DYNAMIC PROPERTIES OF STOCHASTIC LINEAR ECONOMETRIC MODELS

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ABSTRACT

It is well known that the solutions of systems of stochastic difference equations that are obtained when the errors are neglected may show widely different patterns from the solutions that take these errors into account. Moreover, the results of a number of Monte Carlo simulation experiments indicate that the stochastic solutions often exhibit realistic time paths of the variables whereas the deterministic solutions generate time paths which are markedly disimilar from the observed paths of the variables. Yet, despite the presumed importance of the disturbance terms, very little effort has been devoted to the analysis of systems of stochastic difference equations that arise in economics. The purpose of this paper is to sketch an analytical procedure that can be used to characterize the stochastic properties of systems of equations and to apply the technique to a simple multiplier-accelerator model of the United States economy.

The analytical methods which are used are based on the spectral representation of a stochastic process. Attention is focussed on the spectra and cross-spectra which are implied by the system of equations. Since the procedure is analytical, it avoids the difficulties associated with the analysis of simulation experiments. In particular the results are exact and are not subject to the sampling variability inherent in simulation experiments.

A simple multiplier-accelerator model is introduced to illustrate this method of analyzing stochastic systems. The parameters of the model are

unnecessary for it is then possible to obtain the theoretical power spectra directly from the system of equations. The purpose of this paper is to outline this analytical approach to the analysis of stochastic linear systems and to illustrate its use in connection with a relatively simple dynamic model.

The paper is structured in the following way. In the next section the final form of a linear econometric model is described using operator notation. In Section 3 the usual methods of determining the dynamic properties of an econometric model are described with reference to the general solution of the system of equations. This provides the necessary background for the discussion in Section 4 of the stochastic properties of linear models. The paper concludes with an application of the technique to a simple multiplier-accelerator model. The computational details and interpretation of the results are described within the context of this model.

2. The Final Form of Linear Econometric Models

Before discussing the dynamic properties of stochastic econometric models, it will be useful to establish some notation and the framework within which the analysis will be conducted. Consider the standard linear econometric model

$$(2.1)\quad By_t + \Gamma z_t = u_t$$

where B and Γ are $g \times g$ and $g \times k$ matrices of coefficients, y is a g-dimensional vector of endogenous variables, z is a k-dimensional vector of predetermined variables, and u is a g-dimensional disturbance vector. Throughout the paper it is assumed that the coefficient matrices B and Γ are known so that the only source of uncertainty arises in connection with the residuals generated by the disturbance process. For expository purposes it is also convenient to assume that all identities have been eliminated from the model. As shown in the Appendix this is by no means necessary.

If the model is truly dynamic the vector of predetermined variables will include lagged values of the endogenous variables. Let the vector of predetermined variables z_t be partitioned such that z_{1t} includes only the lagged endogenous variables and z_{2t} includes only the exogenous variables. Provided Γ is partitioned correspondingly, (2.1) may be rewritten as

$$(2.2)\quad By_t + \Gamma_1 z_{1t} + \Gamma_2 z_{2t} = u_t .$$

If r denotes the maximum lag of any endogenous variable in z_{1t}, the lagged endogenous terms may be written explicitly as

$$(2.3)\quad \Gamma_1 z_{1t} = A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_r y_{t-r}$$

where the $A_j (j=1,r)$ are $g \times g$ matrices. Introducing the lag operator L the system may be expressed as

$$(2.4)\quad A(L)\, y_t = \Gamma_2 z_{2t} + u_t$$

where

$$(2.5)\quad A(L) = B + A_1 L + A_2 L^2 + \ldots + A_r L^r .$$

The final form of the system may be obtained as follows. Let the $g \times g$ λ-matrix $a(\lambda)$ denote the adjoint of $A(\lambda)$ and let $\Delta(\lambda) = |A(\lambda)|$ denote the determinantal polynomial of $A(\lambda)$. Premultiplying (2.4) by $a(L)$ yields the final form of the econometric model:

$$(2.6)\quad ||\Delta(L)||\; y_t = -a(L)\; {}_2\, z_{2t} + a(L)\, u_t$$

where $||\Delta(L)||$ is a matrix with $\Delta(L)$ on the main diagonal and zeros everywhere else. The final form is thus a system of <u>stochastic</u> difference equations, each terms. The usual method of determining the dynamic properties of the solution is to suppress the stochastic part of the solution and to analyze only the deterministic solution. This is equivalent to looking at the expected value of the time path of the endogenous variables of the system given the exogenous variables.

