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FOUNDATIONS OF NON-COOPERATIVE GAMES

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PREFACE

The most important aim of the material presented in this manuscript is the introduction of some new concepts for non-cooperative games. These ideas are closely related to the basic idea of an equilibrium point. Many of them are extensions of this important concept. We are concerned most of the time with characterizing these ideas and with finding conditions for their existence. Some simple applications are also presented.

The manuscript is divided into two parts. The main part, which is self-contained, consists of chapters I, II, IV and V and centers on the use of Kakutani's fixed point theorem in a way similar to Nash's treatment of equilibrium points in [12]. In addition, some intuitive considerations are presented and some simple applications to mixed extensions of finite games are made. The second part consists of chapters III and VI and involves some advanced generalizations of the results of the first part. For this reason the corresponding chapters have been marked by asterisks.

The treatment in the second part extends several important methods designed to deal with equilibrium points, namely: the Fan-Glicksberg fixed point theorem [2] and [6] which generalizes that of Kakutani; the very recent method of intersecting sets having a convex cylinder due to Fan [5] and finally the procedure using the maximum function originated in [13] by Nash and extended by Nikaido-Isoda [16]. Several simple and straightforward applications to mixed extensions of continuous games are presented.

This manuscript is based on work done during my stay at the Econometric Research Program of Princeton University. It includes and extends results presented in several papers there.

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TABLE OF CONTENTS

CHAPTER I

I.1. Two-Person Games: Saddle Points

The theory of games has been built on the famous minimax theorem, due to von Neumann which is related to some special kinds of two-person games.

In order to obtain this result and discuss related topics we begin with the introduction of some fundamental concepts.

Mathematically, a two-person game is defined by

$$\Gamma = \{ \Sigma_1, \Sigma_2; A_1, A_2 \}$$

where Σ_1 and Σ_2 are the strategy sets of the first and second players, respectively and are non-empty, and where the corresponding payoff functions A_1 and A_2 are real functions defined on the product space $\Sigma_1 x \Sigma_2$. Usually such a description is referred to as "normal form" in the literature.

This is an abstract representation of many real situations where conflict of interest of two persons is involved.

The intuitive meaning of this definition is that if the first player chooses his strategy $\sigma_1 \epsilon \Sigma_1$ and the second player chooses his strategy $\sigma_2 \epsilon \Sigma_1$, the play of the game is completely determined and ends with the payoff $A_1(\sigma_1, \sigma_2)$ to the first player, and the payoff $A_2(\sigma_1, \sigma_2)$ to the second player.

It is assumed that both players try to maximize their payoff without bounds. This assumption is motivated by real situations where conflict between two persons is involved. This fact is observed as a natural principle.

The principal problem of the theory of two-person games is to describe a rational or optimal behavior for the players.

A two-person game is specified by the structure of its strategy sets. For example, the finite two-person games are characterized by the finiteness of both strategy sets. They are also called matrix games because the payoff functions can be expressed as a matrix. Another important kind of two-person games used in our exposition has as its strategy sets non-empty sets in a euclidean space.

Thus, it is seen that in general the strategy sets can have a topological structure.

Consider a two-person game

$$\Gamma = \{ \Sigma_1, \Sigma_2; A_1, A_2 \}$$

where the strategy sets are non-empty compact sets, i.e. bounded and closed, in euclidean spaces. Let the payoff functions to be continuous functions. The existence of the following quantities

$$V_i(\Gamma) = \max_{s_i \in \Sigma_i} \min_{s_j \in \Sigma_j} A_i(s_i, s_j) \qquad (j \neq i:1,2)$$

$$V^i(\Gamma) = \min_{s_j \in \Sigma_j} \max_{s_i \in \Sigma_i} A_i(s_i, s_j)$$

is guaranteed by the fact that the functions maximum and minimum taking over a variable of a continuous function on the product space, is continuous.

These amounts are called the inferior value or maximin value and the superior value or minimax value of the player $i(i:1,2)$, respectively.

In what follows, if it is not mentioned to the contrary we assume such properties for the strategy sets and payoff functions for all the games. From an intuitive viewpoint the amount $V_i(\Gamma)$ $(i:1,2)$, is the minimum safe winnings that the player i can obtain independent of the behavior of his opponent.

Indeed, if player i choses the strategy $\sigma_i \in \Sigma_i$, he is assured to obtain at least the amount

$$\min_{s_j \in \Sigma_j} A_i(\sigma_i, s_j) \qquad (j \neq i: 1,2).$$

Then, since this function of minimum winnings of player i depends only on his choices and since he wishes to maximize his safe positions, he is sure of obtaining the amount

$$V_i(\Gamma) = \max_{s_i \in \Sigma_i} \min_{s_j \in \Sigma_j} A_i(s_i, s_j) \qquad (j \neq i: 1,2) .$$

On the other hand the value $V^i(\Gamma)$ (i:1,2) , is the maximum winnings that the opponent $j \neq i$ is able to prevent the player i from getting more independent of the actions of player i .

Again, by the choice $\sigma_i \in \Sigma_i$, the player i can prevent his opponent from making more than

$$\max_{s_j \in \Sigma_j} A_j(s_j, \sigma_i) \qquad (j \neq i: 1,2).$$

This function depends only upon the strategy of player i . If he wishes to minimize the maximum position of his opponent he is able to prevent the position

$$V^i(\Gamma) = \min_{s_j \in \Sigma_j} \max_{s_i \in \Sigma_i} A_i(s_i, s_j) \qquad (j \neq i: 1,2) .$$

A strategy $\bar{\sigma}_i \in \Sigma_i$ is said to be a <u>maximin strategy</u> of the player i (i:1,2) in the two-person game Γ if

$$\min_{s_j \in \Sigma_j} A_i(\bar{\sigma}_i, s_j) = \max_{s_i \in \Sigma_i} \min_{s_j \in \Sigma_j} A_i(s_i, s_j) = V_i(\Gamma) \quad (j \neq i: 1,2)$$

A strategy $\bar{\sigma}_i \in \Sigma_i$ is called a minimax strategy of the player $i(i:1,2)$ in the two-person game Γ if

$$\max_{s_j \in \Sigma_j} A_j(s_j,\bar{\sigma}_i) = \min_{s_i \in \Sigma_i} \max_{s_j \in \Sigma_j} A_j(s_j,s_i) = V^j(\Gamma) \quad (j \neq i:1,2).$$

Obviously the existence of the maximin and minimax values implies the existence of the maximin and minimax strategies. Intuitively, one can see a maximum strategy as a safe defensive behavior and a minimax strategy as a safe attacking behavior.

From the above definitions, we see that the notions of maximin and minimax strategy are very different, nevertheless there is an important kind of two-person game, for which these concepts coincide, namely the constant-sum two-person games. These are characterizated by

$$A_1(\sigma_1,\sigma_2) + A_2(\sigma_1,\sigma_2) = c \qquad \text{for all } \sigma_1 \in \Sigma_1 , \quad \sigma_2 \in \Sigma_2$$

for some real number c . In such a case the constant c is called the sum of two-person game.

If in the constant-sum two-person game we substitute in the definition of the maximin value $V_i(\Gamma)$ for the payoff function A_i, $c-A_j$ we get:

$$V_i(\Gamma) = c-V^j(\Gamma) \qquad (i \neq j:1,2) .$$

By the same substitution in the definition of a maximum strategy of the player i (i:1,2) and the above we show that for this kind of two-person games a strategy of a player is maximin strategy if and only if is a minimax strategy.

Now, in this case, the concept of a maximin or minimax strategy of a player has a stronger interpretation, i.e. it is a strategy which assures by itself the maximin value to the corresponding player and at the same time prevents the opponent from winning more than his corresponding minimax value. In other words it is simultaneously a safe defensive and attacking behavior.

Given a constant-sum two-person game Γ with sum c, we can always associate with it a <u>zero-sum two-person game</u> Γ_o, i.e. a constant-sum two-person game with zero sum:

$$\Gamma_o = \{ \Sigma_1, \Sigma_2; A_1, A_2 - c \}$$

whose structure is the same of the primitive game Γ. Indeed, we can consider that before or after each play the second player pays the amount c. Thus, the strategic behavior of both players in such games is the same. Therefore, we will now examine only zero-sum two-person games.

This kind of two-person games with zero-sum can be seen as those situations where there is a direct conflict of interest between the players. In the zero-sum two-person games, one can eliminate the description of the payoff function of the second player. Such zero-sum games will be described by

$$\Gamma = \{ \Sigma_1, \Sigma_2; A_1 \} .$$

Motivated by this asymmetry in notation, the first player is usually regarded as the defensive player. That is, the first player is concerned with maximizing his position and the second player is generally considered as the attacking player, preventing the desires of the first player.

This new representation of a zero-sum two-person game allows a more simple description by omitting the character of defensiveness of the second player. It is usual to assign the maximin strategy only to the first player in the same way as the minimax strategy is only fixed to the second player. Moreover, the corresponding values are referred to as those of the first player.

For a zero-sum two-person game Γ the maximin and minimax values always fulfill the following relation

$$V_1(\Gamma) = \max_{s_1 \in \Sigma_1} \min_{s_2 \in \Sigma_2} A_1(s_1, s_2) \leq \min_{s_2 \in \Sigma_2} \max_{s_1 \in \Sigma_1} A_1(s_1, s_2) = V^1(\Gamma)$$

This is easily proven by taking a maximin strategy $\bar{\sigma}_1 \in \Sigma_1$ and a minimax strategy (which exists because the payoff function is continuous on the product space) $\bar{\sigma}_2 \in \Sigma_2$ and considering the simple inequality

$$\min_{s_2 \in \Sigma_2} A_1(\bar{\sigma}_1, s_2) \leq A_1(\bar{\sigma}_1, \bar{\sigma}_2) \leq \max_{s_1 \in \Sigma_2} A_1(s_1, \bar{\sigma}_2) .$$

This relation expresses the intuitively obvious fact that the smallest safe amount, which the second player is able to limit the first player to is not smaller than the largest safe level that this player can get by himself.

In general, the equality in the above relation does not hold. One may verify this fact by taking a simple example. However, it is very interesting to characterize those zero-sum two-person games for which the strict equality holds. In such a case, the unique amount

$$V(\Gamma) = V_1(\Gamma) = V^1(\Gamma)$$

is said to be a value of the zero-sum two-person game. It is then said that the zero-sum two-person game Γ satisfies the minimax theorem.

For this kind of two-person games we have a satisfactory rational behavior which is referred to as the minimax principle. The rational behavior of the players is completely determined by their respective choices of a maximin and a minimax strategy. This, on one hand guarantees the attainment of the value of the game $V(\Gamma)$ to the first player and $-V(\Gamma)$ to the second player, and on the other hand they prevent their opponents from getting more than this amount. We refer to the maximin and minimax strategies as optimal strategies.

A pair: $(\bar{\sigma}_1, \bar{\sigma}_2)$ of optimal strategies which is characterized by the relation

$$\max_{s_1 \in \Sigma_1} A(s_1, \bar{\sigma}_1) = A(\bar{\sigma}_1, \bar{\sigma}_2) = \min_{s_1 \in \Sigma_2} A(\bar{\sigma}_1, s_2)$$

is called a saddle point of the zero-sum two-person game Γ .

By virtue of the obvious assertions

$$\min_{s_2 \in \Sigma_2} A(\bar{\sigma}_1, s_2) \leq \max_{s_1 \in \Sigma_1} \min_{s_2 \in \Sigma_2} A(s_1, s_2)$$

$$\max_{s_1 \in \Sigma_1} A(s_1, \bar{\sigma}_2) \geq \min_{s_2 \in \Sigma_2} \max_{s_1 \in \Sigma_1} A(s_1, s_2) ,$$

the existence of a saddle point of a zero-sum two-person game Γ assures the validity of the minimax theorem.

Before formulating a wide class of zero-sum two-person games with value we recall some simple concepts.

A set Σ in a euclidean space is said to be a <u>convex set</u> if for any pair of points $\sigma, \bar{\sigma} \in \Sigma$ the point

$$\lambda\sigma + (1-\lambda)\,\bar{\sigma} \in \Sigma \qquad \text{for all} \quad \lambda \in [0,1] .$$

A real function A defined on a convex set Σ of a euclidean space is said to be a <u>convex function</u> if for any pair of points $\sigma , \bar{\sigma} \in \Sigma$

$$A(\lambda\sigma + (1-\lambda)\bar{\sigma}) \leq \lambda A(\sigma) + (1-\lambda)A(\bar{\sigma})$$

for all $\lambda \in [0,1]$. Analogously, a function A is called a <u>concave function</u> if for any pair of points $\sigma , \bar{\sigma} \in \Sigma$

$$A(\lambda\sigma + (1-\lambda)\bar{\sigma}) \geq \lambda A(\sigma) + (1-\lambda)A(\bar{\sigma})$$

for all $\lambda \in [0,1]$.

Furthermore, we will often make use of a more general kind of function.

Given two sets Σ and $\bar{\Sigma}$ in euclidean spaces, a correspondence φ which for each point $\sigma \in \Sigma$ assigns a non-empty set $\varphi(\sigma) \in \bar{\Sigma}$ is said to be a <u>multivalued function</u>.

A multivalued function φ defined on a set Σ with values in the set Σ is called <u>upper semicontinuous</u> if for any convergent sequences

$$\sigma(k) \to \sigma \qquad \text{and} \qquad \tau(k) \to \tau$$

then for all positive integers $k: \tau(k) \in \varphi(\sigma(k))$
implies $\tau \in \varphi(\sigma)$

Equivalently a multivalued function φ is upper semicontinuous if and only if its <u>graph</u>

$$G_{\varphi} = \{(\sigma,\tau): \Sigma x \bar{\Sigma} : \tau \epsilon\ \varphi(\sigma)\} \subset \Sigma x \bar{\Sigma}$$

is closed in the product space.

An initial strong characterization of such two-person games, due to Kakutani[7] is given in the following minimax theorem.

THEOREM I.1: Let $\Gamma = \{\Sigma_1, \Sigma_2; A\}$ be a zero-sum two-person game such that the strategy sets Σ_1 and Σ_2 are non-empty, compact, convex sets in euclidean spaces and the payoff function A is continuous in the variable $(\sigma_1,\sigma_2)\epsilon\Sigma_1 x\Sigma_2$ concave with respect to $\sigma_1\epsilon\Sigma_1$ for fixed $\sigma_2\epsilon\Sigma_2$ and convex with respect to $\sigma_2\epsilon\Sigma_2$ for fixed $\sigma_1\epsilon\Sigma_1$.

Then, the game Γ has the value:

$$V(\Gamma) = V_1(\Gamma) = \max_{s_1\epsilon\Sigma_1} \min_{s_2\epsilon\Sigma_2} A(s_1,s_2) = \min_{s_2\epsilon\Sigma_2} \max_{s_1\epsilon\Sigma_1} A(s_1,s_2) = V^1(\Gamma)$$

PROOF: Because the product set of compact and convex sets in euclidean spaces is compact and convex in an euclidean space, then the set $\Sigma_1 x\Sigma_2$ is compact and convex. By the continuity of the payoff function A on the product space $\Sigma_1 x\Sigma_2$, we obtain that for any pair of points $\sigma_1\epsilon\Sigma_1$ and $\sigma_2\epsilon\Sigma_2$ the following sets

$$\varphi_1(\sigma_2) = \{\tau_1\epsilon\Sigma_1 : A(\tau_1,\sigma_2) = \max_{s_1\epsilon\Sigma_1} A(s_1,\sigma_2)\}$$

and

$$\varphi_2(\sigma_1) = \{\tau_2\epsilon\Sigma_2 : A(\sigma_1,\tau_2) = \min_{s_2\epsilon\Sigma_2} A(\sigma_1,s_2)\}$$

are non-empty. On the other hand, by the concavity of the payoff function with respect to the variable $\sigma_1\epsilon\Sigma_1$, for fixed $\sigma_2\epsilon\Sigma_2$, the set $\varphi_1(\sigma_2)$ is convex. Let τ_1, $\bar{\tau}_1$ be any couple of points belonging to the set $\varphi_1(\sigma_2)\subseteq \Sigma_2$, then

$$A(\tau_1,\sigma_2) = A(\bar{\tau}_1,\sigma_2) = \max_{s_1\epsilon\Sigma_1} A(s_1,\sigma_2)$$

and therefore by the concavity

$$A(\lambda\tau_1+(1-\lambda)\bar{\tau}_1,\sigma_2) \geq \lambda A(\tau_1,\sigma_2)+(1-\lambda)A(\bar{\tau}_1,\sigma_2) = \max_{s_1\epsilon\Sigma_1} A(s_1,\sigma_2)$$

for all $\lambda\epsilon[0,1]$. But this inequality is an equality since in the third part we have the maximum amount of the function $A(\sigma_1,\sigma_2)$ for fixed $\sigma_2\epsilon\Sigma_2$. Thus

$$\lambda\tau_1 + (1-\lambda)\bar{\tau}_1 \,\epsilon\,\varphi_1(\sigma_2) \quad \text{for all} \quad \lambda\epsilon[0,1] .$$

Similarly, since the payoff function is convex in the variable $\sigma_2\epsilon\Sigma_2$ for fixed $\sigma_1\epsilon\Sigma_1$, one then can prove the convexity of the set $\varphi_2(\sigma_1)\subseteq \Sigma_2$.

We now consider the multivalued function

$$\varphi : \Sigma_1 x\Sigma_2 \to \Sigma_1 x\Sigma_2$$

defined by

$$\varphi(\sigma_1,\sigma_2) = \varphi_1(\sigma_2) x\ \varphi_2(\sigma_1) .$$

Let

$$\sigma(k) \to \sigma \qquad \text{and} \qquad \tau(k) \to \tau$$

be two converging sequences in the set $\Sigma_1 x \Sigma_2$ with the property that for any positive integer $k: \tau(k) \epsilon \varphi(\sigma(k))$, that is

$$A(\tau_1(k),\sigma_2(k)) = \max_{s_1 \epsilon \Sigma_1} A(s_1,\sigma_2(k))$$

and

$$A(\sigma_1(k),\tau_2(k)) = \min_{s_2 \epsilon \Sigma_2} A(\sigma_1(k),s_2) \quad .$$

By the continuity of the payoff function, the maximum over $s_1 \epsilon \Sigma_1$ of the payoff function A is also a continuous function defined over the set Σ_2. Therefore the following sequence converges

$$\max_{s_1 \epsilon \Sigma_1} A(s_1,\sigma_2(k)) \rightarrow \max_{s_1 \epsilon \Sigma_1} A(s_1,\sigma_2) \ .$$

Similarly, the minimun over $s_2 \epsilon \Sigma_2$ of the payoff function A is a continuous function on Σ_1. Thus

$$\min_{s_2 \epsilon \Sigma_2} A(\sigma_1(k),s_2) \rightarrow \min_{s_2 \epsilon \Sigma_2} A(\sigma_1,s_2) \ .$$

By the same property the convergence of the sequences

$$A(\tau_1(k),\sigma_2(k)) \rightarrow A(\tau_1,\sigma_2) \quad \text{and} \quad A(\sigma_1(k),\tau_2(k)) \rightarrow A(\sigma_1,\tau_2)$$

follows.

All these relations, together with the above two equalities give:

$$\max_{s_1 \epsilon \Sigma_1} A(s_1,\sigma_2) = A(\tau_1,\sigma_2)$$

and

$$\min_{s_2 \in \Sigma_2} A(\sigma_1, s_2) = A(\sigma_1, \tau_2) \quad .$$

These show that the point τ belongs to the set $\varphi(\sigma)$, and therefore the upper semicontinuity of the multivaluated function φ is established.

Now, by making use of the Kakutani Fixed Point Theorem; which assures the existence of a fixed point $\bar{\sigma} \epsilon \phi(\bar{\sigma})$ for any upper semicontinuous multivalued function ϕ defined on a non-empty, compact and convex set in a euclidean space Σ having the property that for every $\sigma \epsilon \Sigma$ the set $\phi(\sigma) \subseteq \Sigma$ is convex; the existence of a fixed point $(\bar{\sigma}_1, \bar{\sigma}_2) \epsilon \Sigma_1 x \Sigma_2$ for the multivalued function φ is guaranteed.

Such a fixed point $(\bar{\sigma}_1, \bar{\sigma}_2) \epsilon\, \varphi(\bar{\sigma}_1, \bar{\sigma}_2)$ is a saddle point of the two-person game Γ :

$$\min_{s_2 \in \Sigma_2} A(\bar{\sigma}_1, s_2) = A(\bar{\sigma}_1, \bar{\sigma}_2) = \max_{s_1 \in \Sigma_1} A(s_1, \bar{\sigma}_2) \ ,$$

which guarantees the equality between the maximin and minimax values $V_1(\Gamma)$ and $V^1(\Gamma)$. Q.E.D.

In this proof we have used the property of non-voidness simultaneously for the sets $\varphi_2(\sigma_1)$ and $\varphi_1(\sigma_2)$.

This fact can be reformulated in two different ways using the symmetric description for a zero-sum two-person game. Let $\Gamma = \{ \Sigma_1, \Sigma_2; A, -A \}$ be such a two-person game, one then can describe the mentioned condition by expressing the fact that for each joint strategy (σ_1, σ_2) there is another joint strategy (τ_1, τ_2) which satisfies

$$A(\tau_1,\sigma_2) = \max_{s_1 \in \Sigma_1} A(s_1,\sigma_2)$$

and

$$-A(\sigma_1,\tau_2) = \max_{s_2 \in \Sigma_2} -A(\sigma_1,s_2)$$

that is, for any established compound behavior there is another joint behavior which maximizes the winnings of each player if his opponent abides by the first behavior. We relate this condition as the <u>defense property</u> of the two-person game Γ .

On the other hand, one can describe the above condition by another reformulation: For each joint strategy (σ_1,σ_2) there is another joint strategy (τ_1,τ_2) , for which the equalities

$$-A(\tau_1,\sigma_2) = \min_{s_1 \in \Sigma_1} -A(s_1,\sigma_2)$$

and

$$A(\sigma_1,\tau_2) = \min_{s_2 \in \Sigma_2} A(\sigma_1,s_2)$$

hold true. In other words, for any assumed joint behavior there is another joint behavior which minimizes the position of each player if this player abides by the first one. This property is called the <u>attack property</u> of the two-person game Γ .

An important application of this theorem will be formulated immediately after we introduce some important concepts.

Given a finite two-person game .

$$\Gamma = \{ \Sigma_1,\Sigma_2;A_1,A_2 \}$$

a <u>mixed strategy</u> of the first player is a distribution of probabilities defined over the set Σ_1 , that is, a function

$$x: \Sigma_1 \to [0,1]$$

with

$$x(\sigma_1) \geq 0 \quad \text{for all} \quad \sigma_1 \in \Sigma_1 \quad \text{and} \quad \sum_{s_1 \in \Sigma_1} x(s_1) = 1 \; .$$

Analogously, we defined a mixed strategy y of the second player.

Usually the strategies in $\Sigma_i (i:1,2)$ are called <u>pure strategies</u> to distinguish between the two kinds.

A simple interpretation of the concept of mixed strategy can be this: in a sequence of plays of the two-person game Γ , in which the choices of the strategies used by each player must be done in a random manner, since otherwise the opponent might obtain certain information of his behavior, which is not desirable in any instance. So the behavior can be described as a distribution of probability on the strategy sets of both players.

Following this interpretation the corresponding payoff functions A_1 and A_2 of both players must be replaced by the respective expectations E_1 and E_2 in the case of mixed strategies.

Thus, given a finite two-person game Γ , let

$$\tilde{\Sigma}_1 = \{x: \Sigma_1 \to [0,1] : x(\sigma_1) \geq 0 \quad \text{for all} \quad \sigma_1 \in \Sigma_1 \quad \text{and} \quad \sum_{s_1 \in \Sigma_1} x(s_1) = 1 \; \}$$

and

$$\tilde{\Sigma}_2 = \{ y:\Sigma_2 \to [0,1]:y(\sigma_2) \geq 0 \quad \text{for all } \sigma_2\epsilon\Sigma_2 \text{ and } \sum_{s_2\epsilon\Sigma_2} y(s_2) = 1 \}$$

by the mixed strategy sets of the first and second player, respectively.

The sets $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are non-empty, compact and convex sets in euclidean spaces whose corresponding dimension is just the number of elements of the respective strategy set Σ_1 and Σ_2 minus one.

