STOCHASTIC PROPERTIES OF THE KLEIN-GOLDBERGER MODEL

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ABSTRACT

After the coefficients of an econometric model have been estimated, it is often of interest to determine the dynamic properties of the resulting system of equations. Two general methods of analysis are often employed in this connection. If the system of equations is linear in the endogenous variables (or if a linear approximation can be obtained), then the techniques applicable to systems of linear difference equations with constant coefficients can be employed to determine the properties of the time paths of the endogenous variables. With this approach the fact that the equations are not exact is ignored and the system is analyzed as if it were a deterministic system. The second general approach involves the study of simulation experiments in which the stochastic nature of the equation system is explicitly recognized.

The Klein-Goldberger model of the United States economy has been subjected to both types of analysis. What has not been considered, however, is the use of analytical procedures to determine the stochastic properties of the model. In this paper, techniques that have been developed in connection with the study of random functions and "modern" time-series analysis are used to analyze the "stochastic dynamics" of the Klein-Goldberger model.

Two linearized versions of the model, one in which tax yields are assumed to be exogenous and one in which tax-yield equations are included, are analyzed. The spectrum matrix of the endogenous variables is derived and the implied power spectra and coherence, gain, and phase relationships of selected variables are discussed. Several interesting conclusions emerge from the analysis. (i) The power spectra

of the endogenous variables in both models have the "typical spectral shape" in which power decreases as frequency increases. (ii) The power spectra of the first differences of the endogenous variables differ between the two models. The exogenous tax yield model might be called a long-swing model in that the first-differenced power spectra exhibit relative peaks near one-twentieth of a cycle per year. With endogenous tax yields, the model is more nearly a business cycle model in which the relative maxima of the power spectra correspond to a four- or five-year cycle. (iii) The phase differences implied by the model are in close agreement with previously published simulation results. (iv) If the tax equations are regarded as automatic stabilizers, the differences between the two models are in general agreement with the theoretical work on stabilization policy in dynamic systems. In particular, the variance is reduced as a result of the introduction of the tax equations, but the frequency of oscillation is increased.

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1. Introduction

Over the past several decades a considerable amount of effort has been devoted to the solution of various problems associated with the estimation of parameters of econometric models. Much less attention, however, has been given to the analysis of the dynamic properties of the specific structures obtained from the estimation procedure. Moreover, the systems analysis which has been performed has, for the most part, been concerned with the deterministic system obtained by suppressing the disturbance vector -- the sine qua non of the estimation problem -- which appears in the basic formulation of an econometric model.

The simulation experiments described by Haavelmo [9], Fisher [4], and Adelman and Adelman [1], to mention only a few, indicate that the time paths generated by a model with and without the disturbance vector can be considerably different. These studies implicitly suggest that the disturbance process imparts a certain amount of realism to the time paths of the variables in the model. Despite the presumed importance of the disturbances, little has been done to introduce analytical procedures to evaluate the stochastic properties of econometric models.

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The purpose of this paper is to apply some of the techniques developed in connection with time series analysis to the Klein-Goldberger econometric model of the United States [7,12] to determine the dynamic properties of the stochastic response of the system of equations. Specifically, the spectrum matrix of the model is derived and the implied power spectra, coherences, and phase relationships of selected endogenous variables are explored. The spectrum-matrix approach has been described in some detail elsewhere [10], so only a brief summary is given in the next section. In Section 3 the two versions of the model which will be analyzed here are described and the characteristic roots are determined. In Section 4 the power spectra are derived and the phase relationships are examined in Section 5.

The Klein-Goldberger model has been selected for an experimental application of the spectrum-matrix approach for two reasons. First, it is a moderately large system and should provide a reasonable test of the computational feasibility of the spectrum-matrix approach. In addition, this model has been subjected to extensive simulation experiments [1] and the simulation results provide an interesting comparison with the results obtained from the analytical procedure.

2. The Spectrum-Matrix Approach

Consider the standard linear econometric model

$$(2.1) \quad By_t + \Gamma z_t = u_t$$

where B and Γ are coefficient matrices, y is a vector of endogenous variables, z is a vector of predetermined variables, and u is a vector of disturbances. The analysis that follows is applicable to either the structural form of the system (2.1) or the reduced form given by

$$(2.2) \quad y_t = \Pi z_t + v_t$$

where $\Pi = -B^{-1}\Gamma$ and $v_t = B^{-1} u_t$. For convenience, the reduced form will be considered throughout with no loss of generality.