Two kinds of information are obtained from the deterministic system. The values of the roots of the determinantal equation yield information about the modulus and periodicity of the transient response. If the roots are all less than unity in absolute value, the system is stable and approaches the particular solution from any set of initial conditions. If complex roots occur this is usually taken as an indication that the system will tend to oscillate. The periodicity and rate of damping of the sinusoidal components contributed by complex roots can be ascertained from these roots. Dynamic multipliers may be calculated to determine the response of the endogenous variables to changes in the exogenous variables.

There is little doubt that these methods provide interesting and useful information about the system of equations. For short-term forecasting and the formulation of discretionary stabilization policy, these techniques may provide a sufficient characterization of the dynamic properties of the model. If, however, the longer-term properties of the model are to be investigated, it may not be reasonable to disregard the impact of the disturbance terms on the time paths of the endogenous variables.[5] Neither of the above techniques provides information about the magnitude or correlation properties of deviations from the

[5]As shown in [5], disregarding the disturbance terms may be quite misleading. For example, stabilization policies designed to increase the stability of the system by reducing the modulus of the roots may in fact increase the variance of the system.

$$(4.2) \qquad u_{jt} = \int_{-\pi}^{\pi} e^{i\omega t} U_j(\omega)\, d\omega \qquad (j=1,2,\ldots,g)$$

since, by definition, u_{js} and $U_j(\omega)$ are a Fourier-transform pair.[8] This last expression, referred to as the spectral representation of the process, can be written in matrix form as

$$(4.3) \qquad u_t = \int_{-\pi}^{\pi} e^{i\omega t} U(\omega)\, d\omega$$

where $U(\omega)$, which is referred to as the kernel of the process, is the g-dimensional vector of stochastic functions defined by (4.1).

Similar relationships hold for the covariance functions of the disturbance process. Let $\gamma_{jk}(s)$ denote the covariance of u_{jt} and u_{kt-s} and consider the functions

$$(4.4) \qquad f_{jk}(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} e^{-i\omega s}\, \gamma_{jk}(s) \qquad (j,k=1,2,\ldots,g)$$

defined on $-\pi \leq \omega \leq \pi$. It again follows that

$$(4.5) \qquad \gamma_{jk}(s) = \int_{-\pi}^{\pi} e^{i\omega s} f_{jk}(\omega)\, d\omega \qquad (j,k=1,2,\ldots,g)$$

which is referred to as the spectral representation of the covariance functions. The kernel of this last transformation, $f_{jk}(\omega)$, is the cross-spectrum relating the two disturbance processes u_{jt} and u_{kt} .

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[8] That (4.1) implies (4.2) may be seen by substituting for $U_j(\omega)$ from (4.1) into (4.2) and noting that

$$\int_{-\pi}^{\pi} e^{i\omega(t-s)}\, d\omega = \begin{cases} 2\pi & t = s \\ 0 & t \neq s \end{cases} .$$

Therefore $e^{i\omega t} U_j(\omega)$ can be thought of as a generating function of u_{jt} where integration replaces the more usual differentiation procedure.

