Issued Under Office of Naval Research
Contract No. Nonr- 1858(16)

# COMPOSITION OF GENERAL SUM GAMES

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Econometric Research Program
Research Memorandum No. 74
April 1965

The research described in this paper was supported partially by the Office of Naval Research and partially by The Carnegie Corporation of New York.

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## § 0. Introduction

In the following pages we generalize von Neumann and Morgenstern's theory of composition ([ N and M], Chapter IX) to general sum games. We do not consider "outside sources"(1) and thus our results are the same as their results for the "normal zone". We emphasize that our results can in no sense be regarded as new; we merely reformulate and strengthen part of the results in [N and M], Chapter IX. Our only contribution is, perhaps, that we succeed to obtain a uniform theory for solutions of a composition.

## § 1. Notation and Preliminaries

Let $N = \{1,\ldots,n\}$ be a set with $n$ members, (identified with the first $n$ natural numbers). A characteristic function is a non-negative real function $v$ defined on the subsets of $N$ which satisfies

(1.1) $v(\emptyset) = 0$ , where $\emptyset$ is the empty set,

(1.2) $v(\{i\}) = 0$ , for all $i \in N$ , and

(1.3) $v(S) \geq 0$ , for all $S \subset N$ .

The pair $G = (N,v)$ is an n-person game. The members of $N$ are called players. Subsets of $N$ are called coalitions. An individually rational payoff vector (i.r.p.v.) is an n-tuple of real numbers $x = (x_1,\ldots,x_n)$ which satisfies

(1) Since we do not assume superadditivity it seems that there is no point in introducing "outside sources".

## ABSTRACT

A description of the structure of the von Neumann and Morgenstern solutions of a composition of general sum cooperative games is given. We do not assume superadditivity and thus we succeed to get a uniform theory. Essentially we get in our formulation the results which were obtained by von Neumann and Morgenstern for the "normal zone".

A solution of a composition determines uniquely the transfer of money between the component games and is the cartesian product of its projections on the payoff spaces of the component games. The transfer is limited only by the requirement that for each payoff vector in the solution, both the two sets of players of the components have non empty effective subsets.

(1.4) $x_i \geq 0$ , $i = 1,\ldots,n$ (individual rationality), and

(1.5) $\sum_{i=1}^{n} x_i = v(N)$

An i.r.p.v. represents a possible outcome of $G$. We remark that when $G$ is superadditive then (1.4) and (1.5) yield the well-known definition of imputation. The set of all the i.r.p.v.'s of $G$ is denoted by $A(G)$ . We call a coalition $S$ effective for the i.r.p.v. $x$ if

(1.6) $\sum_{i \in S} x_i \leq v(S)$ .

Let $x, y \in A(G)$. $x$ dominates $y$ via a coalition $S$ (written $x \succ_S y$) if:

(1.7) $S$ is effective for $x$ , and

(1.8) $x_i > y_i$ for all $i \in S$ .

$x$ dominates $y$ (written $x \succ y$) if there exists a coalition $S \neq \emptyset$ such that $x \succ_S y$ . A solution of $G$ is a subset $V$ of $A(G)$ which satisfies:

(1.9) if $x,y \in V$ then $x$ does not dominate $y$, and

(1.10) if $y \notin V$ then there exists an $x \in V$ such that $x \succ y$ .

Let $t \leq v(N)$ . We define a game $G(t) = (N, v_t)$ by

(1.11) $v_t(S) = \begin{cases} v(S), & S \neq N \\ v(N)-t, & S = N \end{cases}$

Let $x \in A(G(t))$. $x$ will be called detached if

(1.12) $\sum_{i \in S} x_i \geq v(S)$ , for all $S \subset N$ .

$x$ is fully detached if

(1.13) $\sum_{i \in S} x_i > v(S)$ , for all $S \neq \emptyset$ .

Let $G_1 = (N_1, v_1)$ and $G_2 = (N_2, v_2)$ be two games, not necessarily with the same number of players, with disjoint sets of players, that is $N_1 \cap N_2 = 0$. The <u>composition</u> of $G_1$ and $G_2$ is the game $G = (N,v)$ where:

(1.14) $N = N_1 \cup N_2$, and

(1.15) $v(S) = v_1(S \cap N_1) + v_2(S \cap N_2)$, for all $S \subset N$.