If the model is dynamic, the vector of predetermined variables will include lagged values of the endogenous variables. Introducing the lag operator L defined by $Ly_t = y_{t-1}$, the reduced form can be written as

$$(2.3) \quad [I-\Pi_1 L-\Pi_2 L^2-\ldots-\Pi_r L^r]\, y_t = \Pi_o x_t + v_t$$

where $\Pi_1, \Pi_2, \ldots, \Pi_r$ are square matrices of coefficients of $y_{t-1}, y_{t-2}, \ldots, y_{t-r}$; and Π_o is the matrix of coefficients of the exogenous variables x_t . Now it is clear that (2.3) is a system of stochastic difference equations. A formal solution of this system is obtained in the following way [2,5,10,13]. Let $A(L)$ denote the operator matrix shown in (2.3) as $[I-\Pi_1 L-\ldots-\Pi_r L^r]$, let $a(L)$ denote the adjoint of $A(L)$ and let $\Delta(L)$ denote the determinantal polynomial of $A(L)$ ($= |A(L)| = |a(L)|$). Then the solution of (2.3) is given by

$$(2.4) \quad y_t = K\, \Lambda^t + \frac{a(L)}{\Delta(L)}\, \Pi_o\, x_t + \frac{a(L)}{\Delta(L)}\, v_t \quad .$$

In this solution K is a matrix of polynomials in t (of degree zero if there are no repeated roots), Λ is the column vector of roots of the determinantal equation,[1] and $a(L)/\Delta(L)$ is a matrix of rational functions of the lag operator.

In what follows attention will be focussed on the stochastic response of the system given by the last term in the solution shown in (2.4). A natural way to characterize the stochastic response is by means of the <u>spectrum matrix</u> of the system [10,14]. In order to derive the spectrum matrix it is convenient to

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[1]The convention $\Lambda^t = [\lambda_1^t, \ldots, \lambda_m^t]$ is used here.

introduce the spectrum representation of a stochastic process. Suppose that the disturbance process v_t is wide-sense stationary, that is, the means, variances, and lagged covariances are invariant to translation in time. Then the spectrum representation of v_t is

$$(2.5)\quad v_t = \int_{-\pi}^{\pi} e^{i\omega t} dV(\omega)$$

where $dV(\omega)$ is a vector of stochastic set functions defined by

$$(2.6)\quad dV(\omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} e^{-i\omega s} v_s \quad .$$

Now the spectrum matrix of the disturbance process is defined by

$$(2.7)\quad f(\omega) = E[dV(\omega)\ dV^*(\omega)] = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} e^{-i\omega s}\ \Gamma(s)$$

where $\Gamma(s) = E[v_t\ v^*_{t-s}]$ is the covariance matrix of lag s of the residuals and $*$ denotes the conjugate transpose operation.

The spectrum matrix of the system of equations can now be obtained by first substituting (2.5) into (2.4) to obtain the spectrum representation of the stochastic response of the system:

$$(2.8)\quad \tilde{y}_t = \frac{a(L)}{\Delta(L)} \int_{-\pi}^{\pi} e^{-\omega t}\ dV(\omega) = \int_{-\pi}^{\pi} e^{i\omega t}\ T(\omega)\ dV(\omega) \ .$$

The second equality results from interchanging the order of the operations and defining the <u>transfer matrix</u> $T(\omega)$ as the resulting matrix of rational functions in ω obtained by operating on $e^{i\omega t}$ by $a(L)/\Delta(L)$. This interchange of operations is permissible provided $[\Delta(L)]^{-1} e^{i\omega t}$ converges which will be the case if the roots of the determinantal equation lie outside the unit circle. By comparing (2.8) and (2.5), it is clear that the spectrum representation of the stochastic response of the system is

$$(2.9)\quad dY(\omega) = T(\omega)\, dV(\omega)$$

and from (2.7) it follows that the spectrum matrix of the system is

$$(2.10)\quad F(\omega) = E[T(\omega)\, dV(\omega)\, dV^*(\omega)\, T^*(\omega)] = T(\omega)\, f(\omega)\, T^*(\omega)\ .$$

The steps involved in computing the spectrum matrix of a system may be summarized as follows.

(a) Determine the spectrum matrix of the disturbance process $f(\omega)$. If, for example, the disturbance process has the property that

$$\Gamma(s) = \begin{cases} \Sigma & s = 0 \\ 0 & s \neq 0 \end{cases},$$

then the spectrum matrix of the residuals is

$$f(\omega) = \frac{1}{2\pi}\, \Sigma$$

as may be seen from (2.7).

(b) Determine the transfer matrix $T(\omega)$. A computationally simple scheme for evaluating $T(\omega)$ follows from the fact that

$$T(\omega) = \frac{a(\omega)}{\Delta(\omega)} = [I - \Pi_1 e^{-i\omega} - \dots - \Pi_r e^{-ir\omega}]^{-1}\ .$$

Thus the transfer matrix may be obtained by inverting for given values of ω the complex-valued matrix on the right-hand side of this equation. It should be noted that these expressions are valid provided the system is stable so the roots of the determinantal equation should be evaluated before proceeding with this step.

(c) The spectrum matrix $F(\omega)$ is then obtained by performing the matrix multiplications shown in (2.10). A common procedure is to evaluate the spectrum matrix at a set of equi-spaced frequency points on the interval $[0,\pi]$.

