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STRUCTURES IN LOGISTICS OPERATIONS

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Preface. We present an abstract scheme for studying certain aspects of logistics, followed by two appendices, illustrating more concrete ideas. While this seems the most appropriate logical order of exposition, it is not the order in which the ideas were conceived. Rather, the general scheme was inferred from specific examples. For this reason, it is suggested that the reader look at the appendices first (which are self-contained), whence it is hoped the abstract scheme will take on more realism and urgency.

Preliminaries.

We seek to mathematize notions of logistics in order to organize and summarize a variety of problems, approaches, and techniques. Our descriptions provide certain points of view in inventory and production control, etc.; other viewpoints are also needed in the total consideration of physical operations (as J. Tukey has put it in another context, "too many have sought a 'single truth' as vigorously as Henry George sought a 'single tax'").

Consider a physical operation engaged in, among other things, the transfer and/or alteration of certain material from one set of external sources (suppliers) to another set of external sinks (consumers). We take logistics to be concerned solely with the presence, location, and flow of such material. Equipment, processes, and other material which are employed in the transfer and/or alteration of the material of interest are taken for granted, without further inquiry into their physical nature, and are characterized solely in terms of the inputs and outputs of the material of interest in their employment.

We suppose the logistics of material of a given interest in a physical operation can be described "microscopically" to any desired detail by accounting, factually, for the presence, location, and flow of the material over time. Often, such a microscopic description can

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be deduced from physical circumstances of the operation, and more generic descriptions of the conditions of logistic interest can be similarly derived. But usually, such descriptions, in operations of moderate size and complexity, exceed both the current bounds of effective computability and the intuitive grasp of those who would understand them. This dilemma suggests a major goal — to develop "macroscopic" descriptions out of microscopic ones which can be both computable and within the reach of intuition, even at the sacrifice of "fine structure" in the descriptions.

Our point of view is to be statistical. We seek to replace discrete and exact measurements of the presence, location, and flow of material by summary averages, variabilities, correlations, etc., and to replace intractable combinatorial problems by cruder, tractable statistical ones. Our underlying model is to be, eventually, a stochastic process, in which decision makers are embedded, acquiring information and acting upon it over time. It will be unusual that a decision maker know the entire situation of the operation at any given time, or, indeed, that he know the structure of the underlying stochastic process, except as it is partially revealed through sample statistics which are available. We shall be interested in his problems.

Systems

We mathematize the notion of system (as an extension of the notion of abstract relation) as follows: A <u>system</u> is a subset of a product set.

<u>Interpretation</u>: If $x = (x_1, x_2, \ldots, x_n) \in S$, a system, then $x_1, x_2, \ldots, x_n$ are <u>measurements</u> of <u>attributes</u> $1, 2, \ldots, n$ (of some conceived operation) which can co-exist.

<u>Notational Extension</u>: If a system is written $S = \{x|p(x)\}$, where $p(x)$ is some proposition in "variables" $x_1, x_2, \ldots, x_n$, then a tacit understanding of "$S = \{x|\ldots\}$" allows the system S to be represented by $p(x)$; in this case the system has been described in terms of "variables and necessary relations" (though not uniquely).

Discrete Measurements and Concepts

We introduce and briefly discuss some building blocks for a microscopic logistic model. Their forms are, ultimately a matter of taste;

some are quite natural, while others may appear a bit contrived at first glance. It is hoped, in any case, that their uses and conveniences will justify (in the reader) their definitions.

Time. We suppose time is partitioned into discrete (connected) periods — say weeks, months, three second intervals, etc.

Inventories. We suppose all material of interest is classified into a finite number of commodities, determined by intrinsic properties of the material and/or its spatial location. Labeling the commodities $1, 2, \ldots, m$, we denote the inventory of commodity j at the end of time period t by $i_j(t)$. The entire list of inventories, a vector $i(t) = (i_1(t), i_2(t), \ldots, i_m(t))$, is called the inventory profile of period t. Let I be the set of all possible inventory profiles. We assume that all material of interest in the operation at the end of each time period is accounted for in each inventory profile.

Activities. We suppose all material of interest in the operation is either processed from one inventory profile to the next by

1) conservative processes, such as
   - storage
   - transportation
   - transformation

or processed into or out of inventory profiles by

2) nonconservative processes, such as
   - supply by external sources
   - consumption by external sinks.

The entire list of activities of these processes, for time period t will be denoted by a vector $a(t)$, and called the activity profile for period t. Let A be the set of all possible activity profiles. We assume that the disposition of every bit of material in inventory profile $i(t-1)$, either processed into $i(t)$ or consumed by external sinks, is accounted for by activity profile $a(t)$. Similarly, $i(t)$ arises from $a(t)$. Thus, there are strong inner consistency requirements on any sequence

$$\ldots, i(t-1), a(t), i(t), a(t+1), \ldots$$

which purports to be measurements of the location and flow of material in a physical operation. We call these consistency requirements the material balance of the operation. They determine a system, ordinarily called the "input-output system" of the operation.

Agents. The physical operation is conducted by means of agents, internal and external, who, in concert, select an activity profile for each time period. We distinguish one agent, called 0, as environment, and label the others $1, 2, \ldots, n$. It will be supposed that each agent is given exclusive and autonomous jurisdiction over a subset of the processes of the operation. Let the product space A (of all activity profiles) be partitioned into subproduct spaces $A_o, A_1, \ldots, A_n$; we suppose at each time period t, each agent, say k, selects a subvector $a_k(t) \in A_k$, and the activity profile $a(t) = (a_o(t), a_1(t), \ldots, a_n(t))$ results. For convenience, we shall handle the operation's total relationship with external sources and sinks through the single agent 0.

Requisitions. Agents communicate with each other (and with the external world through agent 0) by means of requisitions. Usually, a requisition is taken to be a request for a certain activity; we will find it convenient to generalize as follows: a requisition is a subset of activity profiles, interpreted to mean that any member of the subset is authorized (by the requisitioning agent). A vector $r(t) = r_o(t), r_1(t), \ldots, r_n(t))$, where $r_k(t) \subset A$, for each agent k, is called the requisition profile for period t. Let the set of all possible requisition profiles be denoted by R.

Observable Sequences. We are now in a position to formulate an underlying model of observable events in the conduct of a physical operation. We suppose, in each time period that, first, the agents select, by simultaneous choice, a requisition profile, say $r(t)$, and second, by simultaneous choice, an activity profile, $a(t)$. If $a(t) \in r_k(t)$, each k, then $i(t)$ is determined by material balance; if $a(t) \notin r_k(t)$, some k, we shall terminate our consideration of the operation. Thus, recursively, we obtain a system whose elements are sequences (possibly terminating), of the form,

$$\ldots, i(t-1), r(t), a(t), i(t), r(t+1), a(t+1), \ldots$$

of observable events in the operation.

