form are homogeneous functions of the same degree and if $\mu = f_i x_i \neq 0$ then $1/\mu$ is an integrating factor</u>.

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