The interpretation of the results of the computation is best left for consideration within the context of the Klein-Goldberger model. At this juncture, it might be pointed out that the power spectra of the endogenous variables are contained on the main diagonal of the spectrum matrix and the cross-spectra are given by the off-diagonal elements. These spectra and cross-spectra are the expected or theoretical values of the estimates that would be obtained by simulating the model and then analyzing the results using spectrum estimation techniques. The analytical procedure sketched above has two main advantages over the simulation approach. First, the simulation procedures are relatively expensive in terms of computer time. Second, and what is more important, the simulation results are subject to sampling variability. The analytical derivation of the spectrum matrix eliminates the uncertainty associated with the sampling variability of the simulation estimates.

3. The Spectrum Matrix of the Klein-Goldberger Model

The experimental application of the spectrum-matrix approach to stochastic system analysis which is described here is based on two linearized versions of the Klein-Goldberger Model of the United States economy. The model itself and the estimation procedures and results are described in Klein and Goldberger [12]. The two linearized versions of the model are taken directly from Goldberger [7]. The basic difference between the two models is that in one case tax receipts are assumed to be exogenous while in the other case tax receipts are endogenous variables. In the former case the model consists of twenty-one equations, fifteen of which are behavioral (stochastic) relationships. In the endogenous tax yield model there are twenty-five equations. The first twenty-one variables in Table 3.1 are endogenous in the latter model.

The first step in the computation of the spectrum matrix of the endogenous variables is the estimation of the spectrum matrix of the residuals. As described in [7], the linearized reduced form of the Klein-Goldberger model is given by

$$(3.1) \quad \dot{y}_t = \Pi \dot{z}_t + \dot{v}_t$$

where $\dot{y}_t$ denotes the derivative of y with respect to time. The matrix of coefficients Π is obtained by differentiating the nonlinear relationships and evaluating the partial derivatives which are functions of the predetermined variables at the sample means of these predeterimined variables. The discrete analogue of (3.1) which is used here is

$$(3.1') \quad \Delta y_t = \Pi \Delta z_t + \Delta v_t$$

where Δ, the difference operator, replaces the differential operator.

The residuals in (3.1'), Δv_t, were estimated from

$$(3.2) \quad \Delta \hat{v}_t = \Delta y_t - \hat{\Pi} \Delta z_t$$

by inserting the observed values for Δy_t and Δz_t and the estimate of Π obtained in the linearization. A second set of residuals was also obtained from

$$(3.2') \quad \hat{v}_t = y_t - \hat{\Pi} z_t$$

in a similar way. The first set of residuals is appropriate for a description of the first differences of the variables whereas the second set can be used to describe the levels of the variables.

In order to decide whether to use the residuals of (3.2) or those of (3.2'), the power spectrum of each set of residuals was estimated to determine which set was most nearly serially uncorrelated. With only eighteen (3.2) or twenty (3.2') observations on each series of residuals, no more than the general outline of the spectrum can be discerned; nevertheless it was felt that the estimated spectra would give some evidence as to the "whiteness" of the residuals. Although the

TABLE 3.1: Endogenous Variables in the Klein-Goldberger Model

Number	Abbreviation	Brief Description
1	C	Consumer expenditures
2	I	Gross private domestic investment
3	S_c	Corporate Savings
4	P_c	Corporate Profits
5	D	Depreciation
6	W_1	Private wage bill
7	N_w	Number of wage and salary earners
8	w	Index of hourly wages
9	F_I	Imports
10	R_1	Farm income
11	P_R	Farm price level
12	L_1	Household liquid assets
13	L_2	Business liquid assets
14	i_L	Long-term interest rate
15	i_s	Short-term interest rate
16	K	Capital stock
17	S_B	Corporate surplus
18	P	Nonwage nonfarm income
19	M	National income
20	p	Price level
21	Y	Gross National Product
22	T_E	Indirect taxes
23	T_W	Wage taxes less transfers
24	T_C	Corporate taxes
25	T_M	Nonwage nonfarm noncorporate taxes less transfers

estimated power spectra were not conclusive, they suggested that the residuals from levels (3.2') were white in most cases. The notable exceptions to this statement are the residuals from the equations describing corporate savings, depreciation, form income, farm price level, household liquid assets, the long-term interest rate, and the indirect taxes. In all of these cases, the spectrum densities of the residuals had a predominantly downward sloping spectrum and were significantly different from white noise at the 95 percent level of significance. In the long-term interest rate, and two of the tax equations --- corporate taxes and nonwage nonfarm noncorporate taxes less transfers --- there is a concentration of power near $\frac{1}{4}$ of a cycle per year which indicates that the model is not entirely successful in capturing the business-cycle dynamics of the sample period.