For every pair

$$x\epsilon\tilde{\Sigma}_1 \quad \text{and} \quad y\epsilon\tilde{\Sigma}_2 ,$$

let

$$E_1(x,y) \quad \text{and} \quad E_2(x,y)$$

be the respective expectations which are given by

$$E_i(x,y) = \sum_{s_1\epsilon\Sigma_1} \sum_{s_2\epsilon\Sigma_2} A_i(s_1,s_2)\, x(s_1)y(s_2) \qquad (i:1,2) \quad ,$$

then the two-person game, defined by

$$\tilde{\Gamma} = \{ \tilde{\Sigma}_1,\tilde{\Sigma}_2;E_1,E_2 \}$$

is said to be the mixed extension of Γ .

We can also describe this as

$$E_i(x,y) = \sum_{\sigma_1\epsilon\Sigma_1} E_i(\sigma_1,y)x(\sigma_1) = \sum_{\sigma_2\epsilon\Sigma_2} E_i(x, \sigma_2)y(\sigma_2) \qquad (i:1,2)$$

where $E_i(\sigma_1,y)$ indicates the expectation function for the distribution of probability

$$\bar{x}(\tau_1) = \begin{cases} 1 & \text{if } \tau_1 = \sigma_1 \\ 0 & \text{otherwise} \end{cases}$$

of the first player. Analogously for the expectation function $E_i(x,\sigma_2)$ with respect to the second player. The fundamental minimax theorem of the theory of games, due to von Neumann [14] can now be obtained as an immediate consequence of the above theorem.

<u>THEOREM I.2</u>: Let $\Gamma = \{ \Sigma_1, \Sigma_2; A \}$ be a finite zero-sum two-person game, then the mixed extension $\tilde{\Gamma} = \{ \tilde{\Sigma}_1, \tilde{\Sigma}_2; E \}$ has a value.

<u>PROOF</u>: The mixed strategy sets $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are non-empty, compact and convex in euclidean spaces.

Consider an arbitrary fixed $y \epsilon \tilde{\Sigma}_2$. Then for any pair $x, \bar{x} \epsilon \tilde{\Sigma}_1$ we have:

$$E(\lambda x+(1-\lambda)\bar{x},y) = \sum_{\sigma_1 \epsilon \Sigma_1} \left[\sum_{2 \epsilon \Sigma_2} A(\sigma_1,\sigma_2)y(\sigma_2) \right] (x(\sigma_1)+(1-\lambda)\bar{x}(\sigma_2))$$

$$= \lambda E(x,y) + (1-\lambda)E(\bar{x},y)$$

for all real number λ. This says that the function E is <u>linear</u>(o) with respect to the variable $x \epsilon \tilde{\Sigma}_1$. In particular this equality holds for values of λ restricted to the interval $[0,1]$. Therefore, the function E is concave in the variable $x \epsilon \tilde{\Sigma}_1$ for fixed $y \epsilon \tilde{\Sigma}_2$.

Analogously, for fixed $x \epsilon \tilde{\Sigma}_1$ the function E is linear with respect to the variable y; for any pair $y, \bar{y} \epsilon \tilde{\Sigma}_2$

$$E(x,\lambda y+(1-\lambda)\bar{y}) = \lambda E(x,y) + (1-\lambda)E(x,\bar{y}) ,$$

(o) By simplicity we will use term linear as the restriction of a linear function.

for all real number λ . Hence the function E is convex in $y \in \tilde{\Sigma}_2$ for fixed $x \in \tilde{\Sigma}_1$.

On the other hand, since the expectation function E is bilinear, that is, linear in each variable for any fixed mixed strategy of the remaining variable; it is a continuous function on the product space $\tilde{\Sigma}_1 x \tilde{\Sigma}_2$.

Thus, all the requirements of the preceding theorem are satisfied for the mixed extension $\tilde{\Gamma}$, and therefore the existence of the value of $\tilde{\Gamma}$ is guaranteed. Q.E.D.

An interesting <u>interchangeability property</u> of saddle points is formulated as follows:

<u>THEOREM I.3</u>: Let $\Gamma = \{ \Sigma_1, \Sigma_2; A \}$ be an arbitrary zero-sum two-person game where $(\bar{\sigma}_1, \bar{\sigma}_2)$ and $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ are two saddle points. Then, $(\bar{\sigma}_1, \tilde{\sigma}_2)$ and $(\tilde{\sigma}_1, \bar{\sigma}_2)$ are also saddle points of Γ .

<u>PROOF:</u> By hypothesis we have

$$V(\Gamma) = \max_{s_1 \in \Sigma_1} A(s_1, \bar{\sigma}_2) = A(\bar{\sigma}_1, \bar{\sigma}_2) = \min_{s_2 \in \Sigma_2} A(\bar{\sigma}_1, s_2)$$

and

$$V(\Gamma) = \max_{s_1 \in \Sigma_1} A(s_1, \tilde{\sigma}_2) = A(\tilde{\sigma}_1, \tilde{\sigma}_2) = \min_{s_2 \in \Sigma_2} A(\tilde{\sigma}_1, s_2) \quad .$$

Hence, an immediate consequence of the first equality is

$$A(\tilde{\sigma}_1, \bar{\sigma}_2) \leq A(\bar{\sigma}_1, \bar{\sigma}_2) \leq A(\bar{\sigma}_1, \tilde{\sigma}_2) \quad .$$

Similarly from the second relation arises

$$A(\bar{\sigma}_1, \tilde{\sigma}_2) \leq A(\tilde{\sigma}_1, \tilde{\sigma}_2) \leq A(\tilde{\sigma}_1, \bar{\sigma}_2) \quad .$$

Therefore

$$V(\Gamma) = A(\tilde{\sigma}_1,\bar{\sigma}_2) = A(\bar{\sigma}_1,\tilde{\sigma}_2) \quad .$$

On the other hand, by using the first part of the first equality and the last part of the second equality, we obtain the property of being a saddle point of Γ for the point $(\tilde{\sigma}_1,\bar{\sigma}_2)$. Analogously, by taking the other extreme terms of the mentioned equalities we have verified that the point $(\bar{\sigma}_1,\tilde{\sigma}_2)$ is a saddle point of Γ. Q.E.D.

Besides the interchangeability property of the saddle points in a zero-sum two-person game, we have gotten the equivalence property between them, that is, the payoffs on any saddle points are equal.

With this property of interchangeability, one then has that if either one or both players change their strategies from their component of an established saddle point to another, it will not be any modification on the payoffs and the strategic situations of both players.

For the zero-sum two-person games which have been previously considered the set of all saddle points has a very simple structure. This formulation is given as follows.

THEOREM I.4: Let $\Gamma = \{\Sigma_1,\Sigma_2;A\}$ be a zero-sum two-person game such that the strategy sets Σ_1 and Σ_2 are non-empty, compact, convex sets in euclidean spaces and the payoff function A is continuous in the variable $(\sigma_1,\sigma_2)\epsilon\Sigma_1 x\Sigma_2$, concave with respect to $\sigma_1\epsilon\Sigma_1$ for fixed $\sigma_2\epsilon\Sigma_2$ and convex with respect to $\sigma_2\epsilon\Sigma_2$ for fixed $\sigma_1\epsilon\Sigma_1$. Then the set of saddle points is non-empty, compact and convex.

PROOF: The non-voidness of the set of saddle points of Γ is assured by the theorem I.1.

Consider a convergent sequence of saddle points

$$(\bar{\sigma}_1(k),\bar{\sigma}_2(k)) \to (\bar{\sigma}_1,\bar{\sigma}_2) \quad ,$$

which are characterized by

$$\max_{s_1\in\Sigma_1} A(s_1,\bar{\sigma}_2(k)) = A(\bar{\sigma}_1(k),\bar{\sigma}_2(k)) = \min_{s_2\in\Sigma_2} A(s_2,\bar{\sigma}_2(k)) \ .$$

By the continuity of the payoff function, the maximun over $s_1\in\Sigma_1$ of A is also continuous function over the compact set Σ_2. Hence we have

$$\max_{s_1\in\Sigma_1} A(s_1,\bar{\sigma}_2(k)) \to \max_{s_1\in\Sigma_1} A(s_1,\bar{\sigma}_2) \ .$$

Analogously, for the minimun function

$$\min_{s_2\in\Sigma_2} A(\bar{\sigma}_1(k),s_2) \to \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2) \ .$$

Therefore by using the convergence of the sequence

$$A(\bar{\sigma}_1(k),\ \bar{\sigma}_2(k)) \to A(\bar{\sigma}_1,\bar{\sigma}_2)$$

together with the preceding two convergences we obtain

$$\max_{s_1\in\Sigma_1} A(s_1,\bar{\sigma}_2) = A(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2) \ .$$

This indicates that the joint strategy $(\bar{\sigma}_1,\bar{\sigma}_2)$ is a saddle point of Γ, i.e. the set of saddle points of the zero-sum two-person game Γ is closed, and therefore compact since it is a subset of the compact set $\Sigma_1 x\Sigma_2$.

We now will prove the convexity of such a set. First of all, let us consider two important simple cases.

Let $(\bar{\sigma}_1,\bar{\sigma}_2)$ and $(\bar{\sigma}_1,\tilde{\sigma}_2)$ be two saddle points of Γ, which satisfy the conditions

$$\max_{s_1\in\Sigma_1} A(s_1,\bar{\sigma}_2) = A(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2)$$

and

$$\max_{s_1\in\Sigma_1} A(s_1,\tilde{\sigma}_2) = A(\bar{\sigma}_1,\tilde{\sigma}_2) = \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2) .$$

By the convexity of the payoff function with respect to $\sigma_2\in\Sigma_2$ for fixed $\sigma_1\in\Sigma_1$, the following condition obviously holds:

$$A(\bar{\sigma}_1,\lambda\bar{\sigma}_2+(1-\lambda)\tilde{\sigma}_2) \leq \lambda A(\bar{\sigma}_1,\bar{\sigma}_2) + (1-\lambda)A(\bar{\sigma}_1,\tilde{\sigma}_2) = \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2) .$$

for all $\lambda\in[0,1]$.

This inequality must be actually the strict equality since in the third term there appears the minimun amount of the payoff function on $\bar{\sigma}_1\in\Sigma_1$ over the set Σ_2. Thus, for each point $(\bar{\sigma}_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2)$ with $\lambda\in[0,1]$:

$$A(\bar{\sigma}_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2)= \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2)$$

Let $\sigma_1 \epsilon \Sigma_1$ be any strategy of the first player. Then, by the definition of saddle points $(\bar{\sigma}_1,\bar{\sigma}_2)$ and $(\bar{\sigma}_1,\tilde{\sigma}_2)$ the following inequalities hold:

$$A(\sigma_1,\bar{\sigma}_2) \leq A(\bar{\sigma}_1,\bar{\sigma}_2) = V(\Gamma) \quad \text{and} \quad A(\sigma_1,\tilde{\sigma}_2) \leq A(\bar{\sigma}_1,\tilde{\sigma}_2) = V(\Gamma) \quad .$$

Again from the convexity of the payoff function in the variable $\sigma_2 \epsilon \Sigma_2$, on the point $(\sigma_1,\lambda\bar{\sigma}_2+(1-\lambda)\tilde{\sigma}_2)$ with $\lambda\epsilon[0,1]$ we obtain:

$$A(\sigma_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2) \leq \lambda A(\sigma_1,\bar{\sigma}_2) + (1-\lambda)A(\sigma_1,\tilde{\sigma}_2) \quad .$$

Together with the last relations, this condition determines

$$A(\sigma_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2) \leq A(\bar{\sigma}_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2) \; ,$$

This implies

$$\max_{s_1 \epsilon \Sigma_1} A(s_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2) = A(\bar{\sigma}_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2) \; ,$$

Therefore for each $\lambda\epsilon[0,1]$, the point $(\bar{\sigma}_1,\lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2)$ is saddle point of Γ.

In a similar fashion, let $(\bar{\sigma}_1,\bar{\sigma}_2)$ and $(\tilde{\sigma}_1,\bar{\sigma}_2)$ be two saddle points of Γ, for which

$$\max_{s_1 \epsilon \Sigma_1} A(s_1,\bar{\sigma}_2) = A(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_2 \epsilon \Sigma_2} A(\bar{\sigma}_1,s_2)$$

and

$$\max_{s_1 \epsilon \Sigma_1} A(s_1,\bar{\sigma}_2) = A(\tilde{\sigma}_1,\bar{\sigma}_2) = \min_{s_2 \epsilon \Sigma_2} A(\tilde{\sigma}_1,s_2) \quad .$$

The concavity of the payoff function in the variable $\sigma_1 \in \Sigma_1$ for fixed $\sigma_2 \in \Sigma_2$ gives:

$$A(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, \bar{\sigma}_2) = \max_{s_1 \in \Sigma_1} A(s_1, \bar{\sigma}_2)$$

for all $\lambda \in [0,1]$.

On the other hand, for any arbitrary $\sigma_2 \in \Sigma_2$ by virtue of the definition of saddle point, one, now has

$$A(\bar{\sigma}_1, \sigma_2) \geq A(\bar{\sigma}_1, \bar{\sigma}_2) \quad \text{and} \quad A(\tilde{\sigma}_1, \sigma_2) \geq A(\tilde{\sigma}_1, \bar{\sigma}_2) \quad .$$

This together with the concavity of the payoff function with respect to $\sigma_1 \in \Sigma_1$ determines, at the point $(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, \sigma_2)$ with $\lambda \in [0,1]$,

$$A(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, \bar{\sigma}_2) \leq A(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, \sigma_2)$$

which is equivalent to

$$A(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, \bar{\sigma}_2) = \min_{s_2 \in \Sigma_2} A(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, s_2) \quad .$$

Hence, the point $(\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1, \bar{\sigma}_2)$ with $\lambda \in [0,1]$ is a saddle point Γ .

With the two above assertions, we will now directly show the convexity of the set of saddle points.

Let $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2)$ and $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2)$ be two saddle points of Γ , then by virtue of the interchangeability property given in the preceding theorem, the points

$$(\bar{\sigma}_1, \tilde{\sigma}_2) \quad \text{and} \quad (\tilde{\sigma}_1, \bar{\sigma}_2)$$

are saddle points of Γ .

Now, let λ be any arbitrary value in $[0,1]$, then the assertion of the first simple case considered before, applied to the saddle points $(\bar{\sigma}_1,\bar{\sigma}_2)$ and $(\bar{\sigma}_1,\tilde{\sigma}_2)$ assures that the following is a saddle point: $(\bar{\sigma}_1, \lambda\bar{\sigma}_2+(1-\lambda)\tilde{\sigma}_2)$.

Similarly, by the same argument applied to the saddle points $(\tilde{\sigma}_1,\bar{\sigma}_2)$ and $(\tilde{\sigma}_1,\tilde{\sigma}_2)$ implies that $(\tilde{\sigma}_1, \lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2)$ is also a saddle point of Γ.

Hence, since both points

$$(\bar{\sigma}_1, \lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2) \quad \text{and} \quad (\tilde{\sigma}_1, \lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2)$$

are saddle points of Γ, the assertion of the second simple case just considered for these joint strategies guarantees the property of saddle point for

$$\sigma_\lambda = \lambda\bar{\sigma} + (1-\lambda)\tilde{\sigma} = (\lambda\bar{\sigma}_1 + (1-\lambda)\tilde{\sigma}_1 \; , \; \lambda\bar{\sigma}_2 + (1-\lambda)\tilde{\sigma}_2 \;) \quad .$$

This implies the condition of convexity of the set of saddle points of Γ. Q.E.D.

An immediate consequence of this result is obtained by remembering that for the mixed extension $\tilde{\Gamma}$ of a finite zero-sum two-person game Γ , the mixed strategy sets are non-empty, compact and convex sets in euclidean spaces and the expectation function is bilinear and thus continuous. Therefore, the sets of saddle points of $\tilde{\Gamma}$ are non-empty, compact and convex sets.

Before formulating a possible exact extension of the above results for two-person games we are going to introduce some important concepts and intuitive observations.

Given a two-person game $\Gamma = \{ \Sigma_1,\Sigma_2;A_1,A_2 \}$, one can intuitively observe it as the simultaneous superposition of the following zero-sum two-person games

$$\Gamma_1 = \{ \Sigma_1, \Sigma_2; A_1 \} \quad \text{and} \quad \Gamma_2 = \{ \Sigma_2, \Sigma_1; A_2 \}$$

which are referred to the first and second player, respectively.

We recall that the second player in Γ_2 has the role of first player. From the corresponding definitions of the maximin and minimax value of the two-person games Γ, Γ_1 and Γ_2, we have:

$$V_1(\Gamma) = V_1(\Gamma_1) \quad , \quad V^1(\Gamma) = V^1(\Gamma_1)$$

and

$$V_2(\Gamma) = V_1(\Gamma_2) \quad , \quad V^2(\Gamma) = V^1(\Gamma_2)$$

where $V_i(\Gamma)$ expresses the maximin and $V^i(\Gamma)$ is the minimax values of the player i:1,2 in the two-person game Γ, and $V_1(\Gamma_1)$, $V_1(\Gamma_2)$ are the maximin values of the first and second players in the respective Γ_1 and Γ_2. Analogously, the minimax values are $V^1(\Gamma_1)$ and $V^1(\Gamma_2)$.

This dual description can be done by considering that each player wishes to maximize his sure position independently of the acts of the remaining player. Indeed, the fact that a player behaves with a whole independence of the behavior of the other player is equivalent to expressing that each player uses a maximin strategy in Γ, which is also maximin strategy in the corresponding associated zero-sum two-person game Γ_1 or Γ_2.

Obviously, a joint strategy whose first component is a maximin strategy and the second component is a maximin strategy in the corresponding associated zero-sum two-person games, always exists for the class of two-person games now considered.

There also always exists for such two-person games a joint strategy whose first and second components are both minimax strategies in the corresponding associated zero-sum two-person games.

Actually, such maximin strategies might be minimax strategies, in the other associated zero-sum two-person game. Moreover might be components of some saddle points in such associate two-person games. For this kind of two-person game it is again provided to be a satisfactory rational behavior for situations which involve non-cooperative behavior between the players; in other words, where cooperation is not permitted by the rules of the two-person game.

A joint strategy $(\bar{\sigma}_1, \bar{\sigma}_2)$ is said to be a <u>double saddle point</u> of the two-person game $\Gamma = \{ \Sigma_1, \Sigma_2; A_1, A_2 \}$ if is a saddle point of both associated zero-sum two-person games, i.e.:

$$V(\Gamma_1) = \max_{s_1 \in \Sigma_1} A_1(s_1, \bar{\sigma}_2) = A_1(\bar{\sigma}_1, \bar{\sigma}_2) = \min_{s_2 \in \Sigma_2} A_1(\bar{\sigma}_1, s_2)$$

and

$$V(\Gamma_2) = \max_{s_2 \in \Sigma} A_2(\bar{\sigma}_1, s_2) = A_2(\bar{\sigma}_1, \bar{\sigma}_2) = \min_{s_1 \in \Sigma_1} A_2(s_1, \bar{\sigma}_2) \quad .$$

In such a case:

$$V(\Gamma_1) = V_1(\Gamma) = V^1(\Gamma)$$

and

$$V(\Gamma_2) = V_2(\Gamma) = V^2(\Gamma) \quad .$$

These amounts constitute a <u>value of the two-person game Γ</u>, which now is defined by the vector

$$V(\Gamma) = (V(\Gamma_1) \ , \ V(\Gamma_2)) \ .$$

Again, in this generalized description, one may easily show that the existence of the value of the two-person game Γ under consideration is equivalent to asking for the existence of a double saddle point.

Using these concepts, we will now formulate an extension of the first theorem for the kind of two-person game under consideration.

<u>THEOREM I.5</u>: Let $\Gamma = \{ \Sigma_1, \Sigma_2; A_1, A_2 \}$ be a two-person game such that the strategy sets Σ_1 and Σ_2 are non-empty, compact, convex sets in euclidean spaces; the payoff functions A_1 and A_2 are continuous in the variable $(\sigma_1, \sigma_2) \epsilon \Sigma_1 x \Sigma_2$; for fixed $\sigma_2 \epsilon \Sigma_2$, A_1 is concave and A_2 is convex with respect to $\sigma_1 \epsilon \Sigma_1$; for fixed $\sigma_1 \epsilon \Sigma_1$, A_1 is convex and A_2 is concave with respect to $\sigma_2 \epsilon \Sigma_2$.
Then, if for each joint strategy $(\sigma_1, \sigma_2) \epsilon \Sigma_1 x \Sigma_2$ there is another one $(\tau_1, \tau_2) \epsilon \Sigma_1 x \Sigma_2$ with :

$$A_1(\tau_1, \sigma_2) = \max_{s_1 \epsilon \Sigma_1} A_1(s_1, \sigma_2) \ , \quad A_2(\tau_1, \sigma_2) = \min_{s_1 \epsilon \Sigma_1} A_2(s_1, \sigma_2)$$

and

$$A_1(\sigma_1, \tau_2) = \min_{s_2 \epsilon \Sigma_2} A_1(\sigma_1, s_2) \ , \quad A_2(\sigma_1, \tau_2) = \max_{s_2 \epsilon \Sigma_2} A_2(\sigma_1, s_2)$$

the two-person game Γ has a double saddle point.

<u>PROOF</u>: As before, the product space $\Sigma_1 x \Sigma_2$ is non-empty, compact and convex in an euclidean space, and therefore by the last condition for any joint strategy $(\sigma_1, \sigma_2) \epsilon \Sigma_1 x \Sigma_2$, the following sets

$$\varphi_1(\sigma_2) = \{\tau_1 \epsilon \ _1 : A_1(\tau_1,\sigma_2) = \max_{s_1\epsilon\Sigma_1} A_1(s_1,\sigma_2) \text{ and } A_2(\tau_1,\sigma_2) = \min_{s_1\epsilon\Sigma_1} A_2(s_1,\sigma_2)\}$$

and

$$\varphi_2(\sigma_1) = \{\tau_2 \epsilon \ _2 : A_1(\sigma_1,\tau_2) = \min_{s_2\epsilon\Sigma_2} A_1(\sigma_1,s_2) \text{ and } A_2(\sigma_1,\tau_2) = \max_{s_2\epsilon\Sigma_2} A_2(\sigma_1,s_2)\}$$

are non-empty. According to the concavity of the payoff function A_1 and the convexity of A_2 with respect to the variable $\sigma_1\epsilon\Sigma_1$, for fixed $\sigma_2\epsilon\Sigma_2$, the set $\varphi_1(\sigma_2)$ is convex. Let us consider τ_1, $\bar{\tau}_1$ be any pair of strategies of the first player belonging to the set $\varphi_1(\sigma_2) \subseteq \Sigma_1$, then

$$A_1(\lambda\tau_1+(1-\lambda)\bar{\tau}_1,\sigma_2) = \lambda A_1(\tau_1,\sigma_2) + (1-\lambda)A_1(\bar{\tau}_1,\sigma_2) = \max_{s_1\epsilon\Sigma_1} A_1(s_1,\sigma_2)$$

and

$$A_2(\lambda\tau_1+(1-\lambda)\bar{\tau}_1,\sigma_2) = \lambda A_2(\tau_1,\sigma_2) + (1-\lambda)A_2(\bar{\tau}_1,\sigma_2) = \min_{s_1\epsilon\Sigma_1} A_2(s_1,\sigma_2)$$

for all $\lambda\epsilon[0,1]$. Consequently

$$\lambda\tau_1 + (1-\lambda)\bar{\tau}_1 \epsilon\, \varphi_1(\sigma_2) \qquad \text{for all} \qquad \lambda\epsilon[0,1] .$$

The condition of convexity of the payoff function A_1 and the concavity of A_2 in the variable $\sigma_2\epsilon\Sigma_2$, for fixed $\sigma_1\epsilon\Sigma_1$, assures the property of convexity of the set $\varphi_2(\sigma_1) \subseteq \Sigma_2$.

