FISCAL POLICY, MATHEMATICAL PROGRAMMING AND TWO-PERSON ZERO-SUM GAME THEORY

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ABSTRACT

Various techniques of mathematical programming are applied to a simple fiscal policy problem. First the problem is stated as a linear program. Then the decomposition principle is applied to a decentralized version of the fiscal policy problem. The equivalence of a linear programming formulation of the problem and the formulation as a two-person zero-sum game is shown. Finally, the rate of change of the value of the game is investigated. The linear program is cast into dynamic programming formulation in an appendix. Eventually the coefficients of the constraints are assumed to be stochastic.

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1. Introductory Remarks

It seems that mathematical programming techniques have not yet been given the attention they deserve in public finance.[1] In this paper linear programming and two-person zero-sum game theory are used to help determine an optimal fiscal policy. The problem is then cast into dynamic and stochastic programming framework. This is done in an appendix because for economists, trained in public finance and not familiar with mathematical programming, it will be more difficult to understand than the first five sections of this paper. The techniques used are general instruments applied to optimization problems of a special kind. The example of fiscal policy chosen is merely one among many possible applications of these four approaches. There are other fiscal policy questions whose solutions could be aided by use of these methods.

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[1]There are: E. Fossati, For the Application of Linear Programming to Public Finance, Public Finance, 2 (1957), p. 118 ff; J. Cutolo, An Application of Linear Programming to Public Finance, Public Finance, 3 (1958), p. 216 ff; D. Dosser, Linear Programming and Public Finance, Public Finance, 5 (1960), p. 51 ff; C. Shoup, Linear Programming and Public Finance, Finanzarchiv, 22 (1962/63), p. 464 ff; D. Dosser, Notes on Linear Programming and Public Finance, Finanzarchiv, 23 (1963), p. 279 ff; K.K. Kurihara, A Linear Programming Fiscal-Policy Model of Capacity Growth, Public Finance, 20 (1965), p. 272 ff. And in a somewhat different context: Ch. J. Hitch and R. N. McKean, The Economics of Defense in the Nuclear Age, New York 1965.

As a result it will be shown that a fiscal-policy mix, or in special cases just one policy measure has to be used, to stimulate private investment. Searching for optimal investment stimulation policies, the government has to evaluate costs and benefits of various policy measures. This point does not seem to be stressed enough in traditional literature on fiscal policy. The arbitrary numbers taken in the following example of unemployed skilled and unskilled labor and the optimal budget loss suggest that various employment policies may for a particular country be practically infeasible because of their relatively high budget loss.

At first sight, the problem analyzed seems to be artificial because of the arbitrary coefficients in the linear program. The present paper is not, however, intended as an evaluation of the coefficients of some econometric model of the state. Instead the aim is to show the broad usefulness of linear programming and two-person zero-sum game theory in public finance. Estimating the structural parameters of this model is not the purpose of this paper. Nevertheless it has to be mentioned that this whole approach is only feasible if the coefficients can be estimated.

These few remarks on the coefficients of the linear program have already touched on one of the major problems involved. Four main difficulties arise when the methods discussed are used in a more realistic context (1) what to maximize and (2) what coefficients to attach to the constraints; (3) how many constraints are there, and (4) are the variables truly independent of each other. In the present example, only the second, third and fourth difficulties arise. The second one has already been mentioned and disposed of and will again be referred to in the appendix (8), the third one will be referred to in section (2), the fourth is simply assumed away in this particular example but can be solved in particular cases at least, as is mentioned in section (3).

2. Optimal Investment Stimulation: A Linear Programming Model

Our model is an extension of the Dosser-Shoup-type models in public finance in the sense that more activities and constraints are given. If one assumes that the government and/or the central bank can influence private investment activity,[2] and therefore employment, by different kinds of tax, interest, subsidies, and depreciation policies, then linear programming provides a policy mix which in the degenerate case contains fewer policy measures for achieving the goal. As we shall see presently, there are two ways in which this goal may be formulated; either a budget is given and has to be allocated optimally between aims, or, the aims are given and the minimum budget loss necessary to reach these aims has to be determined.

One of the crucial assumptions of the model is that different policy measures have different consequences for various kinds of unemployed labor (skilled and unskilled labor) in a particular economy. For example, a general interest reduction of a given percentage by the central bank on new investment increases the present value of investments to be undertaken. It is therefore likely to result in the substitution of new equipment for old; generally investments with a high share of interest costs will be expanded. But this implies especially that skilled labor will be newly employed. It is also likely that this effect will not be weakened by adverse effects resulting from lower income out of interest payments. A tax rate cut in contrast normally cannot be applied to new investment alone because gains also accrue to past investments.

[2] C. F. C. Brown, Tax Incentives for Investment, American Economic Review, Papers and Proceedings, 52 (1962), p. 335 ff; R. A. Musgrave, The Theory of Public Finance, A Study in Public Economy, New York, 1959, p. 336 ff.

The effect of such a measure would therefore be of the pure income type. Higher profit income would not necessarily mean increased investment. In so far as consumer expenditures then would rise, this easily could lead primarily to new employment of unskilled labor. An investment tax credit on the other hand, which gives a subsidy to firms that invest in new equipment, tends to shorten the life of assets because it is available for every new investment. A greater influence on the employment of skilled or the employment of unskilled labor does not seem to be present. Finally a policy that provides an initial allowance with repayment over the lifetime of the equipment leads to a substitution of long-term for short-term investment and thus produces the desired increase in investment activity. The reason is again the effect on the present value of newly undertaken investments. The longer the lifetime, the higher the gain in interest which is either not to be paid or, is obtained for this amount of money which is now no longer needed for the particular investment, and therefore the higher the present value of the particular investment. Underlying these considerations is the further assumption that there exists no considerable amount of selffinancing. Then multiplier effects will lead to a general pickup of economic activity which in turn may result in new employment of unemployed workers and offset in the long run the initial tax loss to a certain extent.

But an increase in the present value of a planned investment does not necessarily imply higher actual investment. A further assumption therefore has to be introduced. The investment activity in this particular economy before adoption of the fiscal policy measures is so high that the present value of a newly planned investment would be zero, unless the government introduces a investment stimulating measure. This means that there exists an income equilibrium at underemployment.

Turning to the programming model, let y_1 = the level of tax loss resulting from a 10 percent interest cut, y_2 = the level of tax loss resulting from a 10 percent tax cut, y_3 = the level of tax loss resulting from a 10 percent tax credit, y_4 = the level of tax loss resulting from a 10 percent initial allowance with repayment over the lifetime of the investment. Furthermore for simplicity suppose there are only two kinds of unemployed labor: skilled labor (b_1) and unskilled labor (b_2). In a more realistic context this assumption can easily be dropped. Then one can formulate the policy goal in either of two ways: (1) full-employment of both kinds of unemployed labor with an undetermined tax loss, or (2) maximization of employment with a budget loss, fixed ex ante. In the following the first approach is used most.