States. We shall find it convenient to define a state of the operation as a truncated sequence, or a history, say $s(t)$, where,

$$s(t) = (\ldots, r(t-1), a(t-1), i(t-1), r(t), a(t), i(t));$$

let S be the set of all possible states of the operation. (We shall admit terminated states, written $(\ldots, r(t), a(t), T)$, in S.)

Information. Internal agents in the operation are partially informed about the state of the operation at their occasions of choice. We formulate their states of information as follows:

> At the occasion of a requisition choice, each agent $k \neq 0$ is informed that state $s(t)$ is contained in some subset, say F_k, of all possible states S; F_k is to depend only on $s(t)$, not t.
>
> At the occasion of an activity choice, each agent $k \neq 0$ is informed that $(s(t), r(t+1))$ is contained in some subset, say G_k, of the product set $S \times R$; G_k is also independent of t.

In contrast, agent 0, environment, is fully informed of the state of the operation and the succeeding requisition profile at its occasions of choice, and, in addition, informed of the time period, itself. This latter distinction is crucial, for it embeds the agent environment in real time and forbids this knowledge to internal agents of the operation.

This completes our informal description of a microscopic logistic model, via the introduction and description of its building blocks. For convenience, we summarize it below.

Microscopic Logistic Model — Axiomatic Description

A microscopic logistic model is determined by a collection of sets,

$$N = \{1, 2, \ldots, n\}, I, A_0, A_1, \ldots, A_n,$$

with defined sets,

$$A = A_0 \times A_1 \times \ldots \times A_n, \quad R = \underbrace{2^A \times 2^A \times \ldots \times 2^A}_{(n+1 \text{ sets})}, \quad S = \ldots \times R \times A \times I$$

and a set of mappings

$$F: N \times S \longrightarrow 2^S, \quad G: N \times S \times R \longrightarrow 2^{S \times R}, \quad H: S \times R \times A \longrightarrow S.$$

In this description, we interpret N as the set of internal agents, I, A, R as the sets of inventory, activity, requisition profiles, S as the set of states, F and G as information specifications at requisition and activity choices, and H the material balance-termination rule of the operation.

Stochastic Logistic Models

It remains to complete our description to choices made by the internal agents of a physical operation in response to specific environmental events; we seek, then, to study such situations through the assumption that environmental events are stochastic in origin, inducing stochastic behavior throughout the operation.

Policies. Physical operations can conceivably be conducted entirely by caprice; we shall assume, however, that a given information specification will lead to a unique choice by an internal agent at any occasion of either requisition of activity choice in the operation. Simply put, we assume there exists a mapping from information to choice, and call such a mapping a policy; specifically, we suppose, for agent $k \neq 0$, a requisition policy, p_k, and activity policy q_k, which provides correspondences,

$$p_k: 2^S \longrightarrow 2^A, \quad q_k: 2^{S \times R} \longrightarrow A_k .$$

We also use this notation to indicate policies for agent 0, letting T be the set of all integers (time periods)

$$p_0: T \times S \longrightarrow 2^A, \quad q_0: T \times S \times R \longrightarrow A_0 .$$

Given an initial state, $s(0)$, then, a set of policies $p_0, p_1, p_2, \ldots, p_n, q_0, q_1, q_2, \ldots, q_n$ will determine, recursively, the states $s(1), s(2), \ldots,$ and thereby, the logistics of the operation. Suppose, now, that P_0 and Q_0 are stochastic variables, taking values in the sets of environmental requisition and activity policies; then P_0 and Q_0 induce, through the recursion described above, a stochastic process. It is in the statistical descriptions of such stochastic processes that we pose and seek answers to problems of decision makers in the operation.

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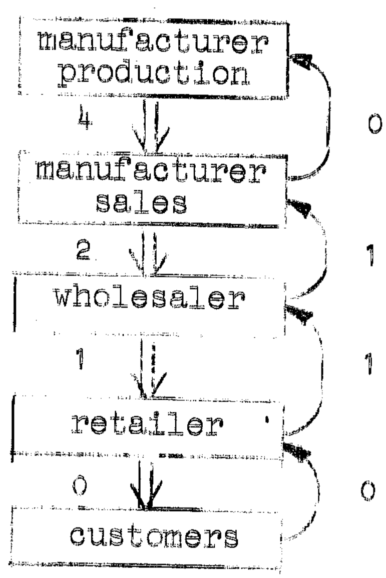
APPENDIX 1. A LOGISTIC MODEL

A Laboratory Operation

We consider a "laboratory" multi-echelon operation of a simple nature to illustrate our approach. The image of a manufacturer-retailer-wholesaler is a convenient one though countless other examples will do as well. We suppose a manufacturer, a dealer, and a retailer, as a single operation, dealing with a single product. It is supposed the product is a "shelf item", and that customers will not back order — substitutes being readily available in other brands, etc. For convenience, we will say that all events take place in the operation on "Monday morning", and use weekly periods to monitor the operation. A diagram suggesting the operation follows, the double arrows indicating flow of the product, the single arrows, the counter flow of orders, from component to component.



The numbers along side the arrows indicate the number of time periods required for that particular flow. For example, shipments from manufacturer to wholesaler require two weeks, while one week is required for a retailer's order to reach the wholesaler. The four weeks shipment time from production to sales in the manufacturing organization is the "manufacturing cycle".

On "Monday morning" we suppose the following sequence of events, in each component:

1. An order is placed for that week.

2. The shipments coming into that component that week arrive.

3. The orders coming into that component that week arrive (these are the whole week's sales, in the case of the retailer).

4. A shipment is made for that week out of inventory on hand (sales made to customers during the week in the case of the retailer).

In the event that a shipment is different than that requested by its corresponding order, we also suppose an immediate notification of the exact amount of the shipment, so that a complete knowledge of inventories in transit is available. As set up, our operation requires that a component make its order for the week, before finding out what demands are placed on it. Clearly we could allow the order to be made at the end of the week, but this would be equivalent (calling that "next week's" order) to the scheme set up. We shall also suppose, on the "requisition end" of the operation that the production department of the manufacturer always meets, exactly, the orders placed on it by the sales department.