implied by the model will now be derived. Returning to the complete solution of the linear econometric model given by (3.2), the particular solution corresponding to the disturbance terms can now be written as

$$(4.10) \qquad y_t = \frac{a(L)}{\Delta(L)} \int_{-\pi}^{\pi} e^{i\omega t}\, U(\omega)\, d\omega$$

where u_t has now been replaced by its spectral representation. Interchanging the order of the operations in this expression leads directly to

$$(4.11) \qquad y_t = \int_{-\pi}^{\pi} e^{i\omega t}\, T(\omega)\, U(\omega)\, d\omega$$

where $e^{i\omega t}\, T(\omega) = e^{i\omega t} a(\omega)/\Delta(\omega)$ is the $g \times g$ transfer matrix obtained by operating on $e^{i\omega t}$ by $a(L)/\Delta(L)$.[9] This interchange of operations involved in going from (4.10) to (4.11) is permissible provided that each of the elements in the matrix $a(L)/\Delta(L)\, e^{i\omega t}$ converges absolutely. This will be true if the roots of the determinantal equation $x^n \Delta(\frac{1}{x}) = 0$ are of modulus less than one so that the system is stable. Provided this is the case, this last expression indicates that the kernel of the y_t process is $Y(\omega) = T(\omega)\, U(\omega)$.

The spectral matrix $F(\omega) = [F_{ij}(\omega)]$ of the endogenous variables is now obtained using (4.9) with the appropriate substitutions:

$$(4.12) \quad F(\omega) = E[Y(\omega)\, Y^*(\omega)] = E[T(\omega)\, U(\omega)\, U^*(\omega)\, T^*(\omega)] = T(\omega)\, f(\omega)\, T^*(\omega) .$$

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[9] Specifically, each element of the $g \times g$ matrix $a(L)/\Delta(L)$ is a rational function in L, a typical element being

$$\frac{a_0 + a_1 L + \ldots + a_p L^p}{b_0 + b_1 L + \ldots + b_n L^n} ,$$

for example. Operating on $e^{i\omega t}$ with this rational function yields

$$e^{i\omega t} \left[\frac{a_0 + a_1 e^{-i\omega} + \ldots + a_p e^{-ip\omega}}{b_0 + b_1 e^{-i\omega} + \ldots + b_n e^{-in\omega}}\right]$$

provided the power series expansion in $z = e^{-i\omega}$ of the denominator converges absolutely. As indicated in the text, this is true if the system is stable.

A more general assumption about the disturbances is that they are generated by the process

$$(4.15) \qquad u_t = H_0 v_t + H_1 v_{t-1} + \ldots + H_s v_{t-s}$$

where v_t is a g-dimensional vector of serially independent random variables with contemporaneous variance-covariance matrix Σ'. In this case the spectral matrix of the residuals is

$$(4.16) \quad E[U(\omega)\ U^*(\omega)] = E[H(\omega)\ V(\omega)\ V^*(\omega)\ H^*(\omega)] = \frac{1}{2\pi} H(\omega)\ \Sigma'\ H^*(\omega)$$

where $H(\omega) = H_0 + H_1 e^{-i\omega} + \ldots + H_s e^{-i\omega s}$. The spectral matrix of the endogenous variables under this assumption is obtained by replacing $\Sigma/2\pi$ by $H(\omega)\ \Sigma'\ H^*(\omega)/2\pi$ in (4.14). These two specifications of the properties of the disturbance processes are sufficient to describe most of the linear models which have appeared in the econometrics literature.[12]

5. A Multiplier-Accelerator Model

To illustrate the spectral-matrix approach to the analysis of dynamic models, a simple multiplier-accelerator model will now be considered. The model consists of three equations:

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[12]The case in which the disturbances u_t have an autoregressive structure has already been incorporated in the original specification of the model. Suppose that the model is

$$By_t + \Gamma z_t = u_t$$

$$u_t = H\, u_{t-1} + v_t$$

where the disturbance vector v_t is serially uncorrelated. By premultiplying by $I - HL$ the model satisfies the conditions underlying the derivation of (4.16) and (4.17).