A plausible assumption must be made about the numerical size of the coefficients; i.e. how much employment of skilled and unskilled labor, respectively, is produced by a one dollar tax loss. The dimension of the coefficients is man years of skilled and unskilled labor per dollar tax loss, respectively. It will be assumed that one dollar tax loss resulting from applying a 10 percent interest cut employs 3/10,000 man years of skilled labor and 2/10,000 man years of unskilled labor. As pointed out earlier these different coefficients reflect the stimulation of capital intensive investments. One may in a loose way infer, the higher the coefficient for skilled labor, the more capital intensive the investment. The respective coefficients for a 10 percent tax cut are 1/10,000 and 2/5,000. The underlying assumption is that the resulting income effect will lead to employment of more unskilled than skilled workers. The coefficients for a tax credit are 2/10,000 and 2/10,000. The assumption that underlies this is that the general shortening of the life of assets has no decisive influence on either one of the two kinds of employment. Finally it is assumed that the initial allowance, with repayment over the life time, has a larger employment effect on unskilled than on skilled labor. This assumption may be unrealistic. But nothing prevents our changing it in a different model. The respective numbers of skilled and unskilled labor are 10^6 and 2.10^6. These very strong assumptions on the size of the coefficients will be dropped in the Appendix (8).

A scale factor has now to be introduced. First, the coefficients and the numbers of unemployed are multiplied by 10,000; second, the numbers of unemployed are divided by 10^9. The following dual linear programming problem results:

$$\text{Min } Z = y_1 + y_2 + y_3 + y_4$$

$$\text{s.t.} \quad 3y_1 + y_2 + 2y_3 + y_4 = 10$$

$$2y_1 + 4y_2 + 2y_3 + 3y_4 = 20$$

$$y_1 \geqq 0, \quad y_2 \geqq 0, \quad y_3 \geqq 0, \quad y_4 \geqq 0 .$$

The dual maximization problem is:

$$\text{Max } Z' = 10x_1 + 20x_2$$

$$\text{s.t.} \quad 3x_1 + 2x_2 \leqq 1$$

$$x_1 + 4x_2 \leqq 1$$

$$2x_1 + 2x_2 \leqq 1$$

$$x_1 + 3x_2 \leqq 1$$

x_1, x_2 are unrestricted.

The budget loss is to be minimized in the primal, subject to employing the given amount of unemployed skilled and unskilled workers. The coefficients in the contraints show the various employment effects per dollar tax loss.

The total value of employment of skilled and unskilled workers (to the government) is to be maximized in the dual. x_1 and x_2 are the "prices" or marginal values of skilled and unskilled labor. x_3, x_4, x_5, x_6 are the opportunity "values" of using the different policies. For the marginal values to be optimal they must equal the marginal cost of policy j per dollar tax loss and the opportunity "values" must become zero. As will be seen presently the marginal value of policy one (interest cut) and two (tax rate cut) is equal to their respective one dollar tax loss. In the remaining two cases (tax credit and initial allowance) the value is less than the budget loss. These latter policies will therefore have to be abandoned in the given simple example.

There exists no direct relationship between these dual prices and the market prices of skilled and unskilled labor. The dual prices are determined by the coefficients of the linear program, i.e. by the coefficients of the constraints while the market prices are determined by the market forces. This shows the importance of the dual prices for public budgeting. It is only their relative size to other public expenditure and tax programs which is of interest.

To solve this linear program one uses the simplex method. In order to put the problem in canonical form, the slack variables y_5 and y_6 are introduced into the primal problem,[3] and x_3, x_4, x_5, x_6 into the dual. Combining

[3]For example, W.J. Baumol, Economic Theory and Operations Analysis, Englewood Cliffs, N.J., second edition, 1965, p. 116 ff; D. Gale, The Theory of Linear Economic Models, New York, 1960, p. 97 ff. For an inductive proof of the simplex algorithm see G. B. Dantzig, Linear Programming and Extensions, Princeton, 1963, p. 120 ff.

both the primal and the dual in one simplex table both programs are solved simultaneously.

Table I:

			x_1	x_2	
Z'	=	0	10	20	
x_3	=	1	-3	-2	y_1
x_4	=	1	-1	-4*	y_2
x_5	=	1	-2	-2	y_3
x_6		1	-1	-3	y_4
		= Z	$= -y_5$	$= -y_6$	

The maximization problem will be solved. Clearly 4* is then the pivot.

Table II:

			x_1	x_4	
Z'	=	5	5	-5	
x_3	=	1/2	-5/2*	1/2	y_1
x_2	=	1/4	-1/4	-1/4	y_6
x_5	=	1/2	-3/2	-1/2	y_3
x_6	=	1/4	-1/4	3/4	y_4
		= Z	$= -y_5$	$= -y_2$	

One more pivot step produces the result.

Table III:

			x_3	x_4	
Z'	=	6	-2	-4	
x_1	=	1/5	-2/5	1/5	y_5
x_2	=	1/5	-1/10	-3/10	y_6
x_5	=	1/5	3/5	1/5	y_3
x_6	=	1/5	1/10	7/10	y_4
		Z	$= -y_1$	$= -y_2$	

The solutions to the primal and to the dual can now be read off the table. They are:

$$y_1 = 2 \quad y_2 = 4 \qquad Z = 6\ ; \qquad x_1 = 1/5 \quad x_2 = 1/5 \qquad Z' = 6 \ .$$

The scale factor introduced above was 10^9. y_1, y_2, and Z have to be multiplied by this number. The final result is

$$y_1 = 2.10^9 \quad y_2 = 4.10^9 \qquad Z = 6.10^9\ ; \qquad x_1 = 1/5 \quad x_2 = 1/5 \qquad Z' = 6.10^9 \ .$$

The government has an optimal tax loss of $\$6.10^9$. It uses only policy one (interest cut) and policy two (tax cut). The dual maximal value of the employed labor is of course $\$6.10^9$.

The value of 1/10,000 man years of skilled labor to be employed is 1/5; 1/10,000 man years of unskilled labor has the same value to the government. One man year of skilled and unskilled labor is therefore worth \$2,000, respectively, to the government.

A slight change in the coefficients produces a degenerate solution which means that a single policy has to be used by the government. In a more realistic example this would mean that there are fewer optimal policies than in the non-degenerate case. Consider then the following dual linear programming problem:

$$\text{Min } Z = y_1 + y_2 + y_3 + y_4$$

$$\text{s.t.} \quad \begin{aligned} y_1 + 4y_2 + 2y_3 + 2_y &= 10.10^9 \\ 2y_1 + 4y_2 + 4y_3 + y_4 &= 20.10^9 \end{aligned}$$

$$y_1 \geqq 0 \,,\; y_2 \geqq 0 \,,\; y_3 \geqq 0 \,,\; y_4 \geqq 0 \;.$$

One can see at once that policy y_2 will be used. Its employment effect for skilled and unskilled labor dominates all other policies. The optimal feasible solution to this problem is:

$$y_2 = 5.10^9 \,,\; y_1 = y_3 = y_4 = 0 \;;\; Z = 5.10^9 \,.$$

The dual has the solution:

$$x_2 = 1/4 \,,\; x_1 = x_3 = x_4 = 0 \;;\; Z' = 5.10^9 \,.$$

The difference between non-degenerate and degenerate solutions may be a rather artificial one, because even small changes in one or more coefficients may bring about degeneracy. But small changes of coefficients become almost meaningless if projected to economic reality.

The solutions to these dual linear programs show another remarkable result. The absolute size of the budget loss may seem to be practically infeasible for a particular country. If the assumed coefficients should be realistic, the whole investment stilulation program could be infeasible. Either the true coefficients are greater than those used above, or, different fiscal policy measures to reach full employment must be adopted.