The operation, through time, then, is determined by the appearance of seven numbers each week

— the customer demand (on the retailer)

— for each of the three components

the amount shipped

the amount ordered.

A Representation of the Logistic Operation

In order to describe the operation in action, we will list each of these seven numbers for each period, and in addition, list the presence and location of all material in the operation. For convenience, this list is organized from the point of view of each of the three components, and what they know at any particular time period, using the symbols

D: Demand during week

S: Shipments during week

I: Inventory in transit at beginning of week (we denote T_1, T_2, ... as inventory to arrive in 1, 2, ... weeks)

R: Order placed (R for requisition) at beginning of week

and displaying the list as follows:

	Retailer					Wholesaler						Manufacturer						
Week	D	S	I	T_1	R	$\bar{D}$	$\bar{S}$	$\bar{I}$	$\bar{T}_1$	$\bar{T}_2$	$\bar{R}$	$\bar{\bar{D}}$	$\bar{\bar{S}}$	$\bar{\bar{I}}$	$\bar{\bar{T}}_1$	$\bar{\bar{T}}_2$	$\bar{\bar{T}}_3$	$\bar{\bar{R}}$
1	23	13	13	17	11	14	14	21	32	28	16	17	17	21	17	13	19	15
2	12	12	17	11	14	11	11	39	28	16	14	16	16	21	13	19	15	21
3	21	16	16	14	17	14	14	56	16	14	8	14	14	18	19	15	21	14
⋮																		
t	D_t	S_t	I_t	T_{1t}	R_t	$\bar{D}_t$	$\bar{S}_t$	$\bar{I}_t$	$\bar{T}_{1t}$	$\bar{T}_{2t}$	$\bar{R}_t$	$\bar{\bar{D}}_t$	$\bar{\bar{S}}_t$	$\bar{\bar{I}}_t$	$\bar{\bar{T}}_{1t}$	$\bar{\bar{T}}_{2t}$	$\bar{\bar{T}}_{3t}$	$\bar{\bar{R}}_t$

(since there is no time delay in the order placed by sales on the production department, $\bar{\bar{R}}_t = \bar{\bar{T}}_{4t}$). These numbers have internal consistencies, due to material balances, and the time delays in the operation.

The shipment decisions are quite trivial. Each component, in the natural course of events, would try to fill an order, and failing that, do as well as possible; i.e., we have "shipping policies",

$$S_t = \min(D_t, I_t), \quad \bar{S}_t = \min(\bar{D}_t, \bar{I}_t), \quad \bar{\bar{S}}_t = \min(\bar{\bar{D}}_t, \bar{\bar{I}}_t) .$$

We note in passing, that in many situations, the shipping decision is not trivial, where "allocation problems" exist. Were our laboratory operation to contain several wholesalers, and for each of these, several retailers, a component with more orders than inventory on hand would be faced with the problem of allocating the available inventory.

The ordering decisions are not trivial. Indeed, decisions of this kind are at the root of a large number of problems in logistics. While such decisions can be made by caprice, and the operation will proceed through time, we shall not consider caprical behavior. Rather, we consider "ordering policies" — rules which supply an unambiguous order, out of current or past data available to a component at any point of time. Some simple examples of ordering policies are:

1. A "two bin" policy: when inventory falls below a given level, say I^*, order an amount R^*. I.e.,

$$R_t = \begin{cases} R^* & \text{if } I_t < I^* \\ 0 & \text{if } I_t \geq I^* \end{cases} .$$

2. A two bin policy based on a running average: let the average of the past ten weeks be defined as the current sales rate; when inventory falls below 4 times the sales rate, order twice the sales rate. I.e., let

$$S_t^* = \frac{1}{10} \sum_{j=1}^{10} S_{i-j} = \frac{1}{10}(S_{i-1} + S_{i-2} + \cdots + S_{i-10})$$

and

$$R_t = \begin{cases} 2S_t^* & \text{if } I_t < 4S_t^* \\ 0 & \text{if } I_t \geq 4S_t^* \end{cases} .$$

3. A "fixed cycle" policy with a buffer: attempt to maintain 3 times the current sales rate (see example 2 above) in inventory at the beginning of each week. For the retailer, say, the estimated inventory, before ordering, at time $t + 2$ will be

$$I_t - 2S_t^* + T_{1t}$$

while the goal is $3S_t^*$. The order required is

$$3S_t^* - (I_t - 2S_t^* + T_{1t}) = 5S_t^* - I_t - T_{1t}$$

if positive; thus the policy is

$$R_t = \max(0, 5S_t^* - I_t - T_{1t}) .$$

These policies are illustrative, rather than suggestions of "good" policies. The third one, particularly, has certain inherent instabilities that make it dangerous, which is taken up in Appendix 2.

In order to illustrate our laboratory operation in action, we will pick some ordering policies arbitrarily for the three components. We suppose each component attempts to keep two week's of buffer inventory on hand by using a running average of past demands, and adjusting orders to maintain an estimated two week's lead on future demands. And, simply for ease in computing a history of the operation in action, we will take, in each case, a running average of the order-shipment cycle time for each component plus two — i.e., a four-week average for the retailer, a five-week average for the wholesaler, and a six-week average for the manufacturer. The retailer's policy is, then

$$R_t = \max(0,\ D_{t-1} + D_{t-2} + D_{t-3} + D_{t-4} - I_t - T_{1t}) .$$

Nominally, the wholesaler would order

$$\bar{D}_{t-1} + \bar{D}_{t-2} + \bar{D}_{t-3} + \bar{D}_{t-4} + \bar{D}_{t-5} - \bar{I}_t - \bar{T}_{1t} - \bar{T}_{2t}$$

if positive, but, since the retailer is immediately notified of short shipments, if they occur, and these are accounted for in his ordering ($D_{t-1} = R_{t-2}$, etc.), the sum of retailer's demands on the wholesaler would have the effect of counting unfilled orders (between the retailer and wholesaler) more than once — the best estimate of the retailer's past demands is, then,

$$\bar{D}_{t-1} + \bar{S}_{t-2} + \bar{S}_{t-3} + \bar{S}_{t-4} + \bar{S}_{t-5} ,$$

for the shortages in past shipments are accumulated in $\bar{D}_{t-1}$. Thus, we take

$$\bar{R}_t = \max(0,\ \bar{D}_{t-1} + \bar{S}_{t-2} + \bar{S}_{t-3} + \bar{S}_{t-4} + \bar{S}_{t-5} - \bar{I}_t - \bar{T}_{1t} - \bar{T}_{2t})$$

and, similarly, for the manufacturer, we take

$$\bar{\bar{R}}_t = \max(0,\ \bar{\bar{D}}_{t-1} + \bar{\bar{S}}_{t-2} + \bar{\bar{S}}_{t-3} + \bar{\bar{S}}_{t-4} + \bar{\bar{S}}_{t-5} + \bar{\bar{S}}_{t-6} - \bar{\bar{I}}_t - \bar{\bar{T}}_{1t} - \bar{\bar{T}}_{2t} - \bar{\bar{T}}_{3t}) .$$