Consequently, the convex set

$$\varphi(\sigma_1,\sigma_2) = \varphi_1(\sigma_2) x\, \varphi_2(\sigma_1)$$

for each pair $(\sigma_1,\sigma_2)\epsilon\Sigma_1 x\Sigma_2$ defines the corresponding multivalued function

$$\varphi:\ \Sigma_1 x\Sigma_2 \to \Sigma_1 x\Sigma_2 \quad .$$

Consider two converging sequences of joint strategies

$$\sigma(k) \to \sigma \qquad \text{and} \qquad \tau(k) \to \tau$$

which satisfy the condition that for each positive integer k: $\tau(k) \epsilon\ \varphi(\sigma(k))$. This fact implies that for all k

$$A_1(\tau_1(k),\sigma_2(k)) = \max_{s_1 \epsilon \Sigma_1} A_1(s_1,\sigma_2(k)) \ , \qquad A_2(\tau_1(k),\sigma_2(k)) = \min_{s_1 \epsilon \Sigma_1} A_2(s_1,\sigma_2(k))$$

and

$$A_2(\sigma_1(k),\tau_2(k)) = \min_{s_2 \epsilon \Sigma_2} A_1(\sigma_1(k),s_2) \ , \qquad A_2(\sigma_1(k),\tau_2(k)) = \max_{s_2 \epsilon \Sigma_2} A_2(\sigma_1(k),s_2) \ .$$

The continuity of the payoff functions A_1 and A_2 with respect to the variable $(\sigma_1,\sigma_2) \epsilon \Sigma_1 x \Sigma_2$ guarantees the convergence of the eight sequences just represented. These eight sequences converge to their respective values of the payoff functions obtained by replacing in their respective places the point $(\sigma_1(k),\sigma_2(k))$ by (σ_1,σ_2) and the joint strategy $(\tau_1(k),\tau_2(k))$ by (τ_1,τ_2).

According to what has just been established, the following relations are completely satisfied

$$A_1(\tau_1,\sigma_2) = \max_{s_1 \epsilon \Sigma_1} A_1(s_1,\sigma_2) \ , \quad A_2(\tau_1,\sigma_2) = \min_{s_1 \epsilon \Sigma_1} A_2(s_1,\sigma_2)$$

and

$$A_1(\sigma_1,\tau_2) = \min_{s_2 \epsilon \Sigma_2} A_1(\sigma_1,s_2) \ , \quad A_2(\sigma_1,\tau_2) = \max_{s_2 \epsilon \Sigma_2} A_2(\sigma_1,s_2) \ ,$$

These express the fact that the joint strategy τ belongs to the set $\varphi(\sigma_1,\sigma_2)$. Hence, the multivalued function φ is upper-semicontinuous.

Then, the Kakutani Fixed Point Theorem applied to this multivalued function assures the existence of a fixed point $(\bar{\sigma}_1,\bar{\sigma}_2)\in \varphi(\bar{\sigma}_1,\bar{\sigma}_2)$. This is equivalent to the fulfillment of the following conditions

$$\max_{s_1\in\Sigma_1} A_1(s_1,\bar{\sigma}_2) = A_1(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_2\in\Sigma_2} A_1(\bar{\sigma}_1,s_2)$$

and

$$\max_{s_2\in\Sigma_2} A_2(\bar{\sigma}_1,s_2) = A_2(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_1\in\Sigma_1} A_2(s_1,\bar{\sigma}_2) \quad .$$

Thus, such a point is a double saddle point of the two-person game Γ . Q.E.D.

Before continuing the exposition, let us note a very simple fact. Given an arbitrary strategy of the second player $\sigma_2\in\Sigma_2$, consider the set of strategies of the first player $\tau_1\in\Sigma_1$ for which the payoff function A_1 is a maximum and the set of those points where the payoff function of the second player reaches the minimun for such given strategy $\sigma_2\in\Sigma_2$. Then, the last condition in the preceding theorem means that the intersection of such sets with those analogous sets for the second player is non-empty. We note that if in the theorem one changes the mentioned condition to the new requirement, i.e. the identity between the corresponding sets; the theorem turns trivial. Indeed, the double saddle points of the two-person game Γ are completely determined by those saddle points assured by the theorem I.1 of the associated zero-sum two-person games Γ_1 or Γ_2 .

An example of a special class of two-person games for which the last condition of the above theorem holds is given by the following general expression

$$\Gamma = \{ \Sigma_1, \Sigma_2; A \ , \ -cA+d \} \quad .$$

where the payoff of the second player is obtained by multiplying the payoff function of the first player by a non-negative real number c and then by adding a real number d .

Indeed, one obviously can see by the shape of the payoff function of the second player that the latter condition of the theorem I.5 is completely satisfied.

Then, of course, the zero-sum two-person games are obtained by taking in the preceding kind of two-person games, the following real numbers: $c = 1$ and $d = 0$. Thus, one must observe that the above theorem is a straightforward generalization for two-person games of the result related in theorem I.1.

The strong condition on the form of the payoff functions appearing in this theorem could seem to be a new special restriction of the payoff functions. However, the existence of a joint strategy $(\tau_1, \tau_2) \epsilon \Sigma_1 x \Sigma_2$, for a given $(\sigma_1, \sigma_2) \epsilon \Sigma_1 x \Sigma_2$, which satisfies the requirements

$$A_1(\tau_1, \sigma_2) = \max_{s_1 \epsilon \Sigma_1} A_1(s_1, \sigma_2)$$

and

$$A_2(\sigma_1, \tau_2) = \max_{s_2 \epsilon \Sigma_2} A_2(\sigma_1, s_2)$$

simply, expresses the fact that the two-person game Γ has the defense property. The condition that for any $(\sigma_1, \sigma_2) \epsilon \Sigma_1 x \Sigma_2$ there is another joint strategy $(\tau_1, \tau_2) \epsilon \Sigma_1 x \Sigma_2$ with the properties

$$A_1(\sigma_1, \tau_2) = \min_{s_2 \in \Sigma_2} A_1(\sigma_1, s_2)$$

and

$$A_2(\tau_1, \sigma_2) = \min_{s_1 \in \Sigma_1} A_2(s_1, \sigma_2)$$

can be looked at as the attack property of the two-person Γ .

So one can reformulate the above theorem by saying that, a two-person game with the respective concavity and concavity conditions has a double saddle point if it possesses the defense and attack property.

As an immediate consequence of the theorem I.5, one obtains the following result for the mixed extension of a finite two-person game:

<u>THEOREM I.6</u>: Let $\Gamma = \{ \Sigma_1, \Sigma_2; A_1, A_2 \}$ be a finite two-person game. Then, if for any pair of mixed strategies $x \in \tilde{\Sigma}_1$ and $y \in \tilde{\Sigma}_2$, there is another pair $\bar{x} \in \tilde{\Sigma}_1$ and $\bar{y} \in \tilde{\Sigma}_2$ with the property

$$E_1(\bar{x}, y) = \max_{\mu \in \tilde{\Sigma}_1} E_1(u, y) \quad , \quad E_2(\bar{x}, y) = \min_{\mu \in \tilde{\Sigma}_1} E_2(u, y)$$

and

$$E_1(x, \bar{y}) = \min_{\mu \in \tilde{\Sigma}_2} E_1(x, u) \quad , \quad E_2(x, y) = \max_{\mu \in \tilde{\Sigma}_2} E_2(x, u) \; ,$$

the mixed extension $\tilde{\Gamma} = \{ \tilde{\Sigma}_1, \tilde{\Sigma}_2; E_1, E_2 \}$ has a double saddle point.

<u>PROOF</u>: Again, the mixed strategy sets $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are non-empty, compact and convex in euclidean spaces. The expectation functions E_1 and E_2 are, of course, continuous, bilinear and the conditions of the convexity and concavity of theorem I.5 are thus satisfied for the mixed extension $\tilde{\Gamma}$. Moreover, the

new condition, automatically assures the fulfillment of the defense and attack properties for $\tilde{\Gamma}$. Thus, the existence of a saddle point of $\tilde{\Gamma}$ is guaranteed. Q.E.D.

Considering the previous example, the mixed extension

$$\tilde{\Gamma} = \{ \tilde{\Sigma}_1, \tilde{\Sigma}_2; E, -cE+d \}$$

of any finite two-person game

$$\Gamma = \{ \Sigma_1, \Sigma_2; A, -cA+d \}$$

where c is a non-negative real number and d is a real number, always has a double saddle point.

Theorem I.2 is a special case of the latter result (with $c = 1$ and $d = 0$).

We will now consider the question of interchangeability of saddle points in two-person games.

Given a two-person game Γ which satisfies the requirements of the above theorem, it can be described in a symmetric form, that is, as superposition of its associate zero-sum two-person games. This formulation permits us to express the set of double saddle points of our two-person game Γ as the non-void intersection of the sets of saddle points of the associated zero-sum two-person games. As an immediate consequence of theorem I.4, we find that the set of double saddle points of Γ is non-empty, compact and convex since the intersections of compact, convex sets is also compact and convex. Theorem I.3, guarantees the interchangeability of the joint strategies in the set of double saddle points of Γ .

These conditions are wholly satisfied by the mixed extension of any finite two-person games.

I.2 N-person Games: Equilibrium Points

The class of two-person games discussed above is basic in the theory of games. We will now enlarge the class of two-person games to a wider class, where we will incorporate the theoretical representations of real situations where more than two persons are involved.

We define an n-person game (in normal form) by

$$\Gamma = \{\Sigma_1,\dots,\Sigma_n;\ A_1,\dots,A_n\}$$

where the strategy set of pure strategies Σ_i of the player $i \in N = \{1,\dots,n\}$ is a non-empty set and where the payoff function of the player $i \in N$ is a real function defined on the produce space $\Sigma = \Sigma_1 \times \dots \times \Sigma_n$.

Unless there is not an explicit mention to the contrary, we suppose that the strategy sets are non-empty, compact sets in a euclidean space.

An n-person game is said to be finite if all the strategy sets are finite. Given a finite n-person game $\Gamma = \{\Sigma_1,\dots,\Sigma_n;\ A_1,\dots,A_n\}$ then the n-person game $\tilde{\Gamma} = \{\tilde{\Sigma}_1,\dots,\tilde{\Sigma}_n;\ E_1,\dots,E_n\}$ where $\tilde{\Sigma}_i$ indicates the distribution of probability set of the player $i \in N = \{1,\dots,n\}$ and E_i corresponding expectation function, that is, for each $x_j \in \tilde{\Sigma}_j$ and $j \in N$:

$$E_i(x_1,\dots,x_n) = \sum_{\sigma_1 \in \Sigma_1} \dots \sum_{\sigma_n \in \Sigma_n} A_i(\sigma_1,\dots,\sigma_i)\, x_1(\sigma_1)\dots x_n(\sigma_n),$$

is called the mixed extension of Γ .

The expectation function E_i of player $i \in N$ can be described by

$$E_i(x_1,\dots,x_n) = \sum_{\sigma_{i_1} \in \Sigma_{i_1}} \cdot\cdot \sum_{\sigma_{i_r} \in \Sigma_{i_r}} E_i(x_1,\dots,\sigma_{i_1},\dots,\sigma_{i_r},\dots,x_n)\, x_{i_1}(\sigma_{i_1})\dots x_{i_r}(\sigma_{i_r})$$

where $E_i(X_1, \ldots, \sigma_{i_1}, \ldots, \sigma_{i_r}, \ldots, X_n)$ indicates the expectation function for the compound distribution of probability $\bar{X}_{i_1}, \ldots, \bar{X}_{i_r}$ on the product space $\Sigma_{i_1} x \ldots x \Sigma_{i_r}$, formed by

$$\bar{X}_j(\tau_j) = \begin{cases} 1 & \tau_j = \sigma_j \\ 0 & \text{otherwise} \end{cases}$$

for the player $j \in \{i_1, \ldots, i_r\} \subseteq N$.

Making the choice $\sigma_i \in \Sigma_i$ for each player $i \in N$, the joint strategy $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ represents a play of the game. For this joint strategy the amount $A_i(\sigma_1, \ldots, \sigma_n)$ expresses the winnings of player $i \in N$ in the game Γ for the above play.

It is observed that in real situations, generally all the players wish to behave in such a way as to maximize their respective positions without bounds. This remark is similar to the observation for two-person games.

One of the most important questions in the theory of games is to describe a rational or optimal behavior of the players in an n-person game. Of course, this general problem obviously involves greater complexity than the special examination for two-person games described in the previous paragraph. These solutions have a satisfactory heuristic meaning. One of the most important reasons for the great increase of difficulty in the examination of n-person games is the great increase in the number of simple strategy sets. Indeed in a two-person game each player has only one player as opponent, but in the general case "the opponent" of any given player is the remaining players. We will now divide the opposition into several groups. First we group together those players with a similar aptitude toward the fixed player; that is, for instance, the friends of the fixed player form one group, those antagonistic players with respect to him form another group, and finally those players indifferent to him in a last group. Furthermore, the behavior of such groups may be regarded as <u>cooperative</u> if in the real game where abstraction Γ permits cooperation among the players. On the other hand if this coordination is forbidden by the real situation then the complete game is refered to as <u>non-cooperative</u>.

From a heuristic point of view, the knowledge of these aforementioned groups of players associated with each player in an n-person game, determines the "structure" of the game. This formally is completely specified by a special function.

We will now introduce some new concepts to add intuitive meaning to our exposition of n-person games. We will apply these new concepts to special games. Of course, as can be assumed, these new concepts are built on the preceding investigations and they are close generalizations of those concepts of two-person games already considered.

The principal motive of the remaining part of this first chapter, is the introduction and examination of a few of those concepts which are related to the non-cooperative formulation only.

Let us consider an n-person game $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ with the usual properties, that is, with all strategy sets non-empty and compact in euclidean spaces and with all the payoff functions continuous, where only the non-cooperative behavior between the players is allowed. Then, by having a more simple picture of the situation in the original n-person game, one may see a player embeded in an associated two-person game, from which one could formulate the new concepts by using those important assumptions already obtained in the preceding paragraph.

Of course, the choice of the associated two-person game depends upon certain assumptions. However, from all those posible two-person games for the player $i \epsilon N = \{1,\ldots,n\}$ we now consider it natural to define the following associated zero-sum two-person game,

$$\Gamma_i = \{\Sigma_i, \Sigma_{N-\{i\}}; A_i\}$$

where the role of the second player is now played by the set of players $N-\{i\}$ with the strategy set given by the product space

$$\sigma_{N-\{i\}} \epsilon \Sigma_{N-\{i\}} = \underset{j \epsilon N-\{i\}}{\times} \Sigma_j .$$

We recall that the sets Σ_i and $\Sigma_{N-\{i\}}$ are identical as are the corresponding members σ_i and $\sigma_{N-\{i\}}$ for simplicity, in this case we will use the first notation.

We note that formally this game can be maintained even if the set $N-\{i\}$ becomes void. This situation arises when the original game has only the player i. In this case, the set $\Sigma_\emptyset$ is represented by only one element which can be interpreted as the strategy of doing nothing (clearly optimal).

We now have to assign a role to the second player in the associated zero-sum two-person game Γ_i of player $i \epsilon N$. Each criterion adopted will determine a new concept for the total behavior in the whole game Γ.

We will concern ourselves first with the special case for which the second player in each associated game Γ_i is seen as an "indifferent" player with respect to the first player. This means that the second player neither helps nor attacks the corresponding player $i \epsilon N$. Of course, the payoff of the first player still depends upon the choice of the second player. For this "pseudo behavior" of the second player it is natural to assume that the first player will maximize his position on the choice of the second player.

Formally, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1,\ldots,\bar{\sigma}_n) \in \Sigma$ is a positive very simple equilibrium point or concisely an equilibrium point of the n-person game $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$, if for each associated game $\Gamma_i = \{\Sigma_i, \Sigma_{N-\{i\}}, A_i\}$ of the player $i \in N$:

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{N-\{i\}}) .$$

The necessity of the adjective positive for equilibrium points will appear later.

Of course, one could point out the uselessness of the introduction of the associated game Γ_i for the player $i \in N$, in the above definition. However, we prefer at this point and in the subsequent discussion to use it, since, this game will be a very useful tool in the introduction of new concepts in our exposition. Nevertheless, for simplicity, we will not mention it in the proof of the theorems.

We recall that this new concept has been gotten, by taking the first part of the definition of the saddle point for each associated game Γ_i .

From an intuitive viewpoint, an equilibrium point is seen as a rule of behavior for which each player assures the maximum posible position, if in each instance all the other players abide by it.

Let $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2)$ be a positive very simple equilibrium point of a zero-sum two-person game $\Gamma = \{\Sigma_1, \Sigma_2; A\}$, where the associated games are given by

$$\Gamma_1 = \{\Sigma_1, \Sigma_2; A\} \quad \text{and} \quad \Gamma_2 = \{\Sigma_2, \Sigma_1; -A\} .$$

This first associated game satisfies the equality

$$A(\bar{\sigma}_1,\bar{\sigma}_2) = \max_{s_1 \in \Sigma_1} A(s_1,\bar{\sigma}_2)$$

and for the second one, we have

$$-A(\bar{\sigma}_1,\bar{\sigma}_2) = \max_{s_2 \in \Sigma_2} -A(\bar{\sigma}_1,s_2) \quad .$$

Thus, the equilibrium point $\bar{\sigma}$ is a saddle point of game Γ and conversely.

This fact is one of the most important reasons for the introduction of the concept of positive very simple equilibrium point. On this concept is built a wide part of the modern theory of mathmatical-economics. The concept of equilibrium point was introduced by Nash in [12] where he also shows in the theory of n-person games the usefulness of the Kakutani's fixed theorem. The following general theorem considers the question of existence of such points for the games under consideration.

<u>THEOREM I.7:</u> Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex set in an euclidean space and his payoff function A_i is continuous in the variable $\sigma = (\sigma_1,\ldots,\sigma_n) \in \Sigma$ and concave with respect to the variable $\sigma_i \in \Sigma_i$ for fixed $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$. Then, the game Γ has a positive very simple equilibrium point.

<u>PROOF:</u> Consider for an arbitrary point $\sigma = (\sigma_1,\ldots,\sigma_n)$ belonging to the non-empty, compact and convex product space $\Sigma = \Sigma_1 \text{ x} \ldots \text{x } \Sigma_n$, and for a player $i \in N = \{1,\ldots,n\}$ the following non-empty set

$$\varphi_i(\sigma) = \{\tau\epsilon\Sigma:\ A_i(\tau_i,\sigma_{N-\{i\}}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{N-\{i\}})\} .$$

Now, let us demonstrate convexity of this set. Let $\tau,\bar{\tau}$ be two points belonging to the set $\varphi_i(\sigma)$, which satisfy the equality

$$A_i(\tau_i,\sigma_{N-\{i\}}) = A_i(\tau_i,\sigma_{N-\{i\}}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{N-\{i\}}) .$$

For the point $\tau_\lambda = \lambda\tau + (1-\lambda)\bar{\tau}\epsilon\Sigma$ where $\lambda\epsilon[0,1]$, by the concavity of the payoff function A_i with respect to the variable $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$, the following relation holds true:

$$A_i(\lambda\tau_i + (1-\lambda)\bar{\tau}_i,\ \sigma_{N-\{i\}}) \geq \lambda A_i(\tau_i,\sigma_{N-\{i\}} + (1-\lambda)\ A_i(\bar{\tau}_i,\sigma_{N-\{i\}}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i\sigma_{N-\{i\}}) .$$

Since in the last term of this relation appears the maximum amount of the payoff function calculated on the set Σ_i for $\sigma_{N-\{i\}}\in \Sigma_{N-\{i\}}$, strict equality must hold,

$$A_i(\lambda\tau_i + (1-\lambda)\bar{\tau}_i\ ,\ \sigma_{N-\{i\}}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\ \sigma_{N-\{i\}}) .$$

Hence, the set $\varphi_i(\sigma)$ is convex, since all the strategy sets are convex. Now, for an $\sigma\epsilon\Sigma$ consider the intersection of all those sets

$$\varphi(\sigma) = \bigcap_{i\epsilon N} \varphi_i(\sigma) .$$

This is non-empty, because the point $\tau\epsilon\Sigma$ obtained by taking each component

$\tau_i \epsilon \Sigma_i$ where the payoff function A_i reaches its maximum, is an element of all the sets $\varphi_i(\sigma)$. Moreover, because it is the intersection of convex sets, it itself is convex.

We now define the multivalued function

$$\varphi : \Sigma \to \Sigma$$

determined by the set $\varphi(\sigma)$ for any $\sigma \epsilon \Sigma$.

Let

$$\sigma(k) \to \sigma \quad \text{and} \quad \tau(k) \to \tau$$

be two converging sequences of point in the product space Σ , which satisfy the condition that for each positive integer k: $\tau(k) \in \varphi(\sigma(k))$. Thus, we have for all k and for all player $i \epsilon N$

$$A_i(\tau_i(k) , \sigma_{N-\{i\}}(k)) = \max_{s_i \epsilon \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}(k)) .$$

The condition of continuity of the payoff function A_i with respect of the variable $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$, implies the convergence of the following sequences of real numbers

$$A_i(\tau_i(k) , \sigma_{N-\{i\}}(k)) \to A_i(\tau_i, \sigma_{N-\{i\}})$$

and

$$\max_{s_i \epsilon \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}(k)) \to \max_{s_i \epsilon \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}) ,$$

since, the sequence of points $(\tau_i(k), \sigma_{N-\{i\}}(k))$ of the product space Σ converges to the joint strategy $(\tau_i, \sigma_{N-\{i\}}) \in \Sigma$.

An immediate consequence of these two facts is the equality

$$A_i(\tau_i, \sigma_{N-\{i\}}) = \max_{s_i \epsilon \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}) .$$

From this we deduce that the point τ is an element of the set $\varphi(\sigma)$. Thus, the upper-semicontinuity of the multivalued function is shown.

Therefore, by a straight application of the Kakutani fixed point theorem to the multi valuated function φ, the existence of fixed point $\bar{\sigma} \in \varphi(\bar{\sigma})$ is assured. For this joint strategy for each player $i \in N$ we have:

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{N-\{i\}}) ,$$

which is the definition of a positive very simple equilibrium point of the game Γ. (Q.E.D.)

We note that this theorem was proven because, with respect to the concept of positive very simple equilibrium point, Γ had the defense property, that is, for each joint strategy $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ there is another point $\tau \in \Sigma$ such that for each player $i \in N$

$$A_i(\tau_i, \sigma_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}) .$$

Intuitively speaking, this means that for each established compound behavior in the game Γ there is another one which maximizes the position of each player if all the other players abide by the first behavior.

Having the preceding characterization of equilibrium points, it is natural to ask about the equivalence property of positive very simple equilibrium points. Unfortunately, the answer to this question in the general case is negative, as one can easily verify. Then lack of this property, is seen by some authors as critical of the theory. Indeed, generally the equilib rium points are not equivalent, which means in other words, that the payoff of a player in different positive very simple equilibrium points cannot be coincident.

One could consider for an established positive very simple equilibrium point each player for which his payoff is less than the maximum of his payoff function on all the positive equilibrium points, as an "unsatisfied" player. Thus, an unsatisfied player will try to change from such a positive very simple equilibrium point in order to get another one where his payoffs are better, and therefore the first positive point will be destroyed. From this fact one can say that a positive very simple equilibrium point can be "unstable".

An important consequence of the preceding theorem is expressed in the following.

THEOREM I.8: Let $\Gamma = \{\Sigma_1,\dots,\Sigma_n, A_1,\dots,A_n\}$ be a finite n-person game, then the mixed extension $\tilde{\Gamma} = \{\tilde{\Sigma}_1,\dots,\tilde{\Sigma}_n, E_1,\dots,E_n\}$ has a positive very simple equilibrium point.

PROOF: The mixed strategy set $\tilde{\Sigma}_i$ for player $i \in N$ is non-empty, compact and convex in an euclidean space. On the other hand, the expectation function E_i of player $i \in N$ is a multilinear function, that is, it is a linear function with respect to each variable $x_j \in \tilde{\Sigma}_j$ with $j \in N$, for fixed $x_{N-\{i\}} \in \bigtimes_{k \in N-\{i\}} \tilde{\Sigma}_k$, and therefore concave in $x_i \in \tilde{\Sigma}_i$. Moreover, by this latter property all the expectation functions are continuous in the variable $x = (x_1,\dots,x_n)$. Thus, all the conditions of the theorem I.7 are satisfied, which assures the existence of a positive very simple equilibrium point of the mixed extension $\tilde{\Gamma}$. (Q.E.D.)

