STABILIZATION POLICY IN LINEAR STOCHASTIC SYSTEMS

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ABSTRACT

This paper is concerned with the description and evaluation of linear stabilization policy in linear stochastic systems. It is suggested that the power spectrum provides a useful approach to the characterization of alternative stabilization policies. For a second order autoregressive scheme, iso-frequency and iso-variance loci for the stochastic response are constructed and compared with similar loci which have been constructed for the transient response of the system. Several observations on minimum-variance policies are also included.

1.0 Introduction

In order to evaluate alternative stabilization policies, it is necessary to ascertain the properties of the system into which these policies are to be introduced. Since this is very difficult in economics, the usual practice is to postulate simple systems which are amenable to analysis by general methods, and then to consider the impact of various stabilization policies on these systems. Such an approach led Baumol, for example, to conclude that "policies - automatic or not - which appear to be properly designed may very well turn out to aggrevate fluctuations" [2, p. 21]. This somewhat pessimistic conclusion was suggested by an analysis of the transient response of a deterministic linear system. The question naturally arises as to whether results similar to those derived by Baumol also hold for stochastic systems.

The purpose of this paper is to extend the tools and results derived by Baumol to a linear stochastic system. Since attention will be focussed on the stochastic response of the system, this paper can be viewed as an elaboration of some of the problems discussed by Friedman [5] in connection with stabilization policy. In the next section some of the properties of stochastic linear systems and linear stabilization policies are described. Methods which may be used to evaluate alternative stabilization policies are considered in Section 3.0. The paper concludes with several comments on the implications of minimum-variance stabilization policies.

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2.0 Linear Systems and Stabilization Policies

For purposes of exposition, we follow Baumol in considering one of the most familiar linear dynamic systems - the multiplier-accelerator interaction model. Without going into detail, we note that under certain conditions the time path of income may be considered to be generated by

$$Y_t + b\,Y_{t-1} + c\,Y_{t-2} = G_t + A_t + u_t \qquad (1)$$

where Y_t, G_t, and A_t denote total income, government expenditure, and autonomous expenditure in period t, u_t denotes a stochastic disturbance, and b and c are the parameters of the system. This generating equation can be thought of as arising from theoretical considerations or it can be thought of as the final form of the income equation in a linear econometric model.[1]

The stochastic term in (1) is, in general, a linear combination of the stochastic terms in the individual behavioral relationships such as the consumption and investment equations. The disturbance term is here regarded as the variability arising from the randomness of the decisions described by the behavioral relationships and not from specification or estimation error. For this reason it is important for purposes of analysis to retain this stochastic term in the solution for the time path of output.

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[1] There is, of course, no a priori reason to suppose that the generating equation is of order two. For ease of exposition the second order case will be discussed in what follows and the points at which the argument can be generalized to higher-order systems will be pointed out.

2.1 Linear Stabilization Policies

The government expenditure term in Equation (1) is assumed to include only that portion of government expenditure which is oriented toward stabilization policy; the remainder is subsumed by the autonomous expenditure term or is implicitly accounted for by the structure of the system. As Baumol [2] has suggested a number of plausible government expenditure policies can be approximated by a linear stabilization scheme such as

$$G_t = g_1 Y_{t-1} + g_2 Y_{t-2} + B_t + v_t \qquad (2)$$

where g_1 and g_2 are the parameters of the policy and v_t is a stochastic disturbance. The time dependent term B_t in (2) is introduced to characterize such full-employment policies as

$$G_t = g(Y_F - Y_{t-1}) = - g Y_{t-1} + g Y_F \qquad (2')$$

where Y_F denotes full-employment income. The necessity to date B results from the fact that in this example the full-employment level of income may be changing over time.

