A GAME THEORETICAL APPROACH

TO SOME SITUATIONS IN OPINION MAKING

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Econometric Research Program
Research Memorandum No. 86
May 1967

The research described in this
paper was supported by ONR Contract No. Nonr. 1858(16).
NR 047-086

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ABSTRACT

A game theoretical approach of situations in opinion making is presented. The opinions of the members are supposed to be expressed under internal and external pressures which take into account the sincerety of the members and their authority influence.

Different cases are considered and conditions for the existence of equilibrium points are derived. Furthermore in the last section, by using the contraction principle in metric spaces, a sufficient condition for the existence of one equilibrium point in a general game is presented. Some examples are outlined.

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1. Introduction

In the last decade game theoretical, as well as applied economics has been concerned with an increasing number of topics, giving rise to a large number of different types of applications. In all of them the process starts with the observation of the behavior of an individual in trying to solve a problem. Provided that this problem can be represented by a theoretical model, the individuals then try to obtain a solution by a strictly mathematical approach. But in many cases the economic situation has a special relationship with the bahavior of the people or institutions considered. In some circumstances the economic problem is a direct consequence of the manner in which people act and this behavior may depend on other factors which have little or no connection with the economic aspects. Thus, in general the economic aspect of a problem is a reflection of the aggregate behavior of the people involved, and hence much of the economic theory should be devoted to group rather than individual making.

The study of group decision making forms a substantial proportion of modern work in the social sciences. This can be seen in both empirical and theoretical contributions. Many of the theoretical studies are connected with the theory of games, which, in a broad sense, may be seen as a part of group decision making. Generally, in a theoretical approach it is assumed that the behavior of individuals is established according to some principles which reflect some general intuitive ideas.

Independent of the mechanism and the rules within which the final decision is made, a group always has a period of discussions and deliberations whose duration is closely connected to the particular situation. Indeed, the decision reached by a group is a direct consequence of this period of discussion. In practice an analysis of this step is almost impossible. Nevertheless in spite of all of the difficulties involved we can say that in situations involving conflict the <u>opinions</u> of the members play an important role. Here, the term opinion is understood in a broad sense and without any reference to a specific definition. Consider for example, the committee appointed by a government to determine a new tax law which will be enacted by a simple majority of votes. The final decision, that is, the new tax rate, is reached by the voting system. All the previous debates corresponding to the period of discussion and the final vote, can be seen as the opinions of the corresponding members of the committee. The respective processes by which these opinions are arrived at can be very complicated and moreover, they can be unrealistic or even wrong from a purely technical point of view.

This note is devoted to the presenatation of a theoretical model based on game theory, concerning some conflict aspects arising in situations in which conflicting opinions are present.

In situations involving group decisions, the opinions of the members are expressed with more or less force in a continuous way during the discussion period. Furthermore, these opinions can be changed as a result of the arguments that arise during that period. Of course, we are not able to incorporate this important aspect in our examination, and therefore, for reasons of simplicity we assume that the opinions of all the members are given simultaneously without any reference to the previous deliberations.

From here on we consider the group of people to be denoted by $N = \{1,\ldots,n\}$ whose members will be called players. Thus, each player $i \in N$ expresses a possible opinion denoted by σ_i, chosen from the set of all his possible opinions which is specified by Σ_i.

Having this description, we are interested in describing some situation of conflict of a group, which arises from the susceptibility of each player expressing his opinion in comparison to the remaining players. First of all, it is natural to assume that one can divide all the effects that a player is submitted to into internal and external which are independent. Thus, the internal effect is considered only in relation to himself. At this point, we suppose that it can be measured as a function of the deviation from the adopted opinion and his intrinsic opinion. In other words, if for player $i \in N$, $p_i \in \Sigma_i$ indicates his own opinion on the matter under discussion, and if he adopts the opinion $\sigma_i \in \Sigma_i$, he will then be under the effect, expressed by the utility function P_i, of the deviation "$\sigma_i - p_i$". This may be considered the pressure of his own sincerity. At this point we do note that depending on the problem, such a function can be measured in monetary terms. For example, in a political-economical situation the amount of money that a player gets from an outside agent, under the deviation of his own criterium, can be expressed. Of course, depending on the situation, it may be expressed in terms of external pressure. This example suggests that the external effect indicates the influence that each player has on the group. In general, we can assume that ω_i^j is the influence coefficient of player $i \in N$ with respect to player $j \in N$, which can be interpreted as the weight of player's $j \in N$ opinion on player $i \in N$. In other words, it measures the authority influence of player $j \in N$ over player $i \in N$.

Having these coefficients, we now consider that a part of the external influence for a player depends on the deviation from his expressed opinion and the "weighted average" with respect to his own influence coefficients of all players expressed opinions. This effect can be observed in an intuitive point of view as the average pressure on this player. Let Q_i be the function of player $i \in N$ measuring this effect. Finally, we assume that there is another pressure of the whole group on each player which is due to the unequality of the weighted opinions. This effect can heuristically reflect the dispersive pressure on player $i \in N$, which will be described by the function R_i.

The principal objective here is to guarantee the existence of a joint behavior having an intuitive background of the group without cooperation connections between the players and under all the previous conditions.

2. Precise Formulation.

The previous commentary immediately formulates a game which involves the conflict situation of expressing the opinions under the hypothesis just considered. But in order to completely determine the game we must describe in a more accurate way the preceding comments. For this reason, we consider that the subject under discussion has many extremal points. Let $M = \{1,\ldots,m\}$ be the set of these. Then the set of all permitted opinions for each player is defined by the simplex

$$S_{m-1} = \{(s_1,\ldots,s_m) \in R^m: \sum_{i=1}^{m} s_i = 1 \text{ and } s_i \geq 0 \text{ for each } i \in M\}$$

where R^m indicates the cartesian product $\underset{i \in M}{X} R$ of the real line R. Then, the intrinsic opinion p_i of player $i \in N = \{1,\ldots,n\}$ is a well defined point

$$p_i = (p_{i1},\ldots,p_{im}) \in \Sigma_i = S_{m-1} .$$

Because the difference of opinions

$$\sigma_i - p_i = (\sigma_{i1} - p_{i1}, \ldots, \sigma_{im} - p_{im})$$

always belongs to the cartesian product $[-1,1]^m = \underset{i\in M}{X}[-1,1]$, then the internal pressure is determined by a real function

$$P_i = [-1,1]^m \rightarrow R$$

which is regarded in utility terms.