We now return to consider the associated zero-sum two-person game $\Gamma_i = \{\Sigma_i, \Sigma_{N-\{i\}}; A_i\}$ for the player $i \in N$ in the n-person game Γ.

The concept of positive very simple equilibrium point has been obtained as an immediate consequence of the assumption adopted on the behavior of both players in the associated zero-sum two-person games. We were looking at the second player like an indifferent one, and the remaining player as a normal first player in a zero-sum two-person game, that is, he wishes to maximize his position as far as is possible.

Actually, since this view point is unsymmetrical, one can formulate the dual description, that is, to consider the first player as apathetic on his own winning and the remaining player as a normal second player in a zero-sum two-person game. Then, for this "pathological behavior" of the first player, one can assume that the second player will minimize, on the strategy chosen of his opposition, the payoff to the first player.

From these considerations, one can exactly introduce the following precise concept.

Given an n-person game $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$, a joint strategy $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ is called a <u>negative very simple equilibrium point</u> of the game Γ if for each associated game $\Gamma_i = \{\Sigma_i, \Sigma_{N-\{i\}}, A_i\}$ for the player $i \in N$:

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\bar{\sigma}_i, s_{N-\{i\}}) \quad .$$

We note that these equalities are naturally obtained, by considering only the second term in the definition of saddle point for the game Γ_i of each player $i \in N$.

Intuitively speaking, a negative very simple equilibrium point is a behavior rule which is such that if some players change from it, then the position of some other player can be improved.

Motivated by this fact we have introduced the adjective negative for such a point. Another similar remark will be considered later.

Now, we will show that these negative very simple equilibrium points, for a zero-sum two-person game can be transformed into saddle points, as the positive very simple equilibrium points were.

Let $\Gamma = \{\Sigma_1,\Sigma_2; A\}$ be a zero-sum two-person game with $\bar{\sigma} = (\bar{\sigma}_1,\bar{\sigma}_2)\in\Sigma_1 x\Sigma_2$ as a negative very simple equilibrium point, then consider the associate games which are

$$\Gamma_1 = \{\Sigma_1,\Sigma_2; A\} \qquad \text{and} \quad \Gamma_2 = \{\Sigma_2,\Sigma_1; -A\} \quad .$$

The first game assures the equality

$$A(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_2\in\Sigma_2} A(\bar{\sigma}_1,s_2)$$

and the second determines the relation

$$-A(\bar{\sigma}_1,\bar{\sigma}_2) = \min_{s_1\in\Sigma_1} -A(s_1,\bar{\sigma}_2) \quad ,$$

and therefore the point $\bar{\sigma} \in \Sigma_1 x\Sigma_2$ is a saddle point of the zero-sum two-person game Γ, and conversely.

A characterization of such points is formulated in the following existence theorem.

<u>THEOREM I.9:</u> Let $\Gamma = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an n-person game such that the strategy set Σ_i of player $i\in N = \{1,\dots,n\}$ is non-empty, compact and convex set in an an euclidean space, and his payoff function A_i is continuous in the variable $\sigma = (\sigma_1,\dots,\sigma_n) \in \Sigma$, and convex with respect to the variable $\sigma_{N-\{i\}}\in \Sigma_{N-\{i\}}$ for fixed $\sigma_i \in \Sigma_i$. Then, if for each joint strategy $\sigma\in\Sigma$ there is another one $\tau\in\Sigma$ such that for all $i\in N$

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) ,$$

the game Γ has a negative very simple equilibrium point.

PROOF: For any point $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ of the non-empty compact and convex product space Σ, and each player $i \in N$, the set

$$\varphi_i(\sigma) = \{\tau \in \Sigma: A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}})\}$$

is obviously non-empty.

For the question of the important property of convexity of this set, let us consider two arbitrary elements τ and $\bar{\tau}$ of the set $\varphi_i(\sigma)$, which satisfies:

$$A_i(\sigma_i, \tau_{N-\{i\}}) = A_i(\sigma_i, \bar{\tau}_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

Then, the convexity property of the payoff function A_i in the variable $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$ for fixed $\sigma_i \in \Sigma_i$, guarantees for each $\lambda \in [0,1]$

$$A_i(\sigma_i, \lambda\tau_{N-\{i\}} + (1-\lambda)\bar{\tau}_{N-\{i\}}) \leq \lambda A_i(\sigma_i, \tau_{N-\{i\}}) + (1-\lambda)A_i(\sigma_i, \bar{\tau}_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

The strict equality must hold since the last term is the minimum value of the payoff function on the set $\Sigma_{N-\{i\}}$ with $\sigma_i \in \Sigma_i$ fixed. Thus, for any real number $\lambda \in [0,1]$, one, then has:

$$A_i(\sigma_i, \lambda\tau_{N-\{i\}} + (1-\lambda)\bar{\tau}_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

This implies that the point $\lambda\tau + (1-\lambda)\bar{\tau}$ is a member of the set $\varphi_i(\sigma)$.

Consider for an arbitrary joint strategy $\sigma\epsilon\Sigma$ the intersection of those sets

$$\varphi(\sigma) = \bigcap_{i\epsilon N} \varphi_i(\sigma) ,$$

which determine a multivalued function

$$\varphi: \Sigma \to \Sigma .$$

On one hand, for each point $\sigma\epsilon\Sigma$ the set $\varphi(\sigma)$ is non-empty by virtue of the last condition, and on the other hand is convex because it is the intersection of convex sets.

We are now going to prove the upper-semicontinuity of the above multivalued function. Let

$$\sigma(k) \to \sigma \qquad \text{and} \quad \tau(k) \to \tau$$

be two arbitrary converging sequences of elements of the product space Σ, such that for each positive integer k : $\tau(k) \epsilon \varphi(\sigma(k))$. This means that for each $i\epsilon N$, the following relation holds:

$$A_i(\sigma_i(k), \tau_{N-\{i\}}(k)) = \min_{s_{N-\{i\}}\epsilon \Sigma_{N-\{i\}}} A_i (\sigma_i(k), s_{N-\{i\}})$$

for any k . By virtue of the continuity of the payoff function A_i , of player $i\epsilon N$, with respect to the variable $\sigma\epsilon\Sigma$, and because the sequence of points $(\sigma_i(k), \tau_{N-\{i\}}(k))$ of the product space Σ , converges to the joint strategy $(\sigma_i, \tau_{N-\{i\}}) \epsilon \Sigma$, then the two sequences of real numbers having as general terms the members of the preceding equality converge.

$$A_i(\sigma_i(k) , \tau_{N-\{i\}}(k)) \to A_i(\sigma_i, \tau_{N-\{i\}})$$

and

$$\min_{s_{N-\{i\}}\epsilon \Sigma_{N-\{i\}}} A_i(\sigma_i(k) , s_{N-\{i\}}) \to \min_{s_{N-\{i\}}\epsilon \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

Thus, for each $i \epsilon N$, one has

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) \quad .$$

This expresses the fact that the point τ belongs to the set $\varphi(\sigma)$, and therefore the multivalued function φ is upper-semicontinuous.

In accordance with the Kakutani theorem there exists a fixed point $\bar{\sigma} \epsilon \varphi(\bar{\sigma})$ of the multivalued function φ , for which

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}} A_i(\bar{\sigma}_i, s_{N-\{i\}})$$

for all $i \epsilon N$.

This is the definition of a negative very simple equilibrium point for the game Γ . (Q.E.D.)

The strong condition assumed in this theorem which assures for any point $\sigma \epsilon \Sigma$ the existence of another joint strategy $\tau \epsilon \Sigma$ such that for each player $i \epsilon N$

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}})$$

is called the <u>attack property</u> of the game Γ with respect to the negative very simple equilibrium point concept.

For any accepted behavior of the players in the game Γ there is another one which minimizes the winning of each player if this considered player abides by the first one. This is a heuristic interpretation of the attack property.

Again, the equilvalence property for negative very simple equilibrium points in n-person games is not usually satisfied.

A special case of the above result comes from mixed extension of finite games.

<u>THEOREM I.10:</u> Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n, A_1,\ldots,A_n\}$ be a finite n-person game, such that the expectation function E_i of any player $i\in N$ is linear in the variable

$$x_{N-\{i\}} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in X_{N-\{i\}} = \tilde{\Sigma}_1 x\ldots x\tilde{\Sigma}_{i-1} x\tilde{\Sigma}_{i+1} x\ldots x\tilde{\Sigma}_n$$

for fixed $x_i\in\tilde{\Sigma}_i$. Then, if for each $x\in X = \underset{i\in N}{\times}\tilde{\Sigma}_i$ there is another $y\in X$ such that for all $i\in N$:

$$E_i(x_i,y_{N-\{i\}}) = \min_{s_{N-\{i\}}\in \Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}}) ,$$

the mixed extension $\tilde{\Gamma} = \{\tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; E_1,\ldots,E_n\}$ has a negative very simple equilibrium point.

<u>PROOF:</u> In the mixed extension game $\tilde{\Gamma}$, the mixed strategy set $\tilde{\Sigma}_i$ belonging to the player $i\in N$ is non-empty, compact and convex in a euclidean space. Because the expectation function E_i is linear in the variable $x_{N-\{i\}}\in X_{N-\{i\}}$, for fixed $x_i \in \tilde{\Sigma}_i$, it is also convex in such a variable.

Let x be an arbitrary element of the product space X , then, because the product space $X_{N-\{i\}}$ can be considered wider than $\Sigma_{N-\{i\}}$, we have for an arbitrary player $i\in N$

$$\min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}}) \geq \min_{u_{N-\{i\}}\in X_{N-\{i\}}} E_i(x_i,u_{N-\{i\}}) ,$$

recalling that $E_i(x_i,\sigma_{N-\{i\}})$ indicates the expectation function of the player $i\in N$, for the following element of $X_{N-\{i\}}$:

$$x_{N-\{i\}}(\tau_1,\ldots,\tau_{i-1},\tau_{i+1},\ldots,\tau_n) = \begin{cases} 1 & \text{if } \tau_{N-\{i\}} = \sigma_{N-\{i\}} \\ 0 & \text{otherwise} \end{cases} .$$

On the other hand, because the values $E_i(x_i, u_{N-\{i\}})$ are combinations of values $E_i(x_i, \sigma_{N-\{i\}})$, that is

$$E_i(x_i, u_{N-\{i\}}) = \sum_{\sigma_1 \in \Sigma_1} \cdots \sum_{\sigma_{i-1} \in \Sigma_{i-1}} \sum_{\sigma_{i+1} \in \Sigma_{i+1}} \cdots \sum_{\sigma_n \in \Sigma_n} E_i(x_i, \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots \sigma_n) \cdot$$

$$\cdot \; u_1(\sigma_1) \ldots u_{i-1}(\sigma_{i-1}) \cdot u_{i+1}(\sigma_{i+1}) \ldots u_n(\sigma_n) \; ,$$

the following inequality is immediate

$$\min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}}) \leq \min_{u_{N-\{i\}} \in X_{N-\{i\}}} E_i(x_i, u_{N-\{i\}}) \; ,$$

and therefore both amounts are equal.

Thus, the last condition can be reformulated by a similar one, namely, for each $x \in X$ there is a $y \in X$ such that for each $i \in N$:

$$E_i(x_i, y_{N-\{i\}}) = \min_{u_{N-\{i\}} \in X_{N-\{i\}}} E_i(x_i, u_{N-\{i\}}) \; .$$

This coincides with the latter requirement in the preceding theorem for the mixed extention $\tilde{\Gamma}$. That theorem guarantees the existence of a negative very simple equilibrium point of $\tilde{\Gamma}$. (Q.E.D.)

We note that for finite games, the latter condition in the preceding theorem is equivalent to the following one: for each point x in the product space $X = \tilde{\Sigma}_1 \times \ldots \times \tilde{\Sigma}_n$ there is a joint pure strategy $\sigma \in \Sigma = \Sigma_1 \times \ldots \times \Sigma_N$ such that

$$E_i(x_i, \sigma_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}})$$

for all $i \in N$.

Clearly, this new condition implies the old one. Conversely, if for the point $x \in X$, the corresponding joint mixed strategy $y = (y_1, \ldots, y_n) \in X$ satisfies

$$E_i(x_i,y_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}})$$

for all $i\in N$, then consider for each player $i\in N$ the probability distribution y_i defined on Σ_i corresponding to the joint strategy y .

Let τ_i be an arbitrary pure strategy in Σ_i of the player $i\in N$ for which $y_i(\tau_i) \neq 0$. Such a strategy obviously always exists. Now, we form the point $\tau\in\Sigma$ composed by τ_i in all the coordinates, then for this point we have

$$E_i(x_i,\tau_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}})$$

for all $i\in N$. Indeed, suppose that for an $i\in N$ we had

$$E_i(x_i,\tau_{N-\{i\}}) > \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}}) ,$$

then, since for any $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$:

$$E_i(x_i,\sigma_{N-\{i\}}) \geq \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}})$$

and remembering the expression of the payoff function $E_i(x_i,y_{N-\{i\}})$ and that $y_j(\tau_j)$ for all $j\in N-\{i\}$, it would be

$$E_i(x_i,y_{N-\{i\}}) = \sum_{\sigma_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,\sigma_{N-\{i\}})\, y_{N-\{i\}}(\sigma_{N-\{i\}})$$

$$> \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}}) ,$$

which is a contradiction. Thus, the point $\tau\in\Sigma$ assures the validity of the assertion.

In the previous theorem condition of linearity with respect to the variable $x_{N-\{i\}}$ assigned to the expectation function has been explicitly used. Even though the expectation function is a miltilinear function, that is, a linear function in each variable, it can be a non linear function with respect to the variable $x_{N-\{i\}} \in X_{N-\{i\}}$ for some fixed $x_i \in X_i$.

This fact is illustrated in the following example. Given the three finite sets

$$\Sigma_1 = \Sigma_2 = \Sigma_3 = \{1,2\} ,$$

consider the function A on the product space $\Sigma_1 \times \Sigma_2 \times \Sigma_3$ defined by

$$A(\sigma_1,\sigma_2,\sigma_3) = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 = \sigma_3 \\ 0 & \text{otherwise.} \end{cases}$$

Its corresponding expectation E has the value

$$E(x_1,x_2,x_3) = x_{1,1}\, x_{2,1}\, x_{3,1} + x_{1,2}\, x_{2,2}\, x_{3,2}$$

for the probability distributions $x_i = (x_{i,1}, x_{i,2}) \in \tilde{\Sigma}_i$ with i: 1,2,3.

Let $z_3 = (1,0)$ be an element of the probability distribution set $\tilde{\Sigma}_3$, and let

$$x = (x_1, x_2) = ((1,0), (1,0)) , \quad y = (y_1, y_2) = ((0,1),(0,1))$$

be two points in the product set $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$, then the expectation function on such points has the values

$$E(x,z_3) = 1 \quad \text{and} \quad E(y,z_3) = 0 .$$

On the other hand, consider the point

$$\lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)\, y_1 ,\ \lambda x_2 + (1-\lambda)\, y_2)$$

$$= ((\lambda,1-\lambda), (\lambda,1-\lambda)) \in \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$$

where λ is an arbitrary real number belonging to the segment $[0,1]$. Then on this point the expectation function reaches the value

$$E(\lambda x + (1-\lambda) y_1, z_3) = \lambda^2 ,$$

and therefore

$$\lambda E(x_1, z_3) + (1-\lambda) E(y_1 z_3) = \lambda \neq E(\lambda x_1 + (1-\lambda) y_1 z_3) = \lambda^2 ,$$

which implies the lack of linearality of the expectation function.

In accordance with the above illustration, it seems very natural to ask when the expectation function of the player $i \in N$ is linear in the variable $x_{N-\{i\}} \in X_{N-\{i\}}$. The answer to this question is an immediate consequence of the following:

<u>LEMMA I.11</u>: Let $\Sigma_1, \ldots, \Sigma_n$ be n-non-empty finite sets and let A be a real function defined on the product space $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$. Then, the expectation function E is a linear function with respect to the variable

$$x = (x_1, \ldots, x_n) \in X = \tilde{\Sigma}_1 \times \ldots \times \tilde{\Sigma}_n ,$$

if and only if, the function A is expressible as

$$A(\sigma_1, \ldots, \sigma_n) = a_1(\sigma_1) + \ldots + a_n(\sigma_n)$$

where a_i indicates a function depending only upon the variable $\sigma_i \in \Sigma_i$ with $i \in N = \{1, \ldots, n\}$.

<u>PROOF:</u> First of all, we demonstrate the sufficiency. For a function with such conditions, the expression of the expectation function takes the following form

$$E(x) = E(x_1, \ldots, x_n) = \sum_{i=1}^{n} e_i(x_i)$$

where the function $e_i(x_i)$ is the corresponding expectation of $a_i(\sigma_i)$, that is, $e_i(x_i) = \sum_{\sigma_i \in \Sigma_i} A(\sigma_i)\, x_i(\sigma_i)$, for $i \in N$.

Now, consider two arbitrary points of the product space X :

$$x = (x_1, \dots, x_n) \quad \text{and} \quad y = (y_1, \dots, y_n)$$

and let λ be any real number belonging to the unit segment $[0,1]$, then by using the later equality together with the property of linearity of the expectation function e_i, one has

$$E(\lambda\, x + (1-\lambda) y) = \sum_{i=1}^{n} e_i(\lambda\, x_i + (1-\lambda)\, y_i) = \sum_{i=1}^{n} [\lambda\, e_i(x_i) + (1-\lambda) e_i(y_i)]$$

$$= \lambda \sum_{i=1}^{n} e_i(x_i) + (1-\lambda) \sum_{i=1}^{n} e_i(y_i) = \lambda\, E(x) + (1-\lambda)\, E(y)$$

proving the linearity of the expectation function E.

We now will prove the converse by complete induction on the number n of sets in the product space X.

If n is unity, the necessity is a trivial result. Now, consider the case where n is 2, then the expectation function E is a linear function of the variable $x = (x_1, x_2) \in \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$. In other words, for each pair of elements in the product space $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ and any real number λ such that the point $\lambda\, x + (1-\lambda) y$ is in $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$, we have:

$$E(\lambda\, x + (1-\lambda)\, y) = E(\lambda\, x_1 + (1-\lambda)\, y_1,\ \lambda\, x_2 + (1-\lambda)\, y_2) =$$

$$= \lambda\, E(x_1, x_2) + (1-\lambda)\, E(y_1, y_2) = \lambda\, E(x) + (1-\lambda)\, E(y)\ .$$

On the other hand, the expectation function always is a bilinear function, so:

$$E(\lambda x_1 + (1-\lambda) y_1,\ \lambda x_2 + (1-\lambda) y_2) = \lambda^2 E(x_1, x_2) + (1-\lambda)^2\ E(y_1, y_2)$$
$$+ (1-\lambda)\, \lambda\ (E(x_1, y_2) + E(x_2, y_1))\ .$$

By using both preceding equalities, the following relation between the values of the expectation function holds:

$$E(x_1,x_2) + E(y_1,y_2) = E(x_1,y_2) + E(x_2,y_1) .$$

From here the proof of the necessity for $n = 2$ is straight forward. Indeed, by taking y as a fixed point and x as a variable, the above equality implies that the expectation function has the form:

$$E(x_1,x_2) = e_1(x_1) + e_2(x_2)$$

where by the bilinearity of E, the functions e_1 and e_2 must be linear:

$$e_i(x_i) = \sum_{\sigma_i \epsilon \Sigma_i} a_i(\sigma_i)\, x_i(\sigma_i) \qquad (i = 1,2) .$$

By substituting the expressions of the functions in the later relation, we have

$$\sum_{\sigma_1 \epsilon \Sigma_1} \sum_{\sigma_2 \epsilon \Sigma_2} [A(\sigma_1,\sigma_2) - a_1(\sigma_1) - a_2(\sigma_2)]\, x_1(\sigma_1)\, x_2(\sigma_2) = 0$$

for all $x_1 \epsilon \tilde{\Sigma}_1$ and $x_2 \epsilon \tilde{\Sigma}_2$. Therefore, we deduce

$$A(\sigma_1,\sigma_2) = a_1(\sigma_1) + a_2(\sigma_2)$$

for every point $(\sigma_1,\sigma_2) \epsilon\ \Sigma_1 \times \Sigma_2$. Indeed, if for a $(\sigma_1,\sigma_2) \epsilon\ \Sigma_1 x \Sigma_2$ we had $A(\sigma_1,\sigma_2) \neq a_1(\sigma_1) + a_2(\sigma_2)$ then by considering the probability distribution $\bar{x}_1 \epsilon X_1$ which assigns the unity to $\sigma_1 \epsilon \Sigma_1$ and $\bar{x}_2 \epsilon X_2$ which for the point $\sigma_2 \epsilon \Sigma_1$ is one, the last equality would be false. This contradiction proves the validity mentioned assertion.

Now, for an arbitrary n , consider the expectation function defined by the expression

$$E(x_1,\dots,x_n) = \sum_{\sigma_n \epsilon \Sigma_n} E(x_1,\dots,x_{n-1}, \sigma_n)\, x_n(\sigma_n) .$$

Since the expectation function E is linear with respect to the variable $(x_1,\dots,x_n) \epsilon X$ it is obviously linear in the variable $(x_1,\dots,x_{n-1}) \epsilon X_{N-\{i\}}$ for fixed $x_n \epsilon \tilde{\Sigma}_n$. The induction hypothesis with $n-1$, applied to expectation $E(x_1,\dots,x_{n-1},\sigma_n)$ assures the following form for the function A

$$A(\sigma_1,\dots,\sigma_n) = b_1(\sigma_1,\sigma_n) +\dots+ b_{n-1}(\sigma_1,\sigma_n) .$$

By substituting the values of function A in the expectation function by this latter expression, we have

$$E(x_1,\dots,x_n) = f_1(x_1,x_n) +\dots+ f_{n-1}(x_{n-1},x_n)$$

where f_i indicates the corresponding expectation function of b_i with i: $1,\dots,n-1$, and therefore it is bilinear. We, now will prove that such functions are also linear. For this reason, consider two arbitrary points x and y in the product space $X = \tilde{\Sigma}_1 x \dots x \tilde{\Sigma}_n$ such that for all i different from one or n : $y_i = x_i$. For all real numbers λ such that the point $\lambda\, x + (1-\lambda)\, y$ belongs to the set X , the expectation function, on one hand, is

$$\begin{aligned} E(\lambda x + (1-\lambda)y) &= E(\lambda x_1 + (1-\lambda)y_1,\ x_2,\dots,x_{n-1},\ \lambda\, x_n + (1-\lambda)\, y_n) \\ &= f_1(\lambda x_1 + (1-\lambda)y_1, \lambda x_n + (1-\lambda)y_n) + f_2(x_2, \lambda x_n + (1-\lambda)y_n) \\ &\quad +\dots+ f_{n-1}(x_{n-1}, \lambda x_n + (1-\lambda)\, y_n) . \end{aligned}$$

In analogous fashion, by the linearity of the expectation function, we obtain the following relation

$$\begin{aligned} E(\lambda x + (1-\lambda)y) &= \lambda\, E(x) + (1-\lambda)\, E(y) = \lambda\, E(x_1,\dots,x_n) + (1-\lambda)\, E(y_1,\dots,y_n) \\ &= \lambda\, f_1(x_1,x_n) +\dots+ \lambda\, f_{n-1}(x_{n-1},x_n) + (1-\lambda)\, f_1(x_1,y_n) \\ &\quad +\dots+ (1-\lambda)\, f_{n-1}(x_{n-1},y_n) . \end{aligned}$$

By identification of the respective values in both equalities, the identity

$$f_1(\lambda x_1 + (1-\lambda)y_1, \lambda x_n + (1-\lambda)y_n) = \lambda f_1(x_1 y_1) + (1-\lambda) f_1(x_n, y_n)$$

holds. Thus, the function f_1 is linear with respect to the variable $(x_1, x_n) \in \tilde{\Sigma}_1 \times \tilde{\Sigma}_n$. The considerations demonstrated before gives the following form for the function b_1:

$$b_1(\sigma_1, \sigma_n) = a_1(\sigma_1) + c_1(\sigma_n) .$$

By repeating this process for the remaining i and putting $a_n(\sigma_n) = \sum_{i=1}^{n-1} c_i(\sigma_n)$, we get immediately the fact that the function A is expressible as

$$A(\sigma_1, \ldots, \sigma_n) = a_1(\sigma_1) + \ldots + a_n(\sigma_n) . \quad \text{(Q.E.D.)}$$

Consequently, the expectation function E_i of player $i \in N$ in a finite n-person game $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ is a linear function with respect to the variable $x_{N-\{i\}} \in X_{N-\{i\}}$ if and only if his corresponding payoff function A_i is expressible in the following form

$$A_i(\sigma_1, \ldots, \sigma_n) = a_i^1(\sigma_1, \sigma_i) + \ldots + a_i^{i-1}(\sigma_{i-1}, \sigma_i) + a_i^i(\sigma_i) + a_i^{i+1}(\sigma_{i+1}, \sigma_1) + \ldots + a_i^n(\sigma_i, \sigma_n) .$$

For the mixed extension $\tilde{\Gamma}$ of a finite n-person game Γ having such characteristics, theorem I.10 guarantees the existence of a negative very simple equilibrium point if for each point $x \in X = \tilde{\Sigma}_1 x \ldots x \tilde{\Sigma}_n$ there is a joint pure strategy $\sigma \in \Sigma = \Sigma_1 \, x \ldots x \Sigma$ which satisfies

$$E_i(x_i, \sigma_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}})$$

for all $i \in N$.