Inserting the stabilization policy described by (2) into (1) yields

$$Y_t + b' Y_{t-1} + c' Y_{t-2} = A_t' + u_t' \qquad (3)$$

where $b' = b - g_1$, $c' = c - g_2$, $A_t' = A_t + B_t$, and $u_t' = u_t + v_t$. From this expression it follows that the linear stabilization scheme may effect a change in (a) the autoregressive structure of the system, (b) the level of autonomous expenditure, or (c) the magnitude of the stochastic disturbance term. In order to evaluate the total impact of the policy it is necessary to consider all three of these potential changes. In this paper the major concern is with the first of these, namely, the effect on the time path of output of changes in the autoregressive coefficients of the stochastic difference equation. Since the impact of changes in the level of autonomous expenditure and in the magnitude of the disturbance term is obvious, these two sources of change will not be considered further. This reduces the problem to one of analyzing the alternative time paths of the solution of

$$Y_t + b\,Y_{t-1} + c\,Y_{t-2} = A_t + u_t \tag{4}$$

which correspond to various combinations of the autoregressive parameters b and c.[2]

2.2 The Time Path of Output

In what follows it is assumed that the disturbance terms have mean zero and are serially independent.[3] In addition it is assumed that (4) is a stable system in the sense that the variance of deviations from the equilibrium

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[2] In order to avoid confusion it should be pointed out that only permanent changes in the autoregressive parameters are considered here, i.e., the paper is not concerned with the time path of output generated by an autoregressive scheme with time-dependent coefficients. Nor is the paper concerned with the problem of estimation of the (possibly time-dependent) coefficients from a realization of the process.

[3] This assumption is necessary in order to maintain simplicity. If the disturbance terms are serially correlated, under suitable conditions the system may be transformed to a higher-order system with uncorrelated disturbances. But then the simplicity of the second-order case is surrendered.

time path is finite.[4] A necessary and sufficient condition for stability of a linear system with constant coefficients is that the characteristic roots of the system lie within the unit circle in the complex plane. This will be the case if the autoregressive coefficients satisfy the following inequalities:[5]

$$\begin{aligned} 1 + b + c &> 0 \\ 1 - b + c &> 0 \\ 1 - c &> 0 \end{aligned} \tag{5}$$

The triangular boundary of the stable region in the c - b parameter space is shown in Figure 1. The region to the right of the parabola in this figure corresponds to values of the autoregressive coefficients which yield complex characteristic roots (i.e., $b^2 - 4c < 0$).

Provided the system is stable, the complete solution for Y_t is given by[6]

$$Y_t = \bar{Y}_t + k_1 r_1^t + k_2 r_2^t + \sum_{j=0}^{t-2} \zeta_j u_{t-1} \quad (t = 1, 2, \ldots) \tag{6}$$

where $\bar{Y}_t$ denotes the particular solution corresponding to the autonomous

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[4] The equilibrium level of output Y_t^e is defined as the expected value of income at time t. The variance of income at time t, $\sigma_y^2(t)$, is then given by $E[(Y_t - Y_t^e)^2]$ where E denotes the expectation operator. The system is said to be stable if $\lim_{t \to \infty} \sigma_y^2(t) < \infty$.

[5] These inequalities can be developed directly from the roots of the characteristic equation of (4). See, for example, Goldberg [6].

[6] See, for example, Bartlett [1]. By convention $\sum_{j=0}^{t-2} x_j = 0$ for $t \leq 1$.

expenditure A_t, r_1 and r_2 are the characteristic roots of (4), k_1 and k_2 are arbitrary constants determined by the initial conditions y_o and y_1 , and ζ_j is given by

$$\zeta_j = \frac{r_1^{j+1} - r_2^{j+1}}{r_1 - r_2} \quad . \tag{7}$$

If the characteristic roots of (4) are complex, then (6) may be written as[7]

$$Y_t = D^t[k_1 \cos\theta t + k_2 \sin\theta t] + \sum_{j=0}^{t-2} \zeta_j u_{t-j} \quad (t = 1, 2, \ldots) . \tag{6'}$$

In the non-stochastic formulation of this model, D, the damping factor, is usually taken to be a measure of the stability of the system; the smaller the value assumed by D, the more stable the system. The autoregressive period, θ, is taken as a measure of the periodicity of the system.