The final form is thus a nonhomogenous third-order difference equation. The solution of this equation is

$$(5.5)\quad Y_t = \bar{Y}_t + S_t + (0.66)^t[k_1 \cos 2\pi\, t/9.05 + k_2 \sin 2\pi t/9.05] + k_3(0.25)^t$$

where $\bar{Y}_t$ is a particular solution corresponding to the exogenous variables t and G, S_t is a particular solution corresponding to u_{1t} and u_{2t}, and k_1, k_2, k_3 are arbitrary constants determined by the initial values assumed by Y. From this last expression it follows that the system is stable since the characteristic roots are of modulus less than one. One pair of roots is complex valued so the transient response oscillates over time with a periodicity of 9.05 years.

Given three initial values of income and the time path of government expenditure, the time path of income implied by the model in the absence of disturbances can be calculated from (5.4). This is shown as the dashed line in Figure 5.1 for the sample period 1946-1965. Similarly the time paths of consumption and investment could also be calculated. It should be noted that these time paths indicate what would happen if the disturbance process were "turned off" in 1949 since the 1946, 1947, and 1948 values of income which are used as initial conditions in (5.4) are actual values and not values which would have been observed in the absence of the disturbances which were injected into the system in these three years.

In the presence of random disturbances, the actual time paths of the variables will differ from their expected values given by (5.4). For example, the actual path of income is shown as the solid line in Figure 5.1. Since it is these deviations from the expected values that are of primary interest here, the government expenditure term and the trend in investment will be suppressed throughout the following discussion. This has the effect of eliminating the particular solution $\bar{Y}_t$ from (5.5). After subtracting $\bar{Y}_t$ from the solution, the transient response and the stochastic response S_t are left. The transient response, however, tends to zero as t increases. Provided the system has been started in the distant past, the transient response will be negligible so that what is effectively present after subtracting the particular solution is the stochastic response.[14] The dynamic properties of this stochastic response will now be considered.

As shown in the Appendix, it is not necessary to eliminate the identities from the model before proceeding with the anlaysis of the stochastic response. Therefore the system may be written in matrix form as it now stands.

$$(5.6) \quad \begin{bmatrix} 1-.25L & 0 & -.68L+.17L^2 \\ 0 & 1-.33L & -.49+.49L \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} C_t \\ I_t \\ Y_t \end{bmatrix} \qquad \begin{bmatrix} u_{1t} \\ u_{2t} \\ 0 \end{bmatrix}$$

On the assumption that u_t is serially uncorrelated with a contemporaneous covariance matrix $\sigma_{11} = 91$, $\sigma_{12} = 0$, $\sigma_{22} = 16$, the spectral matrix of the system is given by

$$(5.7) \quad F(\omega) = T(\omega)\, f(\omega)\, T^*(\omega)$$

where

$$(5.8) \quad f(\omega) = \frac{1}{2\pi} \begin{bmatrix} 91 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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[14] It should be noted in particular that Y_{46}, Y_{47}, and Y_{48} can not be used to determine the values of the constants in k_1, k_2, and k_3 in (15.5). This would be

range between 0 and 1/2 in steps of 1/100 cpy. The period of a particular component, which indicates the number of years it takes to complete a cycle, is simply the inverse of the frequency in cpy.

The power spectra of C, I, and Y which are implied by the model are shown in Figure 5.2. The significance and interpretation of the power spectrum is based on the relationship (4.5) which for $s = 0$ simplifies to

$$(5.10) \qquad \gamma_{jj}(0) = \int_{-\pi}^{\pi} F_{jj}(\omega)\, d\omega \,.$$

For real-valued time series, $F_{jj}(\omega)$ is an even function of ω so that the integrand can be doubled and the integral taken over $(0,\pi)$. In this case $2\,F_{jj}(\omega)\,d\omega$ indicates the contribution to the variance of the band of frequencies in the interval $(\omega, \omega + d\omega)$. If the spectrum exhibits a relative peak at ω_0, this means that the band centered on this frequency contributes more to the variance than neighboring frequency bands. For purposes of comparison, recall that the spectrum of a sequence of independent random variables is a constant as shown by (4.13).