On the other hand, for the description of the external pressure it is natural to assume that the influence coefficients ω_i^j is a non-negative real number with the normalized property

$$\sum_{j\in N} \omega_i^j = 1 \qquad \text{for every} \quad i\in N .$$

Now, let Σ be the set of all the opinions of all of the players considered. This is given by the cartesian product

$$\Sigma = \underset{i\in N}{X} \Sigma_i = \underset{i\in N}{X} S_{m-1} .$$

Then, for an arbitrary established joint opinion

$$\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$$

the weighted average of player $i\in N$ with respect to his own influence coefficients is given by

$$E_i(\sigma) = \sum_{j\in N} \omega_i^j \sigma_j = (E_{i1}(\sigma), \ldots, E_{im}(\sigma)) = (\sum_{j\in N} \omega_i^j \sigma_{j1}, \ldots, \sum_{i\in N} \omega_j^i \sigma_{jm}) \in S_{m-1} .$$

This is a vector with the property that

$$0 \leq E_{ik}(\sigma) = \sum_{j\in N} \omega_i^j \sigma_{jk} \leq 1$$

for each component $k\in M$ and each joint opinion $\sigma\in\Sigma$, since

$$0 \leq \omega_i^j \leq 1 \quad \text{and} \quad 0 \leq \sigma_{jk} \leq 1 .$$

The deviation from the weighted average of the expressed opinion $\sigma_i \in \Sigma_i = S_{m-1}$ of player $i \in N$, is given by the difference

$$E_i(\sigma) - \sigma_i \in [-1,1]^m ,$$

and therefore the average pressure on player $i \in N$ is determined by the real function

$$Q_i : [-1,1]^m \to R .$$

Finally, the dispersion of opinions among the players can be measured by the weighted variance of the opinions considered as samples of a random variable, that is, by

$$D_i(\sigma) = (D_{i1}(\sigma),\dots,D_{im}(\sigma)) = \Big(\sum_{j\in N} \omega_i^j \sigma_{j1}^2 - (\sum_{j\in N} \omega_i^j \sigma_{j1})^2, \dots, \sum_{j\in N} \omega_i^j \sigma_{jm}^2 - (\sum_{j\in N} \omega_i^j \sigma_{jm})^2 \Big)$$

$$\in [0,1]^m$$

since the numerical value of the weighted variance always satisfies

$$0 \leq \sum_{j\in N} \omega_i^j \sigma_{jk}^2 - (\sum_{j\in N} \omega_i^j \sigma_{jk})^2 = \sum_{j\in N} \omega_i^j (\sigma_{jk} - \sum_{j\in N} \omega_i^j \sigma_{jk})^2 \leq 1$$

for every $\sigma \in \Sigma$ and every $k \in M$. The influence of this dispersion is defined by the real function

$$R_i : [0,1]^m \to R .$$

With all these elements, we can observe the conflict situation of opinionation as determined by an n-person game, where using the principle of superposition for utilities, the payoff function of a player is related to the <u>sum</u> of the partial pressures functions. Formally, we define the n-person <u>opinion game</u> by

$$\Gamma_{op} = \{\Sigma_1,\dots,\Sigma_n ; U_1,\dots,U_n\}$$

where the strategy set Σ_i of player $i \in N$ is S_{m-1} and his corresponding payoff $U_i : \Sigma \to R$, is defined by

$$U_i(\sigma_1,\dots,\sigma_n) = P_i(\sigma_i - p_i) + Q_i(E_i(\sigma)-\sigma_i) + R_i(D_i(\sigma))$$

for all $\sigma\epsilon\Sigma$. This can be regarded in terms of utility.

Having the form of payoff functions, it is very important that we express some properties of the functions P_i, Q_i and R_i. First of all, it is natural to think that if player $i\epsilon N$ deviates from his own opinion p_i then the contribution of function P_i to the whole utility decreases monotonically. Thus the function P_i reaches the maximum at the point $(0,\dots,0)\epsilon [-1,1]^m$. Analogously the functions Q_i and R_i reach their own maximums at the same point. Furthermore, they must be monotonically decreasing elsewhere, since by deviating from it their corresponding contribution of utility can be seen to decrease.

Now from all the possible ways that this decreasing can be described, we assume for simplicity the convexity of these function. The advantage here is that we will get a well defined concavity property on the utility functions which as it will be seen, is necessary from a mathematical point of view.

The central problem is to characterize a joint opinion based on intuitive considerations and to determine under what kind of conditions it exists.

For the situations under consideration, where coalitions are not permitted, it seems natural to assume that players will adopt a joint opinion such that if any player departs from it, his own utility will decrease, if all the remaining players abide by it. Such a concept corresponds to the notion due to Nash of an equilibrium point introduced in [1] for the opinion game Γ_{op} . Formally, we say that the joint strategy $\bar{\sigma} = (\bar{\sigma}_1,\dots,\bar{\sigma}_n) \epsilon \Sigma$ is an equilibrium opinion of Γ_{op} if

$$U_i(\bar{\sigma}_1,\dots,\bar{\sigma}_{i-1},\bar{\sigma}_i,\bar{\sigma}_{i+1},\dots,\bar{\sigma}_n) = \max_{s_i\epsilon\Sigma_i} U_i(\bar{\sigma}_1,\dots,\bar{\sigma}_{i-1},s_i,\bar{\sigma}_{i+1},\dots,\bar{\sigma}_n)$$

for all the players $i\epsilon N$.

Now, we are interested in answering the second question which involves the existence of such an equilibrium opinion. This will be obtained as a direct consequence of the following auxiliary well known theorem concerning the existence of such points for arbitrary games:

THEOREM 1: Let $\Gamma = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an arbitrary n-person game such that the strategy set Σ_i of player $i\epsilon N$ is a bounded, closed, convex set in a euclidean space and his payoff function A_i is continuous. If $A_i(\sigma_1,\dots,\sigma_n)$ is concave with respect $\sigma_i \epsilon \Sigma_i$ for fixed $\sigma_1 \epsilon \Sigma_1,\dots,$ $\sigma_{i-1} \in \Sigma_{i-1}, \sigma_{i+1} \in \Sigma_{i+1},\dots,\sigma_n \in \Sigma_n$, then Γ has at least one equilibrium point.

Before formulating the central existence theorem for equilibrium opinion points, the following very simple but useful results are recalled:

Proposition 2: If the functions $f,g: S \to R$ are concave, then the sum $f+g: S \to R$ is also concave.

PROOF: By the definition, for any s_1, s_2 belonging to the convex set S and any $\lambda \in [0,1]$, we have

$$f(\lambda s_1 + (1-\lambda)s_2) \geq \lambda f(s_1) + (1-\lambda) f(s_2)$$

and

$$g(\lambda s_1 + (1-\lambda) s_2) \geq \lambda g(s_1) + (1-\lambda) g(s_2) \quad .$$

Hence, by adding them we have;

$$\begin{aligned}(f+g)(\lambda s_1 + (1-\lambda)s_2) &= f(\lambda s_1 + (1-\lambda)s_2) + g(\lambda s_1 + (1-\lambda)s_2) \\ &\geq \lambda(f(s_1) + g(s_1)) + (1-\lambda)(f(s_2) + g(s_2)) = \\ &= \lambda(f+g)(s_1) + (1-\lambda)(f+g)(s_2) ,\end{aligned}$$

and therefore the concavity of the sum function $f+g$ is demonstrated. (Q.E.D.)

Of course, the previous result holds true for an arbitrary number of functions each having this property.

Given functions $g_k : T \to [a,b] \subseteq R$ defined on a convex set $T \subseteq R^n$, with $k = 1,\dots,m$, we define the vector function

$$g = (g_1,\dots,g_m) : T \to [a,b]^m \subseteq R^m$$

by

$$g(s) = (g_1(s),\dots,g_m(s)) \in [a,b]^m$$

for each $s \in T$.

<u>Proposition 3:</u> If the functions

$$f: [a,b]^m \subseteq R^m \to R \quad \text{and} \quad g: T \to [a,b] \subseteq R$$

with $k = 1,\dots,m$ are concave and $f(s_1) \geq f(s_2)$ whenever $s_{1k} \geq s_{2k}$ for every $k=1,\dots,m$; then, the composition function

$$fg = f(g_1,\dots,g_m) : T \to R$$

is concave.