For simplicity in this example only two constraints and 4 variables were used. If this is the case in reality, no application of linear programming techniques to fiscal policy would be necessary because the optimal mix could be seen by inspection. But as already mentioned above there exists no obstacle to introducing many constraints with many variables. This just requires somewhat more computational work. Therefore to introduce the distinction between male and female unemployed or, different age classes of unemployed workers can easily be done. In the same way different monetary and fiscal policy instruments may be added. The program also may be divided into different sectors in an economy or different economic areas. In fact this will be done in a rudimentary way in the next section.

Arguing in this way toward more realism inevitably leads one to a dynamic approach to the problem. This does not necessarily mean a dynamic programming approach, as will be seen in the Appendix (7), but rather some sort of decentralization over time. To show this is not attempted in this paper.

3. Optimal Investment Stimulation: A Decentralized Model

Fiscal policy, in order to be successful, must be used on federal, state and local levels. Federal measures must not be completely neutralized or even hampered by state and local counter policies. In practice very often this harmony of action in economic policy is not reached. The main reason is a certain degree of state and local autonomy of taxation and expenditure. This does not prevent onself from imagining a complete agreement on economic policy aims of the federal and state governments. But even if the federal government and the state governments used their fiscal authority for fighting depressions or inflations this need not necessarily yield an optimal countercyclical policy. If the federal government and the states used their policy measures independently of each other this could mean a suboptimal use of fiscal authority. Again a simple example may help to clarify.

A given country may consist of two states and a federal government. The federal government may have some overall budgetary powers, the state governments may carry out within their boundaries additional taxation and expenditures. There may or may not exist aid from government to states or between states. This would not interfere with the state autonomy defined in the federal constitution.

Both states are assumed to have a current and a capital budget. State I may be less industrialized than state II. The total country may suffer from a depression which caused more unemployment in II than in I. There are again two kinds of unemployed workers, skilled and unskilled. Each state wants now to fight depression by its own budgetary measures, aided, of course, by federal fiscal policy. If the federal government is willing to pump 40 into the economy,[4] the government of state I is willing to spend 10 out of the current and 5 out of the capital budget, the government of II 15 out of the current and 15 out of the capital budget, this will cause different amounts of employment. We may assume that the employment effect of state expenditures and a tax cut is generally lower in I than in II, because I is less industrialized. In addition one may assume that the employment effect for skilled workers is generally higher if there are tax reductions than if expenditures are increased.

[4] For simplicity convenient numbers are used.

A general tax reduction may increase the present value of newly undertaken investments and provide new investible funds. On the other hand new expenditure programs may favor more unskilled workers if they are mainly used for road repair and a generally better road system. The necessary countercyclical federal action may have a relatively higher employment effect in state II because of possible agglomeration effects.

If

x_1 = level of new employment of skilled labor in state I

x_2 = level of new employment of unskilled labor in state I

x_3 = level of new employment of skilled labor in state II

x_4 = level of new employment of unskilled labor in state II

let the following linear program be given:[5]

$$\text{Max } E = x_1 + x_2 + x_3 + 2x_4$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 2x_2 + \tfrac{1}{2}x_3 + x_4 &\leq 40 \\ x_1 + 3x_2 &\leq 10 \\ 2x_1 + x_2 &\leq 5 \\ \tfrac{1}{2}x_3 + 2x_4 &\leq 15 \\ x_3 + \tfrac{3}{2}x_4 &\leq 15 \end{aligned}$$

$$x_1 \geq 0 \; ; \; x_2 \geq 0 \; ; \; x_3 \geq 0 \; ; \; x_4 \geq 0 \; .$$

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[5] The reader may realize that we adopted now for reason of convenience the second possible approach of formulation of the fiscal policy problem.

This linear programming problem can be solved by using the simplex method. But because of its special structure the decomposition algorithm[6] can be applied, too. The decomposition principle implies the idea of solving the subprograms separately and providing the optimal solutions to the subprograms to the central authority for use in the master program. The central executive then checks whether these subsolutions are optimal also given the master constraints. The master program uses convex combinations of the basic solutions to the subprograms. Using the dual prices of the master program new coefficients for the objective functions of the sub-programs are computed. The subprograms provide then new optimal solutions which go fractionally into a changed master program. This process is terminated when a shift of the scarce common factor from one state to another can no longer bring about an increase in value of the master objective function.

Denoting the basic solutions of the subprograms[7] as X^*_r, $r = 1,\ldots,4$, and X^*_q, $q = 4,\ldots,8$, the following master program can be derived:

$$\begin{aligned} \text{Max } E' &= 1(0.\mu_1) + 1(0.\mu_1) + 1(0.\nu_1) + 2(0.\nu_1) \\ \text{s.t.}\quad & 1(0.\mu_1) + 2(0.\mu_1) + \tfrac{1}{2}(0.\nu_1) + 1(0.\nu_1) \leq 40 \\ & \mu_1 = 1 \\ & \nu_1 = 1 \end{aligned}$$

or,

$$\begin{aligned} \text{Max } E' &= 0\mu_1 + 0\nu_1 \\ \text{s.t.}\quad & 0\mu_1 + 0\nu_1 \leq 40 \\ & \mu_1 = 1 \\ & \nu_1 = 1 \;. \end{aligned}$$

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[6] C.f. G. B. Dantzig, op. cit., p. 448 ff.; W. J. Baumol, T. Fabian, Decomposition Pricing for Decentralization and External Economies, Management Science, II (1964) p. 1ff.

[7] The reader may verify that each subprogram has four different basic feasible solutions. The first are of course zero activities.

Under the assumption that the feasible sets of the subprograms are bounded and convex, any X_i solving $A_1X = b_1$ or, $A_2X = b_2$ can be represented by a convex combination of the extreme points of the set of feasible solutions to the subprograms.[8] X_i, X, b_i are vectors, A_1, A_2 matrices. Hence given any basic feasible solution to the subprograms the master program can be derived which contains now the weighted averages X^*_r and X^*_q as variables. The unknown weights μ_i and ν_i have to be determined.

The solution of this first master program is obviously $E' = 0$, $\mu_1 = 1$, $\nu_1 = 1$, $W = 40$ (slack variable). This simply means there is no net increase in employment and therefore the total federal budget (and the state budgets) remain untouched. Clearly the dual prices or, simplex multipliers, are $\Pi = 0$, $\bar{\Pi}_1 = 0$, $\bar{\Pi}_2 = 0$, Π denoting the value of the federal budget dollar, $\bar{\Pi}_1$ denoting the marginal value of transferring some of the federal budget dollars to state I and $\bar{\Pi}_2$ denoting the marginal value of a transfer of federal budget dollars to state II. This is a feasible solution, but certainly not an optimal one. This will be shown by the next few steps.

Using the provisional dual prices, revised state unit value figures must be calculated. The idea behind this procedure is to account for the using up of the federal budget dollar by the state programs. We know that every employed worker requires state and federal measures. Because the use of these federal budget dollars by the states is different, they impose different costs on the federal budget. In order to reach an optimal allocation of the federal budget dollar, these different cost figures must go into the evaluation procedure. With the given subprograms and the given master solution it turns out that the

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[8] C. f. G. B. Dantzig, op. cit., p. 449.

dual price of the federal budget dollar is zero, of course, because the state activities are zero.