But for the stochastic case, the solution for the time path of output given in (6) indicates that the general solution is composed of three parts: the particular solution, a transient response, and a random response. In a stable system the transient response is zero in the limit so that asymptotically the deviations y_t from equilibrium take the form of an infinite weighted sum of random variables:

$$y_t = \sum_{j=0}^{\infty} \zeta_j u_{t-j} . \tag{8}$$

This suggests that in a stochastic system it may not be entirely appropriate to evaluate alternative stabilization policies on the basis of their effect on

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[7] This is described in some detail in Kendall [7].

the transient response alone. Indeed, it is shown below that in certain instances such a procedure will be quite misleading.

3.0 Evaluation of Alternative Stabilization Policies

As suggested above, simple stabilization policies can be viewed as effecting changes in the coefficients b and c of the autoregressive process described by (4). We now wish to compare the alternative time paths corresponding to different combinations of these coefficients. It is important to note that these alternative solutions may be dominated initially by the transient response of the process. Thus if short-term considerations dominate in the selection of alternative stabilization policies, the transient response may be extremely important. However, if stabilization policies once initiated are changed only relatively infrequently, the properties of the asymptotic stochastic solution (8) will be of interest.

3.1 Stochastic Properties of Stabilization Policies

In order to describe the asymptotic characteristics of the deviations from equilibrium it is necessary to determine the impact of changes in b and c on the coefficients ζ_j in (8) and hence on the infinite sum of random variables. This is facilitated by assuming that u_t is generated by a stationary stochastic process with power spectrum $f_u(\omega)$. Since we are dealing with stable systems with constant coefficients, it follows that (8) defines a stationary stochastic process with the power spectrum

$$f_y(\omega) = |T(\omega)|^2 f_u(\omega) \qquad (-\pi \leq \omega \leq \pi) \tag{9}$$

where $T(\omega)$, the transfer function of the autoregressive process, is given by

$$T(\omega) = (1 + a e^{-i\omega} + b e^{-i2\omega})^{-1} . \tag{10}$$

In what follows it is assumed that u_t is serially uncorrelated so that $f_u(\omega) = \frac{\sigma_u^2}{2\pi}$. We now propose to evaluate alternative stabilization policies on the basis of their effect on the power spectrum of the process.

More specifically, in the case of the second-order system it seems reasonable to consider the effect of changes in the autoregressive coefficients on the variance and "spectrum period" of the output. This can be accomplished quite easily graphically by constructing iso-variance and iso-frequency loci. The variance of the output is obtained by integrating the spectrum which in this case yields

$$\sigma_y^2 = \int_{-\pi}^{\pi} f_y(\omega) d\omega = \frac{(1 + c)}{(1 - c)\{(1 + b)^2 - c^2\}} \sigma_u^2 . \tag{11}$$

The iso-variance loci in Figure 2 were constructed by setting $\sigma_y^2/\sigma_u^2 = k$ (a constant) and graphing the resulting equation in b and c. In the first and second quadrants of Figure 2, curves intersecting the ordinate at higher values correspond to higher values of the ratio σ_y^2/σ_u^2. The iso-variance loci in the third and fourth quadrants of the figure are reflections about the abscissa of the first- and second-quadrant loci.

It is apparent from this figure that as $|b|$ increases, the variance ratio increases. This is an interesting result which would not have been anticipated on the basis of an analysis of the transient response of the system. For in the case of the transient, changes in $|b|$ have no affect on the stability of the system as it is usually defined (i.e., D, the damping factor, is not a function of b). Another interesting feature of Figure 2 is that for certain values of the parameters a reduction in the value of c (with b unchanged) <u>increases</u> the variance ratio. This means that it is possible that attempts to make the system more stable in the sense of reducing the time it takes for the transient response to damp out may actually increase the variance of the system!

The iso-frequency loci of Figure 3 are constructed in the following way. The frequency at which the spectrum exhibits a relative peak is obtained by solving

$$\frac{d f_y(\omega)}{d\omega} = 0 \qquad (12)$$

for $\hat{\omega}$, say. This yields

$$\hat{\omega} = \cos^{-1} \frac{-b(1+c)}{4c} \qquad (13)$$

for the second-order autoregressive process. The iso-frequency loci are then obtained by setting $\hat{\omega} = k$ (a constant) and graphing the resulting equation in b and c. The curves in the first quadrant (Q I) correspond to peak frequencies between 1/4 and 1/2 cycle per period (cpp) and those in the fourth quadrant (Q IV) correspond to frequencies between 0 and 1/4 cpp.