<u>PROOF:</u> By the definition of concavity for the functions g_k, we have that for any pair of points t_1 and t_2 belonging to the convex set T, we have

$$g_k(\lambda t_1 + (1-\lambda)\, t_2) \geq \lambda\, g_k(t_1) + (1-\lambda)\, g_k(t_2)$$

for any $\lambda \in [0,1]$. From this by the last condition on the function f, for the point $\lambda t_1 + (1-\lambda)t_2 \in T$ we have,

$$\begin{aligned} f(g(\lambda t_1 + (1-\lambda)t_2)) &= f(g_1(\lambda t_1 + (1-\lambda)t_2),\dots,g_m(\lambda t_1 + (1-\lambda)\, t_2)) \\ &\geq f(\lambda g_1(t_1) + (1-\lambda)\, g_m(t_2),\dots,\lambda\, g_m(t_1)+(1-\lambda)g_m(t)) \\ &= f(\lambda\, g(t_1) + (1-\lambda)\, g\,(t_2)) \quad . \end{aligned}$$

Finally, by virtue of the concavity property of function f, we have for such points

$$f(\lambda g(t_1) + (1-\lambda)\ g\ (t_2)) \geq \lambda f\ (g(t_1)) + (1-\lambda)\ f(g(t_2))$$

which implies

$$f(g(\lambda t_1 + (1-\lambda)t_2)) \geq \lambda f(g\ (t_1)) + (1-\lambda)\ f\ (g\ (t_2)) \quad .$$

Thus, the concavity of the composition function is proven. (Q.E.D.)

A very close result regarding similar conditions is formulated as follows:

Proposition 4: If the functions $g_k: T \to [a,b] \subseteq R$ with $k=1,\ldots,m$ are convex and

$$f: [a,b]^m \subseteq R^m \to R$$

is concave such that

$$f(s_1) \geq f(s_2)$$

whenever $s_{1k} \leq s_{2k}$ for every $k=1,\ldots,m$. Then the composition function

$$fg = f(g_1,\ldots,g_m) : T \to R$$

is concave.

PROOF: Again by the definition of convexity for the functions g_k, we have

$$g_k(\lambda\ t_1 + (1-\lambda)\ t_2) \leq \lambda\ g_k(t_1) + (1-\lambda)\ g_k(t_2)$$

for every pair of points t_1, t_2 belonging to the convex set T, and for all $\lambda\epsilon[0,1]$. Then, by virtue of the additional condition on function f, at this point, we have,

$$\begin{aligned} f(g(\lambda t_1 + (1-\lambda)t_2)) &= f(g_1(\lambda t_1 + (1-\lambda)t_2),\ldots,g_m(\lambda t_1 + (1-\lambda)\ t_2)) \\ &\geq f(\lambda g_1(t_1) + (1-\lambda)\ g_1(t_2),\ldots,\lambda\ g_m(t_1) + (1-\lambda)\ g_m(t_2)) \\ &= f(\lambda\ g(t_1) + (1-\lambda)\ g\ (t_2))\ , \end{aligned}$$

and therefore from the concavity of f, it follows that

$$f(g(\lambda t_1 + (1-\lambda)t_2)) \geq \lambda f(g(t_1)) + (1-\lambda)\ f\ (g(t_2))$$

which implies the concavity of the composition function fg. (Q.E.D.)

Having these very simple auxiliary results, the existence of the equilibrium point for the opinion game will be obtained as a direct consequence of first theorem. This is formulated as follows.

THEOREM 5: If for every player $i\epsilon N$ the continuous functions P_i, Q_i and R_i are concave such that

$$R_i(s_1) \geq R_i(s_2)$$

whenever $s_{1k} \leq s_{2k}$ for every $k = 1,\ldots,m$, then, the opinion game Γ_{op} has at least one equilibrium opinion.

PROOF: Consider for an arbitrary player $i\epsilon N$ and a given joint opinion $(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) \epsilon \underset{j\epsilon N-\{i\}}{X} S_{m-1}$ the corresponding functions P_i, Q_i and R_i. Then, for the first function, for a pair of points σ_i and $\bar{\sigma}_i$ belonging to the strategy set $\Sigma_i = S_{m-1}$ of player $i\epsilon N$ and for any real number λ in the unit interval $[0,1]$ from concavity we have

$$P_i(\lambda\sigma_i + (1-\lambda)\bar{\sigma}_i - p_i) = P_i(\lambda(\sigma_i - p_i) + (1-\lambda)(\bar{\sigma}_i - p_i))$$

$$\geq \lambda P_i(\sigma_i - p_i) + (1-\lambda) P_i(\bar{\sigma}_i - p_i)$$

and therefore calling $\alpha_i(\sigma_i) = \sigma_i - p_i \epsilon[-1,1]^m$ the composition function

$$P_i\alpha_i : S_{m-1} \to R$$

is concave with respect to the variable $\sigma_i \epsilon \Sigma_i$. On the other hand for the second function,

$$Q_i(\sum_{j\epsilon N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)(\lambda\sigma_i + (1-\lambda)\bar{\sigma}_i)) =$$

$$= Q_i(\lambda(\sum_{j\epsilon N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)\sigma_i) + (1-\lambda)(\sum_{j\epsilon N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)\bar{\sigma}_i) \geq$$

$$\lambda Q_i(\sum_{j\epsilon N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)\sigma_i) + (1-\lambda) Q_i(\sum_{j\epsilon N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)\bar{\sigma}_i) .$$

Then, by calling

$$\beta_i(\sigma_i) = \sum_{j\in N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)\sigma_i \in [-1,1]^m \quad ,$$

it follows that the composition function

$$Q_i\beta_i : S_{m-1} \to R$$

is concave with respect to the variable $\sigma_i \in \Sigma_i$. Finally, we examine the remaining function. First of all, let us consider for $k: 1,\ldots,m$

$$D_{ik}(\sigma_1,\ldots,\sigma_{i-1}, \sigma_i, \sigma_{i+1},\ldots,\sigma_m) =$$

$$= [\sum_{j\in N-\{i\}} \omega_i^j \sigma_{jk}^2 - (\sum_{j\in N-\{i\}} \omega_i^j \sigma_{jk})^2] - 2\omega_i^i(\sum_{j\in N-\{i\}} \omega_i^j \sigma_{jk})\sigma_{ik} + \omega_i^i(1-\omega_i^i)\sigma_{ik}^2$$

which can be written as

$$\gamma_{ik}(\sigma_{ik}) = D_{ik}(\sigma_1,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\ldots,\sigma_m) = a_{ik} + b_{ik}\sigma_{ik} + c_{ik}\sigma_{ik}^2$$

if we put

$$a_{ik} = \sum_{j\in N-\{i\}} \omega_i^j \sigma_{jk} - (\sum_{j\in N-\{i\}} \omega_i^j \sigma_{jk})^2$$

and

$$b_{ik} = -2\omega_i^i (\sum_{j\in N-\{i\}} \omega_i^j \sigma_{jk})$$

$$c_{ik} = \omega_i^i (1-\omega_i^i) \quad .$$

The function $\gamma_{ik} : [0,1] \to [0,1]$ is convex because the constant and linear functions and the quadratic functions with the coefficient $c_{ik} \geq 0$ are convex, and γ_{ik} is the sum of these. Define for player $i\in N$ the function

$$\gamma_i: [0,1]^m \to [0,1]^m$$

by

$$\gamma_i(\sigma_i) = (\gamma_{i1}(\sigma_{i1}),\ldots,\gamma_{im}(\sigma_{im}))$$

for all the points of $[0,1]^m$. Thus, the restriction of the composition function

$$R_i\ \gamma_i|S_{m-1}: S_{m-1} \rightarrow R$$

is concave with respect to the variable $\sigma_i \epsilon \Sigma_i = S_{m-1}$. Indeed, by virtue of concavity and the additional property on function R_i, proposition 4 applied to the function R_i and $\gamma_i = (\gamma_{i1},\ldots,\gamma_{im})$ guarantees the assertion.