The revised unit values for the objective functions of the states, are then given by

$$R_{11} = 1 - \Pi = 1 \qquad S_{11} = 1 - \frac{1}{2}\ \Pi = 1$$

$$R_{12} = 1 - 2\Pi = 1 \qquad S_{12} = 2 - \quad \Pi = 2$$

The unit values of employed labor remain unchanged.

To obtain $E^2 > 0$ at least one of the two optimal solutions to the subprograms has to be employed in a new master program. One may select X_3^* of state I, for it contributes to net employment as may be seen by evaluating:

$$R_1 = R_{11} X_1 + R_{12} X_2 = 1 + 3 > \bar{\Pi}_1 = 0 .$$

The new master program will be

$$\text{Max } E^2 = 0\mu_1 + 4\mu_3 + 0\nu_1$$

$$\begin{aligned} \text{s.t.} \qquad 0\mu_1 + 7\mu_3 + 0\nu_1 + W &= 40 \\ \mu_1 + \mu_3 &= 1 \\ \nu_1 &= 1 . \end{aligned}$$

The solution to this master program is:

$$E^2 = 4, \quad \mu_3 = 1, \quad \nu_1 = 1, \quad W = 33 ,$$
$$\Pi = 0, \quad \bar{\Pi}_1 = 4, \quad \bar{\Pi}_2 = 0 .$$

Since Π is still zero, the subprograms need not be recomputed with new unit value fitures. Instead, the proposal X_6^* of the second subprogram has to be introduced into the master program, because $S_1 = 6S_{11} + 6S_{12} = 18 > \bar{\Pi}_2 = 0$.

Hence the third master program is:

$$\text{Max } E^3 = 0\mu_1 + 4\mu_3 + 0\nu_1 + 18\nu_3$$

$$\begin{aligned} \text{s.t.} \quad & 0\mu_1 + 7\mu_3 + 0\nu_1 + 9\nu_3 + W = 40 \\ & \mu_1 + \mu_3 = 1 \\ & \nu_1 + \nu_3 = 1 \end{aligned}$$

The solution to this program is $E^3 = 22,\ \mu_3 = 1,\ \nu_3 = 1,\ W = 24$,

$$\bar{\Pi}_1 = 4,\quad \bar{\Pi}_2 = 18,\quad \Pi = 0 .$$

The unit values in the subprograms must not be revised because still $\Pi = 0$; i.e. the public budget dollar is still a free good. Furthermore a comparison between R_1 and $\bar{\Pi}_1$, and S_1 and $\bar{\Pi}_2$, shows the following:

$$R_1 = 1 + 3 = 4 = \bar{\Pi}_1$$

$$S_1 = 6 + 12 = 18 = \bar{\Pi}_2$$

Because a further shift of the federal budget dollars from state I to state II and vice versa could not bring about any new increase in the total value of employment, the program is terminated[9] and must be optimal.

In this simple example the master constraint was not binding, the federal budget dollar was not a scarce resource and had therefore a zero price. This in turn led to the result, that independent maximization of employment in both states did not provide any distortions in the allocation of the federal budget dollars.

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[9] For a rigorous proof see: G. B. Dantzig, op.cit., p. 452, W. J. Baumol, T. Fabian, op. cit., p. 21 ff.

In other cases where the federal budget deficit is limited and the budget dollars become a scarce factor, and where there are additional constraints the result may be different.[10] Then both states have to compete for the scarce federal budget dollars and the new allocation of funds leads to an optimal set of variables which is different from the initial optimal solutions to the subprograms under independent maximization. This shows a very strong case then for coordinated action between the federal government, the state governments and the local authorities.

This analysis leads to the problem of externalities which are present in any macroeconomic fiscal policy setting. As Baumol and others[11] have shown these problems may be solved in a decomposable linear or nonlinear program.

4. Optimal Investment Stimulation: A Two-Person Zero Sum Game Model

In this section it will be shown that the linear programming problem of section 2 is equivalent to a two-person zero-sum game. Game theory is here used as a mathematical device not as a model of economic behavior. Economic policy in general and the relations between government and the private sector of an economy especially are a rich source of application of game theory as a model of behavior. But to demonstrate this is not intended in this paper. The adjusted minimization problem was:

$$\text{Min } Z = y_1 + y_2 + y_3 + y_4$$

$$\text{s.t.} \quad 3y_1 + y_2 + 2y_3 + y_4 \geqq 10$$

$$2y_1 + 4y_2 + 2y_3 + 3y_4 \geqq 20$$

$$y_1 \geqq 0, \quad y_2 \geqq 0, \quad y_3 \geqq 0, \quad y_4 \geqq 0 .$$

[10] Compare the example of W. J. Baumol, T. Fabian, op.cit., p. 24 ff.

[11] C.f. W.J. Baumol, T. Fabian, op.cit., p. 18 ff., and Ch. J. Hitch, R. N. McKean, op.cit., p. 396 ff.

Because all coefficients are of the same sign and the values of the different policies in the objective function are equal to one, one can use a simplified method to transform this linear program into a two-person zero-sum game.[12] The system has to be rewritten by dividing each row i by the coefficients b_i's. The new program is:

$$\text{Min } A = y_1 + y_2 + y_3 + y_4$$

$$\text{s.t.} \quad 3/10y_1 + 1/10y_2 + 2/10y_3 + 1/10y_4 = 1$$

$$2/20y_1 + 4/20y_2 + 2/20y_3 + 3/20y_4 = 1$$

$$y_1 \geqq 0 \, , \; y_2 \geqq 0 \, , \; y_3 \geqq 0 \, , \; y_4 \geqq 0 \, .$$

But this minimization problem equals exactly the maximization problem of the row player (maximin-player). This can be shown easily.

Assume a 2X2 zero-sum game matrix:

a_{11}	a_{12}
a_{21}	a_{22}

Player one, the row player, wants to maximize the value of the game, V ;

$$\text{Max } V$$

$$\text{s.t.} \quad a_{11}x_1 + a_{21}x_2 \geqq V$$

$$a_{12}x_1 + a_{22}x_2 \geqq V$$

$$x_1 + x_2 = 1$$

$$x_1 \geqq 0, \; x_2 \geqq 0 \, .$$

[12] J. v. Neumann and O. Morgenstern, Theory of Games and Economic Behaviour, Third ed., Princeton, 1953, p. 85 ff.; R.D. Luce and H. Raiffa, Games and Decisions, New York, 1957, p. 423.

It can be shown that this maximization problem is equal to a minimization problem. By dividing the constraints by V and writing $y'_j = x_j/V$ it is seen that player one can obtain at least V if there is a vector y', where $y'_j \geqq 0$, for i=1,2, and $\sum_j y'_j = 1/V$, such that:

$$\begin{aligned} \text{Min} \quad & y'_1 + y'_2 \\ \text{s.t.} \quad & a_{11}y'_1 + a_{21}y'_2 \geqq 1 \\ & a_{12}y'_1 + a_{22}y'_2 \geqq 1 \\ & y'_1 + y'_2 = 1/V \\ & y'_1 \geqq 0 , \quad y'_2 \geqq 0 . \end{aligned}$$

This minimization problem has the same structure as the minimization problem of section 1. It is therefore allowed to write the original linear programming problem in matrix form. The solution of this game requires of course that the equal signs are valid, i.e. that the difference between $\frac{1}{\sum_j y_j}$ and the value of the game equals zero.[13] Player one faces the following game with the strategy set $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, player two with strategies (β_1, β_2).