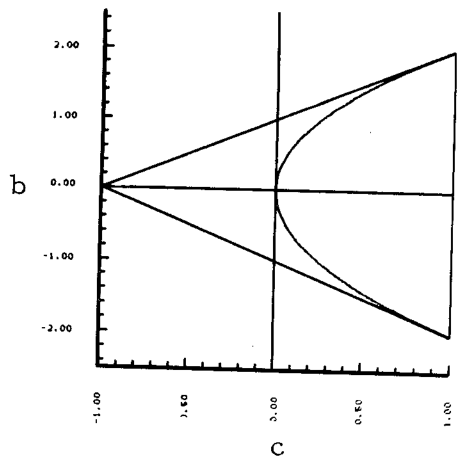

Figure 1. Stability Region for $y_t + by_{t-1} + cy_{t-2} = u_t$

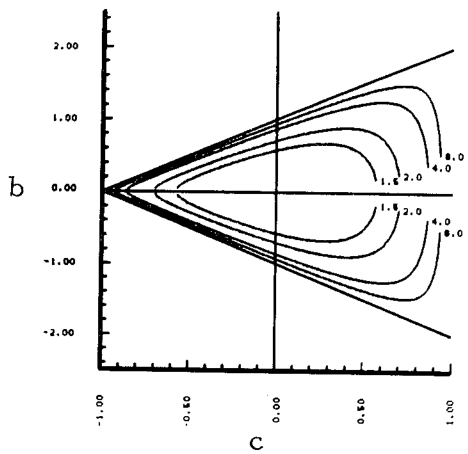

Figure 2. Iso-Variance Loci for $y_t + by_{t-1} + cy_{t-2} = u_t$

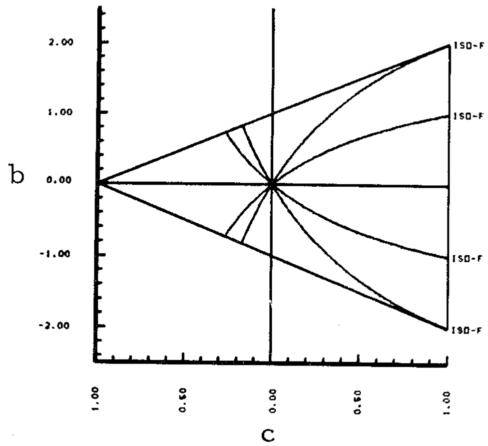

Figure 3. Iso-Frequency Loci for $y_t + by_{t-1} + cy_{t-2} = u_t$

It will be noted that these loci do not cover the entire region of stability in these two quadrants. The region below the 0 cpp locus in Q IV contains values of the autoregressive coefficients for which the power spectrum decreases monotonically with frequency and in the region above the 1/2 cpp locus in Q I the spectrum increases monotonically with frequency.

These loci in the first and fourth quadrants are quite similar to the iso-frequency loci constructed by Baumol [2] for the transient response of the second-order system. It is readily verifiable that as b increases (c held constant) the peak in the spectrum is shifted to a higher frequency and as c increases (b held constant) the peak is shifted to a higher or lower frequency depending on whether b is less than or greater than zero. These statements are valid provided the spectrum exhibits a relative peak on the open interval (0, 1/2) cpp.

The loci in the second and third quadrants of Figure 3 correspond to relative minima of the power spectrum.[8] For values of the autoregressive parameters in quadrant two the power spectrum exhibits a minimum on the interval (0, 1/4) cpp and in quadrant three the power spectrum exhibits a minimum on the interval (1/4, 1/2) cpp. In both of these quadrants the characteristic roots of the difference equation are real and less than one in absolute value.

This completes the characterization of the parameter space in terms of the shape of the power spectrum and the variance ratio. By combining

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[8] It is fairly straightforward to verify that $d^2 f_y(\omega)/d\omega^2 \lesseqgtr 0$ as $c \gtreqless 0$.