Given, now, a past history, and a future sequence of customer demands these ordering (and shipping) policies completely determine the future history of the operation. E.g., the following data are sufficient to determine the future (weeks, 1, 2, ...):

Week	D	S	I	T_1	R	$\bar{D}$	$\bar{S}$	$\bar{I}$	$\bar{T}_1$	$\bar{T}_2$	$\bar{R}$	$\bar{\bar{D}}$	$\bar{\bar{S}}$	$\bar{\bar{I}}$	$\bar{\bar{T}}_1$	$\bar{\bar{T}}_2$	$\bar{\bar{T}}_3$	$\bar{\bar{R}}$
-5													10					
-4							15						20					
-3	25						20						20					
-2							15						20					
-1							25						10					
0	10	10	30	15	30	15	15	20	10	20	40	20	20	25	25	45	10	15
1	10																	
2	20																	
3	15																	
4	25																	
5	20																	
.	.																	
.	.																	
.	.																	

Using these policies (and extending the sequence of customer's demands), we carry out ten weeks of operation, in illustration, to get an array of data as follows:

Week	D	S	I	T_1	R	$\bar{D}$	$\bar{S}$	$\bar{I}$	$\bar{T}_1$	$\bar{T}_2$	$\bar{R}$	$\bar{\bar{D}}$	$\bar{\bar{S}}$	$\bar{\bar{I}}$	$\bar{\bar{T}}_1$	$\bar{\bar{T}}_2$	$\bar{\bar{T}}_3$	$\bar{\bar{R}}$
-5													10					
-4							15						20					
-3	25						20						20					
-2	15						15						20					
-1	20						25						10					
0	10	10	30	15	30	15	15	20	10	20	40	20	20	25	25	45	10	15
1	10	10	35	15	20	30	15	15	20	30	25	40	30	30	45	10	15	0
2	20	20	40	20	0	20	20	20	30	25	30	25	25	45	10	15	0	60
3	15	15	40	0	20	0	0	30	25	30	5	30	30	30	15	0	60	20
4	25	25	25	20	10	20	20	55	30	5	0	5	5	15	0	60	20	40
5	20	20	20	10	40	10	10	65	5	0	0	0	0	10	60	20	40	0
6	15	10	10	40	30	40	40	60	0	0	5	0	0	70	20	40	0	0
7	25	25	40	20	15	30	20	20	0	5	65	5	5	90	40	0	0	0
8	10	10	35	0	50	15	0	0	5	65	30	65	65	125	0	0	0	0
9	15	15	25	5	40	50	5	5	65	30	20	30	30	60	0	0	0	45
10	25	15	15	40	10	40	40	65	30	20	5	20	20	30	0	0	45	40

The transition from one row of data to the next is most easily seen by reference to what the columns "stand for" in the operation — for example, inventory in transit two weeks from arrival in one row automatically becomes the inventory in transit one week from arrival, a shipment, $\bar{\bar{S}}_t$, by the manufacturer is, simultaneously, inventory in transit two weeks from arrival $\bar{T}_{2t}$, for the wholesaler, etc. To reflect these transitions, and, again, to guard against ambiguities, we list a sequence of computations, for reference, which carry one row into the next:

Given all data of the table for $t - 1,\ t - 2,\ \ldots,$

D_t is given arbitrarily

$$\bar{\bar{T}}_{3t} = \bar{\bar{R}}_{t-1}$$

$$\bar{\bar{T}}_{2t} = \bar{\bar{T}}_{3,t-1}$$

$$\bar{\bar{T}}_{1t} = \bar{\bar{T}}_{2,t-1}$$

$$\bar{\bar{I}}_t = \bar{\bar{I}}_{t-1} - \bar{\bar{S}}_{t-1} + \bar{\bar{T}}_{1,t-1}$$

$$\bar{\bar{D}}_t = \bar{R}_{t-1}$$

$$*\bar{\bar{R}}_t = \max(0,\ \bar{\bar{D}}_{t-1} + \bar{\bar{S}}_{t-2} + \bar{\bar{S}}_{t-3} + \bar{\bar{S}}_{t-4} + \bar{\bar{S}}_{t-5} + \bar{\bar{S}}_{t-6} - \bar{\bar{I}}_t - \bar{\bar{T}}_{1t} - \bar{\bar{T}}_{2t} - \bar{\bar{T}}_{3t})$$

$$*\bar{\bar{S}}_t = \min(\bar{\bar{D}}_t,\ \bar{\bar{I}}_t)$$

$$\bar{T}_{2t} = \bar{\bar{S}}_t$$

$$\bar{T}_{1t} = \bar{T}_{2,t-1}$$

$$\bar{I}_t = \bar{I}_{t-1} - \bar{S}_{t-1} + \bar{T}_{1,t-1}$$

$$\bar{D}_t = R_{t-1}$$

$$*\bar{R}_t = \max(0,\ \bar{D}_{t-1} + \bar{S}_{t-2} + \bar{S}_{t-3} + \bar{S}_{t-4} + \bar{S}_{t-5} - \bar{I}_t - \bar{T}_{1t} - \bar{T}_{2t})$$