	β_1	β_2
α_1	3/10	2/20
α_2	1/10	4/20
α_3	2/10	2/20
α_4	1/10	3/20

It is easy to see that row three in the matrix game is majorized by row one, and row four is majorized by row two; i.e. the majorizing row has at least as big

[13] J. v. Neumann and O. Morgenstern, op.cit., p. 143 ff.

values as the majorized row. The problem is therefore reduced to a two by two matrix which can easily be solved.

If $\alpha^0 = (\alpha_1^0, \alpha_2^0)$ is player one's unique maximin strategy, and the value of the game is V, the numerical values for the example are $\alpha^0 = (1/3, 2/3)$; $V^{-1} = 1/6$ or $V = 6$. Similarly for player two $\beta^0 = (1/3, 2/3)$; $V^{-1} = 1/6$ or $V = 6$. This result is equal to the solution of the original linear programming problem. The equilvalence between the linear programming problem and the corresponding two-person zero-sum game and vice versa is shown.[14] Both methods can therefore be transformed into each other.

The equivalence is certainly a mathematical one. An economic interpretation does make little sense and may be a little strained in this particular example. The government minimaximizes the value of the game which is, as shown in the above example, equal to minimizing costs in the linear programming problem. The zero-sum assumption underlying the technique seems to be existent only in border cases, where a revolutionary population acts contrary to the government. Otherwise it should be also in the interest of the individual citizen to reach full employment and this at minimum costs. The assumption, as it is sometimes made, that the government minimaximizes against nature is also not very convincing, at least in this particular example; for entrepreneurs act decidedly, strategically. But this nevertheless can be looked upon by the government as uncertain and therefore as a game against nature.

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[14] Of course in an intuitive way only. For rigorous proof see: G. B. Dantzig, op.cit., p. 286 ff.

5. Marginal Values of the Original Linear Programming Problem and the Derived Game

In this section we are concerned with the rate of change of value, i.e. the marginal value,[15] of the matrix game as the entries of the game are varied.

The marginal value of the preceding game matrix G in respect to the augmented matrix H is defined as:

$$\frac{\partial \, \Delta(G)}{\partial H} = \lim_{\alpha \to 0^+} \frac{\Delta(G + \alpha H) - \Delta(G)}{\alpha} \, .$$

Furthermore the optimal strategies of player one are:

$$Y^0(G) = \{ y \epsilon \mathbf{Y} \mid yGx \geqq \Delta(G) \quad \text{if} \quad x \epsilon X \} ,$$

and for player two:

$$X^0(G) = \{ x \epsilon X \mid yGx \leqq \Delta(G) \quad \text{if} \quad y \epsilon Y \} .$$

Y and X are the sets of probability vectors in E_m and E_n, such that

$$Y = \{ y = (y_i) \mid y_i \geqq 0 \, , \; \sum_i y_i = 1 \} ,$$

$$X = \{ x = (x_j) \mid x_j \leqq 0 \, , \; \sum_j x_j = 1 \} .$$

Player one selects a $y \epsilon Y$, player two selects a $x \epsilon X$, each in ignorance of the other's choice, and then, 2 pays 1 the amount $yGx = \sum_{i,j} y_i g_{ij} x_j$.

The Main Theorem[16] of matrix games tells how to behave in this situation:

<u>THEOREM</u>: Given a matrix G, there exists a number,

$$\Delta(G) = \max_{y \epsilon Y} \min_{x \epsilon X} yGx = \min_{x \epsilon X} \max_{y \epsilon Y} yGx ,$$

and non empty sets of $Y^0(G)$ and $X^0(G)$.

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[15] C.f. H.D. Mills, Marginal Values of Matrix Games and Linear Programs, <u>Linear Inequalities and Related Systems</u>, ed. by H. W. Kuhn and A. W. Tucker, Annals of Mathematics Studies, 38, Princeton, 1956, p. 183ff.

[16] J. v. Neumann and O. Morgenstern, <u>op.cit.</u> p. 153 ff.

Mill's Theorem[17] (1) states that:

THEOREM: The marginal value of the game G with respect to H is the value of the unconstrained matrix game H where the players are restricted to their sets of optimal strategies in the game G . That is

$$\frac{\partial\, \Delta(G)}{\partial H} = \max_{y \in Y^0(G)} \; \min_{x \in X^0(G)} \; yHx \; .$$

In the example G is the following matrix:

$$G = \begin{pmatrix} 3/10 & 1/10 \\ 1/10 & 1/5 \end{pmatrix} \text{ and } y^0 = (1/3,\ 2/3), \quad x^0 = (1/3,\ 2/3) \; .$$

If

$$H = \begin{pmatrix} 3/100 & 1/100 \\ 1/100 & 1/50 \end{pmatrix} \text{ and } (G + \alpha H) = \begin{pmatrix} 33/100 & 11/100 \\ 11/100 & 11/50 \end{pmatrix} \text{ where } \alpha = 1,$$

then

$$\frac{\partial\, \Delta(G)}{\partial H} = 1/60 \; .$$

Reconverted into the linear programming problem this equals $6/11.10^9$. This number has to be subtracted from 6.10^9 to obtain the new optimal value of the linear program. The marginal value of the game increases, as the optimal value of the linear programming problem decreases. This seems to be intuitively clear, because higher coefficients in the linear programming problem mean that one dollar tax loss of activity i produces now employment of a higher amount than before; consequently the total tax loss decreases.

[17] H. D. Mills, op. cit., p. 184 ff.

The value of the game decreases, and therefore the optimal value of the dual linear program increases, if

$$(G-\alpha H) = \begin{pmatrix} 27/100 & 9/100 \\ 9/100 & 9/50 \end{pmatrix}, \quad \text{therefore } H = \begin{pmatrix} 3/100 & 1/100 \\ 1/100 & 1/50 \end{pmatrix} \text{ and } \alpha = 1 .$$

Then

$$\frac{\partial \; \Delta(G)}{\partial H} \quad 1/60 .$$

The optimal value of the linear program increases by $2/3.10^9$, i.e. from 6.10^9 to $20/3.10^9$. This can be explained by the smaller employment effect of a dollar tax loss.

The corollary to Mills' theorem (1) states:

COROLLARY: The marginal value of the game G with respect to a single entry is, for $i = 1,\dots,m$ and $j = 1,\dots,n$:

$$\frac{\partial \; \Delta(G)}{\partial g_{ij}^{+}} = \left(\max_{y \in Y^0(G)} y_i \right) \left(\min_{x \in X^0(G)} x_j \right)$$

and

$$\frac{\partial \; \Delta(G)}{\partial g_{ij}^{-}} = \left(\min_{y \in Y^0(G)} y_i \right) \left(\max_{x \in X^0(G)} x_j \right) ,$$

where

$$\frac{\partial \; \Delta(G)}{\partial g_{ij}^{+}} \quad \text{and} \quad \frac{\partial \; \Delta(G)}{\partial g_{ij}^{-}}$$

are positive or negative directional derivatives.