Figures 2 and 3 the stochastic properties of any particular stabilization policy can be described. Since the power spectrum is defined for any stationary stochastic process, the ideas described here for the second-order system apply equally well to higher-order linear systems.[9] It will be recalled, however, that these iso-frequency and iso-variance loci are relevant for a comparison of the asymptotic properties of alternative stabilization policies. The question of the speed with which the power spectrum converges to its asymptotic value will now be considered.

3.2 Speed of Convergence

Returning to Equation (6), it is seen that the variance of income after a change in the autoregressive parameters at time t = 0 is given by

$$\sigma_y^2(t) = E[(y_t - E[y_t])^2] = \sigma_u^2 \sum_{j=0}^{t-2} \zeta_j^2 \quad (t = 1, 2, \ldots) \qquad (14)$$

provided the disturbance terms are serially uncorrelated. The expression for the variance of y given in (11) is the limit of $\sigma_y^2(t)$ as t approaches infinity. Again, from Equation (6) the power spectrum of the system immediately following the policy change is given by

$$f_y(\omega, t) = \left| \sum_{j=0}^{t-2} \zeta_j e^{-i\omega j} \right|^2 \frac{\sigma_u^2}{2\pi} . \qquad (15)$$

[9] In higher-order systems, the power spectrum may, of course, have more than one relative peak.

The expression for $f_y(\omega)$ given in (9) is the limit of $f_y(\omega, t)$ as t approaches infinity. We now consider the speed with which $\sigma_y^2(t)$ and $f_y(\omega, t)$ converge to their asymptotic values.

Although it may be possible to construct a general argument, we proceed here by considering a specific example. Suppose that $a = -1.1$ and $b = 0.5$. The variance ratio (σ_y^2/σ_u^2) converges to 2.885 and from Figure 4 it is seen that the convergence is quite rapid. By the third period after the stabilization policy is instituted the variance is more than 95 percent of its asymptotic value. The power spectrum converges somewhat less rapidly in this case. The first five power spectra $f(\omega, t)$, $(t = 1, 5)$, are shown in Figure 5, the second five $f(\omega, t)$, $(t = 6, 10)$, are shown in Figure 6, and the asymptotic spectrum $f_y(\omega)$ is shown in Figure 7. Although the spectrum converges somewhat more slowly than the variance in this example, the general shape of the asymptotic spectrum emerges within four periods after the postulated policy change.

This example illustrates the point that the power spectrum may be a quite useful theoretical tool of analysis even in the first few periods immediately following a policy change. Moreover, the variance ratio is seen to converge quickly to its asymptotic value. This indicates that even for short-run policy considerations the stochastic terms in the solution for the time path of the output of the system are both important and amenable to (approximate) analysis. We can thus have some confidence that policies designed on the basis of the asymptotic properties of the system will have the desired effect fairly quickly after they have been initiated.