Actually, it is interesting to illustrate by an example, a class of finite n-person game for which the above condition holds.

By the imposed form on the payoff function A and by the linearity requirement it follows that

$$\min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} E_i(x_i,s_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} \sum_{j\in N} e_i^j(x_i,s_j) = \sum_{j\in N} \min_{s_j\in\Sigma_j} e_i^j(x_i,s_j)$$

for all $i\in N$, where $e_i^j(x_i,s_j)$ indicates the expectation function taken over $\tilde{\Sigma}_i$ of the function $a_i^j(\sigma_i,s_j)$, in an obvious manner. Therefore the above condition is transformed to the asking of the existence of a point $\sigma\in\Sigma$ with

$$\sum_{j\neq i} e_i^j(x_i,\sigma_j) = \sum_{j\neq i} \min_{s_j\in\Sigma_j} e_i^j(x_i,s_j)$$

for all $i\in N$. But this requirement is equivalent to asking for each $x\in X$ the existence of a joint strategy $\sigma\in\Sigma$ which for all $i\in N$ and all $j\neq i$

$$e_i^j(x_i,\sigma_j) = \min_{s_j\in\Sigma_j} e_i^j(x_i,s_j) .$$

Indeed, suppose that for some $i\in N$ and $j\neq i$ we had

$$e_i^j(x_i,\sigma_j) > \min_{s_j\in\Sigma_j} e_i^j(x_i,s_j)$$

then, by adding over $j\neq i$ we would have

$$\sum_{j\neq i} e_i^j(x_i,\sigma_j) > \sum_{j\neq i} \min_{s_j\in\Sigma_j} e_i^j(x_i,s_j)$$

which is absurd because we have assumed the validity of the strict equality in this last equation. Conversely, this latter condition obviously, implies the above requirement.

A simple example of n-person games for which this condition is completely satisfied appears when the payoff function of the player $i \in N$ has the following form

$$A_i(\sigma_1, \ldots, \sigma_n) = a_i^i(\sigma_i) + a_i^{g(i)}(\sigma_i, \sigma_{g(i)})$$

where $g(i) \neq i$ indicates a fixed player for $i \in N$ with the following property: there is not a player $j \in N$ for which there is more than one $i \in N$ satisfying $j = g(i)$.

For such kinds of n-person games, given a point $x = (x_1, \ldots, x_n) \in X$, since for any player $j \in N$ there is at most one player $i \in N$ with $g(i) = j$, by choosing the strategy $\sigma_j \in \Sigma_j$ of the player $j = g(i) \in N$ if $g(i) \neq \emptyset$ such that

$$e_i^{g(i)}(x_i, \sigma_j) = \min_{s_{g(i)} \in \Sigma_{g(i)}} e_i^{g(i)}(x_i, s_{g(i)}) ,$$

and any arbitrary $\sigma_j \in \Sigma_j$ for each player $j \in N$ which does not belong to any $g(i)$, the additional condition remains completely satisfied. Thus, the mixed extension of such an n-person game has a negative very simple equilibrium point, namely, the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ formed of those components $x_j = x_{g(i)} \in \tilde{\Sigma}_j$ with $g(i) \neq \emptyset$ which are optimal mixed strategy in the zero-sum two-person game

$$\tilde{\Gamma}_i = \{\tilde{\Sigma}_i, \tilde{\Sigma}_{g(i)}; e_i^{g(i)}(x_i, x_{g(i)})\}$$

(which could be now considered as the associated game) and any arbitrary $x_j \in \Sigma_j$ belonging to the player $j \in N$ which does not belong to any $g(i)$.

In this kind of n-person game, the player $g(i)$ can be interpreted as the direct opposite of the player $i \in N$. Then, a more special situation appears when both players i and $g(i)$ are considered embeded in a strict competitive situation, that is, if

$$g(g(i)) = i,\ a_i^i(\sigma_i) = 0 \quad \text{and} \quad a_i^{g(i)}(\sigma_i, \sigma_{g(i)}) = -c_i\ a_{g(i)}^i(\sigma_i, \sigma_{g(i)}) + d_i$$

where c_i is a non-negative real number and d_i any arbitrary real number. Of course, for these special games each player $i \epsilon N$ has only one opposite which is the player $g(i) \in N$, and therefore the number of players must be even. The previous theorem guarantees the existence of a negative very simple equilibrium point for such a game. Since each component of a negative very simple equilibrium point is a minimax strategy of the player $g(i)$ in the zero-sum two-person assocated game $\tilde{\Gamma}_i$, by the form of the payoff function, it is also a maximin strategy for himself in his corresponding zero-sum two-person game $\tilde{\Gamma}_{g(i)}$. But, that point is then also a positive very simple equilibrium point of the mixed extension.

This fact induces us to introduce the following new concept involving the two concepts already considered.

Given an n-person game $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n) \in \Sigma$ is said to be a <u>neutral very simple equilibrium point</u> or <u>very simple saddle point</u> of the game Γ if for each associated game $\Gamma_i = \{\Sigma_i, \Sigma_{N-\{i\}}; A_i\}$ of the player $i \epsilon N$:

$$\max_{s_i \epsilon \Sigma_i} A_i(s_i, \bar{\sigma}_{N-\{i\}}) = A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}} A_i(\bar{\sigma}_i, s_{N-\{i\}}) .$$

In other words, $\bar{\sigma} \epsilon \Sigma$ is a saddle point in each associated game, or equivalently, it is simultaneously a positive and negative equilibrium point.

This concept of neutral very simple equilibrium point has been obtained by considering the third possible way of looking at the role of the players into the zero-sum two-person associated game to each player, namely: both players have their respective normal roles, that is, the first player wishes to maximize his sure position independently of the behavior of his opponent and the second player tries to minimize with safety the winnings of the first player.

From a heuristic viewpoint, a neutral very simple equilibrium point is a rule of behavior which is such that if all the players except one remain on it, then his position will not improve. It will be able to decrease and the winnings of the other players could be increased. In other words, it is optimal for any one player in the game Γ .

The introduction of the respective adjective positive, negative and neutral for very simple equilibrium points has been motivated by the following considerations.

If a positive very simple equilibrium point has been established in an n-person game, then in a certain respect the group of players act in a positive manner, because they do not try to attack with their behavior the other participants; in other words, each player has complete confidence in the actions of the remaining members.

On the other hand, if a negative very simple equilibrium point is established, then each player is concentrated on attacking any other participants without considerations of his own winnings. Of course, this form of behavior of the group of the players, has a negative aspect from an intuitive point of view.

Finally, in a neutral very simple equilibrium point each player can be visualized as a player wishing to maximize his own position and at the same time trying to minimize the winning of the other members. This symmetry of the behavior of the players in the game reflects in some sense an aspect of neutrality.

The first result concerning neutral very simple equilibrium points is formulated in the following theorem:

<u>THEOREM I.12:</u> Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game such that the strategy set Σ_i of player $i \in N = \{1,\ldots,n\}$ is non-empty, compact and convex set in a euclidean space, and his payoff function A_i is continuous in the variable $\sigma = (\sigma_1,\ldots,\sigma_n) \in \Sigma$; concave with respect to the variable $\sigma_i \in \Sigma_i$ for fixed $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$; and convex in the variable $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$ for fixed $\sigma_i \in \Sigma_i$. Then, if for each joint strategy $\sigma \in \Sigma$ there is another one $\tau \in \Sigma$ such that for all $i \in N$:

$$A_i(\tau_i, \sigma_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}})$$

and

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) ,$$

the game Γ has a neutral very simple equilibrium point.

<u>PROOF:</u> For any joint strategy $\sigma = (\sigma_1,\ldots,\sigma_n)$ belonging to the non-empty, compact and convex product space Σ, consider for each player $i \in N$ the non-empty set

$$\varphi_i(\sigma) = \{\tau \in \Sigma: A_i(\tau_i, \sigma_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}) \quad \text{and}$$

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}})\} .$$

We now show that such a set is convex. Let τ and $\bar{\tau}$ be any two arbitrary points of the set $\varphi_i(\sigma)$, which satisfy the following equalities

$$A_i(\tau_i, \sigma_{N-\{i\}}) = A_i(\bar{\tau}_i, \sigma_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}})$$

and

$$A_i(\sigma_i, \tau_{N-\{i\}}) = A_i(\sigma_i, \bar{\tau}_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

By the condition of concavity of the payoff function A_i with respect to the variable $\sigma_i \epsilon \Sigma_i$, for any real number $\lambda \epsilon [0,1]$:

$$A_i(\lambda\tau_i + (1-\lambda)\bar{\tau}_i , \sigma_{N-\{i\}}) \geq \lambda \; A_i(\tau_i,\sigma_{N-\{i\}}) + (1-\lambda) \; A_i(\bar{\tau}_i,\sigma_{N-\{i\}}) = \max_{s_i \epsilon \Sigma_i} A_i(s_i,\sigma_{N-\{i\}}),$$

where only the strict equality must hold since the last term is the maximum value of the payoff function over $\Sigma_{N-\{i\}}$.

On the other hand by the convexity of the payoff function A_i with respect to the variable $\sigma_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}$, for any λ in the closed unit interval

$$A_i(\sigma_i,\lambda\tau_{N-\{i\}} + (1-\lambda)\bar{\tau}_{N-\{i\}}) \leq \lambda \; A_i(\sigma_i,\tau_{N-\{i\}}) + (1-\lambda) \; A_i(\sigma_i,\bar{\tau}_{N-\{i\}})$$

$$\leq \min_{s_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}} A_i(\sigma_i,s_{N-\{i\}})$$

where, again, only the strict equality must appear.

Thus, both equalities determines that the point $\lambda\tau + (1-\lambda)\bar{\tau}$ with $\lambda \epsilon [0,1]$ belongs to the set $\varphi_i(\sigma)$, which indicates the convexity of the set $\varphi_i(\sigma)$.

Now, consider the multivalued function

$$\varphi : \; \Sigma \to \Sigma$$

defined by the non-empty and convex intersection of the above sets, that is, for any $\sigma \epsilon \Sigma$

$$\varphi(\sigma) = \bigcap_{i \epsilon N} \varphi_i(\sigma) .$$

For the examination of the upper-semicontinuity property of the multivalued function φ , let us consider

$$\sigma(k) \to \sigma \qquad \text{and} \qquad \tau(k) \to \tau$$

two arbitrary converging sequences of elements of the product space Σ , with the property that for any positive integer k: $\tau(k) \in \varphi(\sigma(k))$. Thus, for each player $i \in N$:

$$A_i(\tau_i(k),\ \sigma_{N-\{i\}}(k)) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}(k))$$

and

$$A_i(\sigma_i(k),\ \tau_{N-\{i\}}(k)) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i(k),\ s_{N-\{i\}}) \ .$$

By the continuity of the payoff function with respect to the variable $\sigma \in \Sigma$, the sequences having as general terms the corresponding four members of the two preceding equalities, converge to those corresponding values of the payoff functions obtained by changing in their respective places the point $(\tau_i(k), \sigma_{N-\{i\}}(k)) \in \Sigma$ to $(\tau_i, \sigma_{N-\{i\}}) \in \Sigma$ and the joint strategy $(\sigma_i(k),\ \tau_{N-\{i\}}(k)) \in \Sigma$ to $(\sigma_i,\ \tau_{N-\{i\}}) \in \Sigma$.

Owing to all these convergences, it is then an immediate consequence that the following equalities hold true for any $i \in N$:

$$A_i(\tau_i, \sigma_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}})$$

and

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i,\ s_{N-\{i\}}) \ ,$$

which guarantee that the joint strategy τ is an element of the set $\varphi(\sigma)$. Thus, the upper-simicontinuity of the multivalued function φ is completely proved.

Then by the Kakutani fixed point theorem, the existence of a fixed point $\bar{\sigma} \in \Sigma$ of the multivalued function φ : $\bar{\sigma} \in \varphi(\bar{\sigma})$ is assured.

At this fixed point, for each player $i \in N$, we have

$$\max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{N-\{i\}}) = A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\bar{\sigma}_i, s_{N-\{i\}}) ,$$

which is by definition an neutral very simple equilibrium point of the game Γ. (Q.E.D.)

Again, the last condition in this theorem establishes the validity of both attack and defense properties in the n-person game Γ with respect to the neutral very simple equilibrium point concept $\sigma \epsilon \Sigma$ that is, for any point $\sigma \epsilon \Sigma$ there is a joint strategy $\tau \epsilon \Sigma$ such that for each player $i \epsilon N$

$$A_i(\tau_i, \sigma_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}})$$

and

$$A_i(\sigma_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

For any established behavior there is another one which maximixes the position of each player if the remaining players abide by the first one and minimizes the winning of each player if this player under consideration abides by the first one. This can be an intuitive interpretation of such a property.

A direct application of the preceding theorem is related to the mixed extension of finite n-person games.

THEOREM I.13: Let $\Gamma = \{\Sigma_1, \ldots, \Sigma_n, A_i, \ldots, A_n\}$ be a finite n-person game, such that the expectation function E_i of any player $i \epsilon N$ is linear in the variable

$$x_{N-\{i\}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{N-\{i\}} = \bigtimes_{j \neq i} \tilde{\Sigma}_j$$

for fixed $x_i \epsilon \tilde{\Sigma}_i$.

Then, if for each point $x \in X = \underset{i \in N}{\times} \tilde{\Sigma}_i$ there is another $y \in X$ such that for all $i \in N$:

$$E_i(y_i, x_{N-\{i\}}) = \max_{s_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}})$$

and

$$E_i(x_i, y_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} (x_i, s_{N-\{i\}}) ,$$

then the mixed extension $\tilde{\Gamma} = \{\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_n; E_1, \ldots, E_n\}$ has a neutral very simple equilibrium point.

<u>PROOF:</u> For any player $i \in N$ in the mixed extension game $\tilde{\Gamma}$, the corresponding mixed strategy set $\tilde{\Sigma}_i$ is non-empty, compact and convex in an euclidean space. On the other hand, since the expectation function E_i of the player $i \in N$ is a linear function with respect to the variable $x_{N-\{i\}} \in X_{N-\{i\}}$, for fixed $x_i \in \tilde{\Sigma}_i$, it is also convex in that variable. Furthermore, for fixed $x_{N-\{i\}} \in X_{N-\{i\}}$, the expectation function E_i is always linear in $x_i \in \tilde{\Sigma}_i$, and therefore concave. Obviously, it is a continuous function in $x \in X$.

Now, by taking into account the following equalities between the maximum and minimum amounts of the expectation function

$$\max_{s_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}}) = \max_{u_i \in \Sigma_i} E_i(u_i, x_{N-\{i\}})$$

and

$$\min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}}) = \min_{u_{N-\{i\}} \in \tilde{\Sigma}_{N-\{i\}}} F_i(x_i, u_{N-\{i\}}) ,$$

the latter condition of the preceding theorem applied to the mixed extension game $\tilde{\Gamma}$ holds. This theorem assures the existence of a neutral very simple equilibrium point for the game $\tilde{\Gamma}$. (Q.E.D.)

As before the last requirement for the mixed extension of a finite n-person game Γ , can be reformulated in an equivalent way, namely: for any point x in the product space $X = \underset{i \in N}{\times} \tilde{\Sigma}_i$ there is a joint strategy $\sigma \in \Sigma = \Sigma_1 x \ldots x \Sigma_n$ such that for all $i \in N$

$$E_i(\sigma_i, x_{N-\{i\}}) = \max_{s_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}})$$

and

$$E_i(x_i, \sigma_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}})$$

Obviously, this new formulation implies the old one. Now, let us consider the converse. If for an arbitrary point $x \in X$, the corresponding joint mixed strategy $y \in X$ satisfies

$$E_i(y_i, x_{N-\{i\}}) = \max_{x_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}})$$

and

$$E_i(x_i, y_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}})$$

for all $i \in N$, then, consider the distribution of probability y_i defined on Σ_i being to the i^{th} component of the joint strategy $y \in X$. Let τ_i be an arbitrary pure strategy of the player $i \in N$ for which $y_i(\tau_i) \neq 0$. Now, we take the point τ in the product space Σ composed of τ_i in each coordinate. Such a point satisfies the new formulation of the condition, that is,

$$E_i(\tau_i, x_{N-\{i\}}) = \max_{s_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}})$$

and

$$E_i(x_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}})$$

for all $i \in N$. Suppose that for some $i \in N$ we had

$$E_i(\tau_i, x_{N-\{i\}}) < \max_{s_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}})$$

or

$$E_i(x_i, \tau_{N-\{i\}}) > \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}}) .$$

Then, because the values $E_i(y_i, x_{N-\{i\}})$ and $E_i(x_i, y_{N-\{i\}})$ are combinations of those corresponding values expressing in the latter inequalities where the maximum and minimum values appear and $y_i(\tau_i) \neq 0$; for some $i \in N$, we would have

$$E_i(y_i, x_{N-\{i\}}) < \max_{s_i \in \Sigma_i} E_i(s_i, x_{N-\{i\}})$$

or

$$E_i(x_i, y_{N-\{i\}}) > \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} E_i(x_i, s_{N-\{i\}}) .$$

This is a contradiction. Thus, the new requirement is equivalent to the above condition.

A very special example of a finite n-person game Γ for which all the conditions in the above theorem are satisfied, is determined by having the payoff functions

$$A_i(\sigma_1, \ldots, \sigma_n) = a_i^{g(i)}(\sigma_i, \sigma_{g(i)})$$

where $g(i)$ indicates the "opposite" player of $i \in N$, for which it is assumed $g(g(i)) = i$ and where the following relationship between the payoff functions holds:

$$a_{g(i)}^{i}(\sigma_i, \sigma_{g(i)}) = - c_i \, a_i^{g(i)}(\sigma_i, \sigma_{g(i)}) + d_i ,$$

where c_i is non-negative real number and d_i any arbitrary real number.

Thus, the previous theorem guarantees the existence of a neutral very simple equilibrium point for the mixed extension $\tilde{\Gamma}$. Such a neutral very simple equilibrium point $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_n) \in X$ is determined by choosing each component $\bar{x}_i \in \tilde{\Sigma}_i$ as an maximin strategy in the zero-sum two-person associated mixed extension game $\tilde{\Gamma}_i = \{\tilde{\Sigma}_i, \tilde{\Sigma}_{g(i)}; E_i\}$ corresponding to the player $i \in N$ of the mixed extension game $\tilde{\Gamma}$.

Having the above results, it is natural to ask about the interchangeability property for neutral very simple equilibrium points of an n-person game Γ . Evidently, the examination of this question has a more complicated character than the corresponding one of two-persons.

Given two arbitrary neutral equilibrium points $\bar{\sigma}$ and $\tilde{\sigma}$ in Σ of Γ , then, consider any point σ^* in the product space Σ formed by having each component $\sigma_i^* \in \Sigma_i$ either coinciding to the coordinate $\bar{\sigma}_i$ or $\tilde{\sigma}_i$ corresponding to the neutral equilibrium points under consideration. Let I be the set of players $i \in N$ for which $\sigma_i^* = \bar{\sigma}_i$, and let $N-I$ be the set of players $i \in N$ with $\sigma_i^* = \tilde{\sigma}_i$. Obviously, if the subset of players I is empty, then the point σ^* coincides with the neutral very simple equilibrium point $\tilde{\sigma}$. Similarly, if $N-I$ is empty, the joint strategy σ^* is identical to $\bar{\sigma}$.

Suppose that both subsets I and $N-I$ are non-empty. Then, if the point σ^* has only one coordinate of one of them, for instance of $\bar{\sigma}$:

$$\sigma^* = (\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{i-1},\bar{\sigma}_i,\tilde{\sigma}_{i+1},\ldots,\tilde{\sigma}_n) \in \Sigma \quad ,$$

then, the theorem I.3 which relates the interchangeability of saddle points of zero-sum two-person games, assures that such a point σ^* is an saddle point of the associated zero-sum two-person game Γ_i of the player $i \in N$. Moreover, that result also guarantees such a property for the point

$$(\bar{\sigma}_1,\dots,\bar{\sigma}_{i-1},\tilde{\sigma}_i,\bar{\sigma}_{i+1},\dots,\bar{\sigma}_n) \in \Sigma \quad .$$

Actually, if the number of the players n of the game Γ is bigger than two, then the joint strategy σ^* has at least two components j and k coinciding with the respective coordinates of $\tilde{\sigma}$. For the player $j\in N$ the payoff function value on the point σ^*, by the definition of neutral very simple equilibrium point of $\tilde{\sigma}\in\Sigma$, satisfies

$$A_i(\bar{\sigma}_i,\tilde{\sigma}_{N-\{i\}}) \geq A_j(\tilde{\sigma}_i,\tilde{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} A_j(\tilde{\sigma}_i,s_{N-\{i\}}) \quad .$$

Unfortunately, this is all the information which is available for the value of the payoff function A_j on the point σ^*, when there is not any other requirements on the payoff functions in the game. Indeed, the serious difficulty arises from the fact that the point σ^* generally does not belong to the subset

$$\underset{\ell\neq j}{\times} \Sigma_\ell \times \{\bar{\sigma}_j\}$$

of the product space Σ, where the minimum of the payoff function A_j is taken as part of the neutral very simple equilibrium point property for the joint strategy $\bar{\sigma}$. Therefore we cannot compare the values of the payoff function A_j on the points σ^* and $\bar{\sigma}$. Of course, this difficulty does not appear in the special case when $n = 2$.

In a more general case, that is, when each of the subsets of players I and $N-I$ have at least two elements, the situation evidently is much more complicated than the simple one already examined. Thus, without further impositions on the payoff functions, the interchangeability property between neutral very simple equilibrium points does not hold true. However, it is interesting to observe that for the special example of mixed extension of the finite n-person game just considered after the theorem I.13 the interchangeability property is satisfied.

Nevertheless, the lack of the mentioned property does not destroy the equivalence property between them, that is, all the payoff functions restricted on the set of neutral very simple equilibrium points are constant functions.

This fact is obtained by using the equivalence property for zero-sum two-person games which was related as an immediate consequence of the interchangeability of the saddle points of such games. This property is described as follows:

<u>THEOREM I.14:</u> Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an arbitrary n-person game, then all the neutral very simple equilibrium points of Γ are equivalent, i.e., for any two neutral very simple equilibrium points $\bar{\sigma}$ and $\tilde{\sigma}$ of Γ we have

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{N-\{i\}}) = A_i(\tilde{\sigma}_i,\tilde{\sigma}_{N-\{i\}})$$

for all $i \in N = \{1,\ldots,n\}$.

<u>PROOF:</u> Consider an arbitrary player $i \in N$ of the game Γ , then in his own zero-sum two-person associated game $\Gamma_i = \{\Sigma_i,\Sigma_{N-\{i\}}, A_i\}$ both points $\bar{\sigma}$ and $\tilde{\sigma}$ are saddle points. Therefore, by the theorem I.3 applied to the game Γ_i , the points

$$(\bar{\sigma}_1,\ldots,\bar{\sigma}_{i-1},\tilde{\sigma}_{i+1},\ldots,\bar{\sigma}_n) \quad \text{and} \quad (\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{i-1},\bar{\sigma}_i,\tilde{\sigma}_{i+1},\ldots,\tilde{\sigma}_n)$$

are saddle points of Γ_i too. This fact implies the following equality of the values of payoff function of the player $i \in N$

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{N-\{i\}}) = A_i(\tilde{\sigma}_i,\tilde{\sigma}_{N-\{i\}}) \quad .$$

Thus, the neutral very simple equilibrium point $\bar{\sigma}$ and $\tilde{\sigma}$ are equivalent. (Q.E.D.)