With respect to the power spectra implied by the multiplier-accelerator model, it is interesting to note that even though the endogenous variables all have the same autoregressive structure as shown by (2.6), the power spectra do have different properties. These power spectra exhibit relative peaks at 3/40, 5/40, and 4/40 of a cycle per year. Since these peak frequencies correspond to oscillations of 13 1/3, 8, and 10 years, this model might be called a "major-cycle" model. This is not to say that the 8- to 13- year cycle is the only fluctuation that would be visible if the model were simulated, but it is the predominant oscillation in terms of its contribution to the variance of the series.

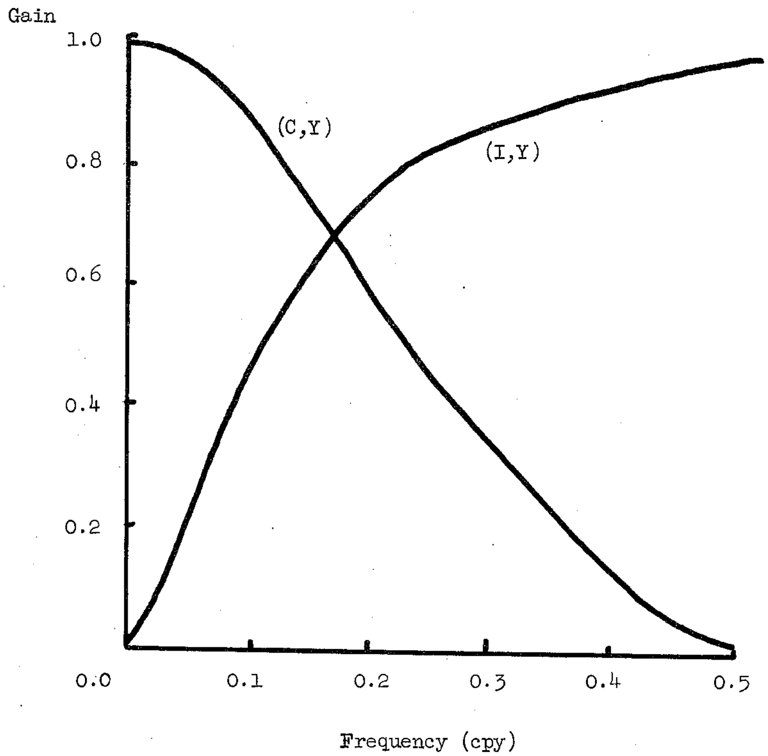

Figure 5.4 Gain of Consumption on Income and Investment on Income

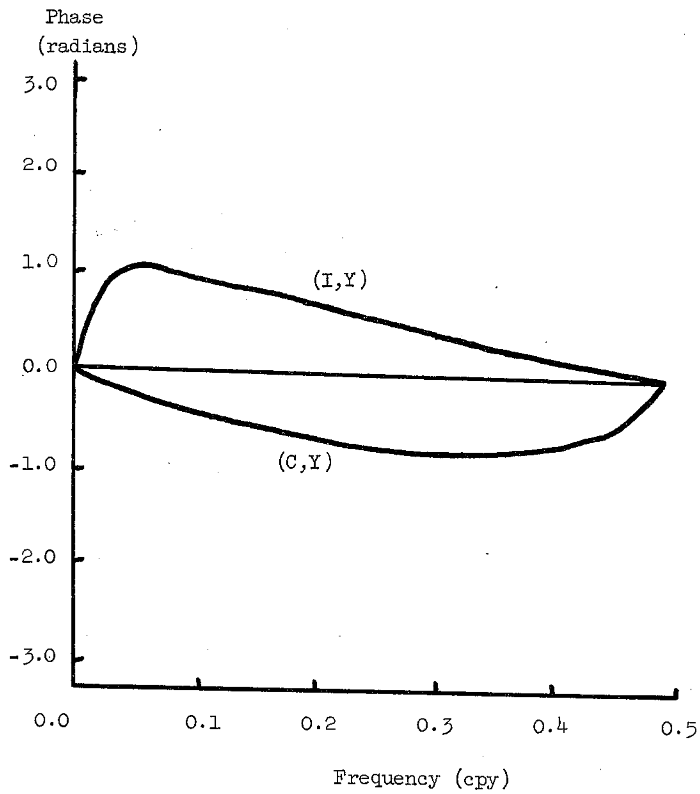

Figure 5.5 Phase difference (in radians) between Consumption and Income and Investment and Income

considered in isolation yield the same relationships in the frequency domain as does the system (5.1) - (5.3)? To answer this, it is necessary to analyze the consumption function in some detail. Suppose that the system is

$$(5.15) \qquad C_t = .68\, Y_{t-1} + u_t$$

$$(5.16) \qquad u_t = .25\, u_{t-1} + \epsilon_t$$

where it is assumed that ϵ_t is serially independent with mean zero and variance $\sigma^2 = 91$, Y_t and u_{t-s} are independent for all t and s, and the power spectrum of Y_t is the same as that shown in Figure 5.2.

Using the techniques described in the preceding section, it is a simple matter to show that these assumptions, together with (5.15) and (5.16) imply the following relationships.

$$(5.17) \qquad F_{uu}(\omega) = |1-.25\, e^{-i\omega}|^{-2}\, \sigma^2/2\pi$$

$$(5.18) \qquad F_{CC}(\omega) = (.68)^2\, F_{YY}(\omega) + F_{uu}(\omega)$$

$$(5.19) \qquad C^2_{CY}(\omega) = [1 + F_{uu}(\omega)/(.68)^2\, F_{YY}(\omega)]^{-1}$$

$$(5.20) \qquad G_{CY}(\omega) = .68$$

$$(5.21) \qquad \varphi_{CY}(\omega) = -\,\omega$$

The power spectrum of consumption in this alternative model, shown in Figure 5.6, has the same general shape as that implied by the system of equations. It is, however, somewhat less peaked and the power decreases less rapidly as frequency increases. The coherence relationship as shown in Figure 5.7 is similar in appearance to the system coherence shown in Figure 5.3. Here again the coherence between consumption and income decreases less rapidly than in the system of equations.