$$*\bar{S}_t = \min(\bar{D}_t,\ \bar{I}_t)$$

$$T_{1t} = \bar{S}_t$$

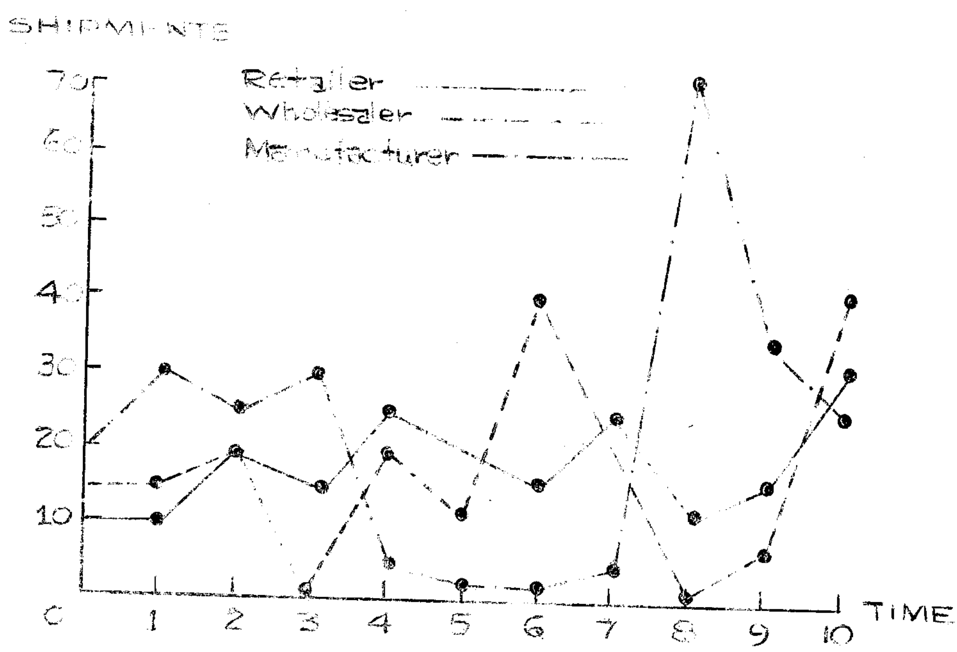
$$I_t = I_{t-1} - S_{t-1} + T_{1,t-1}$$

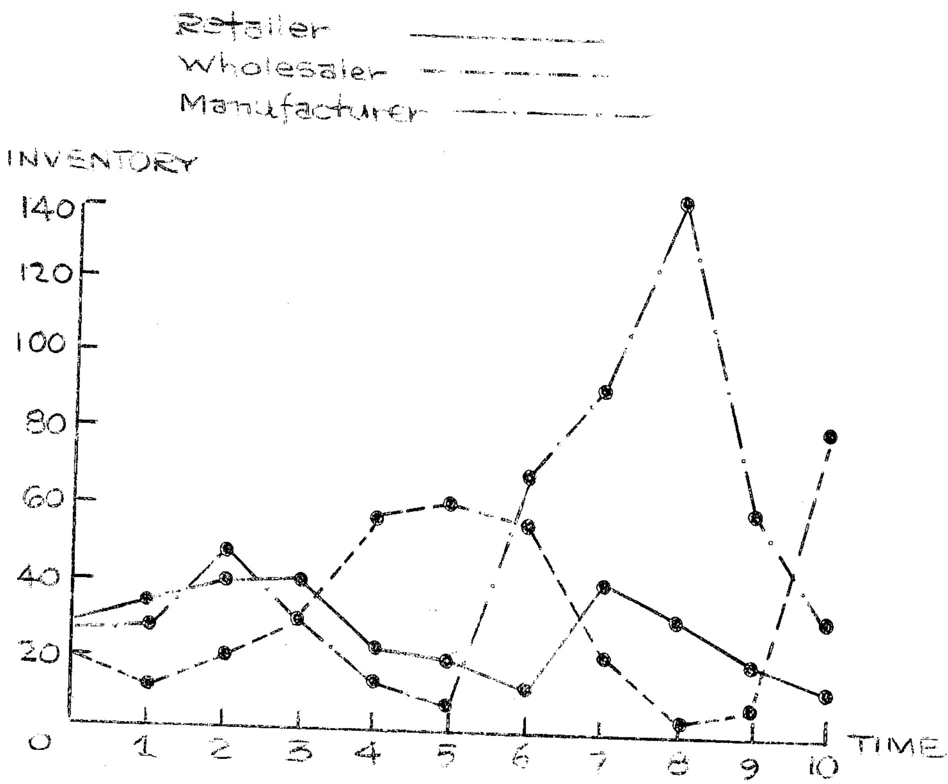
$$*R_t = \max(0,\ D_{t-1} + D_{t-2} + D_{t-3} + D_{t-4} - I_t - T_{1t})$$

$$*S_t = \min(D_t,\ I_t)\ .$$

The six instances of choice in the operation in each time period are denoted by asterisks — their formulas are the "variables" of the operation. Notice this sequence is, indeed, "computable", for it is arranged so that no data is used to define other data until it has been defined itself. (This history has no significance in itself, beyond illustrating our scheme for organizing and describing the operation — the left hand column of customer demands is simply a "haphazard" sequence.)

Columns of such arrays are time series of events occurring in the operation which can be diagrammed in familiar ways. One can often get a better visualization of certain aspects of the history by plotting such time series simultaneously. For example, we plot time series of shipments and inventories, below, comparing them among the three components.





It is easy to see a phenomenon in these diagrams which seems to be generally true (even though this particular illustration does not prove anything in itself). The further removed a component is from the source of external demands, in an operation of this kind, the more violent are the internal demands placed upon it. In each of these diagrams the manufacturer's swings are the greatest, the retailer's the least. Indeed, the sequence of demands $(\bar{\bar{R}})$ on the production department of the manufacturer is even more erratic than that of the sales department $(\bar{\bar{D}})$.

The analogy of this operation to a convoy of trucks seems useful for visualizing, in a loose way, this kind of phenomena. As the lead truck of a convoy varies its speed, the whole convoy expands and contracts, accordion fashion. A small variation in the lead truck speed is converted into larger and larger variations in trucks farther back in the convoy. If we imagine a correspondence between the trucks and the components of our laboratory operation, with the speed of a truck corresponding to the rate of "flow of material" through a component, there are various ways of making this analogy more precise. Some trucks in a convoy may be large and ponderous, others small and nimble, drivers may have different reaction times, etc.

Two Experiments With The Laboratory Operation

Whereas we simply used a haphazard sequence of customer demands in illustrating the description of our laboratory operation in action, we now consider a bit more purposeful sequences. The number of questions we might ask abstractly about the laboratory operation is boundless. In any specific situation, there will be some sensible few. For example, if we are concerned with a suboperation — say the manufacturer — we might know what types of ordering and shipping policies are used in the rest of the operation. Then, we may want to "try out", in some way, various possible policies for the suboperation, given the other policies. Or, if we are concerned with the operation as a whole, we might want to "try out", in some way, integrated, correlated, policies for the suboperations, etc.

In illustration, we consider two simple experiments in our laboratory operation, using the policies defined above. When customer orders are constant, the operation becomes completely stable — each row of data is duplicated in the next row, etc. In fact, every entry in a row will be this constant demand except the three inventory figures, which will be twice the constant demand. We ask two questions now:

1. When the operation is stable at a constant customer demand of 10, how does it behave if the demand goes to 15 and stays there?