From a heuristic viewpoint the lack of interchangeability for neutral very simple equilibrium point should not be seen as a deficient characteristic, since the important property on which the stability of a neutral very simple equilibrium point is based is the equivalence property. Indeed, if some players change their corresponding components from an established neutral very simple equilibrium point, then, their payoffs will be decreased and the payoffs of the remaining players will increase. Of course, they will not have interest in this new position.

An interesting property of the structure of the set of neutral very simple equilibrium points for games described in the theorem I.12 is formulated in the following result.

THEOREM I.15: Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n;\ A_1,\ldots,A_n\}$ be an n-person game such that the strategy set Σ_i of player $i\in N = \{1,\ldots,n\}$ is a non-empty, compact and convex set in a euclidean space, and his payoff function A_i is continuous in the variable $\sigma\in\Sigma$; concave with respect to the variable $\sigma_i\in\Sigma_i$ for fixed $\sigma_{N-\{i\}}\in\Sigma_{N-\{i\}}$, and convex in the variable $\sigma_{N-\{i\}}\in\Sigma_{N-\{i\}}$ for fixed $\sigma_i\in\Sigma_i$. Then, if for each joint strategy $\sigma\in\Sigma$ there is another one $\tau\in\Sigma$ such that for all $i\in N$

$$A_i(\tau_i,\sigma_{N-\{i\}}) = \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{N-\{i\}})$$

and

$$A_i(\sigma_i,\tau_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} A_i(\sigma_i,s_{N-\{i\}}) ,$$

the set of neutral very simple equilibrium points of Γ is non-empty, compact and convex.

PROOF: The non-emptyness of the set of neutral very simple equilibrium points of the game Γ is determined by the theorem I.12.

Now, we are going to prove the compactness.

Let

$$\sigma(k) \to \sigma$$

be an arbitrary converging sequence of neutral equilibrium points of Γ. Then, for any non-negative integer k, we have

$$\max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}(k)) = A_i(\sigma_i(k), \sigma_{N-\{i\}}(k)) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i(k), s_{N-\{i\}})$$

for all $i \in N$. By the continuity of the payoff function A_i in the variable $\sigma \in \Sigma$ defined on the compact set Σ, the following convergences for each $i \in N$ hold:

$$\max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}(k)) \to \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}) ,$$

and

$$\min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i(k), s_{N-\{i\}}) \to \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}}) .$$

Therefore, by using the convergence of sequence

$$A_i(\sigma_i(k), \sigma_{N-\{i\}}(k)) \to A_i(\sigma_i, \sigma_{N-\{i\}}) ,$$

together with the preceding relations one obtains

$$\max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{N-\{i\}}) = A_i(\sigma_i, \sigma_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\sigma_i, s_{N-\{i\}})$$

for all $i \in N$; which is the definition of the neutral very simple equilibrium point of the game Γ for the point σ in the product space Σ . Hence, the set of neutral very simple equilibrium points of the game Γ is compact.

We now examine the convexity of such a set. Let us consider $\bar{\sigma}$ and $\tilde{\sigma}$ be two arbitrary neutral very simple equilibrium points of game Γ , which are saddle points in the zero-sum two-person associate game $\Gamma_i = \{\Sigma_i, \Sigma_{N-\{i\}}; A_i\}$ corresponding to each player $i \in N$. Because the game Γ_i of the player $i \in N$, satisfies all the conditions required by the theorem I.4, then, the convexity of the set of saddle point of Γ_i remains completely guaranteed. Thus, for any real number λ belonging to the unit interval, the point $\lambda\bar{\sigma} + (1-\lambda)\tilde{\sigma} \in \Sigma$ is a saddle point of Γ_i, i.e.:

$$A_i(\lambda\bar{\sigma}_i + (1-\lambda)\,\tilde{\sigma}_i,\ \lambda\bar{\sigma}_{N-\{i\}} + (1-\lambda)\,\tilde{\sigma}_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \lambda\bar{\sigma}_{N-\{i\}} + (1-\lambda)\tilde{\sigma}_{N-\{i\}})$$

$$= \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(\lambda\bar{\sigma}_i + (1-\lambda)\tilde{\sigma}_i, s_{N-\{i\}})$$

for any $i \in N$, which is the definition of neutral equilibrium point for the point $\lambda\bar{\sigma} + (1-\lambda)\tilde{\sigma} \in \Sigma$. Hence, the set of neutral very simple equilibrium points is convex. (Q.E.D.)

An immediate consequence of the previous theorem is obtained for the mixed extension game $\tilde{\Gamma}$ of an finite n-person game Γ having the properties required by theorem I.13: The set of neutral very simple equilibrium points of such a game $\tilde{\Gamma}$ is then a non-empty compact and convex set in an euclidean space.

II.1 e-Simple Equilibrium Points

Once, having the preceding results, we could aim to extend these concepts by using a more general description of the associated zero-sum two-person game for each player.

One of the possible ways of generalizing those concepts will be examined in this section. What follows is essentially based on seeing the second player in the associated zero-sum two-person game corresponding to each player, divided into two parts which have associated suitable roles, and thereby introduce a simple structure for the game.

Let us consider an n-person game $\Gamma = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ having the usual properties, that is, where the strategy set of player $i \in N$ is a non-empty and compact set in a euclidean space and his payoff function is a continuous function in the product space. Furthermore, the real situation represented by the game Γ allows only non-cooperative behavior among the players.

Any arbitrary player in the game Γ is considered as embedded in the associated game $\Gamma_i = \{ \Sigma_i, \Sigma_{N-\{i\}}; A_i \}$, where the roles of both players generally do not reflect the roles of the corresponding players in the zero-sum two-person game, since such roles are imposed arbitrarily. However, we recall that this viewpoint is of fundamental importance in the theory, because each criterion adopted to answer the question of the roles of the players in the associated game will immediately determine a new concept for the total behavior in the game Γ.

Now, consider for the player $i \in N$ the second player in the associated game which is composed of the set of players $N-\{i\}$, divided into two disjoint sets of players, namely:

$$N-\{i\} = e(i) \cup f(i) \quad .$$

The "subplayer" of the second player formed by the set of players $f(i)$ always has an indifferent character with respect to the first player, in the sense already specified. Therefore one can describe the conflict situation between player $i \in N$ and the corresponding "sub-opposer" player $e(i)$ depending on the choice of the subplayer $f(i)$ in the associated game Γ_i. Of course, all these considerations made on the associated game Γ_i may confuse the precise formulation, which is very important to keep straight. From the formulated assumption, new concepts can be obtained.

More precisely, a function

$$\underline{e}: N \rightarrow \underline{P}_N x \underline{P}_N$$

which for each player $i \in N = \{i, \ldots, n\}$ of the n-person game Γ assigns two disjoint subsets of players

$$\underline{e}(i) = (e(i),\ f(i)), \quad \text{with} \quad e(i) \in \underline{P}_N \quad \text{and} \quad f(i) \in \underline{P}_N$$

belonging to the set of all subsets of $N: \underline{P}_N$, is said to be a <u>simple structure-function</u> for the game Γ if

$$e(i) \cup f(i) = N-\{i\} \quad .$$

Furthermore, the set of players $f(i)$ is called the <u>indifferent coalition</u> corresponding to the player $i \in N$.

An n-person game Γ having an associated simple structure function $\underline{e}$ [that is, $(\Gamma, \underline{e})$] will be indicated by $\Gamma_{\underline{e}}$ and for the purpose of simplicity $\Gamma_{\underline{e}}$ is said to be a game also.

In the remaining pages of this chapter we deal exclusively with games having an associated simple structure, even though we do not mention it explicitly.

As was stated before it is adequate to decompose the associated game of the player $i \in N$ into a family of associated zero-sum two-person games, which depend on the different choices of the indifferent subplayer $f(i)$ and where the role of second player is assumed by the subplayer $e(i)$. Formally, for the player $i \in N = \{1, \ldots, n\}$ and a joint strategy for the set of players $f(i)$:

$$\sigma_{f(i)} \in \Sigma_{f(i)} = \underset{j \in f(i)}{\times} \Sigma_j$$

we define the <u>$\sigma_{f(i)}$-associated zero-sum two-person game</u> with respect to the game $\Gamma_{\underline{e}}$ with simple structure function $\underline{e}$, by

$$\Gamma_i(\sigma_{f(i)}) = \{ \Sigma_i, \Sigma_{e(i)}; A_i(\sigma_i, \sigma_{e(i)}, \sigma_{f(i)}) \}$$

where the strategy set of the second player is given by the product

$$\sigma_{e(i)} \in \Sigma_{e(i)} = \underset{j \in e(i)}{\times} \Sigma_j \quad .$$

We recall that when the set $f(i)$ is void, the product set Σ_{ϕ} is represented by only one element: the "o-tuple", then, formally the "o-tuple"-associated zero-sum two-person game represents the associated zero-sum two-person game Γ_i . Furthermore, in the special case where the set $e(i)$ is empty, any $\sigma_{N-\{i\}}$-associated game represents a one-person game formed by the player $i \in N$.

Having just introduced, for each player in the n-person game $\Gamma_{\underline{e}}$ with simple structure function $\underline{e}$, the role of indifferent player (with respect to $i \in N$), to each element in the subset of players $f(i)$, we must assign the other role to the subplayer $e(i)$. Actually, if the player $e(i)$ is considered also as an indifferent player with respect to player $i \in N$, we might try to extend the concept of positive very simple equilibrium point by considering that the first player $i \in N$ in the associated game determined by the choice of his indifferent coalition $f(i)$ maximizes his position by the choice of the second player $e(i)$.

A point $\bar{\sigma} = (\bar{\sigma}_1,\ldots,\bar{\sigma}_n)$ belonging to the product space Σ is said to be an <u>e-positive simple equilibrium point</u> of the n-person game $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ with simple structure function $\underline{e}$, if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_i,\Sigma_{e(i)},A_i\}$ of the player $i \in N$:

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) .$$

But, of course, this definition completely coincides with the old definition of positive very simple equilibrium point for the game Γ . Thus, it does not have any advantages, and therefore such a new concept is useless.

This arose because the behavior of the set of players $N-\{i\}$ has been associated under the consideration that the set $e(i)$ acts after the indifferent coalition $f(i)$, where both are considered indifferent with respect to player $i \in N$. But, of course, this representation is completely equivalent to considering the joint behavior of the set $N-\{i\}$ in an indifferent manner.

On the other hand, by assigning the normal behavior of the second player to the set of players $e(i)$ in the associated game determined by the choice of the indifferent coalition of the player $i \in N$, one can generalize the concept of negative very simple equilibrium point by assuming in the introduction, this latter player is apathetic with respect to his own position.

Because of this, we refer to the set of players $e(i)$ corresponding to the player $i \in N$, in the game $\Gamma_{\underline{e}}$ with structure function $\underline{e}$ as his <u>antagonistic coalition.</u>

Rigorously speaking, given an n-person game $\Gamma_{\underline{e}} = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with the simple structure function $\underline{e}$, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n) \in \Sigma$ is said to be an $\underline{e}$-negative simple equilibrium point of the game $\Gamma_{\underline{e}}$, if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_i, \Sigma_{e(i)}; A_i \}$ of the player $i \in N = \{ 1, \ldots, n \}$:

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) \;.^*$$

*We recall that formally if either $e(i) = \phi$ or $f(i) = \phi$ then $A_i(\sigma_i, \sigma_{e(i)}, \sigma_{f(i)})$ is $A_i(\sigma_i, \sigma_{N-\{i\}})$ and if $e(i) = \phi$

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} (\sigma_i, s_{e(i)}, \sigma_{f(i)}) = A_i(\sigma_i, \sigma_{f(i)})$$

A special case of these points arises when the simple structure function $\underline{e}$ of the game Γ is determined by $\underline{e}(i) = (N-\{i\},\phi)$ for all player $i \in N$. In such a case the concepts of $\underline{e}$-negative simple equilibrium point and negative very simple equilibrium point coincide.

An $\underline{e}$-negative simple equilibrium point is a rule of behavior with the property that if some or all players of the members of some antagonistic coalition $e(i)$ of a player $i \in N$, change from it, then the winnings of the player $i \in N$ will be increased if his indifferent coalition still abides by it. This can be a possible interpretation of an $\underline{e}$-negative simple equilibrium point. Of course, if the players changing from such a point, are simultaneously members of the antagonistic coalitions of several other players, then the positions of all these players will be increased. We also note that these kind of points do not satisfy the equivalence property.

The following theorem concerning the existence of $\underline{e}$-negative simple equilibrium points is a direct extension of theorem I.9 for the class of games usually under consideration.

<u>THEOREM II.1</u>: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \in N = \{1,\ldots,n\}$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i is continuous in the variable $\sigma \in \Sigma$ and convex with respect to the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)}) \in \Sigma_i x \Sigma_{f(i)}$. Then, if for each joint strategy $\sigma \in \Sigma$ there is another one $\tau \in \Sigma$ such that for all $i \in N$

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)}) ,$$

the game Γ_e has an e-negative simple equilibrium point.

PROOF: For any point $\sigma = (\sigma_i,\sigma_{e(i)},\sigma_{f(i)})\in\Sigma$ of the non-empty, compact and convex product space Σ , let

$$\varphi_i(\sigma) = \{ \tau\in\Sigma: A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)}) \}$$

be a non-empty set of points in the product space Σ corresponding to player $i\in N$. We remark that for those $i\in N$, which have their respective antagonistic coalitions empty, the set $\varphi_i(\sigma)$ for any point $\sigma\in\Sigma$ is the whole product space Σ .

Let τ and $\bar{\tau}$ be any two arbitrary elements of the set $\varphi_i(\sigma)$. Hence, by definition of such a set, the following equalities hold true:

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = A_i(\sigma_i,\bar{\tau}_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)})$$

which imply the relation

$$A_i(\sigma_i,\lambda\tau_{e(i)}+(1-\lambda)\bar{\tau}_{e(i)},\sigma_{f(i)}) \leq \lambda A_i(\sigma_i,\tau_{e(i)}, \sigma_{f(i)}) + (1-\lambda)A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)})$$

$$= \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)})$$

for any real number $\lambda\in[0,1]$, by virtue of the convexity property of the payoff function A_i with respect to the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)})\in\Sigma_i x\Sigma_{f(i)}$. In this last equality we must have only the strict equality sign, since the latter amount is the mininum value of the payoff function considered over the product set $\Sigma_{e(i)}$. Thus, for any real number, $\lambda\epsilon[0,1]$, we have

$$A_i(\sigma_i,\lambda\tau_{e(i)}+(1-\lambda)\bar{\tau}_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) ,$$

which expresses the fact that the point $\lambda\tau+(1-\lambda)\bar{\tau}$ is an element of the set $\varphi_i(\sigma)$, since all the strategy sets are convex. Hence, $\varphi_i(\sigma)$ is a convex set.

Now, let us consider the multivalued function

$$\varphi : \Sigma\rightarrow\Sigma$$

defined by the convex set

$$\varphi(\sigma) \bigcap_{i\epsilon N} \varphi_i(\sigma),$$

for any $\sigma\epsilon\Sigma$, which is non-empty by virtue of the last condition of the hypothesis.

Now, we consider the question of upper-semicontinuity of the multivalued function φ . Let

$$\sigma(k)\rightarrow\sigma \quad \text{and} \quad \tau(k)\rightarrow\tau$$

be two arbitrary converging sequences of members in the product space Σ having the property that $\tau(k)\epsilon\varphi(\sigma(k))$ for any positive integer k . Hence, for any $i\epsilon N$ we equivalently have

$$A_i(\sigma_i(k),\tau_{e(i)}(k),\sigma_{f(i)}(k)) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i(k),s_{e(i)},\sigma_{f(i)}(k))$$

for all k . By the continuity of the payoff function A_i of player $i\epsilon N$, in the product space Σ , the two sequences of real numbers having the general terms identified as those parts of the second equality, converge respectively to the values of the payoff functions obtained by replacing respectively the point $(\sigma_i(k), \tau_{e(k)}(k), \sigma_{f(i)}(k))$ by $(\sigma_i, \tau_{e(i)}, \sigma_{f(i)})$. In accordance with this observation, the following equality for each player $i \in N$ is completely satisfied

$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}) .$$

Thus, the point τ is an element of the set $\varphi(\sigma)$, which implies the upper-semicontinuity property of the multivalued function φ .

As an immediate consequence of Kakutani's fixed point theorem applied to the semicontinuous function φ we get the existence of a fixed point $\bar{\sigma} \epsilon \varphi(\bar{\sigma})$. That is,

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all player $i \in N$. This is the definition of e-negative simple equilibrium point for the joint strategy $\bar{\sigma} \in \Sigma$. (Q.E.D.)

The last condition in the above theorem which assures for any point $\sigma \in \Sigma$ the existence of another joint strategy $\tau \in \Sigma$ such that for each player $i \in N$

$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$

can be considered as the attack property with respect to the concept of an e-negative simple equilibrium point for the game Γ .

Each established behavior has a new one which minimizes the position of any player if, in each instance, this considered player together with his indifferent coalition abide by the old original behavior. This is a simple intuitive interpretation of that attack property.

Using this theorem, one can immediately characterize such points for mixed extensions of finite n-person games. This formulation is related in the following result.

THEOREM II.2: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ be a finite n-person game with simple structure function e such that the expectation function E_i of player $i\in N$ is linear in the variable

$$x_{e(i)}\in X_{e(i)} = \underset{\in e(i)}{\times} \tilde{\Sigma}_j$$

for fixed

$$(X_i,x_{f(i)})\in X_i x X_{f(i)} = \tilde{\Sigma}_i \times \underset{j\in f(i)}{\times} \tilde{\Sigma}_j \quad .$$

If for each $x\in X = \underset{j\in N}{\times} \tilde{\Sigma}_j$ there is another $y\in X$ such that for all $i\in N$:

$$E_i(x_i,y_{e(i)},x_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(x_i,s_{e(i)},x_{f(i)}) \ ,$$

then the mixed extension $\tilde{\Gamma}_e = \{\tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;E_1,\ldots,E_n\}$ has an e-negative simple equilibrium point.

PROOF: The mixed strategy set $\tilde{\Sigma}_i$ of the player $i\in N$ in the mixed extension game $\tilde{\Gamma}$ is non-empty, compact and convex in a euclidean space. The expectation function E_i of player $i\in N$ is a linear function with respect to the variable $x_{e(i)}\in X_{e(i)}$ for fixed $(x_i,x_{f(i)})\in X_i x X_{f(i)}$, which is also convex.

On the other hand, since the payoff function E_i is a multilinear function, then it is obviously continuous with respect to the variable $x \in X$.

Finally, by virtue of the validity of the following equality between the minimum values

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_i, s_{e(i)}, x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} E_i(x_i, u_{e(i)}, x_{f(i)})$$

the attack property for the mixed extension game $\tilde{\Gamma}_{\underline{e}}$ is completely satisfied.

Therefore, the preceding theorem guarantees the existence of an $\underline{e}$-negative simple equilibrium point for the game $\tilde{\Gamma}_{\underline{e}}$ (Q.E.D.)

The last condition imposed on the previous result is equivalent to the following requirement for each point x belonging to the product space X: There is a joint pure strategy $\sigma \in \Sigma = \Sigma_1 x \ldots x \Sigma_n$ such that for all $i \in N$

$$E_i(s_i, \sigma_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_i, s_{e(i)}, x_{f(i)}) .$$

The proof of this assertion can be carried out in a way completely analogously to those we have already done. For this reason we do not repeat here such a demonstration.

The class of all the finite n-person game $\Gamma_{\underline{e}}$, for which the expectation function E_i of each player $i \in N$ is a linear function with respect to the variable $x_{e(i)} \in X_{e(i)}$ of his antagonistic coalition, is characterized by the payoff function of player $i \in N$ formed by

$$A_i(\sigma_i, \sigma_{e(i)}, \sigma_{f(i)}) = a_i^i(\sigma_i, \sigma_{f(i)}) + \sum_{j \in e(i)} a_i^j(\sigma_i, \sigma_j, \sigma_{f(i)}) ,$$

which is an immediate consequence of the lemma I.11.

Actually, for this kind of finite n-person games, the attack property requirement can be formulated as follows: for any point in the product space $x \in X = \underset{j \in N}{\times} \tilde{\Sigma}_j$ there is a joint pure strategy $\sigma \in \Sigma = \underset{j \in N}{\times} \Sigma_j$ for which:

$$\sum_{j \in e(i)} e_i^j(x_i, \sigma_j, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} \sum_{j \in e(i)} e_i^j(x_i, s_j, x_{f(i)})$$

for all players $i \in N$ with $e(i) \neq \phi$, where e_i^j indicates the expectation function of the function a_i^j associated with the player $i \in N$.

But such a requirement imposed on the expectation functions by virtue of the equality

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} \sum_{j \in e(i)} e_i^j(x_i, s_j, x_{f(i)}) = \sum_{j \in e(i)} \min_{s_j \in \Sigma_j} e_i^j(x_i, s_j, x_{f(i)})$$

for each player $i \in N$ with antagonistic coalition $e(i)$ non-empty, can be immediately transformed into a more simple statement. Namely, for any $x \in X$ there is an $\sigma \in \Sigma$ such that

$$\sum_{j \in e(i)} e_i^j(x_i, \sigma_j, x_{f(i)}) = \sum_{j \in e(i)} \min_{s_j \in \Sigma_j} e_i^j(x_i, s_j, x_{f(i)})$$

for all $i \in N$ with $e(i) \neq \phi$.

Furthermore, we can see easily that this condition is equivalent to having for each $x \in X$ an $\sigma \in \Sigma$ which for all $i \in N$ with $e(i) \neq \phi$, we have

$$e_i^j(x_i, \sigma_j, x_{f(i)}) = \min_{s_j \in \Sigma_j} e_i^j(x_i, s_j, x_{f(i)})$$

for all $j \in e(i)$.

It is interesting to illustrate by an example one of the games which satisfies this very simple form of the attack property.

Let $\underline{e}$ be a simple structure function of the finite n-person game which is characterized by the property that each player $j \in N$ belongs at most to only one antagonistic coalition. Let the subset J be the set of all players $j \in N$ for which there exists an $i \in N$ with: $j \in e(i)$ i.e.

$$J = \{ j \in N : j \in e(i) \text{ for some } i \in N \} .$$

Let I be the subset of all players with their non-empty antagonistic coalitions. Evidently such a simple structure function always exists.

Then, any game with a simple structure having the described property and the form of the payoff functions already mentioned, fulfills the attack condition. Indeed, for an arbitrary point x in the product space X, consider for each player $j \in J$ a pure strategy $\sigma_j \in \Sigma_j$ for which the equality

$$e_i^j(x_i, \sigma_j, x_{f(i)}) = \min_{s_j \in \Sigma_j} e_i^j(x_i, s_j, x_{f(i)})$$

holds true, for the player $i \in I$ with $j \in e(i)$. Such a pure strategy obviously exists.

Now, by forming the joint pure strategy $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ composed of those coordinates $\sigma_j \in \Sigma_j$ for each $j \in J$ and any arbitrary component $\sigma_j \in \Sigma_j$ for each $j \in N - J$, we obtain the existence of a point satisfying the attack property for $x \in X$.

Therefore, by the above theorem the mixed extension $\tilde{\Gamma}_{\underline{e}}$ of any finite n-person game $\Gamma_{\underline{e}}$ having such a simple structure function and such payoff functions, has an $\underline{e}$-negative simple equilibrium point.

We will now extend the concepts of $\underline{e}$-simple equilibrium points.

Given an n-person game $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ with the simple structure function $\underline{e}$, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1,\ldots,\bar{\sigma}_n)\in\Sigma$ is called an $\underline{e}$-simple neutral equilibrium point or $\underline{e}$-simple saddle point of the game $\Gamma_{\underline{e}}$ if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_i,\Sigma_{e(i)};A_i\}$ for the player $i\in N = \{1,\ldots,n\}$:

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)}) ,$$

that is, if it is simultaneously an $\underline{e}$-positive and an $\underline{e}$-negative simple equilibrium point. In other words, if for any $\bar{\sigma}_{f(i)}$-associated game it is a saddle point.