The similarity between the single-equation model and the system model ends at this point. The gain of consumption on income is a constant in the single-equation model, whereas the system gain has a shape which is consistent with the permanent income hypothesis. The phase relationships implied by the two models are also different. A simple time lag is implied by the consumption equation alone whereas the system phase relationship is somewhat more complicated. These considerations indicate that while the simple consumption function alone is not consistent with a permanent-income explanation of consumption, the relationship between consumption and income in the system is consistent with the permanent income hypothesis.

While this multiplier-accelerator model is an extremely simple system, it does illustrate the spectral-matrix approach and the results which can be obtained from the analysis. In particular, it provides a framework within which it is possible to pursue an analysis of the stochastic properties of an econometric model. As well, it may be helpful in understanding the spectral representation of stochastic processes.

6. Conclusion

This paper has concerned itself with an analytical method which can be used to characterize the dynamic properties of linear stochastic systems. The motivation for the paper was twofold. First, the solution of an equation or system of equations with the disturbance terms suppressed may be quite different from the solution which is obtained when the errors are taken into account. Second, an analysis of the results obtained from a simulation of the system with and without disturbances is subject to sampling variability so that the results are not exact. Although in some instances it is possible to determine the limits of sampling error, it was felt that an analytical approach would be desirable.

It was suggested that the spectrum matrix of the system provides a useful characterization of the stochastic properties of the model. The information which is needed to derive the spectrum matrix of the system is exactly the same as that needed to perform a simulation of the model. However, the spectrum matrix and the quantities derived from it such as the power spectra, coherences, and phase relationships, are exact so that the sampling variability associated with the analysis of simulation experiments is avoided.