If one changes G to $(G + \alpha H)$, where

$$(G+\alpha H) = \begin{pmatrix} 3/10 & 3/20 \\ 1/10 & 1/5 \end{pmatrix} \quad \text{and} \quad (G-\alpha H) = \begin{pmatrix} 3/10 & 1/20 \\ 1/10 & 1/5 \end{pmatrix}$$

and

$$H = \begin{pmatrix} 0 & 1/20 \\ 0 & 0 \end{pmatrix}, \quad \alpha = 1,$$

then

$$\frac{\partial\,\Delta(G)}{\partial g_{12+}} = 2/9.1/20 = \frac{1}{90} \quad \text{and} \quad \frac{\partial\,\Delta(G)}{\partial g_{12-}} = 2/9.1/20 = 1/90\ .$$

The value of the game is increased or decreased by this amount. This means that the optimal value of the linear program is decreased by $(90/15-90/16).10^9 = 3/8.10^9$, and increased by $(90/14-90/15).10^9 = 3/7.10^9$, respectively.

6. Conclusion

As a result of this simple fiscal policy example, the importance of the coefficients in this linear programming problem, or its counterpart, the two-person zero-sum matrix game, has been shown. With empirical coefficients it can happen that there exists only one degenerate solution or fewer solutions than there are constraints. In this case the government uses a different policy mix. It is possible that the necessary budget loss, required to employ all labor, exceeds the current size of the public budget. In this case different policies may have to be used. The introduction of decentralized decision making suggests cases for which it is necessary to reach fully coordinated action. The decomposition principle shows a possibility of how to deal with external effects in government, state and local economic policy. The analysis shows further that there are at least two methods available to treat such problems, the linear programming approach and the two-person zero-sum game theory. Which method to apply depends on the given problem. Finally Mills' theory of marginal values gives some insight into the rate of change of the optimal values of a game. Some considerations in the appendix will show that a dynamic programming formulation of the problem may not always be feasible and, the determinateness of the coefficients may be abandoned and replaced by stochastic coefficients.

gives the optimal amount of tax loss at each stage. Given these notions the simple linear programming problem of an optimal fiscal policy mix can be cast into a dynamic linear programming problem which, as will be seen, requires much more computation than the simplex method. But it must be pointed out now, that the usefulness of dynamic programming does normally not lie in replacing linear programming but in providing a new method of determining global maxima for linear as well as non-linear programming problems.

7.2 Formulation of the Dynamic Programming Problem

The linear programming problem of section 2 had the form:

$$\text{Min} \quad Z = \sum_{j=1}^{4} y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{4} a_{1j} y_j = b_1 \qquad (1)$$

$$\sum_{j=1}^{4} a_{2j} y_j = b_2$$

$$y_j \geq 0 \,, \qquad j = 1,\ldots,4 \,.$$

This linear programming problem can be considered as a 4 stage decision problem where at stage j the optimal value of y_j, the optimal tax loss resulting from policy j, is selected. The parameters are the vectors $\{\beta_1\}$ and $\{\beta_2\}$, the stage variables y_1 through y_4. When y_j is specified, then $a_{1j}y_j$ man years of unemployed skilled labor and $a_{2j}y_j$ man years of unskilled labor will be employed by using policy j. It will be assumed throughout this section that the state parameters and the state variables can assume only integral values.

Appendix:

7.0 A Dynamic Programming Formulation of the Simple Fiscal Policy Model[18]

7.1 Introductory Remarks

Dynamic Programming is based on some simple concepts:

(1) One dimensional (1 constraint) or multidimensional (more than 1 constraint) dynamic programming problems are completely described at every stage (policy) by their state parameters. Each stage corresponds to a different policy. The designation of these parameters is arbitrary. The less there are the easier the problems are to solve. Thus it is advantageous to decrease the number of state parameters to that point where any further simplification would destroy the utility of a given model. The state parameter in our given problem is the number of unemployed workers still existing after k policies have already been applied.

(2) At every stage there is a given state function which depends on the state parameters. This function is defined as the optimal value of the objective function given these state parameters. In the given problem the state function would be the optimal (minimal) tax loss for each given state (amount of unemployment).

(3) The optimal values of the stage variables for stage k, given the state parameters at stage k, are represented by the policy function. The stage variable in our example is the amount of tax loss, i.e. y_j. The policy function

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[18] R. Bellman, Dynamic Programming, Princeton, 1957, passim. R. A. Howard, Dynamic Programming, Management Science, 12 (1966), p. 317 ff.; G. Hadley, Nonlinear and Dynamic Programming, Reading, 1964, p. 350 ff.

The computational procedure for solving this dynamic programming problem is the following. Denote by Z^* the absolute minimum of (2),

$$Z^* = \min_{y_1,\ldots,y_4} \left\{ \sum_{j=1}^{4} c_j y_j \right\} , \tag{2}$$

where the minimzation is carried out over non-negative y_j which satisfy the constraints

$$\sum_{j=1}^{4} a_{ij} y_j = b_i , \qquad \text{for } i=1,2. \tag{3}$$

Suppose now one selects one value of y_n and, holding y_n fixed, maximizes Z over $y_1,\ldots,y_{n-1}$. If this procedure is done for every allowable value of y_n , then, finally Z^* will be the largest of these z values and the corresponding set of y_j will maximize Z . This computational procedure in fact is the essence of dynamic programming.

In equation form, first, select a value of y_n and compute

$$\min_{y_1,\ldots,y_{n-1}} \left\{ \sum_{j=1}^{n} c_j y_j \right\} = c_n y_n + \min_{y_1,\ldots,y_{n-1}} \sum_{j=1}^{n-1} c_j y_j . \tag{4.1}$$

Once y_n has been chosen $y_1,\ldots,y_{n-1}$ must be restricted to non-negative integers which satisfy the constraints

$$\sum_{j=1}^{n-1} a_{1j} y_j = b_1 - a_{1n} y_n$$

and

$$\sum_{j=1}^{n-1} a_{2j} y_j = b_2 - a_{2n} y_n . \tag{4.2}$$

The relationship between y_n and $\min\limits_{y_1,\ldots,y_{n-1}} \sum\limits_{j=1}^{n-1} c_j y_j$ is straightforward.

Because of (4.2) it will be defined as:

$$\Lambda_{n-1}(b_1-a_{1n}y_n, b_2-a_{2n}y_n) = \min_{y_1,\ldots,y_{n-1}} \sum_{j=1}^{n-1} c_j y_j \quad , \tag{5}$$

where the minimization again is carried out for non-negative integers $y_1,\ldots,y_{n-1}$ satisfying the constraints. If this is computed for every allowable value of y_n, then the minimum will be

$$Z^* = \min_{y_n} [\, c_n y_n + \Lambda_{n-1}(b_1-a_{1n}y_n, b_2 a_{2n}y_n)] \tag{6}$$

with values of y_n from $0,1,\ldots,\delta_n$, where

$$\delta_n = \min\left\{ \frac{\xi_1}{a_{1n}} , \frac{\xi_2}{a_{2n}} \right\} \quad \text{and} \quad \xi_i = b_i - a_{in}y_n$$

for $i=1,2$.

This procedure determines the state function, i.e. an optimal value of the objective function at a given stage (countercyclical policy); and it determines simultaneously the optimal value of y_n, y_n^* (policy function). The problem would then be solved if one knew $\Lambda_{n-1}(b_1-a_{1n}y_n, b_2-a_{2n}y_n)$.

Because the a_i, b_i, y_j are non-negative integers, $\Lambda_{n-1}(b_1-a_{1n}y_n, b_2-a_{2n}y_n)$ can be written as $\Lambda_{n-1}(\xi_1,\xi_2)$ where ξ_1,ξ_2 are non-negative integers.