Now, because for fixed $(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots\sigma_n)$ the payoff function A_i as a function of the variable $\sigma i \epsilon \Sigma$ is given by the sum

$$A_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\ldots,\sigma_n) = P_i\alpha_i(\sigma_i) + Q_i\beta_i(\sigma_i) + R_i\gamma_i\ |S_{m-1}(\sigma_i) ,$$

it follows that it is concave, and therefore the last condition of theorem 1 for the opinion game is satisfied.

Remembering that the strategy sets $\Sigma_i = S_{m-1}$ are bounded, closed and convex and that the payoff function A_i is continuous with respect to the product variable, since it is the sum of compositions of continuous functions.

Thus, theorem 1 guarantees the existence of an equilibrium opinion for game Γ_{op}. (Q.E.D.)

At this point, we do point out that in the previous theorem the condition that the functions P_i, Q_i and R_i of player $i \epsilon N$ reach their respective maximum at the point $(0,\ldots,0) \epsilon R^m$ is not used, and therefore, it is not necessary for the existence of an equilibrium opinion. But, such a property should be considered in the model since it is related to the intuitive considerations mentioned.

Now, we are going to present a very simple example, to which we will apply the above result.

First of all, we consider that the number m of extreme opinions involved in the conflict is two. Thus, the strategy set of each player in the game is given by $S_1 \subseteq R^2$. Then, the functions of player $i \epsilon N$

$$P_i, Q_i : [-1,1] \times [-1,1] \to R$$

are given by the corresponding

$$P_i(s_1,s_2) = \alpha_i + \frac{\gamma_i}{k_i} s_1^{k_i} \quad \text{and} \quad Q_i(s_1,s_2) = \beta_i + \frac{\delta_i}{\bar{k}_i} s_1^{\bar{k}_i}$$

where $\alpha_i, \beta_i, \gamma_i$ and δ_i are real numbers, k_i and $\bar{k}_i$ integers, and the function $R_i \equiv 0$. These functions are independent of the second variable which does not determine any loss of generality. Indeed, this can be done in the general formulation, since one of the coordinates of S_{m-1} is completely determined by the knowledge of the others. Now, in order to establish the concavity of such functions, we must choose the numbers γ_i and δ_i non-positive and the integers k_i and $\bar{k}_i$ even. Since s^k with k even is a convex function and by the independence on the variable s_2 , both functions P_i and Q_i are concave. Therefore by theorem 1, the opinion game having these, has at least one equilibrium opinion. We point out that such functions satisfy the important condition of reaching their maximum at the point whose first coordinate is zero.

Finally, we remark that even though theorem 1 holds true with a weaker condition on the form of the payoff functions, namely, the quasi-concavity, this cannot be applied to the functions P_i, Q_i and R_i , since the sum of quasi-concave functions is not, in general quasi-concave.

3. Some applications to mixed extensions.

This section is devoted to some of the applications of the results formulated in the previous paragraph concerning existence of an equilibrium opinion. They are related to opinion situations with repetition, that is, where the players involved express their opinions, a very large number of times which can itself be varied. Thus, we consider the relative frequency of such opinions, and the expectations of the winnings of the players in the opinion conflict. Of course this consideration is quite vague since with the expressed concepts one can realize several different formulations, as we will show in the remaining part of this section. We will examine only three approaches involving different rules of the opinion conflict under consideration.

First of all, we consider only the case where the number of extreme opinions is finite, and the players are constrained to choose one element from them. An example of this is illustrated in voting where the extreme opinions are those classically used, namely: for, against and abstain. Of course, one can illustrate more complicated opinion situations where the number of the extreme opinions is larger than three.

Let $S = \{1,\ldots,m\}$ be the set of all those extreme opinions. Assuming that the pressures involved are the same kind as just mentioned, then, for player $i\epsilon N$ they are measured by the same functions P_i, Q_i and R_i. But, now they are defined differently according to the situation under consideration. It seems natural for the subsequent discussion to define the first of them, measuring the internal effect, by

$$P_i : \Sigma_i \times \Sigma_i = S \times S \rightarrow R .$$

The opinion of player $i\epsilon N$ is determined by the fixed second coordinate $p_i \epsilon \Sigma_i$. This function reaches the maximum on $\sigma_i = p_i$. The external pressure is now given by the superposition of partial effect expressed by the functions

$$Q_i : [-(m-1), m-1] \rightarrow R$$

whose argument is again

$$-(m-1) \leq E_i(\sigma) - \sigma_i \leq m-1$$

for each $\sigma = (\sigma_1,\ldots,\sigma_n) \epsilon \Sigma = \underset{j\epsilon N}{X} \Sigma_i = S^n$, and by the functions

$$R_i : [0, (m-1)^2] \rightarrow R$$

since its argument

$$D_i(\sigma) = \sum_{j\in N} \omega_i^j \sigma_j^2 - (\sum_{j\in N} \omega_i^j \sigma_j)^2 = \sum_{j\in N} \omega_i^j (\sigma_j - \sum_{j\in N} \omega_i^j \sigma_j)^2$$

for every joint opinion $\sigma = (\sigma_1,\dots,\sigma_n) \in \Sigma = S^n$ now satisfies:

$$0 \leq D_i(\sigma) \leq (m-1)^2 .$$

Again, by the intuitive considerations we assume that the maxima of these functions are reached at the point zero . Having these functions, then, the opinion game $\Gamma_{f,op} = \{\Sigma_1,\dots, \Sigma_n; U_1,\dots,U_n\}$ describing the conflict situation is given by the strategy sets $\Sigma_i = S$ and the payoff functions

$$U_i(\sigma_1,\dots,\sigma_n;p_i) = P_i(\sigma_i;p_i) + Q_i(E_i(\sigma)-\sigma_i) + R_i(D_i(\sigma))$$

where for simplicity $p_i\in\Sigma_i$ appears explicitly, it is a finite game.

Our attention is not concentrated on the opinion game $\Gamma_{f,op}$ which describes the opinion conflict under the assumption that it does occur only once, but in the results of many repetitions of this game $\Gamma_{f,op}$.

In this simpler examination, we suppose that the intrinsic opinion $p_i \in \Sigma_i$ of player $i\in N$ is kept constant, that is it does not change.

Since it is important to consider only the statistical behavior of the opinion, let $x_i(\sigma_i)$ be the probability that player $i\in N$ plays the opinion $\sigma_i\in\Sigma_i$, that is, the relative frequence of this opinion. Then, if the game is repeated a very large number of times, we can represent by $\tilde{\Sigma}_i = S^{m-1}$ the set of all the available probabilities of player $i\in N$, which will be called a <u>mixed strategy</u> differenciating from the strategy $\sigma_i\in\Sigma_i$ which will now be called <u>pure strategy</u>.