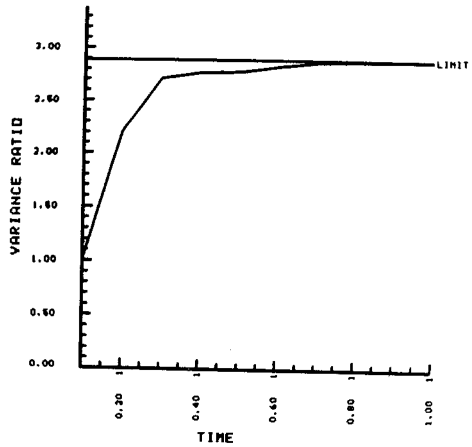

Figure 4.
Var $[y_t]$/ Var $[u_t]$; t = 1, 10

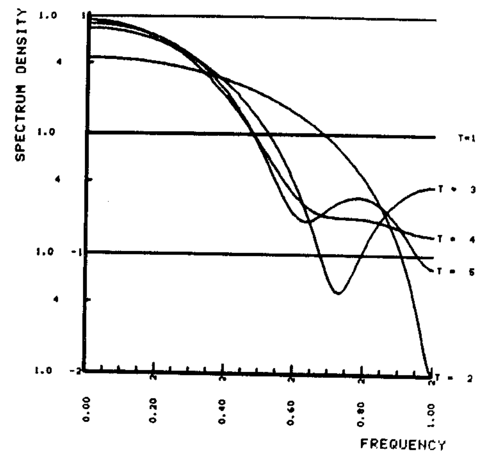

Figure 5.
$|T(\omega,t)|^2$; t = 1,5

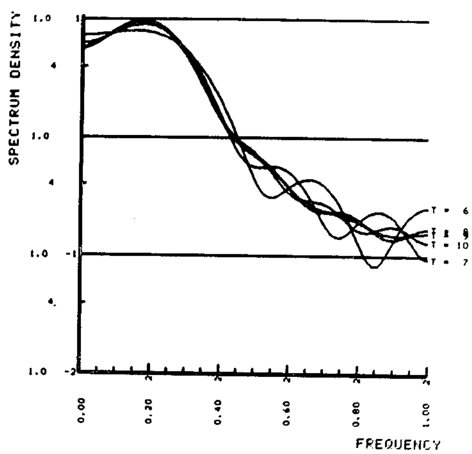

Figure 6.
$|T(\omega,t)|^2$; t = 6, 10

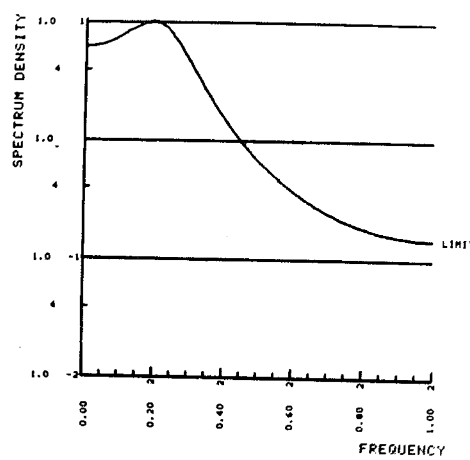

Figure 7.
$|T(\omega)|^2$

4.0 Minimum-Variance Stabilization Policies

Up to this point this paper has been concerned with the description of the stochastic properties of different linear stabilization schemes within the context of a simple model of income determination. Having gotten this far, it may be of some interest to consider briefly the question of the design of stabilization policy within the context of this model. In particular, suppose that the variance of the deviations from the equilibrium income path is to be minimized. Setting the partial derivatives of the expression for the disturbance variance

$$V(b, c) = \frac{1 + c}{(1 - c)\{(1 + c)^2 - b^2\}} \tag{16}$$

equal to zero yields $b = c = 0$ as the first-order conditions for (16) to assume a minimum. This indicates that in order to minimize the variance of the deviations from equilibrium, the coefficients of the linear stabilization scheme should be chosen so as to destroy the autoregressive structure of the system.

There are, however, two possibly undesirable features of this policy. First, the deviations of income from its equilibrium will be uncorrelated and, in fact, equal to the random shocks administered to the system.[10] The fact that no smoothing of these shocks occurs may not be particularly desirable. It may well be, for example, that it is easier to adjust to slowly

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[10] Under appropriate conditions, this result is true of higher-order autoregressive systems as well. For a discussion of this point in a somewhat different context the reader is referred to Bode and Shannon [3] and Box and Jenkins [4].

varying fluctuations than to completely random movements. This is, of course, the idea underlying Baumol's implicit suggestion [2] that low-frequency variations are less undesirable than high-frequency oscillations. This example indicates that there is a trade-off between the shape of the spectrum and the variance of the output. A policy which is oriented toward smoothing the disturbances is not a minimum-variance policy.