An i.r.p.v. $z \in A(G)$ is a real function defined on $N$ which satisfies (1.4) and (1.5); its restrictions to $N_1$ and to $N_2$ will be denoted by $z^1$ and $z^2$ respectively. Clearly $z^1 \in A(G_1(t))$ and $z^2 \in A(G_2(-t))$, where $t = t(z)$ is defined by

(1.16) $$t(z) = v_1(N_1) - \sum_{i \in N_1} z_i$$

$t(z)$ is the amount that is transferred from $N_1$ to $N_2$ when the payments to the members of $N$ are specified by $z$.

## § 2. <u>The Structure of the Solutions of a Composition</u>

Let $G_i = (N_i, v_i)$, $i = 1,2$, be two games with disjoint sets of players, and let $G = (N,v)$ be the composition of $G_1$ and $G_2$.

<u>Lemma</u> 2.1 <u>Let</u> $z \in A(G)$. <u>If</u> $S$ <u>is effective for</u> $z$ <u>then</u>[(2)] $S \cap N_1$ <u>or</u> $S \cap N_2$ <u>is effective for</u> $z$.

<u>Proof</u>: (1.6) and (1.15)

<u>Corollary</u> 2.2 <u>If</u> $z, \bar{z} \in A(G)$ <u>and</u> $z \succ_S \bar{z}$ <u>then</u> $z \succ_{S \cap N_1} \bar{z}$ <u>or</u> $z \succ_{S \cap N_2} \bar{z}$.

Corollary 2.3 <u>If</u> $z \in A(G)$ <u>then there is an</u> $S_1 \subset N_1$, $S_1 \neq \emptyset$, <u>or there</u> is an $S_2 \subset N_2$, $S_2 \neq \emptyset$, <u>which is effective</u> for $z$.

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(2) The possibility that both these coalitions are effective for $z$ is not excluded.

Proof: (1.5) and Lemma 2.1

Lemma 2.4 If $V$ is a solution for $G$ and $z, \bar{z} \in V$ then $t(z) = t(\bar{z})$.

Proof: Suppose, per absurdum, that $t(z) < t(\bar{z})$. Let

$e = v(N) - \sum_{i \in N_1} \bar{z}_i - \sum_{i \in N_2} z_i$. $e > 0$. Define $z^* \in A(G)$ by:

$z^*_i = \bar{z}_i + \frac{e}{|N|}$ for $i \in N_1$, and $z^*_i = z_i + \frac{e}{|N|}$ for $i \in N_2$, where $|N|$ is the number of players of $N$. We now distinguish the following two possibilities

(2.1) $z^* \in V$

It follows from corollary 2.3 and the definition of $z^*$ that $z^*$ dominates $z$ or $\bar{z}$, a contradiction.

(2.2) $z^* \notin V$.

There exists a $z^{**} \in V$ that dominates $z^*$. By Corollary 2.2 and the definition of $z^*$, $z^{**}$ dominates $z$ or $\bar{z}$, a contradiction.

Lemma 2.5 Let $V$ be a solution of $G$. If $z \in V$ then $z^1$ and $z^2$ are not fully detached.

Proof: Assume, per absurdum, that $z^1$ is fully detached (see (1.13)). Hence there is an $i \in N_1$ such that $z^1_i > 0$. Choose $0 < e \leq z^1_i$ and define $\bar{z}$ by:

$$(2.3) \qquad \bar{z}_j = \begin{cases} z_j, & j \in N_1 \text{ and } j \neq i \\ z_i - e, & j = i \\ z_j + \frac{e}{|N_2|}, & j \in N_2 \end{cases}$$

Here $|N_2|$ denotes the number of players in $N_2$. Clearly we can choose e such that $\bar{z}^1$ will be fully detached (with respect to $G_1(t)$, where $t = t(\bar{z})$). By Lemma 2.4 $\bar{z} \notin V$ thus there exists a $z^* \in V$ which dominates $\bar{z}$. By Corollary 2.2 and the definition of $\bar{z}$ $z^* \succ z$, a contradiction. The proof that $z^2$ is not fully detached is similar.