Assuming that the player $i\in N$ plays the mixed strategy $x_i\in\tilde{\Sigma}_i = S^{m-1}$, then his corresponding expectation of the utility payoff is given by

$$\tilde{U}_i(x_1,\dots,x_n;p_i) = \sum_{\sigma_1\in\Sigma_1} \dots \sum_{\sigma_n\in\Sigma_n} U_i(\sigma_1,\dots,\sigma_n)\, x_1(\sigma_1)\dots x_n(\sigma_n)$$

$$= \tilde{P}_i(x,p_i) + \tilde{Q}_i(x) + \tilde{R}_i(x) ,$$

where $x = (x_1,\dots,x_n)$ and where the expectations functions $\tilde{P}_i$ for fixed $p_i\in\Sigma_i$, $\tilde{Q}_i$ and $\tilde{R}_i$ are defined by

$$\tilde{P}_i(x,p_i) = \sum_{\sigma_i\in\Sigma_i} P_i(\sigma_i,P_i)\, x_i(\sigma_i)$$

$$\tilde{Q}_i(x) = \sum_{\sigma_1\in\Sigma_1} \dots \sum_{\sigma_n\in\Sigma_n} Q_i(E_i(\sigma)-\sigma_i)\, x_1(\sigma_1)\dots x_n(\sigma_n)$$

and finally

$$\tilde{R}_i(x) = \sum_{\sigma_1\in\Sigma_1} \dots \sum_{\sigma_n\in\Sigma_n} R_i(D_i(\sigma))\, x_1(\sigma_1)\dots x_n(\sigma_n) .$$

With these new functions, we can express the new opinion conflict by repetition of the original game in the <u>mixed extension</u> opinion game

$$\tilde{\Gamma}_{f,op} = \{\tilde{\Sigma}_1,\dots,\tilde{\Sigma}_n;\ \tilde{U}_1,\dots,\tilde{U}_n\} .$$

Now we consider the problem of an "equilibrium" in this new game. It will have a different intuitive meaning since the strategies now are the probabilities. If the circumstances do not permit cooperation between the players it seems natural to define a <u>statistical equilibrium opinion</u> as a joint strategy $\bar{x} = (\bar{x}_1,\dots,\bar{x}_n) \in \tilde{\Sigma}_1 x\dots x\, \tilde{\Sigma}_n$ such that:

$$\tilde{U}_i(\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},\dots,x_{n,pi}) = \max_{u_i\in\Sigma_i} \tilde{U}_i(\bar{x}_1,\dots\bar{x}_{i-1},u_i,\bar{x}_{i+1},\dots,\bar{x}_{n,pi})$$

for every player $i\in N$.

Once such a point is established, no player has any reason to change his own statistical behavior, assuming that those remaining abide by it.

Now we are interested in knowing if there exist some such point. In a direct way, applying theorem 1 to the mixed extension $\tilde{\Gamma}_{f,op}$ the existence is immediately established. Indeed, because the expectation function $\tilde{U}_i$ of player $i \epsilon N$ is a multivalued function with respect to $x_1,\ldots,x_n$, and the strategy set $\Sigma_i = S_{m-1}$ is bounded, closed and convex, then all the requirements are satisfied.

Now we can assume that the opinion $p_i \epsilon \Sigma_i$ does not remain fixed during the repetitions of the game. Thus one can incorporate a larger class of situations involving opinions where the players are permitted to change their own intrinsic opinions. Of course, the preceding approach where those were fixed then follow as a particular case of this second approach. Since we are concerned with the statistical measure of the utility of players, we describe it only in a statistical way by assigning to player $i \epsilon N$ a probability distribution y_i for the intrinsic opinion defined over the set $\Sigma_i = S = \{1,\ldots,m\}$ of extreme opinions. Now, the intern effect of player $i \epsilon N$ will be a function of two variables, namely: $\sigma_i \epsilon \Sigma_i$ and $p_i \epsilon \Sigma_i$, which is given by P_i. On the other hand, the external effect is not altered, since this new statistical consideration on the change of the intrinsic opinion does not affect their determination.

Thus, if the player $i \epsilon N$ has the probability distribution for the intrinsic opinion, given by y_i then if he chooses the opinion $\sigma_i \epsilon \Sigma_i$ in a play, it is natural that he will expect the amount of internal effect

$$\sum_{p_i \epsilon \Sigma_i} P_i(\sigma_i, p_i)\, y_i(P_i) \quad .$$

The new game expressing such a new conflict situation is therefore given by

$$\approx{\Gamma}_{f,op}(y) \;=\; \{\, \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;\ \approx{U}_1,\ldots,\approx{U}_i,\ldots,\approx{U}_i \}$$

where y indicates the joint distribution $(y_1,\ldots,y_n) \epsilon \tilde{\Sigma}_1 x \ldots x\, \tilde{\Sigma}_n$ and the payoff functions are defined by

$$\tilde{\tilde{U}}_i(x_1,\ldots,x_n,y_i) = \tilde{\tilde{P}}_i(x,y_i) + \tilde{Q}_i(x) + \tilde{R}_i(x)$$

where

$$\tilde{\tilde{P}}_i(x,y_i) = \sum_{p_i\in\Sigma_i} \sum_{\sigma_i\in\Sigma_i} P_i(\sigma_i,p_i)\, x_i(\sigma_i)\, y(p_i)$$

$$= \sum_{p_i\in\Sigma_i} \tilde{P}_i(x,p_i)\, y(p_i) \quad .$$

From the definition of the expectation of utility $\tilde{U}_i$ we have,

$$\tilde{\tilde{U}}_i(x_1,\ldots,x_n,y_i) = \sum_{p_i\in\Sigma_i} \tilde{U}_i(x_1,\ldots,x_n;p_i)\, y_i(p_i) \quad .$$

Again, by the form of the expectations functions, given the joint distribution y, the existence of a statistical equilibrium opinion for game $\tilde{\tilde{\Gamma}}_{f,op}(y)$, that is, a point $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_n) \in \tilde{\Sigma}_1\times\ldots\times\tilde{\Sigma}_n$ such that

$$\tilde{\tilde{U}}_i(\bar{x}_1,\ldots,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},\ldots,\bar{x}_n,y_i) = \max_{u_i\in\tilde{\tilde{\Sigma}}_i} \tilde{\tilde{U}}_i(\bar{x}_1,\ldots,\bar{x}_{i-1},u_i,\bar{x}_{i+1},\ldots,\bar{x}_n,y_i)$$

for every player $i\in N$, it is guaranteed.

We do note that in this approach for each player $i\in N$ the distributions x_i and y_i of the respective probability intrinsic opinion are considered independent, which is from an intuitive point of view, a very strong condition.

The form of the expectations of this second approach, suggests immediately that we consider the situation where the probability of the intrinsic opinion is also a strategy of the players. Thus, now in general, the complete mixed strategy of player $i\in N$ is determined by the variable z_i belonging to the strategy set $\widetilde{\Sigma_i \times \Sigma_i}$. Thus, for a (σ_i,p_i), the number $z_i(\sigma_i,p_i)$ represents the probability that the player $i\in N$ chooses the pure strategy $\sigma_i\in\Sigma_i$ and the intrinsic opinion

$p_i \epsilon \Sigma$. It is interesting to observe that, in some sense, this reflects the statistical <u>tendency</u> in which the player must change his own intrinsic opinion.