2. When the operation is stable at a constant customer demand of 15, how does it behave if the demand goes to 10 and stays there?

The next two pages contain histories for each of these cases, showing the transition from one stable operation to another. The length of time it takes the operation to regain stability may be surprising — over thirty weeks in each instance, and the apparent chaos which takes place in the meantime. For example, the inventory of the manufacturer does go from 20 to 30 in the first case, but it "goes by way of 80"!

History of operation going from constant demand of 10 to constant demand 15

Week	D	S	I	T_1	R	$\bar{D}$	$\bar{S}$	$\bar{I}$	$\bar{T}_1$	$\bar{T}_2$	$\bar{R}$	$\bar{\bar{D}}$	$\bar{\bar{S}}$	$\bar{\bar{I}}$	$\bar{\bar{T}}_1$	$\bar{\bar{T}}_2$	$\bar{\bar{T}}_3$	$\bar{\bar{R}}$
			stable					stable						stable				
0	10	10	20	10	10	10	10	20	10	10	10	10	10	20	10	10	10	10
1	15	15	20	10	10	10	10	20	10	10	10	10	10	20	10	10	10	10
2	15	15	15	10	20	10	10	20	10	10	10	10	10	20	10	10	10	10
3	15	10*	10	20	20	20	20	20	10	10	10	10	10	20	10	10	10	10
4	15	15	20	10	25	20	10*	10	10	10	30	10	10	20	10	10	10	10
5	15	15	15	10	35	25	10*	10	10	20	40	30	20*	20	10	10	10	10
6	15	10*	10	10	40	35	10*	10	20	10	35	40	10*	10	10	10	10	40
7	15	10*	10	20	30	40	20*	20	10	10	45	35	10*	10	10	10	40	30
8	15	15	20	10	30	30	10*	10	10	10	60	45	10*	10	10	40	30	5
9	15	15	15	10	35	30	10*	10	10	10	50	60	10*	10	40	30	5	20
10	15	10*	10	10	40	35	10*	10	10	40	25	50	40*	40	30	5	20	25
11	15	10*	10	10	40	40	10*	10	40	25	10	25	25	30	5	20	25	30
12	15	10*	10	40	10	40	40	40	25	10	15	10	10	10	20	25	30	20
13	15	15	40	10	10	10	10	25	10	15	20	15	15	20	25	30	20	10
14	15	15	35	10	15	10	10	25	15	20	20	20	20	30	30	20	10	15
15	15	15	30	15	15	15	15	30	20	20	10	20	20	40	20	10	15	35
16			stable			15	15	35	20	10	20	10	10	40	10	15	35	30
17						15	15	40	10	20	20	20	20	40	15	35	30	0
18						15	15	35	20	20	0	20	20	35	35	30	0	0
19						15	15	40	20	0	10	0	0	50	30	0	0	25
20						15	15	45	0	10	20	10	10	80	0	0	25	0
21						15	15	30	10	20	15	20	20	70	0	25	0	0
22						15	15	25	20	15	15	15	15	50	25	0	0	5
23						15	15	30	15	15	15	15	15	60	0	0	5	20
24								stable				15	15	45	0	5	20	10
25												15	15	30	5	20	10	10
26												15	15	20	20	10	10	30
27												15	15	25	10	10	30	20
28												15	15	20	10	30	20	10
29												15	15	15	30	20	10	15
30												15	15	30	20	10	15	15
31												15	15	35	10	15	15	15
32												15	15	30	15	15	15	15
33														stable				

*Shipment below requisition.

History of operation going from constant demand of 15 to constant demand of 10

Week	D	S	I	T_1	R
	stable				
0	15	15	30	15	15
1	10	10	30	15	15
2	10	10	35	15	5
3	10	10	40	5	5
4	10	10	35	5	5
5	10	10	30	5	5
6	10	10	25	5	10
7	10	10	20	10	10
8	10	10	20	10	10
9	10	10	20	10	10
10	10	10	20	10	10
11	10	10	20	10	10
12	10	10	20	5	15
13	10	10	15	15	10
14	10	10	20	10	10
	stable				

Week	$\bar{D}$	$\bar{S}$	$\bar{I}$	$\bar{T}_1$	$\bar{T}_2$	$\bar{R}$
	stable					
0	15	15	30	15	15	15
1	15	15	30	15	15	15
2	15	15	30	14	14	14
3	5	5	30	15	15	15
4	5	5	40	15	15	0
5	5	5	50	15	0	0
6	5	5	60	0	0	0
7	10	10	55	0	0	0
8	10	10	45	0	0	0
9	10	10	35	0	0	0
10	10	10	25	0	0	15
11	10	10	15	0	15	15
12	10	5*	5	15	15	15
13	15	15	15	15	15	5
14	10	10	15	15	5	15
15	10	10	20	5	15	10
16	10	10	15	15	10	10
17	10	10	20	10	10	10
18	10	10	20	10	5	15
19	10	10	20	5	10	15
20	10	10	15	10	15	10
21	10	10	15	15	10	10
22	10	10	20	10	10	10
			stable			

Week	$\bar{\bar{D}}$	$\bar{\bar{S}}$	$\bar{\bar{I}}$	$\bar{\bar{T}}_1$	$\bar{\bar{T}}_2$	$\bar{\bar{T}}_3$	$\bar{\bar{R}}$
			stable				
0	15	15	30	15	15	15	15
1	15	15	30	15	15	15	15
2	15	15	30	14	14	14	14
3	15	15	30	14	14	14	14
4	15	15	30	15	15	15	15
5	0	0	30	15	15	15	15
6	0	0	45	15	15	15	0
7	0	0	60	15	15	0	0
8	0	0	75	15	0	0	0
9	0	0	90	0	0	0	0
10	0	0	90	0	0	0	0
11	15	15	90	0	0	0	0
12	15	15	75	0	0	0	0
13	15	15	60	0	0	0	0
14	5	5	45	0	0	0	0
15	15	15	40	0	0	0	10
16	10	10	25	0	0	10	30
17	10	10	15	0	10	30	20
18	10	5*	5	10	30	20	5
19	15	10*	10	30	20	5	0
20	15	15	30	20	5	0	0
21	10	10	35	5	0	0	25
22	10	10	30	0	0	25	5
23	10	10	20	0	25	5	10
24	10	10	10	25	5	10	10
25	10	10	25	5	10	10	15
26	10	10	20	10	10	15	10
27	10	10	20	10	15	10	5
28	10	10	20	15	10	5	10
29	10	10	25	10	5	10	10
30	10	10	25	5	10	10	10
31	10	10	20	10	10	10	10
32							
			stable				

* Shipment below requisition.