Despite these findings the spectrum matrix of residuals which is used in the computations is

$$(3.3)\quad \hat{f}(\omega) = \frac{1}{2\pi}\hat{\Sigma}$$

where $\hat{\Sigma}$ is the estimated covariance matrix of the residuals $\hat{v}_t$. The spectrum matrix of the endogenous variables which is discussed below should therefore be interpreted as describing the stochastic response of the system when uncorrelated disturbances are fed into it. It might be noted in passing that several alternative procedures might have been employed in determining the spectrum matrix of residuals. For example, the estimated spectra and cross-spectra of the residuals could have been used in the subsequent computations. Another alternative would involve transforming the residuals in (3.2') by an autoregressive process of the form

$$(3.4)\quad \hat{v}_t = \hat{\rho}_1\hat{v}_{t-1} + \ldots + \hat{\rho}_s\hat{v}_{t-s} + \epsilon_t$$

where the new residuals $\hat{\epsilon}_t$ are serially uncorrelated. The procedure which is actually used is based on considerations of simplicity and the fact that (3.3) corresponds most nearly to the assumption on the disturbance vector that was employed in estimating the model.

Having determined the spectrum matrix of the residuals, the second preliminary question which must be answered is concerned with the stability of the model. The characteristic roots of the two versions of the Klein-Goldberger model were calculated by first writing the equations as a first-order system by introducing new variables and then evaluating the characteristic roots of the matrix of this first order system [2,5]. The roots of modulus greater than .05 for the exogenous tax yield case are shown in Table 3.2 and in Table 3.3 for the endogenous tax yield model. It will be noted that both models are stable since all of the roots are of modulus less than one. In the exogenous tax yield model, there are five real roots and three pairs of complex roots which are greater than 0.05 in absolute value. The periods of the complex roots are approximately 593, 24, and 9 years for this version of the model. For the endogenous tax yield model there are seven real roots and two pairs of complex roots which are greater than 0.05 in absolute value. For this version of the model the periods of the complex roots are approximately 96 and 6 years.

This completes the preliminary computations and the spectrum matrix of the endogenous variables can now be computed using Equation (2.10). The results will be described in the next two sections: first the power spectra of selected endogenous variables and then the coherence and phase relationships of selected endogenous variables.

TABLE 3.2

Dominant Roots of the Linearized Klein-Goldberger Model:
Exogenous Tax Yield Case

Real Part	Imaginary Part	Modulus	Period
0.9815	0.0	0.9815	
0.9813	±0.0104	0.9814	592.9
0.8504	±0.2239	0.8794	24.4
0.8162	0.0	0.8162	
0.6145	0.0	0.6145	
0.3211	0.0	0.3211	
0.2339	±0.2063	0.3119	8.7
-0.0762	0.0	0.0762	

TABLE 3.3

Dominant Roots of the Linearized Klein-Goldberger Model:
Endogenous Tax Yield Case

Real Part	Imaginary Part	Modulus	Period
0.9941	0.0	0.9941	
0.9815	0.0	0.9815	
0.9776	0.0	0.9776	
0.8483	0.0	0.8483	
0.8139	0.0	0.8139	
0.6046	±0.0396	0.6059	96.1
0.3321	0.0	0.3321	
0.1076	±0.2395	0.2626	5.5
-0.0811	0.0	0.0811	

4. Power Spectra of Selected Endogenous Variables

Before examining the power spectra of selected endogenous variables in the model, several alternative hypotheses which will aid in the interpretation of the results should be formulated. The simplest hypothesis about the power spectra implied by the model is that they are constant over frequency. If this were true, the autoregressive structure of the model would be virtually meaningless since the response to white noise would be a sequence of uncorrelated random variables. A casual inspection of the characteristic roots of the system indicates that this is highly unlikely, however. The fact that there are several real roots which are close to one suggests that an alternative hypothesis is more likely to be true; namely, the endogenous variables all have the "typical spectral shape" [8]. In this case the power spectra would indicate that power decreases smoothly with frequency except possibly at very low frequencies. A third hypothesis of interest is that the power spectra of first differences of the endogenous variables are white. This is similar to the typical-spectral-shape hypothesis but is somewhat stronger for it is possible for a root close to one to dominate the shape of the spectrum and therefore obscure the contribution of higher frequencies to the power spectrum.

The power spectra of consumption, investment, and gross national product are shown in Figure 4.1 for the exogenous tax yield model and in Figure 4.2 for the endogenous tax yield model. The spectra of these variables are representative of the spectra implied by these two models and indicate that the levels of the variables do exhibit the "typical" shape. This is, of course, not surprising, since in both models there are several positive real roots which are close to unity. It is interesting that investment in the exogenous tax model exhibits a relative peak at 1/25th of a cycle per year and a relative peak near 1/6th of a