A second possibly undesirable feature of minimum-variance policies may be illustrated by assuming that autonomous expenditure is constant over time.[11] In this case the comparative static multiplier is $m = (1+b+c)^{-1}$ and the minimum-variance policy reduces the multiplier to unity. This indicates that there is a trade-off between the multiplier and the variance of the deviations from the equilibrium time path of income. This trade-off may be developed more explicitly by examining a "constrained minimum-variance policy" of the following variety: minimize $V(b, c)$ subject to $m = k$ (a constant). For this stabilization policy the parametric first-order conditions are

$$\begin{aligned} b &= \frac{1}{m} - \frac{1}{\sqrt{m}} \\ c &= \frac{1}{\sqrt{m}} - 1 \end{aligned} \tag{17}$$

and the minimum-variance ratio subject to the multiplier constraint is

$$V(m) = m\sqrt{m}(2\sqrt{m} - 1)^{-1}. \tag{18}$$

[11] It seems to be a plausible conjecture that the results similar to those described here can also be derived under the assumption that autonomous expenditure is growing at a constant relative rate per annum, ρ, in which case the comparative static multiplier is

$$m = (1 + b(1+\rho)^{-1} + c(1+\rho)^{-2})^{-1}.$$

The multiplier-variance frontier shown in Figure 8 exhibits the expected property that the variance ratio varies directly with the multiplier. It is also interesting to note that the autoregressive coefficients corresponding to the constrained minimum-variance policy lie in the third quadrant of the parameter space as shown in Figure 9. This means that the power spectrum of the constrained policy exhibits a relative minimum on the interval (1/4, 1/2) cycle per period. As the multiplier is increased, the relative minimum of the spectrum is shifted to a lower frequency as shown by the iso-frequency loci of Figure 3. This finding in some sense reinforces the concern of stabilization policy with the traditional three to four year business cycle for it is in precisely this range that, for plausible values of the multiplier, the spectrum should exhibit a minimum and not a maximum.

5.0 Summary

This paper has been concerned with the properties of stabilization policies in linear stochastic systems. Techniques which may be used to evaluate the stochastic response to changes in the parameters of the system were described. One of the conclusions which emerged from the analysis is that it is not in general valid to assume that the stochastic response and the transient response of a linear system have identical properties with respect to amplitude and periodicity. In fact, it was shown that policies which increase the stability of the system in the sense that they increase the rate

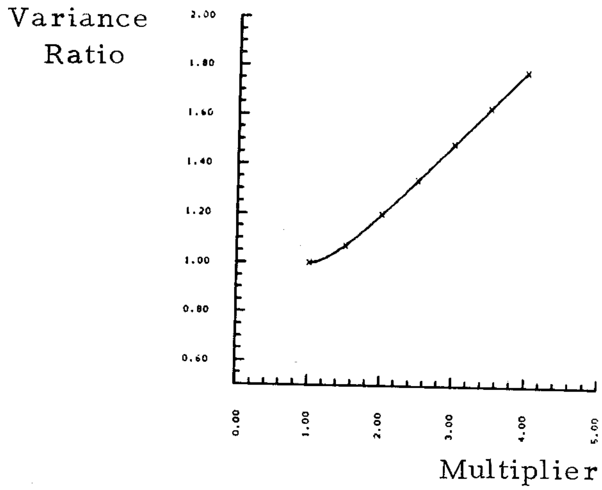

Figure 8. Multiplier-Variance Frontier

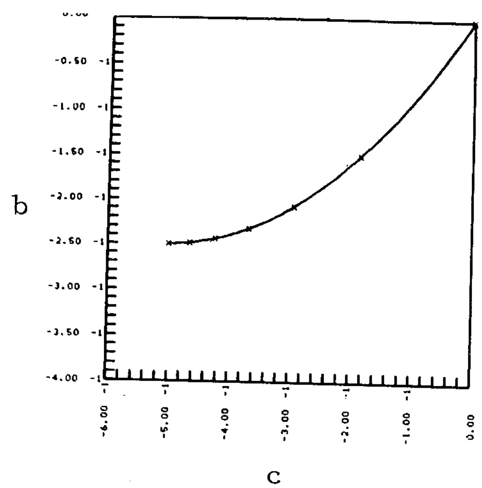

Figure 9. Parameters of the Constrained Minimum-Variance Policy

at which the transient response damps out may actually increase the variance of the time path of income.

In connection with the design of stabilization policies, it was found that there exist two trade-offs: one between the variance and the shape of the spectrum of the disturbances from the equilibrium path of output, and the other between the variance and comparative static multiplier of the system. The objective function on the basis of which policy is designed must therefore explicitly or implicitly include all three of these characteristics to provide meaningful results.

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