In order to compute $\Lambda_{n-1}(\xi_1, \xi_2)$ one proceeds as above and obtains:

$$\Lambda_{n-1}(\xi_1, \xi_2) = \min_{y_{n-1}} [c_{n-1}y_{n-1} + \Lambda_{n-2}(\xi_1-a_{1,n-1}y_{n-1}, \xi_2-a_{2,n-1}, y_{n-1})] \tag{7}$$

ξ_2 ranges from $0, 1, \ldots,$ to b_2 ,

then finally $y_1(\xi_1, \xi_2)$, the value of y_1 which minimizes $c_1 y_1$, can be determined.

The idea may become clearer by using a table, such as table I.

TABLE I

$\xi_1, \bar{\xi}_2$	$\Lambda_1(\xi_1, \bar{\xi}_2)$	$y_1(\xi_1, \bar{\xi}_2)$
0, 0	$\Lambda_1(0, 0)$	$y_1(0, 0)$
1, 0	$\Lambda_1(1, 0)$	$y_2(1, 0)$
2, 0	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .
b_1, 0	$\Lambda_1(b_1, 0)$	. . .
0 , 1	$\Lambda_1(0, 1)$	. . .
1 , 1	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .
b_1, 1	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .

It may turn out that $y_1(\xi_1, \xi_2)$ is not unique. Then there are several optimal solutions, a fact which may be of great importance.

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then finally $y_1(\xi_1, \xi_2)$, the value of y_1 which minimizes $c_1 y_1$, can be determined.

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TABLE I

$\xi_1, \bar{\xi}_2$	$\Lambda_1(\xi_1, \bar{\xi}_2)$	$y_1(\xi_1, \bar{\xi}_2)$
0, 0	$\Lambda_1(0, 0)$	$y_1(0, 0)$
1, 0	$\Lambda_1(1, 0)$	$y_2(1, 0)$
2, 0	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .
b_1, 0	$\Lambda_1(b_1, 0)$	. . .
0 , 1	$\Lambda_1(0, 1)$	. . .
1 , 1	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .
b_1, 1	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .
. .	. . .	. . .

It may turn out that $y_1(\xi_1, \xi_2)$ is not unique. Then there are several optimal solutions, a fact which may be of great importance.

Having obtained $\Lambda_1(\xi_1, \xi_2)$ one proceeds to compute Λ_2, again, for every allowed (integer values!) combination of $\{\xi_1, \xi_2\}$. One uses the familiar formula

$$\Lambda_2(\xi_1,\xi_2) = \min_{0 \le y_2 \le \delta_2} [c_2 y_2 + \Lambda_1(\xi_1 - a_{12} y_2, \xi_2 - a_{22} y_2)] \tag{11}$$

for $\delta_2 = \min\{\frac{\xi_1}{a_{12}}, \frac{\xi_2}{a_{22}}\}$.

y_2 ranges over integers in the interval $0 \le y_2 \le \delta_2$. Because $\Lambda_1(\xi_1, \xi_2)$ was evaluated for every integer combination of $\{\xi_1, \xi_2\}$, $\Lambda_1(\xi_1 - a_{12}y_2, \xi_2 - a_{22}y_2)$ can be easily read off table I. But a table of the same form as table I can be constructed for $\Lambda_2(\xi_1,\xi_2)$ and $y_2(\xi_1,\xi_2)$, for $\Lambda_3(\xi_1,\xi_2)$ and $y_3(\xi_1,\xi_2)$, finally for $\Lambda_4(b_1,b_2)$ and $y_4(b_1,b_2)$. One does not need a table of $\Lambda_4(\xi_1,\xi_2)$ values, because it follows from the preceding considerations that the last maximization is done only for b_1 and b_2 and not all integral combinations of $\{\xi_1,\xi_2\}$.

The optimal set $\{y^*\}$ remains to be determined. This is easily done by looking up the local optima of $y(\xi_1,\xi_2)$. Once y_n^* is known, then the remaining n-1 variables must satisfy

$$\sum_{j=1}^{n-1} a_{ij} y_{ij} = b_i - a_{in} y_n^* , \qquad i=1,2 . \tag{12}$$

Thus $\sum_{j=1}^{n-1} c_j y_j$ has to be minimized for non-negative integers y_j subject to the above constraints (12). But we know that this minimum is simply $\Lambda_{n-1}(b_1 - a_{1n}y_n^*, b_2 - a_{2n}y_n^*)$. We know also that $y_{n-1}(b_1 - a_{1n}y_n^*, b_2 - a_{2n}y_n^*)$ is (are) the minimal value(s) which can be read off the tables. In the same way one finds

$$y^*_{n-i} = y_{n-i}(b_i - \sum_{u=0}^{i-1} a_{1,n-u}y^*_{n-u}, b_2 - \sum_{u=0}^{i-1} a_{2,n-u}y^*_{n-u}) \tag{13}$$

We have now derived a second numerical procedure for solving the linear-programming fiscal-policy model of section 2.

A comparison of both methods, the linear versus the dynamic programming approach, yields two interesting points. (1) The question of multiple optimal solutions; (2) the computational feasibility. As regards point (1) it can turn out that the policy function at one or several stages is not unique, the value function, of course has to be unique by definition. The reason for this non-uniqueness may be explained intuitively. A linear program has multiple optimal solutions when the linear objective function is tangent to one of the linear constraints; or, in a multi-dimensional framework, hyperplanes are tangent to each other. In the same way if there is a non-linear objective function, this function may be tangent at more than one point to the linear (or non-linear) constraints. Dynamic programming has the advantage of providing such a method for computation of these multiple optimal policy functions.

Given a non-unique policy function then there exists a choice between two or more sets of policy functions which all lead to the same unique value function.[2] This fact may be of far-reaching importance in economic policy, because very often economically optimal policies could therefore substantially increase political feasibility of fiscal policy.

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[19] C.f. R. A. Howard (loc. cit., p. 328 ff.) who presents a three stage model of production which allows the adoption of two different optimal policies of allocation over time.

As far as point (2) is concerned, one may encounter serious difficulties in applying the dynamic programming framework to linear and non-linear programming problems.

The number of state parameters may exceed the computational capacity of modern computers. In our problem there are only two state parameters; the first can assume 10+1, the second 20+1 different values. This means that at three stages 3X(11X21) = 693 entries have to be computed. Because every entry y_i can assume different values the amount of computation soon becomes considerable - even in such a simple problem. The computational effort is therefore much higher than in linear programming, in fact, it may soon become unsolvable if the number of state parameters exceeds 3.[20]

8. Formulation of a Simple Stochastic Programming Problem with Random Variables Appearing in the Technological Coefficients

In section 2 the simple linear programming problem was formulated as a "diet problem." The aim was to minimize the budget loss or deficit required to provide employment for a given number of man-years of unemployed, skilled and unskilled, labor. For simplicity it was assumed, among other things, that the employment effects of different fiscal and monetary anticyclical policies were deterministic. This is certainly a strong assumption given the stochastic nature of business and consumer behavior. In the following this assumption will therefore be dropped. Instead of given coefficients a_{ij} it will be assumed, that these coefficients are random variables.

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[20] Compare J. Hadley, op. cit., p. 425.