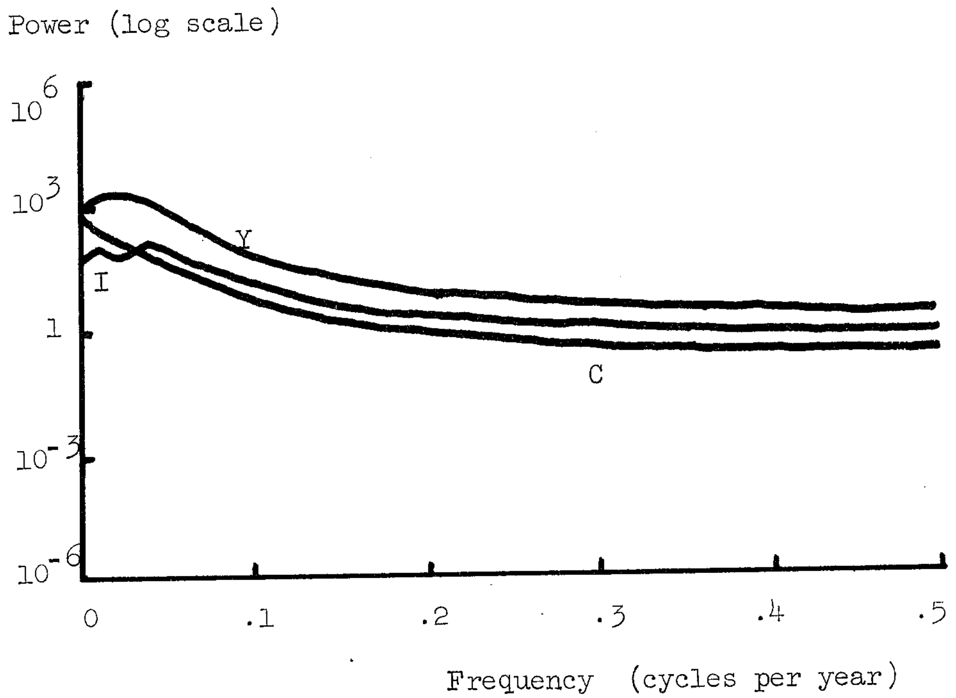

Figure 4.1 Power Spectra of Consumption, Investment, and Gross National Product Implied by the Exogenous Tax Yield Model.

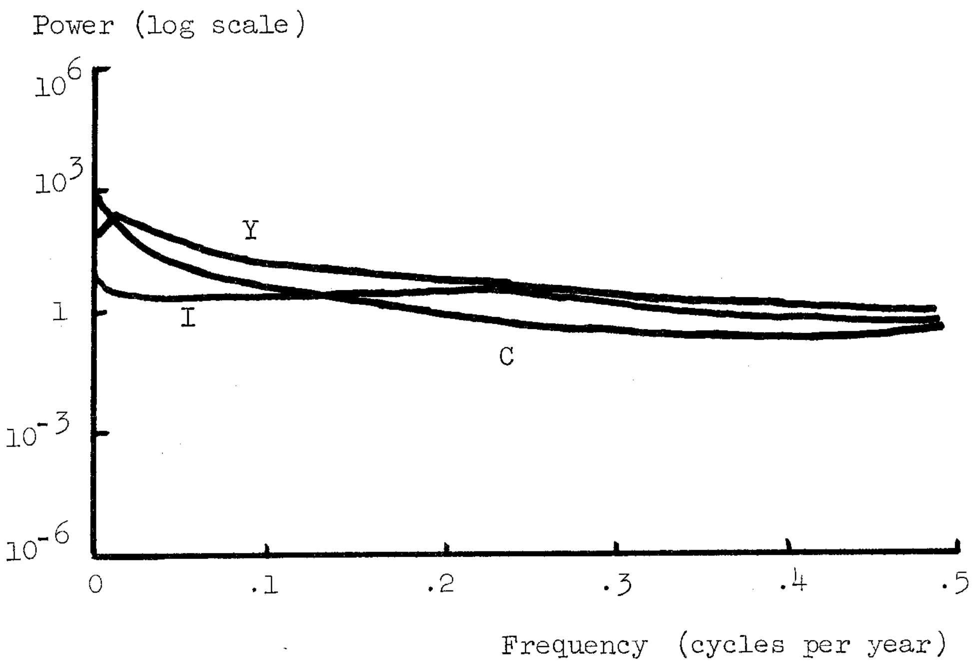

Figure 4.2 Power Spectra of Consumption, Investment, and Gross National Product Implied by the Endogenous Tax Yield Model.

cycle per year in the endogenous tax model, although in the latter case it is barely discernable.

The first differenced power spectra of consumption, investment, and income are shown in Figures 4.3 and 4.4 for these two models. These were obtained by multiplying the power spectra of the levels described above by the gain (squared) of the first-difference filter which is $G^2(\omega) = |1-e^{-i\omega}|^2$. In the exogenous tax yield model, the first-differenced power spectra of these variables exhibit relative peaks at 1/20th of a cycle per year. The relative peaks emerge at much higher frequencies in the endogenous tax model: 1/8th of a cycle per year for consumption, 1/4th of a cycle per year for investment, and 1/5th of a cycle per year for income. These spectra illustrate the point which is exhibited clearly in Table 4.1 that the two models have quite different stochastic properties. Table 4.1 shows location of the relative peaks in the power spectra of all the endogenous variables in each of the two models. A quick glance at the table indicates that in the exogenous tax yield model, the power is concentrated in the low-frequency end of the spectrum whereas in the endogenous tax case, the power is concentrated in the intermediate frequencies.