A special case of $\underline{e}$-simple neutral equilibrium point arises when the simple structure function $\underline{e}$ is given by $\underline{e}(i) = (N-\{i\},\phi)$ for each player $i\in N$. In such a case it is also a neutral very simple equilibrium point.

Another particular case is found when $\underline{e}(i) = (\phi,N-\{i\})$ for all players $i\in N$. For this simple structure function the last equality of the definition of $\underline{e}$-simple saddle points turns trivial, and therefore it coincides with the concept of very simple equilibrium point.

We note that such points do not generally satisfy the equivalence property, that is, it could have different $\underline{e}$-simple saddle points where the values of some payoff functions do not coincide. This arises since the associated games are parametrized by the choice of the joint strategy of the indifferent coalitions.

Intuitively speaking, an $\underline{e}$-simple saddle point is a rule of behavior which, on one hand is such that if the players belonging to some antagonist coalition $e(i)$ of the player $i \in N$, change their behavior, the winnings of player $i \in N$ will be increased, if his corresponding indifferent coalition remain on it. On the other hand, it is such that if all the players except one abide by it, then his position will decrease. Furthermore, if such a player is a member of some antagonistic coalitions of some other players, then the winnings of all these other players will be increased.

In other words, it is optimal for each player and each antagonistic coalition, given the actions of the indifferent coalitions.

A word of caution about this concept should be stated. If a player changes from an e-simple saddle point, one should observe that the positions of those players for which he is seen as an indifferent player might decrease.

A first result about these points is related in the following existence theorem.

THEOREM II.3: Let $\Gamma_e = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with simple structure-function e , such that the strategy set Σ_i of player $i \in N = \{1,\ldots,n\}$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i is continuous in the variable $\sigma \in \Sigma$; convex with respect to the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)}) \in \Sigma_i x \Sigma_{f(i)}$ and concave with respect to the variable $\sigma_i \in \Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)}) \in \Sigma_{e(i)} x \Sigma_{f(i)}$. If for each joint strategy $\sigma \in \Sigma$ there is another one $\tau \in \Sigma$ such that for all $i \in N$

$$A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) ,$$

the game Γ_e has an e-simple saddle point.

PROOF: Given any arbitrary point $\sigma\in\Sigma$ of the non-empty, compact and convex product space Σ , consider the non-empty convex set

$$\varphi_i(\sigma) = \{ \tau\in\Sigma: A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

$$\text{and } A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) \}$$

for player $i\in N$. Indeed, let τ and $\bar{\tau}$ be any two elements belonging to the set $\varphi_i(\sigma)$, then they satisfy the following equalities

$$A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) = A_i(\bar{\tau}_i,\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = A_i(\sigma_i,\bar{\tau}_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) \quad .$$

By virtue of the concavity of the payoff function A_i , with respect to the variable $\sigma_i\in\Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\in\Sigma_{e(i)}x\Sigma_{f(i)}$ for any $\lambda\in [0,1]$, one has

$$A_i(\lambda\tau_i+(1-\lambda)\bar{\tau}_i,\sigma_{e(i)},\sigma_{f(i)}) = \lambda A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)})+(1-\lambda)A_i(\bar{\tau}_i,\sigma_{e(i)},\sigma_{f(i)})$$

$$= \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

since in the last term there appears the maximum value of the payoff function regarded as a function on the set Σ_i .

On the other hand, because of the convexity of the payoff function A_i in the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)})\in\Sigma_i x\Sigma_{f(i)}$ for any real number λ in the unit interval $[0,1]$, we have:

$$A_i(\sigma_i,\lambda\tau_{e(i)}+(1-\lambda)\bar{\tau}_{e(i)},\sigma_{f(i)}) = \lambda A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)})+(1-\lambda)A_i(\sigma_i,\bar{\tau}_{e(i)},\sigma_{f(i)})$$

$$= \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) .$$

Since the last amount represents the mininum value of the payoff function considered over the product set $\Sigma_{e(i)}$.

Hence, such equalities imply that for any arbitrary real number $\lambda\epsilon[0,1]$ the point $\lambda\tau+(1-\lambda)\bar{\tau}$ of the product space Σ is an element of the set $\varphi_i(\sigma)$ of player $i\epsilon N$, since all the strategy sets are convex. Thus the set $\varphi_i(\sigma)$ is convex.

Furthermore, from the last conditions, that is, the defense and attack properties with respect to the concept of e-simple saddle point, we have that, for any point $\sigma\epsilon\Sigma$ the intersection set

$$(\sigma) = \bigcap_{i\epsilon N} \varphi_i(\sigma)$$

is a non-empty set. Therefore, the multivalued function

$$\varphi:\Sigma\rightarrow\Sigma$$

which assigns $\varphi(\sigma)$ to each joint strategy $\sigma\epsilon\Sigma$, remains completely determined.

We now examine the upper-semicontinuity property of this function. Let

$$\sigma(k)\rightarrow\sigma \quad \text{and} \quad \tau(k)\rightarrow\tau$$

be any two arbitrary converging sequences of points belonging to the produce space Σ having the property that for any positive integer $k:\tau(k)\ \ (\sigma(k))$. This means that for each k and each player $i\epsilon N$ the following equalities hold:

$$A_i(\tau_i(k),\sigma_{e(i)}(k),\sigma_{f(i)}(k)) = \max_{s_i \in \Sigma_i} A_i(s_i,\sigma_{e(i)}(k),\sigma_{f(i)}(k))$$

and

$$A_i(\sigma_i(k),\tau_{e(i)}(k),\sigma_{f(i)}(k)) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i(k), s_{e(i)},\sigma_{f(i)}(k)) .$$

But, because of the continuity of the payoff functions, the sequences having as general terms one of the corresponding members of the two preceding equalities, converge to those respective values of the payoff functions obtained by changing in their respective places the point $(\tau_i(k),\tau_{e(i)}(k))$ to $(\tau_i,\tau_{e(i)})$ and the joint strategy $(\sigma_i(k),\sigma_{e(i)}(k),\sigma_{f(i)}(k))$ to $(\sigma_i,\sigma_{e(i)},\sigma_{f(i)})$. We recall that if the antagonistic coalition of a player $i \in N$ is an empty set, then the sequence whose general term is the mininum value of the payoff function over the set $\Sigma_{e(i)}$ completely coincides with the sequence : $\{ A_i(\sigma_i(k),\sigma_{N-\{i\}}(k)) \}$. An analogous remark can be made for the remaining sequences.

Thus, by identifying the limiting values of those corresponding sequences, for each player $i \in N$, we obtain

$$A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)}) .$$

This means that the point τ is a member of the set $\varphi(\sigma)$, and therefore the upper-continuity of the multivalued function is established.

Application of the Kakutani's Fixed Point Theorem to the multivalued function φ, guarantees the existence of a fixed point $\bar{\sigma} \in \varphi(\bar{\sigma})$. That is, for each player $i \in N$:

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \ ,$$

which indicates that the point $\bar{\sigma}$ is an e-simple saddle point of the game Γ. (Q.E.D.)

Similarly to what has been indicated for neutral very simple equilibrium points, the last condition in the preceding theorem, that is, for any point $\sigma\in\Sigma$ in the product space Σ there exists a joint strategy $\tau\in\Sigma$ such that for all $i\in N$:

$$A_i(\tau_i, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)})$$

and

$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$

is related to the attack and defense property for the game Γ_e with simple structure function e, with respect to the e-simple saddle point concept. This can be intuitively interpreted as follows: for any accepted joint behavior among the players in the game Γ, there is another one which, on one hand, maximizes the position of each player if the remaining players abide by the first one and on the other hand, minimizes the winnings of each player, if he together with his indifferent coalition remain in the old one.

An immediate consequence of the previous theorem is obtained for mixed extensions of finite n-person games, and is formulated in the following result.

THEOREM II.4: Let $\Gamma_e = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be a finite n-person game with simple structure function e, such that the expectation function E_i of player $i\in N$ is linear in the variable

$$x_{e(i)}\in X_{e(i)} = \underset{j\in e(i)}{\times} \tilde{\Sigma}_j$$

for fixed

$$(x_i, x_{f(i)}) \in X_i x X_{f(i)} = \tilde{\Sigma}_i \times \underset{j \in f(i)}{\times} \tilde{\Sigma}_j .$$

If for each $x \in X = \underset{j \in N}{\times} \tilde{\Sigma}_j$ there is a joint pure strategy $\sigma \epsilon \Sigma$ such that for all $i \in N$

$$E_i(\sigma_i, x_{e(i)}, x_{f(i)}) = \max_{s_i \in \Sigma_i} E_i(s_i, x_{e(i)}, x_{f(i)})$$

and

$$E_i(x_i, \sigma_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_i, s_{e(i)}, x_{f(i)}) ,$$

then the mixed extension $\tilde{\Gamma}_{\underline{e}} = \{ \tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_n; E_1, \ldots, E_n \}$ has an $\underline{e}$-simple saddle point.

<u>PROOF</u>: The mixed strategy set $\tilde{\Sigma}_i$ of player $i \in N$ in the mixed extension game $\tilde{\Gamma}_{\underline{e}}$ having the simple structure function $\underline{e}$ is non-empty, compact and convex in a euclidean space, and his expectation function E_i is continuous with respect to the variable $x \in X$, since it is a multilinear function. Furthermore, it is a linear function in the variable $x_{e(i)} \in X_{e(i)}$ for fixed $(x_i, x_{f(i)}) \in X_i x X_{f(i)}$. Hence the first requirements of the preceding theorem applied to the mixed extension game $\tilde{\Gamma}_{\underline{e}}$ are completely satisfied.

Finally, since the equalities

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_i, s_{e(i)}, x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} E_i(x_i, u_{e(i)}, x_{f(i)})$$

and

$$\max_{s_i \in \Sigma_i} E_i(s_i, x_{e(i)}, x_{f(i)}) = \max_{u_i \in X_i} E_i(u_i, x_{e(i)}, x_{f(i)})$$

always hold true, for any $x \in X$ and any player $i \in N$, the last condition guarantees the validity of the attack and defense properties for the mixed extension $\tilde{\Gamma}_{\underline{e}}$ with respect to the $\underline{e}$-simple saddle point concept.

Therefore, the preceding theorem applied to the mixed extension game $\tilde{\Gamma}$ assures the existence of an $\underline{e}$-simple saddle point of $\tilde{\Gamma}$. (Q.E.D.)

We recall that the last condition indicated in the above theorem is equivalent to the following one: for any point x in the product space $X = \times_{i \in N} \tilde{\Sigma}_i$ there is another $y \in X$ such that for all $i \in N$:

$$E_i(y_i, x_{e(i)}, x_{f(i)}) = \max_{s_i \in \Sigma_i} E_i(s_i, x_{e(i)}, x_{f(i)})$$

and

$$E_i(x_i, y_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_i, s_{e(i)}, x_{f(i)}) .$$

Of course, this fact can be directly proved as well as that which has been shown for the analogous attack and defense property with respect to the neutral very simple equilibrium point, and therefore, we do not repeat it again. For all those finite n-person games $\Gamma_{\underline{e}}$, which fulfill the linearity requirement of the expectation function E_i in the variable $x_{e(i)} \in X_{e(i)}$ for player $i \in N$, by lemma I.11 his corresponding payoff function must have the following form:

$$A_i(\sigma_i, \sigma_{e(i)}, \sigma_{f(i)}) = a_i^i(\sigma_i, \sigma_{f(i)}) + \sum_{j \in e(i)} a_i^j(\sigma_i, \sigma_j, \sigma_{f(i)}) .$$

Now, on one hand because in the latter term of this expression there appears a sum of functions depending only on one variable $\sigma_j \in \Sigma_j$ of the antagonistic coalition $e(i)$ of the player $i \in N$, and on the other hand, since the mininum and maxinum values of any expectation calculated either on pure strategy set or mixed strategy set coincide, then the attack and defense properties for such a kind of games can be expressed as follows: for each $x \epsilon X$ there is a joint pure strategy $\sigma \epsilon \Sigma$ such that for all player $i \epsilon N$ with $e(i) \neq \phi$

$$e_i^i(\sigma_i, x_{f(i)}) + \sum_{j \epsilon e(i)} e_i^j(\sigma_i, x_{e(i)}, x_{f(i)}) = \max_{s_i \epsilon \Sigma_i} [e_i^i(s_i, x_{f(i)}) + \sum_{j \epsilon e(i)} e_i^j(s_i, x_{e(i)}, x_{f(i)})]$$

and

$$e_i^j(x_i, \sigma_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} e_i^j(x_i, s_{e(i)}, x_{f(i)})$$

for all $j \epsilon e(i)$, where the function e_i^j indicates the expectation function of the function a_i^j .

A special case, which satisfies such attack and defense properties occurs when the simple structure function $\underline{e}$ is determined by $\underline{e}(i) = (g(i), N-(g(i) \cup \{i\}))$ for each player $i \epsilon N$, where $g(i) \subseteq N-\{i\}$ is a subset of players with at most one element. Thus if $j \epsilon g(i)$ then $g(j) = g(g(i)) = \{i\}$.

Actually, for such games $\Gamma_{\underline{e}}$, the payoff function of player $i \epsilon N$ occurs in the following form:

$$A_i(\sigma_i, \sigma_{g(i)}, \sigma_{N-(g(i) \cup \{i\})}) = a_i^i(\sigma_i, \sigma_{N-(g(i) \cup \{i\})}) + a_i^{g(i)}(\sigma_i, \sigma_{g(i)}, \sigma_{N-(g(i) \cup \{i\})}) ,$$

where the function $a_i^{g(i)}$ is identified with zero if the set $g(i)$ is empty.

For any game with such a simple structure function and where the payoff functions of the players are formed as above, with the further property that:

$$a_i^i(\sigma_i, \sigma_{N-(g(i) \cup \{i\})}) \equiv 0$$

and

$$a_i^{g(i)}(\sigma_i, \sigma_{g(i)}, \sigma_{N-(g(i) \cup \{i\})}) = -c_i \; a_{g(i)}^{i}(\sigma_{g(i)}, \sigma_i, \sigma_{N-(\{i\} \cup g(i))}) + d_i$$

for all the players $i \in N$ with $g(i) \neq \phi$, where c_i is any non-negative real number and d_i any real number, the attack and defense properties are completely satisfied. That is shown as follows. Let x be any arbitrary point in the product space X, then, for each player $i \in N$ with $g(i) \neq \phi$, we choose the pure strategy $\sigma_i \in \Sigma_i$ such that

$$e_i^{g(i)}(\sigma_i, x_{g(i)}, x_{N-(g(i) \cup \{i\})}) = \max_{s_i \in \Sigma_i} \; e_i^{g(i)}(s_i, x_{g(i)}, x_{N-(g(i) \cup \{i\})})$$

This obviously exists. Moreover, with respect to the player $j \in g(i)$, on this pure strategy $\sigma_i \in \Sigma_i$ of player $i \in g(j) = g(g(i))$ one has:

$$e_j^{g(i)}(s_j, \sigma_{g(j)}, x_{N-(\{i\} \cup g(i))}) = \min_{s_{g(j)} \in \Sigma_{g(j)}} \; e_j^{g(j)}(x_j, s_{g(j)}, x_{N-(\{i\} \; g(i))}) .$$

Therefore, by choosing a strategy $\sigma_i \in \Sigma_i$ with

$$e_i^{g(i)}(\sigma_i, x_{N-\{i\}}) = \max_{s_i \in \Sigma_i} e_i^{g(i)}(s_i, x_{N-\{i\}})$$

for all the players $i \in N$ with empty antagonistic coalition, we have obtained a joint pure strategy $\sigma \in \Sigma$ for which the defense and attack requirements are completely satisfied by $x \in X$. Thus the validity of the assertion has been established.

II.2 e-Simple Stable Points

We have seen that any new concept about the behavior of the players has been obtained by imposing a role on each associated or $\sigma_{f(i)}$-associated zero-sum person game. The considered intuitive concepts about the behavior of the players in these associated games were essentially those of indifferent and normal.

For the very simple and simple saddle points concepts, one has accepted the normal role of both players in the associated game. Of course, this consideration has an appropriate meaning for any zero-sum two-person game satisfying the maximin theorem. However, the intuitive meaning of a normal player develops difficulties in the general case when the maximin and minimax value do not coincide.

In this case one should assign a role to each player in a zero-sum two-person game, in order to develop the concept of normal player. Of course, here we believe that the more natural role associated with the players considered as normal, is the actuation in accordance with their corresponding maximin and minimax strategies. We recall that if the maximin theorem is valid for a zero-sum two-person game, then such roles assigned to the normal players coincides with the old concepts just considered.

Actually, we can have these considerations for both players separately, since their behavior is completely independent.

Therefore, by using the new viewpoints just introduced regarding the role of a normal player in a two-person game, we will obtain other important kinds of concepts of behavior in a game with an arbitrary number of players.

For a player $i \in N$ in a n-person game $\Gamma_{\underline{e}} = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with the simple structure function $\underline{e}$ with the usual properties consider for any joint strategy $\sigma_{f(i)}\in\Sigma_{f(i)}$ the $\sigma_{f(i)}$-associated two-person game $\Gamma_i(\sigma_{f(i)}) = \{ \Sigma_i,\Sigma_{e(i)};A_i \}$. If player $i\in N$ choses the strategy $\sigma_i\in\Sigma_i$ then, he is assured to obtain independent on the actions of his antagonistic coalition the payoff amount

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)}) \quad .$$

But, since if player $i\in N$ wishes to improve his safe position, he is able to obtain the maximin value of the game $\Gamma_i(\sigma_{f(i)})$:

$$V_i(\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i, s_{e(i)},\sigma_{f(i)}) \ ,$$

by using a maximin strategy. Such a maximin strategy $\sigma_i\in\Sigma_i$ is characterized by satisfying the following equality

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i, s_{e(i)},\sigma_{f(i)}) \ ,$$

These can be transformed to the following

$$F_i(\sigma_i,\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

where F_i indicates the mininum over the product set $\Sigma_{e(i)}$ of the payoff function A_i of player $i\in N$.

Now, if player $i\in N$ plays a maximum strategy in the game $\Gamma_i(\sigma_{f(i)})$, his safe winnings will not be depended on the behavior of his opponent player, which is now his antagonistic coalition. Thus, with respect to player $i\in N$ one can consider the actions of his antagonistic coalitions as omitted, that is, having no effect on his own safe position. From this fact, the antagonistic coalition $e(i)$ of the player $i \in N$ can be seen as the set of players that can enter into a non-cooperative alliance, thus, the behavior of its members is directed toward hurting player $i \in N$. However, it is very important to recall that such a consideration has a dual nature, namely: in reality the players of the antagonistic coalition hurt the corresponding player or the player $i \in N$ sees the members of his antagonistic coalition as players in which he should not have confidence. Of course, in both cases the normal player $i \in N$ should choose a maximin strategy, thus, the description of the behavior of his opponent can be omitted.

Having these facts, one is induced to consider a point formed by all the maximin strategies in the respective associate game which has been introduced in [8]. Formally, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n) \in \Sigma$ is said to be an <u>$\underline{e}$-maximin simple stable point</u> or concisely <u>$\underline{e}_m$-simple stable point</u> of the game $\Gamma_{\underline{e}} = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ with simple structure function $\underline{e}$ if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_i, \Sigma_{e(i)}; A_i\}$ of player $i \in N = \{1, \ldots, n\}$ the strategy $\bar{\sigma}_i \in \Sigma_i$ is a maximin strategy. That is

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(s_i, s_{e(i)}, \sigma_{f(i)}) .$$

The reason for the introduction of the adjective stable is that the safe position of any player is independent of the acts of his respective antagonistic coalition.

We observe that generally any two such points are not equivalent.

The outcome for the player $i \in N$ with respect to the $\underline{e}_m$-simple stable point $\bar{\sigma} \in \Sigma$ in the game $\Gamma_{\underline{e}}$, satisfies

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) \geq \max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)}) = V_i(\bar{\sigma}_{f(i)}) .$$

From an intuitive point of view, an $\underline{e}_m$-simple stable point is a rule of behavior on the one hand assures at least the amount $V_i(\bar{\sigma}_{f(i)})$ to each player, independently of the behavior of his antagonistic coalition and on the other hand such that the value $V_i(\bar{\sigma}_{f(i)})$ is the maximun safety value which the mentioned player is able to get, if in each instance all the players of his indifferent coalition remain on it.

From the definition, we have immediately that a point $\bar{\sigma}\in\Sigma$ is an $\underline{e}_m$-simple stable point of the n-person game $\Gamma_{\underline{e}} = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ if and only if it is a positive very simple equilibrium point of the n-person game

$$\Gamma' = \{ \Sigma_1,\ldots,\Sigma_n;F_1,\ldots,F_n \}$$

obtained by replacing in the game $\Gamma_{\underline{e}}$ the payoff function A_i of player $i\in N$ by the minimun function

$$F_i(\sigma_i,\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) .$$

Hence, in the special case of the simple structure function formed by $\underline{e}(i) = (\phi,N-\{i\})$ for each player $i\in N$, the concepts of $\underline{e}_m$-simple stable point and equilibrium point coincide, and therefore there is complete confidence among all the players at such a point.

Another extreme case arises when the simple structure function is determined by $\underline{e}(i) = (N-\{i\},\phi)$ for player $i\in N$. We note that in this latter extreme case, there always exist an $\underline{e}_m$-simple stable point for the games usually considered.

The following result arises immediately as a direct application of theorem I.7.

<u>THEOREM II.5</u>: Let $\Gamma_{\underline{e}} = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with simple structure function $\underline{e}$ such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a euclidean space and his payoff function A_i continuous in the variable $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$. Then, if each function

$$F_i(\sigma_i, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$

is concave with respect to the variable $\sigma_i \in \Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$, the game $\Gamma_{\underline{e}}$ has an $\underline{e}_m$-simple stable point.

<u>PROOF</u>: Consider the n-person game $\Gamma' = \{ \Sigma_1, \ldots, \Sigma_n; F_1, \ldots, F_n \}$ where the payoff function of player $i \in N$ is the function F_i . This game completely satisfies all the requirements of theorem I.7, since the minimun function F_i is continuous with respect to the variable σ in the product space Σ . Thus, the existence of a very simple equilibrium point $\bar{\sigma} \in \Sigma$ for the game Γ' :

$$F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)}) = \max_{s_i \in \Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)})$$

for all $i \in N$, is guaranteed. Such a point $\bar{\sigma} \in \Sigma$ is an $\underline{e}_m$-simple stable point of the game $\Gamma_{\underline{e}}$. (Q.E.D.)

We note that because the very simple equilibrium points do not satisfy the equivalence property, then the $\underline{e}_m$-simple stable points do not have such a property either.

For the mixed extension of any n-person game the $\underline{e}_m$-simple stable point can be characterized as follows:

<u>THEOREM II.6</u>: Let $\Gamma_{\underline{e}} = \{ \Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n \}$ be a finite n-person game with simple structure function $\underline{e}$. Then the mixed extension $\tilde{\Gamma}_{\underline{e}} = \{ \tilde{\Sigma}_1,\dots,\tilde{\Sigma}_n; E_1,\dots,E_n \}$ has an $\underline{e}_m$-simple stable point.

<u>PROOF</u>: For any arbitrary player $i\in N$, consider the minimun function

$$F_i(x_i,x_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(x_i,s_{e(i)},x_{f(i)})$$

defined on the product space $X = \times_{j\in N} \tilde{\Sigma}_j$. We now examine the concavity of this function. For this reason, let y_i and $\bar{y}_i$ be any two points in the mixed strategy set $\tilde{\Sigma}_i$, then, for a real number λ in the unit interval $[0,1]$, by the multilinearity of the expectation function of player $i\in N$, one has

$$E_i(\lambda y_i+(1-\lambda)\bar{y}_i,x_{e(i)},x_{f(i)})=\lambda E_i(y_i,x_{e(i)},x_{f(i)})+(1-\lambda)E_i(\bar{y}_i,x_{e(i)},x_{f(i)})$$

for every $x\in X$, and therefore by recalling the minimun's property:

$$F_i(\lambda y_i+(1-\lambda)\bar{y}_i,x_{f(i)}) \geq \lambda F_i(y_i,x_{f(i)}) + (1-\lambda)F_i(\bar{y}_i,x_{f(i)})$$

for all $x\in X$. Hence, the minimun function F_i of player $i\in N$ is a concave function with respect to the variable $x_i\in\tilde{\Sigma}_i$ for fixed $x_{f(i)}\in X_{f(i)}$.