A point of interest, in this connection, is in the isolated view of the manufacturer. During the first transition, say, the manufacturer's information indicates (up to period 22) quite a haphazard environment even though, as spectators of the entire operation, we know the situation is very simple. This seems often true — understanding a situation to a great extent, depends on finding a "natural" operation to embed it in. Curiously, such natural operations usually cut across organizations and legal entities, within whose bounds problems are usually stated. For example, a manufacturer may often regard wholesalers as "customers" and find their activity, as "orderers" almost impossible to understand. Yet, an operation in which wholesalers are embedded, while beyond direct control of the manufacturer, may bring an air of reason and predictability into the wholesaler's orders.

APPENDIX 2. STABILITY IN MULTI-ECHELON LOGISTIC OPERATIONS

Preliminaries

In the logistic model of Appendix 1, activities at decision points are carried out on the basis of local information and local reasoning. In such an operation we seek to discover how these local activities, which can be analyzed quite accurately as isolated processes, interrelate and aggregate into an operation as a whole.

We consider an operation of several stock points at various echelons, where each stock point handles a common material. The system originates with an "initial" stock point which supplies several other stock points, each of which supplies still other stock points, etc., until the material reaches "final" stock points which supply, on demand, consumers of the material which are external to the system. Within the system each stock point, in general, is engaged in requisitioning (from a unique stock point), storing, and shipping (to other stock points or consumers) the material. The initial stock point procures the material from sources external to the system, of course.

It is easy to conceive and analyze the average flow of material through the system. The average flow through any stock point must be the sum of the average flow through the stock points it supplies, etc.

It is more difficult to conceive and analyze the statistical variability of the flow of material through the system. The variability at any stock point depends on the variabilities at the stock points it supplies in complex ways, involving the ordering rules employed, the serial correlations of the flows of material, etc.

Our main concern is the conception and analysis of the statistical characteristics of a multi-echelon supply system. We begin with an illustrative statistical analysis of a single stock point, and extend the methods to include a multi-echelon operation.

"Months' of Supply" Ordering Rules

A "k months' of supply" ordering rule, using n months of past data to estimate demand, can be described by the relation

$$R_t = \frac{k}{n} \sum_{a=1}^{n} S_{t-a} - I_t \tag{1}$$

if the right side is non-negative, where

R_t is amount requisitioned in month t,
S_t is amount shipped in month t,
I_t is amount on hand at end of month t.

These quantities must also satisfy a material balance

$$I_{t+1} = I_t + P_t - S_t \tag{2}$$

(for simplicity, we assume no delay in filling requisitions). Equations (1) and (2), regarded as difference equations in an independent sequence $\{S_t\}$ and dependent sequences $\{R_t\}$, $\{I_t\}$, have the solutions [3]

$$R_t = (1 + \frac{k}{n})S_{t-1} - \frac{k}{n} S_{t-1-n} \tag{3}$$

$$I_t = - S_{t-1} + \frac{k}{n} \sum_{a=1}^{n} S_{t-1-a} . \tag{4}$$

When k, n, and the variability of elements of $\{S_j\}$ are such that the right side of (1) is non-negative with high probability, we take (3) and (4) to be good approximations of the operation of this ordering rule.

If the S_t are random variables, then the R_t and I_t are random variables also. The ordering rule transforms a sequence of random variables $\{S_t\}$ into sequences $\{R_t\}$ and $\{I_t\}$ by means of (3) and (4). If the S_t are independent and identical random variables with common variance σ_S^2, then R_t, I_t and $\Delta R_t = R_{t+1} - R_t$ have variances which can be obtained from (3) and (4) as [2]

$$\sigma_R^2 = [(1 + \frac{k}{n})^2 + (\frac{k}{n})^2]\sigma_S^2 \tag{5}$$

$$\sigma_I^2 = (1 + \frac{k^2}{n})\sigma_S^2 \tag{6}$$

$$\sigma_{\Delta R}^2 = 2\sigma_R^2(a)$$

(We need $n \geq 2$ for $\sigma_{\Delta R}^2$). Notice, however, that even though the S_t are uncorrelated, the R_t and I_t will be correlated by the process. We define $\sigma_S^2(a)$, $\sigma_R^2(a)$, $\sigma_I^2(a)$ as the auto-covariances of $\{S_t\}$, $\{R_t\}$. $\{I_t\}$, so

$$\sigma_S^2(a) = \text{Exp}[(S_t - \bar{S})(S_{t-a} - \bar{S})], \quad \bar{S} = \text{Exp } S_t, \quad \text{etc.}$$

and have the general transformation in auto-covariances, from (3) and (4) [1]

$$\sigma_R^2(a) = [(1 + \frac{k}{n})^2 + (\frac{k}{n})^2]\sigma_S^2(a) - (\frac{k}{n})(1 + \frac{k}{n})[\sigma_S^2(a + n) + \sigma_S^2(a - n)] \tag{7}$$

$$\sigma_I^2(a) = (1 + \frac{k^2}{n})\sigma_S^2(a) + \sum_{b=1}^{n} (\frac{k(k-1)}{n} - \frac{b}{n^2})[\sigma_S^2(a + b) + \sigma_S^2(a - b)]. \tag{8}$$

It is easily seen that (5) and (6) are special cases of (7) and (8). We are interested in the general formulation of (7) and (8) because, even though the source of external demands on the system may be uncorrelated, the reflections of these demands in requisitions within the system acquire serial correlations; thus, in general, we must be equipped to handle a serially correlated set of demands in the analysis of a stock point ordering rule.

In equation (7), when $a = 0$, we have

$$\sigma_R^2(0) = [(1 + \frac{k}{n})^2 + (\frac{k}{n})^2]\sigma_S^2(0) - 2(\frac{k}{n})(1 + \frac{k}{n})\sigma_S^2(n)$$

and, since $\sigma_S^2(0) \geq \sigma_S^2(n)$, we have

$$\sigma_R^2(0) \geq [(1 + \frac{k}{n})^2 + (\frac{k}{n})^2]\sigma_S^2(0) - 2(\frac{k}{n})(1 + \frac{k}{n})\sigma_S^2(0)$$

$$= [(1 + \frac{k}{n}) - (\frac{k}{n})]^2\sigma_S^2(0) = \sigma_S^2(0) ;$$

that is, no matter what the serial correlations are in the incoming demands, the variance of the outgoing requisitions is at least as great as the variance of the incoming demands. When the incoming demands are uncorrelated, this "variance amplification factor" is

$$(1 + \frac{k}{n})^2 + (\frac{k}{n})^2 \quad ;$$

when several echelons use this same type ordering rule, the amplification gets worse, through the induced serial correlations, rather than better.