This raises the interesting question of why the two specifications should differ so dramatically in their stochastic implications. Tax equations of the type embodied in the Klein-Goldberger model are traditionally thought of as automatic stabilizers and therefore would be expected to reduce the variance of the endogenous variables. This reduction in variance is achieved in the Klein-Goldberger model since the power spectra implied by the endogenous tax model lie uniformly below the corresponding spectra for the exogenous tax model.[2] However, the reduction

[2]This is true even though the tax equations are not exact and therefore contribute to the variance of other endogenous variables through the disturbances which are introduced into the system by their inclusion.

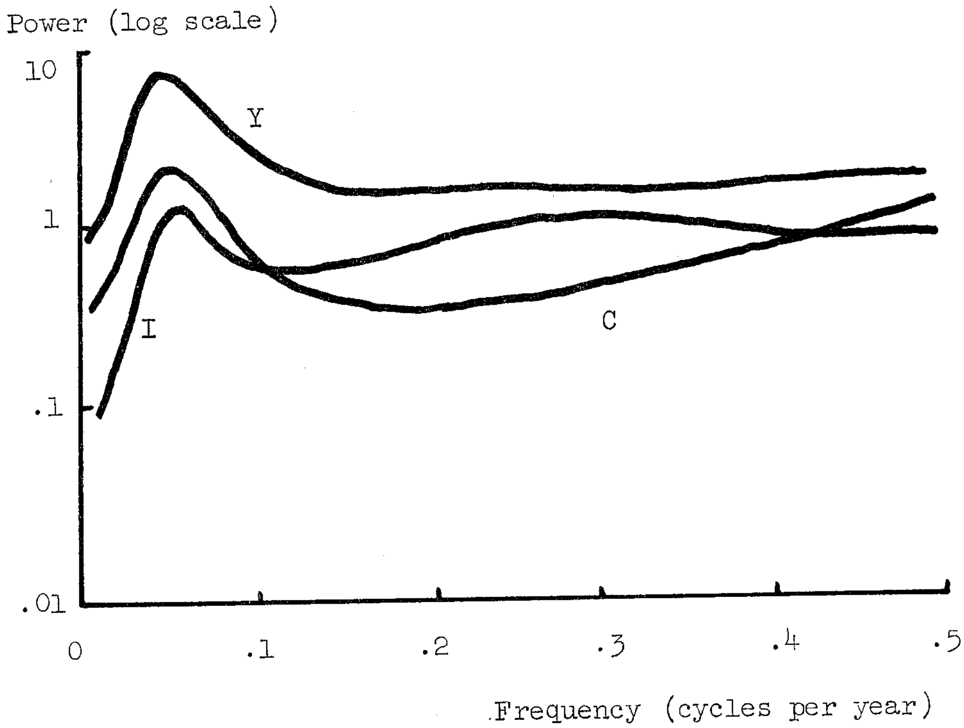

Figure 4.3 First-Differenced Spectra of Consumption, Investment, and Gross National Product Implied by the Exogenous Tax Yield Model.

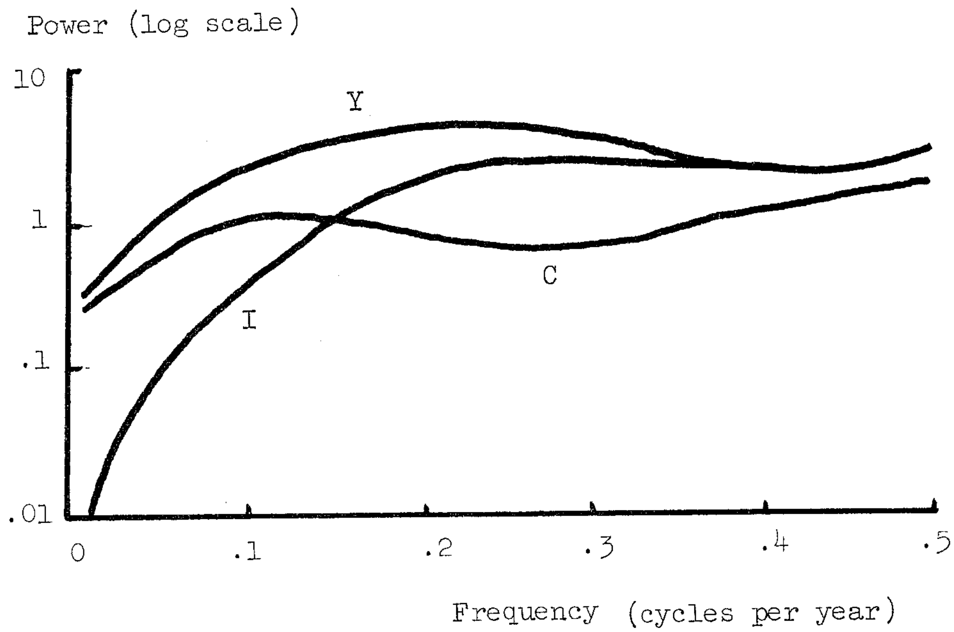

Figure 4.4 First-Differenced Spectra of Consumption, Investment, and Gross National Product Implied by the Endogenous Tax Yield Model.