The approach outlined in the paper was applied to a relatively simple dynamic model. The results obtained from this model were useful in clarifying the interpretation of the method and indicated that the technique is computationally feasible. A final evaluation of the usefulness of the spectrum-matrix approach will have to wait until some experience in its application to large-scale models has been accumulated. At this point, however, it does seem reasonable to conjecture that the approach will be useful in studying the dynamic properties of linear econometric models.

APPENDIX

The Treatment of Identities in Linear Systems

In the text it was assumed that the identities had been eliminated from the system by appropriate substitution. In this appendix it will be shown that this is not an essential restriction and, in fact, the methods described in the text can be applied to systems without eliminating the identities.

Consider a g-equation system in which there are m stochastic equations and $h = g-m$ identities. Without loss of generality the exogenous variables are suppressed throughout the following discussion. The reader may wish to convince himself that this procedure is valid by including a vector of exogenous variables through the first few steps of the exercise. The m stochastic equations can be written as

$$(A.1) \quad A(L)\, y_t = u_t$$

where $A(L)$ is an $m \times g$ matrix of polynomials in the lag operator L, y is a g-dimensional vector of endogenous variables and u is a g-dimensional vector of stochastic disturbances. The set of h identities can similarly be expressed as

$$(A.2) \quad B(L)\, y_t = 0_h$$

where $B(L)$ is $h \times g$ and 0_h is the null vector. It will now be shown that the $g \times g$ spectral matrix of endogenous variables can be obtained by applying the methods described in the text to the system

$$\text{(A.14)} \quad \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}^{-1} .$$

Now, (A.3) can be solved for the endogenous variables in terms of the disturbances:

$$\text{(A.15)} \quad \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} C_1(L) & C_2(L) \\ D_1(L) & D_2(L) \end{bmatrix} \begin{bmatrix} u_t \\ 0_k \end{bmatrix} = \begin{bmatrix} C_1(L)\ u_t \\ D_1(L)\ u_t \end{bmatrix} .$$

The kernel of the y_t process is thus

$$\text{(A.16)} \quad \begin{bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{bmatrix} = \begin{bmatrix} C_1(\omega)\ U(\omega) \\ D_1(\omega)\ U(\omega) \end{bmatrix}$$

so that the spectral matrix of the system is given by

$$\text{(A.17)} \quad F(\omega) = \begin{bmatrix} F_{11}(\omega) & F_{12}(\omega) \\ F_{21}(\omega) & F_{22}(\omega) \end{bmatrix}$$

$$= \begin{bmatrix} C_1(\omega)\ f(\omega)\ C_1^*(\omega) & C_1(\omega)\ f(\omega)\ D_1^*(\omega) \\ D_1(\omega)\ f(\omega)\ C_1^*(\omega) & D_1(\omega)\ f(\omega)\ D_1^*(\omega) \end{bmatrix}$$

It remains to be shown that the elements in this matrix are identical to the expressions given in (A.10) - (A.13). By definition of C_i and D_i $(i=1,2)$, it follows that

$$\text{(A.18)} \quad \text{(a)}\ A_1C_1 + A_2D_1 = I \qquad \text{(c)}\ A_1C_2 + A_2D_2 = 0$$

$$\text{(b)}\ B_1C_2 + B_2D_2 = I \qquad \text{(d)}\ B_1C_1 + B_2D_1 = 0$$

Solving for C_1 from (A.18a) and (A.18d) yields

$$\text{(A.19)} \quad C_1 = [A_1 - A_2B_2^{-1} B_1]^{-1}$$

which is identical to $T(\omega)$ introduced (A.8). This shows that the first element on the main diagonal of (A.17) is identical to (A.10). A similar argument shows that all the elements in the matrix are, in fact, identical to the corresponding expressions in (A.10) - (A.13). Thus it is not necessary to eliminate the identities before proceeding with the computation.