Such a conflict situation is now represented by the game

$$\tilde{\tilde{\Gamma}}_{f,op} = \{\widetilde{\Sigma_1 \times \Sigma_1}, \ldots, \widetilde{\Sigma_n \times \Sigma_n}; \; \tilde{\tilde{U}}_1, \ldots, \tilde{\tilde{U}}_n\}$$

where the mixed strategy set of player $i \epsilon N$ is given by the correlated distribution of the probability set $\widetilde{\Sigma_i \times \Sigma_i}$. This expectation is defined by

$$\tilde{\tilde{U}}_i(z_1, \ldots, z_n) = \tilde{\tilde{P}}_i(z) + \tilde{Q}_i(z) + \tilde{R}_i(z) \;,$$

with

$$\tilde{\tilde{P}}_i(z) = \sum_{(\sigma_i, p_i) \epsilon \Sigma_i \; \Sigma_i} P_i(\sigma_i, p_i) \; z_i(\sigma_i, p_i)$$

and

$$\tilde{Q}_i(z) = \tilde{Q}_i(x) \;, \quad \tilde{R}_i(z) = \tilde{R}_i(x)$$

where $x \epsilon \tilde{\Sigma}_1 \times \ldots \times \tilde{\Sigma}_n$ is the marginal distribution deduced from $z \epsilon \widetilde{\Sigma_1 \times \Sigma_1} \times \ldots \times \widetilde{\Sigma_n \times \Sigma_n}$ with

$$x_i(\sigma_i) = \sum_{p_i \epsilon \Sigma_i} z_i(\sigma_i, p_i) \;,$$

for each player $i \epsilon N$ and each pure strategy $\sigma_i \epsilon \Sigma_i$.

Again, from the multilinearity of the expectation function $\tilde{\tilde{U}}_i$ of player $i \epsilon N$, such a game has a <u>statistical equilibrium opinion</u> $\bar{z}$. Precisely, from theorem 1 it follows that there exists a point $\bar{z} \epsilon \widetilde{\Sigma_1 \times \Sigma_1} \times \ldots \times \widetilde{\Sigma_n \times \Sigma_n}$ such that

$$\tilde{\tilde{U}}_i(\bar{z}_1, \ldots, \bar{z}_{i-1}, \bar{z}_i, \bar{z}_{i+1}, \ldots, \bar{z}_n) = \max_{u_i \epsilon \widetilde{\Sigma_i \times \Sigma_i}} \tilde{\tilde{U}}_i(\bar{z}_1, \ldots, \bar{z}_{i-1}, u_i, \bar{z}_{i+1}, \ldots, \bar{z}_n)$$

for every player $i \epsilon N$.

It is interesting to point out that if here one substitutes for the strategy set $\widetilde{\Sigma_i \times \Sigma_i}$ the product set $\tilde{\Sigma}_i \times \tilde{\Sigma}_i$ which represents the joint independent distribution of probabilities, then the existence of a statistical equilibrium opinion is not assured. Indeed, in such a case the concavity with respect to the variable $(x_i, y_i) \in \tilde{\Sigma}_i \times \tilde{\Sigma}_i$ of function

$$\sum_{\sigma_i \in \Sigma_i} \sum_{p_i \in \Sigma_i} P_i(\sigma_i, p_i)\, x_i(\sigma_i)\, y_i(\sigma_i)$$

is not always guaranteed, and therefore all the requirements of theorem 1 are not satisfied by this game.

All the previous statistical approach has been concerned with the internal and external pressures regarding a simple play, that is determined on pure strategies. The statistical approach has been done only with respect to the repetition of the game. Actually, one can relate the statistical conflict situation considered by assuming the internal and external pressures are related not by a pure strategy but with respect to the mixed strategies. Thus, the effects are due to the statistical behavior of opinions, without any reference to the simple behavior in pure opinions.

This new statistical approach can be formulated by considering the fixed <u>statistical intrinsic opinion</u> $y_i \in \tilde{\Sigma}_i = S_{m-1}$ of player $i \in N$ and the respective internal effect determined by the continuous functions:

$$\bar{P}_i : [-1,1]^m \to R$$

whose argument is the difference $x_i - y_i$ between the statistical behavior x_i and y_i which satisfies

$$x_i - y_i \in [-1,1]^m \quad .$$

In similar fashion, the external statistical effect is measured on the one hand by the continuous function

$$\bar{Q}_i : [-1,1]^m \to R$$

with argument

$$E_i(x) - x_i \in [-1,1]^m$$

and on the other hand by the continuous

$$\bar{R}_i : [0,1]^m \to R$$

which will determine the pressure due to the dispersion

$$D_i(x) \in [0,1]^m .$$

Thus, the new conflict situation is precisely determined by the game

$$\bar{\Gamma}_{f,op} = \{\tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; \bar{U}_1,\ldots,\bar{U}_n\}$$

whose utility functions are given by

$$\bar{U}_i(x_1,\ldots,x_n) = \bar{P}_i(x_i-y_i) + \bar{Q}_i\ (E_i(x) - x_i) + \bar{R}_i\ (D_i(x)) .$$

Now from an intuitive point of view, the functions $\bar{P}_i$, $\bar{Q}_i$ and $\bar{R}_i$ should reach their respective maxima at the point $(0,\ldots,0) \in R^m$. Furthermore, if they are concave such that

$$R_i(z_1) \geq R_i(z_2)$$

whenever $z_{1k} \leq z_{2k}$ for every $k = 1,\ldots,m$ then, the theorem 5 guarantees the existence of a <u>statistical equilibrium opinion</u> $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_n)$ of game $\bar{\Gamma}_{f,op}$, which satisfies

$$\bar{U}_i(\bar{x}_1,\ldots,\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1},\ldots,\bar{x}_n) = \max_{u_i\in\tilde{\Sigma}_i} \bar{U}_i(\bar{x}_i,\ldots,\bar{x}_{i-1},u_i,\bar{x}_{i+1},\ldots,\bar{x}_n)$$

for every player $i \in N$.

4. Uniqueness of the equilibrium opinion.

The concept of equilibrium point for n-person games does not satisfy the equivalence property as the saddle point for zero-sum two-person does, that is, for different equilibrium points the winnings of a player can be different. This lack allows some criticism against the concept of equilibrium point. Of course, such arguments could have no importance if the game under consideration has only one equilibrium point. Thus, it is important to establish under which circumstances a game has only one equilibrium point.

In this final section, we examine this question and apply it to our discussion of opinion making.

Independent of the game represented by the conflict situation of opinion making, we will formulate an answer to the question of uniqueness of equilibrium points for a general game. This result will be obtained using the very useful contraction principle for mapping on metric spaces: Let Σ be an complete metric space and $T : \Sigma \to \Sigma$ a contraction mapping, then there exist a fixed point $\bar{\sigma}\in\Sigma$: $T(\bar{\sigma}) = \bar{\sigma}$. Furthermore, such a point is unique. We recall that given a metric space Σ with distance $d: \Sigma \times \Sigma \to R$, a mapping $T: \Sigma \to \Sigma$ is said to be a contraction if there is a real number $c < 1$ such that

$$d(T(\sigma),T(\tau)) \leq c\, d\,(\sigma,\tau)$$

for every pair $\sigma,\tau\in\Sigma$.