TABLE 4.1: The Location (in years per cycle) of Relative Peaks in the Power Spectra Implied by the Klein-Goldberger Model*

Variable	Exogenous Tax Yield Model: Levels	Exogenous Tax Yield Model: First Differences	Endogenous Tax Yield Model: Levels	Endogenous Tax Yield Model: First Differences
C	-	20	-	8.3
I	25	20	5.9	3.6
S_c	25	20	5.9	-
P_c	25	20	5.6	3.6
D	-	25	-	-
Kl_1	25	20	-	5.6
N_W	25	20	-	4.8
ω	-	25	-	5.0
F_I	-	-	-	-
R_1	-	-	-	-
P_R	-	25	-	-
L_1	-	-	-	-
L_2	-	33	-	5.9
i_L	-	33	-	33
i_3	-	-	-	-
K	-	25	-	5.6
S_B	-	25	-	5.6
P	25	20	5.6	3.8
M	25	20	-	5.0
p	-	25	-	4.8
Y	-	20	-	4.8
T_E			3.6	-
T_W			5.9	-
T_C			5.6	-
T_N			5.9	-

*The entries in the table are the inverses of the frequencies measured to the nearest 1/100 of a cycle per year at which a relative peak emerges. A dash indicates that the power spectrum does not exhibit a relative maximum inside the interval (0,1/2) cycle per year. A blank indicates that the variable is not included in the model.

in variance is accompanied by a change in the shape of the power spectra as indicated by the table, low frequency power being attenuated relatively more than high-frequency power. This is consistent with Baumol's finding [3] that stabilization policies that increase the stability of a system often result in an increase in the frequency of oscillation. Similar results have been derived for simple stochastic systems [11].

5. Coherence and Phase Relationships of Selected Endogenous Variables

The coherence and phase relationships implied by the two versions of the Klein-Goldberger Model are summarized in Table 5.1. The coherences between selected variables andgross national product are shown at two frequencies: 1/20th of a cycle per year and 1/4th of a cycle per year. The phase differences in months between gross national product and selected variables are also shown at these same two frequencies. In the last column of the table the phase relationship obtained from the Adelmans' simulation [1] is also shown.

A number of interesting conclusions emerge from a comparison of these figures. First, the coherence between GNP and the other variables in the frequency band that corresponds to a twenty-year cycle is generally reduced by the introduction of endogenous tax yields. In only one instance, the coherence between GNP and business liquid assets, is the correlation at 1/20th of a cycle per year increased, and in this instance the coherence between these two variables is fairly low in the exogenous tax yield model. On the other hand, the coherences between four-year components is increased in a number of instances by the introduction of the tax equations, one of the more dramatic increases in coherence being that between GNP and business liquid assets.

TABLE 5.1

Coherence and Phase Relationships for Selected Variables in the Linearized Klein-Goldberger Model*

Variable (1)	Exogenous Tax Yield Model: Coherence (2) f=1/20	Coherence (3) f=1/4	Phase (months) (4) f=1/20	Phase (months) (5) f=1/4	Endogenous Tax Yield Model: Coherence (6) f=1/20	Coherence (7) f=1/4	Phase (months) (8) f=1/20	Phase (months) (9) f=1/4	Simulation Phase Relationship (10)
National Income	.99	.97	4.7	0.2	.98	.89	4.2	2.2	Coincident
Consumption	.99	.63	-2.8	0.3	.98	.57	-5.6	-0.1	Coincident
Investment	.99	.76	-0.1	-0.8	.79	.89	9.0	-0.4	Coincident
Private Employee Compensation	.99	.85	-5.7	-2.4	.99	.95	-7.4	-6.3	Coincident
Farm Income	.65	.05	-0.5	6.2	.40	.20	4.2	10.2	Irregular
Nonwage Nonfarm Income	.99	.58	17.1	1.5	.84	.55	42.8	6.8	Coincident
No. of Wage and Salary Earners	.99	.72	7.7	0.3	.97	.86	4.5	0.9	Coincident
Corporate Profits	.99	.72	16.7	2.0	.86	.87	42.5	6.6	Coincident
Corporate Savings	.99	.64	19.9	2.4	.54	.59	40.6	7.2	Coincident
Corporate Surplus	.96	.64	-34.1	-3 6	.55	.59	-13.3	1.2	Coincident
Capital Stock	.98	.71	-42.4	-6.6	.69	.86	-30.6	-5.9	Coincident
Depreciation	.98	.11	-35.7	-3.2	.76	.30	-22.7	-5.3	Leading
Individual Liquid Assets	.76	.15	28.4	23.0	.53	.09	17.5	19.5	Leading
Business Liquid Assets	.49	.03	-21.1	-16.8	.57	.70	2.4	-15.0	Coincident (inverted)
Price Level	.79	.25	-31.0	0.6	.21	.75	-80.1	4.9	Coincident
Agricultural Prices	.78	.18	-30.7	1.0	.19	41	-73 9	4.8	Coincident
Wage Rate	.87	.19	-45.1	-8.8	.43	.61	-5.9	-6.6	Lagging