Let V be a solution of G. By Lemma 2.4 there is a real number $t = t(V)$ such that $t(z) = t$ for all $z \in V$. Define the following sets: $V_i$ is the projection of V on the space whose coordinates are indexed by the members $N_i$, $i = 1, 2$.

<u>Lemma</u> 2.6 $V_1$ <u>is a solution of</u> $G_1(t)$ <u>and</u> $V_2$ <u>is a solution of</u> $G_2(-t)$.

<u>Proof</u>: We shall prove that $V_1$ is a solution of $G_1(t)$. The proof that $V_2$ is a solution of $G_2(-t)$ is similar.

Let $z^1 \in A(G_1(t)) - V_1$. Choose $\bar{z}^2 \in V_2$ and form $z = (z^1, \bar{z}^2)$. $z \notin V$. Hence there exists $\bar{z} \in V$ such that $\bar{z} \succ z$. By Corollary 2.2 and the definition of $z^2$ we must have that $\bar{z}^1 \succ z^1$. Thus $V_1$ is externally stable, i.e. it satisfies (1.10). The fact that $V_1$ is internally stable, i.e., that is satisfies (1.9), follows from the fact that V is.

<u>Lemma</u> 2.7 $V = V_1 \times V_2$

<u>Proof</u>: Let $z, \bar{z} \in V$ and form $z^* = (z^1, \bar{z}^2)$. If $z^* \notin V$ then there exists $z^{**} \in V$ such that $z^{**} \succ z^*$. It follows from Corollary 2.2 and the definition of $z^*$ that $z^{**}$ dominates $z$ or $\bar{z}$, a contradiction.

<u>Lemma</u> 2.8 <u>Let</u> $-v_2(N_2) \leq t \leq v_1(N_1)$ <u>be such that</u> $G_1(t)$ <u>and</u> $G_2(-t)$ <u>have no fully detached</u> i.r.p.v.'s. <u>If</u> $V_1$ <u>is a solution of</u> $G_1(t)$ <u>and</u> $V_2$ <u>is a solution for</u> $G_2(-t)$ <u>then</u> $V = V_1 \times V_2$ <u>is a solution for</u> G.

Proof: It follows from Corollary 2.2 that $V$ is internally stable, i.e., it satisfies (1.9). Let now $z \in A(G)-V$. We distinguish the following possibilities.

(2.4) $$t(z) > t .$$

Let $e = t(z)-t$ and let $\bar{z}^2 \in V_2$. Define $\bar{z} \in A(G)$ by

(2.5) $$\bar{z}_i = \begin{cases} z_i + \frac{e}{|N_1|} , & i \in N_1 \\ \bar{z}_i^2 , & i \in N_2 \end{cases}$$

Here $|N_1|$ is the number of players in $N_1$. If $\bar{z} \in V$, that is $\bar{z}^1 \in V_1$, then, since by the choice of $t$ $\bar{z}^1$ is not fully detached, it has a non empty effective set; it follows that $\bar{z} \in\!\!\dashv z$. If $\bar{z}^1 \notin V_1$ then there exists a $\hat{z}^1 \in V_1$ which dominates it. Let $\hat{z} = (\hat{z}^1, \bar{z}^2)$. $\hat{z} \in\!\!\dashv z$.

(2.6) $$t(z) = t$$

If $z^1 \notin V_1$ then there exists a $\bar{z}^1 \in V$ such that $\bar{z}^1 \in\!\!\dashv z$. Let $\bar{z}^2 \in V_2$. $\bar{z} = (\bar{z}^1, \bar{z}^2)$ dominates $z$. If $z^2 \notin V_2$ the proof that $z$ is dominated by a member of $V$ is similar.

(2.7) $$t(z) < t$$

The treatment of this case is similar to that of (2.5)

We are now able to formulate

Theorem $V$ is a solution of $G$ if and only if it has the following form: $V = V_1 \times V_2$ where $V_1$ and $V_2$ satisfy

(2.8) $V_1$ is a solution of $G_1(t)$ and $V_2$ is a solution of $G_2(-t)$

(2.9) $G_1(t)$ and $G_2(-t)$ have no fully detached i.r.v.p.'s, and

(2.10) $$- v_2(N_2) \le t \le v_1(N_1) .$$

## REFERENCE

[N and M] J. von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton, Princeton University Press, 2nd edition 1947.

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