For simplicity in the subsequent discussion, we will always use the space R^m with the maximum distance, that is with

$$d(\sigma,\tau) = \max_{i\in\{1,\dots,m\}} |\sigma_i - \tau_i| .$$

Given the n-person game $\Gamma = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$, whose strategy set Σ_i for player $i\in N$ is bounded, closed and convex in a euclidean space, if his corresponding payoff function A_i is such that for each fixed $\sigma_1\in\Sigma_1,\ldots,\sigma_{i-1}\in\Sigma_{i-1},\sigma_{i+1}\in\Sigma_{i+1}$ $\sigma_n\in\Sigma_n$, the resulting function $A_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\ldots,\sigma_n)$ with respect to $\sigma_i\in\Sigma_i$ has only a point where the maximum is reached, then we define for such player $i\in N$ the function

$$M_i : \Sigma_1 \times\ldots\times \Sigma_{i-1} \times \Sigma_{i+1}\times\ldots\times \Sigma_n \rightarrow \Sigma_i$$

by

$$M_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) = (M_{i1}(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n),\ldots,$$
$$M_{im}(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n)) = \sigma_i \in R^m$$

where $\sigma_i\in\Sigma_i$ is that unique point of player $i\in N$ for which

$$A_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\ldots,\sigma_n) = \max_{s_i\in\Sigma_i} A_i(\sigma_1,\ldots,\sigma_{i-1},s_i,\sigma_{i+1},\ldots,\sigma_n) .$$

We will now formulate a general result which guarantees the uniqueness of the equilibrium point for a special kind of game.

<u>THEOREM 6:</u> Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an arbitrary n-person game such that the strategy set Σ_i of player $i\in N$ is a bounded, closed set in a euclidean space R^{mi} and his payoff function A_i is such that for each fixed $\sigma_i\in\Sigma_1,\ldots,\sigma_{i-1}\in\Sigma_{i-1},\sigma_{i+1}\in\Sigma_{i+1},\ldots,\sigma_n\in\Sigma_n$ its maximum is reached only at a point. If for each player $i\in N$ there is a real number $c_i < 1$ such that

$$\max_{k\in\{1,\ldots,m_i\}} |M_{ik}(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1}\ldots,\sigma_n) - M_{ik}(\tau_i,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n)|$$
$$< c_i \max_{j\in N-\{i\}} \max_{k=\{1,\ldots,m_j\}} |\sigma_{jk} - \tau_{jk}| ,$$

for every pair

$$(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) \text{ and } (\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n),$$

then the game Γ has a unique equilibrium point.

PROOF: Let us consider on product space $\Sigma = \underset{i\in N}{X} \Sigma_i$ with the maximum distance, the mapping $T : \Sigma \to \Sigma$ given by

$$T(\sigma) = (M_1(\sigma_2,\ldots,\sigma_n),\ldots,M_n(\sigma_1,\ldots,\sigma_{n-1})) \in \Sigma \subseteq R^{m_1}\times\ldots\times R^{m_n}$$

for every joint strategy $\sigma\in\Sigma$. Now, we will show that this is a contraction mapping. From the property on the payoff functions, we have

$$d(T(\sigma),T(\tau)) = \max_{i\in N} \max_{k\in\{1,\ldots,m_i\}} |M_{ik}(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) - M_{ik}(\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n)| <$$

$$\max_{i\in N} [c_i \max_{j\in N-\{i\}} \max_{k\in\{1,\ldots,m_j\}} |\sigma_{jk}-\tau_{jk}|]$$

$$\leq (\max_{j\in N} c_j) \max_{j\in N} \max_{k\in\{1,\ldots,m_j\}} |\sigma_{jk} - \tau_{jk}| = c\ d(\sigma,\tau)$$

where $c = \max_{j\in N} c_j < 1$, for every pair $\sigma,\tau\in\Sigma$, and therefore, T is a contraction mapping. Then, the contraction principle applied to T since the space Σ is complete, guarantees the existence of a unique fixed point $\bar\sigma\in\Sigma$: $T(\bar\sigma) = \bar\sigma$. At such a joint strategy we obtain

$$\bar\sigma_i = M_i(\bar\sigma_1,\ldots,\bar\sigma_{i-1},\bar\sigma_{i+1},\ldots,\bar\sigma_n)$$

for every player $i\in N$. But remembering the definition of functions M_i, it follows that

$$A_i(\bar\sigma_1,\ldots,\bar\sigma_{i-1},\bar\sigma_i,\bar\sigma_{i+1},\ldots,\bar\sigma_n) = \max_{s_i\in\Sigma_i} A_i(\bar\sigma_1,\ldots,\bar\sigma_{i-1},s_i,\bar\sigma_{i+1},\ldots,\bar\sigma_n)$$

for each $i \in N$, which coincides with the equilibrium point definition. The uniqueness of this is an immediate consequence of the uniqueness of the fixed point of the mapping T . (Q.E.D.)

We note that in the previous result the convexity of the strategy sets and the continuity of payoffs functions are not necessary.

We are now interested in applying this theorem to the opinion game in order to obtain the uniqueness of the equilibrium opinion. We only consider the general game introduced in the second section, for which we immediately have the following restricted result.

THEOREM 7: Given that for every player $i \in N$ the continuous functions P_i, Q_i and R_i are strictly concave such that

$$R_i(s_1) \geq R_i(s_2)$$

whenever $s_{ik} \leq s_{2k}$ for every $k = 1,\ldots,m$. If for each player $i \in N$ there is a real number $c_i < 1$ such that*

$$\max_{k \in \{1,\ldots,m\}} |M_{ik}(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) - M_{ik}(\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n)|$$

$$< c_i \max_{j \in N-\{i\}} \max_{k=\{1,\ldots,m\}} |\sigma_{jk} - \tau_{jk}|$$

for every pair

$$(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) \text{ and } (\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n) ,$$

then, the opinion game Γ_{op} has a unique equilibrium opinion.

PROOF: First of all we recall that the maximum of an strictly concave function is reached only at one point. Indeed, suppose that it occurrs on two points, then on the convex combination segment the function would be greater than the the maximum value, which is absurd.

*The function M_i is taken with respect to the utility function $U_i = P_i + Q_i + R_i$.

On the other hand, we point out that all the propositions 2, 3, and 4 remain valid under the substitution of strictly concave functions for concave. Thus, it follows that for each $\sigma_1 \epsilon \Sigma_1, \ldots, \sigma_{i-1} \epsilon \Sigma_{i-1}$ $\sigma_{i+1} \epsilon \Sigma_{i+1}, \ldots, \sigma_n \epsilon \Sigma_n$ the utility function

$$U_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n) = P_i(\sigma_i - p_i) + Q_i\ (E_i(\sigma) - \sigma_i) + R_i(D_i(\sigma))$$

is a strictly concave function with respect to $\sigma_i \epsilon \Sigma_i$, and therefore its maximum reached only at one point.

Then, by the last condition, all the requirements of theorem 6 applied to opinion game Γ_{op} are satisfied, which guarantees the validity of the assertion. (Q.E.D.)

Having these results, we will now illustrate an example of a game with only one equilibrium which can represent an opinion conflict and where certain modificated characteristics are introduced. Because theorem 7 cannot be applied directly to it we will obtain the uniqueness of the equilibrium opinion by using theorem 6.