The phase differences at 1/20 and 1/4 of a cycle per year also indicate that the introduction of the tax equations make a considerable difference in the stochastic properties of the model. Although it is not true in all cases, the business-cycle (1/4 of a cycle per year) leads and lags tend to be increased by the introduction of the tax equations. For example, in the exogenous tax yield model corporate savings tend to lead gross national product by about two months whereas in the endogenous tax case corporate savings lead by about seven months in the business-cycle frequency band. In the low frequency band centered on 1/20 of a cycle per year, the wage rate lags GNP by about 45 months in the exogenous tax model and by only 6 months in the endogenous tax yield case.

A final comparison may be made between the phase differences implied by the model and the results which were obtained from the Adelman simulation of the endogenous tax yield version of the model. Although the simulation was performed on the nonlinear model with a few changes from the original formulation[3], the comparison with the linearized form of the model used here indicates that phase relationships are quite simular. If a phase difference at the business cycle of less than $\pm$ 8 months is considered to be coincident, then only two of the phase relationships are markedly different. In the simulation experiment, depreciation

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*The coherence and phase calculations shown in this table are based on the relationship of gross national product and the corresponding variable shown in column (1) of the tables. The coherence at 1/40 of a cycle per year is shown in column (2) and the coherence at 1/4 of a cycle per year is shown in column (3). The phase differences at these frequencies are shown in columns (4) and (5). A positive entry indicates that the variable in question leads gross national product while a negative entry indicates that the variable lags gnp. The simulation phase relationship is taken from Table 3 of the paper by I. and F. Adelman [1].

[3]Another factor that could contribute to the difference between the results derived here and the simulation results arises from the properties of the disturbance terms that were used in the simulation. Since the model was reduced by successive substitution before simulation, it is not possible to verify that the simulation disturbances had the same covariance matrix as that used in the computations here.

was found to lead GNP while the theoretical relationship indicates that depreciation should be roughly coincident to lagging. In the simulation, it was also found that business liquid assets, when inverted, were coincident with GNP. This implies that business assets should lag (or lead) by one-half of a cycle or twenty-four months. The theoretical lag is only fifteen months. All in all, the simulation results compare quite well with what would have been expected on the basis of the theoretical cross-spectrum analysis.

6. Summary

In this paper the stochastic properties of the Klein-Goldberger model have been analyzed using the spectrum-matrix approach to stochastic systems. Two linearized versions of the model --- one in which taxes are assumed to be exogenous and one in which tax yields are considered to be endogenous variables --- were considered. It was found that the power spectra implied by both of these models have the typical spectral shape in which power decreases with frequency except at the very low frequency end of the spectrum. This was not entirely unexpected for two reasons. First, it was found that both systems of equations have several positive real roots that are close to unity. In addition, since the models were originally estimated from observations on the levels of the variables involved, it is unlikely that they would be able to capture much of the dynamic structure of the economy.

A somewhat surprising conclusion which emerges from this analysis is that the first-differenced power spectra of the exogenous tax yield model are markedly different from those of endogenous tax model. The addition of the tax equations changes the model from a "long swing" model in which the first-differenced spectra have peaks near 1/20 of a cycle per year to a "business cycle" model in which the differenced spectra have relative peaks which correspond to a four-or five-year cycle. The tax relations, traditionally referred to as automatic stabilizers, do reduce the variance of the endogenous variables, but they have the effect of increasing the frequency of oscillation of the predominant cyclical components implied by the model.

The coherence and phase relationships are also markedly different in the two models. In the exogenous tax model, the coherences are highest in the low frequency end of the spectrum. The addition of the tax equations reduces the low-frequency coherence and increases the coherence in the intermediate frequency band around 1/4th of a cycle per year. It was also found that the phase relationships implied by the linearized model are quite similar to the phase differences obtained by applying National Bureau methods to series obtained from a simulation of the model.

The basic merit of the spectrum-matrix approach is that it is an analytical technique which can be used to investigate the dynamic properties of a stochastic system. The results are therefore not subject to sampling variability which often makes it difficult to interpret the results of simulation experiments. Using this method it is possible to derive the stochastic properties directly from the model rather than performing a simulation and then analyzing the results of the simulation. This application of the spectrum-matrix approach to the Klein-Goldberger model indicates the types of questions which can be investigated using this method.

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