Indeed, when incoming demands are uncorrelated and, say, $k = 5$, $n = 10$, we have

$$\sigma_R^2 = 2.5\sigma_S^2, \qquad \sigma_I^2 = 3.5\sigma_S^2 \quad ,$$

and we see that "Months' of Supply" ordering rules are dangerous indeed, in their inadvertent introduction of instability into a supply system. One could better use the extremely simple-minded ordering rule

$$R_t = S_{t-1} \quad ,$$

for then,

$$\sigma_R^2 = \sigma_S^2, \qquad \sigma_I^2 = \sigma_S^2 \quad .$$

An Optimal Class of Ordering Rules

"Months' of Supply" ordering rules introduce statistical instability into a supply system, while the simple rule of ordering the previous month's demands reduces both requistion and inventory variance. We might ask if we can do better — can the variances of requisitioning and inventories be reduced to below that of incoming demands? The paper "Stochastic Properties of Elementary Logistic Components" studies such questions. It establishes three major results, in the case of uncorrelated demands [4]:

1) There exists an ultimate boundary in the $\sigma_R - \sigma_I$ space beyond which improvement (in additional smoothing) is not possible (Theorem 1 - Smoothing Capacity).

2) Ordering rules can be constructed which approach this ultimate performance in a certain asymptotic sense (Theorem 2 - Optimal Policy Class).

3) The effect of time delays in information or material handling on the variance of requisitions and inventories is determined in an asymptotic sense (Theorem 3 - Information Delay).

The first two points can be summarized, approximately, by the following statement: An ordering rule of the form

$$R_t = \begin{cases} \alpha R_{t-1} + (1-\alpha)S_{t-1} & \text{if} \quad I_t \leq I_{max} \\ 0 & \text{if} \quad I_t > I_{max} \end{cases}$$

for $0 < \alpha < 1$, will result, as I_{max} is increased, in variances

$$\sigma_R^2 = \left(\frac{1-\alpha}{1+\alpha}\right)\sigma_S^2, \qquad \sigma_I^2 = \left(\frac{1}{1-\alpha^2}\right)\sigma_S^2 ;$$

these are the best results possible, in the sense that any ordering rule will result in requisition and inventory variances which both equal or exceed the values above for some α.

For comparison, we take $\alpha = .5$ above, and get the following table

Type Policy		σ_R^2/σ_S^2	σ_I^2/σ_S^2
Months' of Supply	$(k = 5, n = 10)$	2.50	3.50
Optimal Policy Class	$(\alpha = .5)$	0.33	1.33

Other members of the optimal policy class will smooth requisitions even more, at the expense of additional inventory variance — for example, with inventory variance ratio at the Months' of Supply value of 3.50, the Optimal Policy Class ordering variance ratio (for $\alpha = .16$) is .09 — decreasing the requisition variance by a factor over 25 with no increase in inventory variance!

A General Method of Analysis

We shall identify stock points by listing the path along which material flows to reach it. First, the initial stock point of a system will be labeled 0, the i^{th} stock point supplied by it by $0i$, the j^{th} in the next echelon supplied by i, by $0ij$, etc. In general, a stock point with a signature

$$0i_1 i_2 \ldots i_n$$

acquires material through n previous stock points 0, $0i_1$, $0i_1i_2$, ..., $0i_1i_2\ldots i_{n-1}$. (Our assumption that each stock point is supplied by a unique other stock point is handy and crucial for this notation.)

Similarly, we will denote shipments, requisitions, and inventories occurring in the system by adjoining these signatures, as

$$R_{0,t}, \quad S_{0i,t}, \quad I_{0ij,t}, \quad \text{etc.}$$

These signatures will allow us to assemble a multi-echelon system out of its components in quite a convenient fashion. For now we assume that inventories are sufficient so that requirements are almost always met, and shipments are synonymous with demands. Then, for each stock point $k = 0i_1i_2\ldots i_n$, we have a pair of relations corresponding to (1) and (2) above — an ordering rule and an inventory balance —

$$R_{k,t} = f_k(R_{k,s}, S_{k,s}, I_{k,s+1} | s < t) \tag{1k}$$

$$I_{k,t+1} = I_{k,t} - R_{k,t} + S_{k,t} \tag{2k}$$

and each stock point, except "final stock points" dealing with external consumers, an "aggregation"

$$S_{k,t} = \sum_{kj} R_{kj,t} \tag{7k}$$

summing over all stock points kj which k supplies. We have not specified the nature of the functions f_k, except to require them to be dependent only on "past information".

The equations of (1k), (2k) and (7k) define a multi-echelon model — delays in information and transit of material are easily handled as minor modifications in these equations. If the functions f_k are taken to be linear, then our model is a discrete stochastic linear system and we have many techniques directly applicable to characterizing and optimizing (through the choice of the f_k's) its operation.

In fact, the two following well known facts [1] allow us to handle auto-covariances in the system in much the same way that average flows can be aggregated.

Fact 1. If $\{R_{k_1,t}\}$ and $\{R_{k_2,t}\}$ are independent random sequences, then, using

$$R_{k,t} = R_{k_1,t} + R_{k_2,t}$$

we have

$$\sigma^2_{R_k}(a) = \sigma^2_{R_{k_1}}(a) + \sigma^2_{R_{k_2}}(a) \quad ;$$

i.e., auto-covariances are additive, just like means and simple variances of independent random variables.

Fact 2. If an ordering rule has the form

$$R_{k,t} = \sum_{a=1}^{\infty} \alpha_a S_{k,t-a}$$

then

$$\sigma^2_{R_k}(a) = \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \alpha_b \alpha_{c-b} \sigma^2_{S_k}(a - c) \ .$$

Thus we begin with the auto-covariances of the demands on the "final" stock points, transform them into the requisition auto-covariances by means of Fact 2, aggregate requisition auto-covariances into demand auto-covariances at the next echelon, and repeat through the system At each stock point, the inventory auto-covariance can also be computed from the incoming demand auto-covariance. In this way, the statistical variability of a multi-echelon system can be analyzed in the same way average flow can be analyzed — the only real difference is that the transformation of average flow through any given stock point is trivially the identity.

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