As what has been considered in the example after theorem 5, we will have situations involving only two extreme opinions. Thus, the strategy set of each player is given by $S_1 \subseteq R^2$. Actually, a strategy of a player will be completely determined by only one coordinate because the remaining one must be equal to one minus the value of the first. One could then represent in a modified way the strategy set of player $i \epsilon N$ by the set $\Sigma_i = [0,1] \subseteq R$. Thus, by this modification, the new mathematical formulation will have a simpler structure. Here, now we must introduce the modified functions which are

$$P_i^m : [-1,1] \rightarrow R$$

whose argument is the difference $\sigma_i - p_i \in -1,1$, which contributes the internal effect,

$$Q_i^m : [-1,1] \rightarrow R$$

with argument. The deviation $E_i(\sigma) - \sigma_i \in [-1,1]$ measures the external pressure and the function R_i^m is considered to be identically zero.

Formally, the conflict situation is given by game modified

$$\Gamma_{op}^m = \{\Sigma_1,\ldots,\Sigma_n;\ U_1,\ldots,U_n\}$$

where the strategy set of player $i\epsilon N$ is $\Sigma_i = [0,1]$ and the payoff function U_i is given by

$$U_i(\sigma_1,\ldots,\sigma_n) = P_i^m(\sigma_i - p_i) + Q_i^m\ (E_i(\sigma) - \sigma_i)\ .$$

Now, we will consider the functions to be defined by

$$P_i(s) = \alpha_i + \frac{\gamma_i}{k_i} s^{k_i}\ , \qquad Q_i(s) = \beta_i + \frac{\delta_i}{\bar{k}_i} s^{\bar{k}_i}\ ,$$

with k_i and $\bar{k}_i$ even and greater than zero. If the non-positive number γ_i and δ_i are not zero, then both functions are strictly concave. The maximum values are established at the point $0 \in [-1,1]$. Furthermore, we consider $\omega_i^i < 1$ for every player $i\epsilon N$.

Given a joint opinion $\sigma_1 \epsilon \Sigma_1,\ldots,\sigma_{i-1} \in \Sigma_{i-1},\ \sigma_{i+1} \epsilon \Sigma_{i+1},\ldots,\sigma_n \epsilon \Sigma_n$, the function $U_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\ldots,\sigma_n) =$

$$= (\alpha_i + \beta_i) + \frac{\gamma_i}{k_i} (\sigma_i - p_i)^{k_i} + \frac{\delta_i}{\bar{k}_i} \Big(\sum_{j\epsilon N-\{i\}} \omega_i^j\ \sigma_j - (1-\omega_i^i)\ \sigma_i\Big)^{\bar{k}_i}$$

is strictly concave with respect to the opinion $\sigma_i \epsilon \Sigma_i = [0,1]$ of player $i\epsilon N$, since it is the sum of strictly concave functions, and therefore its maximum is reached only at one point. Obviously it is a convex combination of the points where the second and last functions reach their respective maximums. Analytically such a point $M_i(\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n)$ is given by

$$M_i(\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n) = \lambda \; p_i + (1-\lambda)\;\frac{\sum\limits_{j\in N-\{i\}} \omega_i^j\sigma_j}{1-\omega_i^i} \in [0,1] \;.$$

Now we are going to determine the value of λ. This is directly obtained from the equation $\frac{\partial U_i}{\partial \sigma_i} = 0$ since the involved functions are differentiable. We have

$$(1-\lambda)^{k_i-1}\,\gamma_i\left[\frac{\sum\limits_{j\in N-\{i\}} \omega_i^j\,\sigma_j}{1-\omega_i^i} - p_i\right]^{k_i-1} - \lambda^{\bar{k}_i-1}\,\delta_i\,(1-\omega_i^i)\;.$$

$$\left[\sum_{j\in N-\{i\}} \omega_i^j\,\sigma_j - (1-\omega_i^i)\,p_i\right]^{\bar{k}_i-1} = 0$$

or

$$\left(\lambda^{-k_i-1/\bar{k}_i-1} - \lambda^{\bar{k}_i-k_i/\bar{k}_i-1}\right)^{k_i-1} = \left\{\frac{\gamma_i}{\delta_i}(1-\omega_i^i)^{k_i}\;\frac{\left[\sum\limits_{j\in N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)p_i\right]^{\bar{k}_i-1}}{\left[\sum\limits_{j\in N-\{i\}} \omega_i^j\sigma_j - (1-\omega_i^i)p_i\right]^{k_i-1}}\right\}\;.$$

Now in the simple case where $\bar{k}_i = k_i$, it reduces to

$$\lambda = \frac{1}{\left(\frac{\delta_i}{\gamma_i}\right)^{1/k_i-1}(1-\omega_i^i)^{k_i/k_i-1} + 1}\;,$$

and therefore

$$M_i(\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n) = \frac{p_i + \left(\frac{\delta_i}{\gamma_i}\right)^{1/k_i-1}(1-\omega_i^i)^{1/k_i-1}\sum\limits_{j\in N-\{i\}} \omega_i^j\sigma_j}{\left(\frac{\delta_i}{\gamma_i}\right)^{1/k_i-1}(1-\omega_i^i)^{k_i/k_i-1} + 1}\;.$$

Hence for two arbitrary points, it follows immediately that

$$M_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) - M_i(\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n) =$$

$$= \frac{\left(\frac{\delta_i}{\gamma_i}\right)^{1/k_i-1} (1-\omega_i^i)^{1/k_i-1} \sum_{j\in N-\{i\}} \omega_i^j (\sigma_j - \tau_j)}{\left(\frac{\delta_i}{\gamma_i}\right)^{1/k_i-1} (1-\omega_i^i)^{k_i/k_i-1} + 1},$$

which implies the existence of $c_i < 1$ such that

$$|M_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) - M_i(\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n)$$

$$< c_i \max_{j\in N-\{i\}} |\sigma_j - \tau_j|$$

for every pair

$$(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) \quad \text{and} \quad (\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n)$$

whenever

$$\frac{\delta_i}{\gamma_i} < \frac{1}{(1-\omega_i^i)} .$$

Thus under these conditions, such a game satisfies all the requirements of theorem 6 and therefore it has only one equilibrium opinion.

Bibliography

[1] Nash, John: Equilibrium Points in N-person Games. Proc. Nat. Accd. Sci. U.S.A. 36 (1950), 48-49.

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Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Princeton University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP
3. REPORT TITLE A GAME THEORETICAL APPROACH TO SOME SITUATIONS IN OPINION MAKING	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Research Memorandum No. 86, May 1967	
5. AUTHOR(S) (Last name, first name, initial) Ezio Marchi	
6. REPORT DATE May 1967	7a. TOTAL NO. OF PAGES 30 7b. NO. OF REFS 1
8a. CONTRACT OR GRANT NO. Nonr 1858(16) b. PROJECT NO. c. Task No. NR 047 086 d.	9a. ORIGINATOR'S REPORT NUMBER(S) Research Memorandum No. 86 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
10. AVAILABILITY/LIMITATION NOTICES Distribution of this document is unliminted.	
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Branch Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

A game theoretical approach of situations in opinion making is presented. The opinions of the members are supposed to be expressed under internal and external pressures which take into account the sincerety of the members and their authority influence.

Different cases are considered and conditions for the existence of equilibrium points are derived. Furthermore in the last section, by using the contraction principle in metric spaces, a sufficient condition for the existence of one equilibrium point in a general game is presented. Some examples are outlined.

DD FORM 1 JAN 64 1473

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Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
game theory						
economics						
opinion making						
decision making						
groups						
intrinsic opinion						
authority influence						
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