It is well known that these employment effects resulting from different anticyclical policies may only, if at all, be predicted randomly. The differences between actual and predicted outcome therefore may be considerable. This fact in turn suggests a re-formulation of the objective function. When the a_{ij} are random variables, then for any given set of y_j it may happen that the total amount of labor i demanded is greater, or, as will be seen later, smaller than b_i. In certain cases this could mean a necessity to work overtime. Nevertheless for the whole economy there exists an employment ceiling - no matter whether we assume 85 percent or 90 percent of total capacity as full-employment level of the economy - which, if exceeded, causes some amount of (demand-pull) inflation. But this in turn could mean considerable social costs in resulting inflationary pressures. So this whole model is based on the lack of knowledge of the precise measures for anticyclical policies.

It will here be assumed that if the actual amount of labor i exceeds by w_i the amount originally available for employment, then a social cost $\pi_i w_i$ is incurred. The social costs of overemployment may - as already mentioned - be incurred even before all the unemployed have found new jobs. But this fact does not influence the general idea of the model.

The linear programming problem with deterministic coefficients had the form:

$$\text{Min } Z = \sum_{j=1}^{4} y_j \qquad (1)$$

$$\text{s.t.} \quad \sum_{j=1}^{4} a_{ij} y_j = b_i , \quad \text{for } i=1,2$$

Suppose now we wish to determine a set of y_j which minimizes the expected overemployment costs. In order to be able to do that the density functions of the a_{ij}, $\Phi_{ij}(a_{ij})$, have to be known. Again, only empirical evidence can show what kind of distribution function fits best, what mean and variance to use. Suppose that the coefficients can be approximated by normal distributions, then $E(a_{ij}) = \mu_{ij}$ is the mean and $E(a_{ij}-\mu_{ij})^2 = \sigma^2_{ij}$ is the variance of $\Phi_{ij}(a_{ij})$. To obtain a solution to the stochastic programming problem the random variables a_{ij} have to be transformed[21] into a set of new random variables $c_i = \sum_{j=1}^{4} a_{ij}y_j$; $i=1,2$. Of course c_i itself consists of the random variables $c_{ij} = a_{ij}y_j$. These new random variables express the employment effects of policies j which cause a y_j dollar tax loss. Determining the density function of c_i, two assumptions must be fulfilled: (1) there are no increasing returns to scale present, or, the random variable denoting the employment effect of policy j given a one dollar tax loss is independent of itself, i.e., for a given j the random variable does not change with different levels of tax loss. (2) there is in addition no (strong) interdependence between the employment effects of different policies. This, of course, is a very strong assumption in a macroeconomic model.

Given these two requirements the density function of c_{ij}, $\Phi^{(y)_j}(c_{ij})$, is the y_j-fold convolution[22] of $\Phi_{ij}(a_{ij})$. It is easy to show that the mean of this new density function is $y_j\mu_{ij} = \hat{\mu}_{ij}$, the variance is $y_j^2\sigma_{ij}^2 = \hat{\sigma}_{ij}$. The density function $\Phi_i(c_i)$ can now be obtained as the convolution of the $\Phi_{ij}^{(y_j)}(c_{ij})$ for fixed i. This procedure is legal because convolution gives the distribution of a sum of independent random variables. The expected value μ_i and the

[21] C.F. K. A. Brownlee, Statistical Theory and Methodology in Science and Engineering, New York, 1965, p. 45 ff.

[22] C. V. N. Feller, An Introduction to Probability Theory and its Applications, New York, London, Sydney, 1966 (Third Printing), p. 250 ff. G. Hadley, op. cit. p. 169.

variance σ_i^2 can easily be computed from $\sum_{j=1}^{4} a_{ij} y_j = \sum_{j=1}^{n} c_{ij} = c_i$.

The stochastic programming problem can be reduced to minimizing the expected costs of inflationary pressure. The objective function is then, for $w_i = c_i - b_i$,

$$\text{Min} \sum_{i}^{2} \pi_i \int_{b_i}^{+\infty} (c_i - b_i)\, \Phi_i(c_i) dc_i \tag{2}$$

$$\text{s.t.} \quad y_j \geq 0 \ .$$

Therefore, if all coefficients of the linear constraints are stochastic, the constraints disappear. They are implicit in the objective function.

There exists a second possibility for formulating the macroeconomic stochastic programming model. Despite the strict equality signs in the constraints, not all the available labor may find employment. The stochastic coefficients may render it impossible to predict exactly the employment effects of a given fiscal and monetary policy program. The social costs arising from the possibility of continuing unemployment must therefore be added to the costs of threatening inflationary pressure. Because it is common-place to estimate these costs in monetary terms they may be added to the government tax loss.

Again, if π_i is the cost of a shortage of labor of type i , if $c_i - b_i = w_i$ is the labor shortage, if k_i is the cost of unemployment, and if $b_i - c_i = u_i$ are the unemployed workers of type i , then the expected cost of continuing unemployment or induced over-employment is:

$$\pi_i \int_{b_i}^{-\infty} (c_i - b_i)\, \Phi_i(c_i)\, d\, c_i + k_i \int_{0}^{b_i} (b_i - c_i)\, \Phi_i(c_i)\, d\, c_i$$

or because,

$$k_i \int_0^{b_i} (b_i - c_i)\, \Phi_i(c_i)\, d\, c_i = k_i(b_i - \mu_i) + $$

$$+ \; k_i \int_{b_i}^{+\infty} (c_i - b_i)\, \Phi(c_i)\, d\, c_i \quad ;$$

the expected cost is

$$\sum_{i=1}^{2} k_i(b_i - \mu_i) + \sum_{i=1}^{2} (k_i + \pi_i) \int_{b_i}^{+\infty} (c_i - b_i)\, \Phi\,(c_i)\, d\, c_i \;.$$

If the tax loss is added, the government faces the following minimization problem:

$$\text{Min} \quad \sum_{j=1}^{4} a_j\, y_j + \sum_{i=1}^{2} k_i(b_i - \mu_i) + \sum_{i=1}^{2} (k_i + \pi_i) \int_{b_i}^{+\infty} (c_i - b_i)\, \Phi(c_i)\, d\, c_i$$

$$\text{s.t.} \quad y_j \geq 0 \;.$$

There are some difficulties involved in this treatment of our original linear programming problem: (1) The density functions of the coefficients must be known. Even if the coefficients are evaluated by estimation methods their variances may be very high. In the worse cases there would not exist any stable parameter at all. For longer run considerations the model therefore does not seem feasible. (2) The independence of a given random variable from different amounts of tax loss and from other different random variables. Large scale economies do seem existent for a given policy. With growing optimism the employment effects of a given policy will probably increase. Also a general tax decrease may have far more favourable employment effects if there exists already an easy money market or vice versa. (3) Finally, even if all these problems did not exist, there would be computational difficulties. Solving the problem by assuming that the c_i are large sums of random variables and applying the central limit theorem, would only yield a local minimum. The reason is that the objective function need not be convex at all. A global maximum only could be found by casting the problem into a dynamic programming framework. But here, as we already know, the computational feasibility is limited.

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13. ABSTRACT

Various techniques of mathematical programming are applied to a simple fiscal policy problem. First the problem is stated as a linear program. Then the decomposition principle is applied to a decentralized version of the fiscal policy problem. The equivalence of a linear programming formulation of the problem and the formulation as a two-person zero-sum game is shown. Finally, the rate of change of the value of the game is investigated. The linear program is cast into dynamic programming formulation in an appendix. Eventually the coefficients of the constraints are assumed to be stochastic.

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