Since, all the requirements of the previous theorem applied to the mixed extension $\tilde{\Gamma}_{\underline{e}}$ are completely satisfied, the existence of an $\underline{e}_m$-simple stable point for $\tilde{\Gamma}_{\underline{e}}$ has been proven. (Q.E.D.)

Because of these results concerning $\underline{e}_m$-simple stable points for those games under consideration, which have been obtained by assigning to the first player in the corresponding game determined by the joint strategy of the indifferent coalition, a normal role it is very natural to introduce the dual description, assigning a new normal role to the second player, without any specification of the first player. That is, considering the second player formed by the antagonistic coalition as a normal player, in the sense that his behavior is in accordance with some minimax strategy in the associated game, without any reference to the first player.

This assumption leads to the following formal definition.

Given an n-person game $\Gamma_{\underline{e}} = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ with the simple structure function $\underline{e}$, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1,\ldots,\bar{\sigma}_n)\in\Sigma$ is said to be an <u>$\underline{e}$-minimax simple stable point</u> or shortly <u>$\underline{e}^m$-simple stable point</u> of the game $\Gamma_{\underline{e}}$ if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_i,\Sigma_{e(i)};A_i\}$ for player $i\in N = \{1,\ldots,n\}$ the joint strategy $\bar{\sigma}_{e(i)}\in\Sigma_{e(i)} = \times_{j\in e(i)} \Sigma_j$ is a minimax strategy, that is

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_i} \max_{s_i\in\Sigma_i} A_i(s_i,s_{e(i)},\sigma_{f(i)}) = V^i(\bar{\sigma}_{f(i)})$$

or equivalently:

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all i N , where G_i indicates the maximum function over the strategy set Σ_i of the payoff function A_i of player $i\in N$.

If an $\underline{e}^m$-simple stable point $\bar{\sigma}\in\Sigma$ has been established in the game $\Gamma_{\underline{e}}$, then the outcome of player $i\in N$ has the property that

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) \leq \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_i\in\Sigma_i} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)}) = V^i(\sigma_{f(i)}) .$$

Usually, these points do not fulfill the equivalence property, as can easily be verified.

Intuitively speaking, an $\underline{e}^m$-simple stable point is a rule of behavior which on the one hand assures to each antagonistic coalition that its corresponding player cannot safely obtain more than $V^i(\bar{\sigma}_{f(i)})$, independently of his own behavior and on the other hand such that the value is the maximun value that the antagonistic coalition will be able safely to limit its corresponding player's behavior, if in each instance all the players of his indifferent coalition abide by it.

Two special cases of $\underline{e}^m$-simple stable points arise immediately, namely: when the simple structure function is given by $\underline{e}(i) = (\phi,N-\{i\})$ for each $i\in N$. In such a case any point is an $\underline{e}^m$-simple stable point, which is in accordance with the heuristic point of view, since no player hurts any other players. On the other hand if we have $\underline{e}(i) = (N-\{i\},\phi)$ then the definition is transformed into

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} \max_{s_i\in\Sigma_i} A_i(s_i,s_{N-\{i\}})$$

for all the players $i\in N$.

Directly from the definition, one immediately has that a point is an $\underline{e}^m$-simple stable point of the game $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ only if it is an $\underline{e}$-negative simple equilibrium point of the n-person game

$$\Gamma'_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; G_1,\ldots, G_n\}$$

obtained from the original game $\Gamma_{\underline{e}}$ by substituting for the payoff function A_i of player $i\in N$ the maximun function

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) .$$

From this simple observation, the following theorem results as an immediate consequence of theorem I.16.

THEOREM II.7: Let $\Gamma_{\underline{e}} = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with simple structure function $\underline{e}$ such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex set in a euclidean space and his payoff function A_i is continuous with respect to the variable $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$, and the function

$$G_i(\sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)})$$

is convex in the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. Then, if for each joint strategy $\sigma \in \Sigma$ there is another one $\tau \in \Sigma$ such that for all $i \in N$

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)}),$$

the game $\Gamma_{\underline{e}}$ has an $\underline{e}^m$-simple stable point.

PROOF: Consider the n-person game $\Gamma''_{\underline{e}} = \{ \Sigma_1, \ldots, \Sigma_n; G_1, \ldots, G_n \}$ with simple structure function $\underline{e}$, where the payoff function of player $i \in N$ is the maximun function G_i. G_i is continuous with respect to the variable $\sigma \in \Sigma$. Thus, such a game fulfills all the requirements of theorem II.1, and the existence of an $\underline{e}$-negative simple equilibrium point $\bar{\sigma} \in \Sigma$:

$$G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all player $i \in N$, for this game $\Gamma'_{\underline{e}}$ is completely guaranteed. Such a point is an $\underline{e}^m$-simple stable point of the game $\Gamma_{\underline{e}}$. (Q.E.D.)

The last condition imposed in the above theorem as well as those corresponding conditions in the previous theorems, can be observed to be the attack property with respect to the concept of $\underline{e}^m$-simple stable point. This can be interpreted by saying that for any established behavior there is another one such that if all the players of the indifferent coalition of any player abide by the first one, the second one is minimax for his antagonistic coalition in the corresponding associated game determined by the choosing of his indifferent coalition.

We should observe a very important fact which is concerned with the special case where the simple structure function in the previous theorem has the indifferent coalitions $f(i)$ for all the players $i \in N$, empty.

In such a case the attack property is transformed to the following condition: there is one $\tau \in \Sigma$ such that

$$\max_{s_i \in \Sigma_i} A_i(s_i, \tau_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{n-\{i\}}} \max_{s_i \in \Sigma_i} A_i(s_i, s_{N-\{i\}})$$

for all players $i \in N$, which is the definition of $\underline{e}^m$-simple stable point for the point τ.

Therefore, the above characterization is completely useless in such a case.

Of course, one might intend an analogous formulation of the above result for this situation. But, unfortunately, this new characterization cannot be done by the techniques used until now, since the multivalued function $\varphi_i(\sigma)$ of all the players $i \in N$ would be completely independent on the variable $\sigma \in \Sigma$.

Moreover, in such a description there is involved the existence of a point at which all the functions G_i reach their minimun on the product space. This is a more complicated question then those just considered, and therefore, we will not deal with it.

An immediate consequence of this theorem, which deals with mixed extension of finite game is related in the following result.

THEOREM II.8: Let $\Gamma_{\underline{e}} = \{ \Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n \}$ be a finite n-person game with simple structure function $\underline{e}$, such that the expectation function E_i of player $i \in N$ is linear in $x_{e(i)} \in X_{e(i)}$ for fixed $(x_i, x_{f(i)}) \in X_i x X_{f(i)}$ and for any x in the product space X there is another one $y \in X$ which satisfies

$$\max_{s_i \in \Sigma_i} E_i(s_i, y_{e(i)}, x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} \max_{s_i \in \Sigma_i} E_i(s_i, u_{e(i)}, x_{f(i)})$$

for all $i \in N$. Then, the mixed extension $\tilde{\Gamma}_{\underline{e}} = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; E_1,\ldots,E_n \}$ has an $\underline{e}^m$-simple stable point.

PROOF: Since expectation function E_i of player $i \in N$ is a linear function with respect to the variable $x_{e(i)} \in X_{e(i)}$ for fixed $(x_i, x_{f(i)}) \in X_i x X_{f(i)}$, then for any arbitrary real number $\lambda \in [0,1]$, one has

$$E_i(x_i, \lambda x_{e(i)} + (1-\lambda)\bar{x}_{e(i)}, x_{f(i)}) = \lambda E_i(x_i, x_{e(i)}, x_{f(i)}) + (1-\lambda) E_i(x_i, \bar{x}_{e(i)}, x_{f(i)})$$

for any pair of points $x_{e(i)}$ and $\bar{x}_{e(i)}$ in $X_{e(i)}$.

Hence, in accordance with the maximun's property, the following inequality holds true

$$G_i(\lambda x_{e(i)} + (1-\lambda)\bar{x}_{e(i)}, x_{f(i)}) \leq G_i(x_{e(i)}, x_{f(i)}) + (1-\lambda)\, G_i(\bar{x}_{e(i)}, x_{f(i)})$$

where the continuous function G_i is given by

$$G_i(x_{e(i)}, x_{f(i)}) = \max_{s_i \in \Sigma_i} E_i(s_i, x_{e(i)}, x_{f(i)}) = \max_{u_i \in \Sigma_i} E_i(u_i, \bar{x}_{e(i)}, x_{f(i)}) ,$$

which expresses the fact that the function G_i is convex with respect to the variable $x_{e(i)} \in X_{e(i)}$ for fixed $x_{f(i)} \in X_{f(i)}$.

Thus, a direct application of the preceding theorem to the mixed extension game $\Gamma_{\underline{e}}$ which satisfies all the corresponding requirements, assures the existence of an $\underline{e}^m$-simple stable point for this game $\tilde{\Gamma}_{\underline{e}}$. (Q.E.D.)

A very special example of a kind of n-person game to which the previous result applies is that just considered after theorem II.4. These are characterized by having the simple structure function $\underline{e}$ with antagonist coalition of all the players formed by at most one element.

Indeed, for any arbitrary point $x \in X$ consider the joint strategy $y \in X$ composed by taking each coordinate $y_j \in \tilde{\Sigma}_j$ for each player $j \in N$ which is a member of one antagonistic coalition $e(i)$ with $i \in N$ as the minimax in the $x_{f(i)}$-associated game, that is,

$$\max_{s_i \in \Sigma_i} e_i^j(s_i, y_j, x_{N-(\{j\} \cup \{i\})}) = \min_{u_j \in X_j} \max_{s_i \in \Sigma_i} e_i^j(s_i, u_j, x_{N-(\{j\} \cup \{i\})})$$

where e_i indicates the expectation of the payoff function

$$A_i(\sigma_i, \sigma_j, \sigma_{N-(\{j\} \cup \{i\})}) = a_i^j(\sigma_i, \sigma_j, \sigma_{N-(\{j\} \cup \{i\})})$$

of player $i \in N$; and any mixed strategy $y_j \in \tilde{\Sigma}_j$ of player $j \in N$ which does not belong to any anticoalition coalition. Such a joint strategy satisfies the attack property with respect to the concept of $\underline{e}^m$-simple stable point on the element $x \in X$.

Therefore, the mixed extension of such a game has at least one $\underline{e}^m$-simple stable point.

These results enable us to extend the new established concepts.

Given a n-person game $\Gamma_{\underline{e}} = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ with the simple structure function $\underline{e}$, a joint strategy $\bar{\sigma} = (\bar{\sigma}_1,\dots,\bar{\sigma}_n)\in\Sigma$ is said to be an <u>$\underline{e}$-simple stable point of the game</u> $\Gamma_{\underline{e}}$, if it is simultaneously an $\underline{e}_m$-simple and $\underline{e}^m$-simple stable point, i.e.:

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)}) = V_i(\bar{\sigma}_{f(i)})$$

and

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_i\in\Sigma_i} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)}) = V^i(\bar{\sigma}_{f(i)})$$

for all the players $i\in N = \{1,\dots,n\}$.

Because neither the $\underline{e}_m$-simple nor the $\underline{e}^m$-simple stable points have the equivalence property, then the above does not have such a property either.

On an $\underline{e}$-simple stable point $\bar{\sigma}\in\Sigma$ the outcome of player $i\in N$ satisfies the following inequality

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) \geq \max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\sigma_{f(i)})$$

$$\leq \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_i\in\Sigma_i} A_i(s_i,s_{e(i)},\sigma_{f(i)}) .$$

An $\underline{e}$-simple stable point is a rule of behavior which is maximin for each player and minimax for his antagonistic coalition in the associated game determined by the choice of his corresponding indifferent coalition.

For the special case where the simple structure function is determined by $\underline{e}(i) = (\phi,N-\{i\})$, the new concept coincides with the concept of positive simple equilibrium point. Another extreme situation arises when $\underline{e}(i) = (N-\{i\},\phi)$. In this case the definition is transformed into the following requirement

$$\min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} A_i(\bar{\sigma}_i, s_{N-\{i\}}) = \max_{s_i\in\Sigma_i} \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} A_i(s_i, s_{N-\{i\}})$$

and

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{N-\{i\}}) = \min_{s_{N-\{i\}}\in\Sigma_{N-\{i\}}} \max_{s_i\in\Sigma_i} A_i(s_i, s_{N-\{i\}})$$

for all the players $i\in N$.

By using the already known minimun function F_i and the maximun function G_i , another equivalent formulation of an $\underline{e}$-simple stable point of the game $\Gamma_{\underline{e}} = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ can be produced by considering simultaneously the following two games

$$\Gamma' = \{ \Sigma_1,\ldots,\Sigma_n;F_1,\ldots,F_n \}$$

and

$$\Gamma_{\underline{e}}'' = \{ \Sigma_1,\ldots,\Sigma_n; G_1,\ldots, G_n \} .$$

Thus, a point is an $\underline{e}$-simple stable point of the game $\Gamma_{\underline{e}}$ if and only if it is a positive simple equilibrium point of the game $\Gamma_{\underline{e}}'$ and is an $\underline{e}$-negative simple equilibrium point of the game $\Gamma_{\underline{e}}''$.

Unfortunately, we do not have in the previous section any existence theorem of this kind of points for the games under consideration, and therefore we must establish a complete new formulation.

<u>THEOREM II.9</u>: Let $\Gamma_{\underline{e}} = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ be an n-person game with simple structure function $\underline{e}$ such that the strategy set Σ_i of player $i\in N$ is a non-empty, compact and convex set in a euclidean space, his payoff function A_i is continuous with respect to the variable $\sigma = (\sigma_1,\dots,\sigma_n)\in\Sigma$, and the function

$$F_i(\sigma_i,\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)},\sigma_{f(i)})$$

is concave with respect to the variable $\sigma_i\in\Sigma_i$ for fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$ and finally the function

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

is convex in the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$.
If for each joint strategy $\sigma\in\Sigma$ there is another one $\tau\in\Sigma$ such that for all $i\in N$

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-simple stable point.

<u>PROOF</u>: For any arbitrary point $\sigma = (\sigma_i,\sigma_{e(i)},\sigma_{f(i)})$ in the non-empty, compact and convex product space Σ , consider the non-empty set

$$\varphi_i(\sigma) = \{ \tau\in\Sigma: F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)}) \quad \text{and}$$

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) \} .$$

Indeed, let τ and $\bar{\tau}$ be two arbitrary elements of the set $\varphi_i(\sigma)$, then one has the following equalities

$$F_i(\tau_i,\sigma_{f(i)}) = F_i(\bar{\tau}_i,\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = G_i(\bar{\tau}_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) \quad ,$$

from which immediately we deduce for any real number $\lambda\in[0,1]$:

$$F_i(\lambda\tau_i+(1-\lambda)\bar{\tau}_i,\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

and

$$G_i(\lambda\tau_{e(i)}+(1-\lambda)\bar{\tau}_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) \quad ,$$

since the minimun function F_i is concave in the variable $\sigma_i\in\Sigma_i$ and the maximun function G_i is convex with respect to $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$. Thus, that the point $\lambda\tau+(1-\lambda)\bar{\tau}\in\Sigma$ is a member of the set $\varphi_i(\sigma)$, since all strategy sets are convex, which implies the convexity of $\varphi_i(\sigma)$.

Now, we define the multivalued function

$$\varphi : \Sigma\to\Sigma$$

by the convex set

$$\varphi(\sigma) = \bigcap_{i\in N} \varphi(\sigma)$$

for each $\sigma\in\Sigma$, which is non-empty by virtue of the last condition. Now, we are going to examine the upper-semicontinuity of such a multivalued function. For this let

$$\sigma(k) \to \sigma \quad \text{and} \quad \tau(k) \to \tau$$

be any two arbitrary converging sequences of elements in the product space having the property that $\tau(k) \in \varphi(\sigma(k))$ for each positive integer k. Equivalently, for each k and each player $i \in N$:

$$F_i(\tau_i(k), \sigma_{f(i)}(k)) = \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}(k))$$

and

$$G_i(\tau_{e(i)}(k), \sigma_{f(i)}(k)) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)}(k)) \quad .$$

By the continuity of the payoff function A_i of player $i \in N$ in the product variable $\sigma \in \Sigma$, the four sequences of the real numbers whose general terms are those respective four members in the last equalities, converge to the values obtained by substituting the point $(\sigma\ (k), \tau_{e(i)}(k), \sigma_{f(i)}(k))$ by $(\sigma_i, \tau_{e(i)}, \sigma_{f(i)})$ in the respective places. Thus, for each player $i \in N$, we have

$$F_i(\tau_i, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)}) \ ,$$

which expresses the fact that the joint point $\tau \in \Sigma$ belongs to the set $\varphi(\sigma)$. Thus, the multivalued function φ is upper-semicontinuous, and therefore satisfies all the requirements of Kakutani's Fixed Point Theorem. Then, the existence of a fixed point $\bar{\sigma} \in \varphi(\bar{\sigma})$ is completely guaranteed. i.e.

$$F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)}) = \max_{s_i \in \Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\in N$, which is in accordance with the definition of e-simple stable point of the game Γ_e. (Q.E.D.)

For any accepted behavior among the players, there is another behavior which is such that, if all the players of the indifferent coalition of any player abide by the first one, the second one is minimax for his antagonistic coalition and maximin for himself in the associated game determined by the choosing of the indifferent coalition. This is a possible interpretation of the last condition in the above theorem, which is observed as the attack and defense property with respect to the concept of e-simple stable points.

Again, we point out the uselessness of the previous result in the special case where the simple structure function is such that every indifferent coalition is empty. Indeed, the attack and defense requirement turns into the thesis of the theorem. As was remarked before for e^m-simple stable points, we will not examine the treatment of such a question.

By applying this theorem to the mixed extension of a finite n-person game, one immediately obtains the following result:

THEOREM II.10: Let $\Gamma_e = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ be a finite n-person game with simple structure-function e, such that the expectation function E_i of player $i\in N$ is linear in $x_{e(i)}\in X_{e(i)}$ for fixed $(x_i,x_{f(i)})\in X_i \times X_{f(i)}$ and for any x in the product space X there is another $y\in X$ which satisfies

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(y_i,s_{e(i)},x_{f(i)}) = \max_{u_i\in X_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(u_i,s_{e(i)},x_{f(i)})$$

and

$$\max_{s_i\in\Sigma_i} E_i(s_i,y_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} \max_{s_i\in\Sigma_i} E_i(s_i,u_{e(i)},x_{f(i)})$$

for all $i\in N$. Then, the mixed extension $\tilde{\Gamma} = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;E_1,\ldots,E_n \}$ has an e-simple stable point.

PROOF: Since the expectation function E_i of player $i\in N$ is a linear function with respect to the variable $x_{e(i)}\in X_{e(i)}$ for fixed $(x_i,x_{f(i)})\in X_i x X_{f(i)}$, for any arbitrary real number λ in the interval $[0,1]$, one has

$$E_i(x_i,\lambda x_{e(i)}+(1-\lambda)\bar{x}_{e(i)},x_{f(i)}) = \lambda E_i(x_i,x_{e(i)},x_{f(i)})+(1-\lambda)E_i(x_i,\bar{x}_{e(i)},x_{f(i)})$$

for any pair of points $x_{e(i)}$ and $\bar{x}_{e(i)}$ in $X_{e(i)}$. By considering the maximin property, the following relation arises immediately:

$$G_i(\lambda x_{e(i)}+(1-\lambda)\bar{x}_{e(i)},x_{f(i)}) \leq \lambda\, G_i(x_{e(i)},x_{f(i)})+(1-\lambda)\, G_i(\bar{x}_{e(i)},x_{f(i)})$$

where the continuous function G_i is defined by

$$G_i(x_{e(i)},x_{f(i)}) = \max_{s_i\in\Sigma_i} E_i(s_i,x_{e(i)},x_{f(i)}) = \max_{u_i\in\tilde{\Sigma}_i} E_i(u_i,x_{e(i)},x_{f(i)}) .$$

Then, the convexity of the function G_i in the variable $x_{e(i)}\in X_{e(i)}$ for fixed $(x_i,x_{f(i)})\in X_i x X_{f(i)}$ remains completely determined.

On the other hand, by the multilinearity of the expectation function E_i , for any arbitrary real number λ in the unit interval $[0,1]$, one obtains:

$$E_i(\lambda x_i+(1-\lambda)\bar{x}_i,x_{e(i)},x_{f(i)})=\lambda E_i(x_i,x_{e(i)},x_{f(i)})+(1-\lambda)E_i(\bar{x}_i,x_{e(i)},x_{f(i)})$$

for any pair of mixed strategies x_i and $\bar{x}_i$ in X_i from which, in accordance with the minimun property the following equality holds

$$F_i(\lambda x_i+(1-\lambda)\bar{x}_i,x_{f(i)}) \geq \lambda F_i(x_i,x_{f(i)})+(1-\lambda)F_i(\bar{x}_i,x_{f(i)})$$

where the continuous function F_i is given by

$$F_i(x_i,x_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(x_i,s_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} E_i(x_i,u_{e(i)},x_{f(i)}) ,$$

which determines the concavity property of the function F_i with respect to the variable $x_i\in X_i$ for fixed $(x_{e(i)},x_{f(i)})\in X_{e(i)}\times X_{f(i)}$.

Thus, all the requirements of the previous theorem applied to the mixed extension game $\tilde{\Gamma}_{\underline{e}}$ are completely satisfied, and therefore the existence of an e-simple stable point of $\tilde{\Gamma}_{\underline{e}}$ is guaranteed. (Q.E.D.)

Having the preceding results, it is interesting to observe that the concept of e-simple stable points includes as a special case of the e-simple saddle point for the games under consideration. Indeed, if a point $\bar{\sigma}\in\Sigma$ is a saddle point in the corresponding $\bar{\sigma}_{f(i)}$-associated game of all the players $i\in N$, i.e.

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

then, by remembering the inequality

$$\max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)}) \leq \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_i\in\Sigma_i} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

which is always true, one obtains the relations

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(s_i, s_{e(i)}, \bar{\sigma}_{f(i)})$$

and

$$\max_{s_i\in\Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_i\in\Sigma_i} A_i(s_i, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for each player $i\in N$. Thus, such a point $\bar{\sigma}\in\Sigma$ is an e-simple stable point.

Actually, one can describe another characterization for such special points complementary to that expressed in theorem II.3, as a consequence of theorem II.9. A simple formulation of this is expressed in the following result:

THEOREM II.11: Let $\Gamma_e = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with simple structure function e such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex set in a euclidean space, his payoff function A_i continuous with respect to the variable $\sigma\in\Sigma$; convex with respect to the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $(\sigma_i, \sigma_{f(i)})\in\Sigma_i x\Sigma_{f(i)}$ and concave with respect to the variable $\sigma_i\in\Sigma_i$ for fixed $(\sigma_{e(i)}, \sigma_{f(i)})\in\Sigma_{e(i)} x\Sigma_{f(i)}$. Then, if for each joint strategy $\sigma\in\Sigma$ there is another one $\tau\in\Sigma$ such that for all $i\in N$:

$$\max_{s_i\in\Sigma_i} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)}) = A_i(\tau_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)}) ,$$

the game Γ_e has an e-simple saddle point.

PROOF: By virtue of the concavity of the payoff function A_i of player $i\in N$ in the variable $\sigma_i\in\Sigma_i$ for fixed $(\sigma_{e(i)}, \sigma_{f(i)})\in\Sigma_{e(i)} x\Sigma_{f(i)}$, for any arbitrary real number λ in the unit interval $[0,1]$:

$$A_i(\lambda\sigma_i+(1-\lambda)\bar{\sigma}_i,\sigma_{e(i)},\sigma_{f(i)}) \geq \lambda A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)})+(1-\lambda)A_i(\bar{\sigma}_i,\sigma_{e(i)},\sigma_{f(i)})$$

for any pair of strategies σ_i and $\bar{\sigma}_i$ in Σ_i. From this inequality, by using the property of the minimun, the validity of the relation

$$F_i(\lambda\sigma_i+(1-\lambda)\bar{\sigma}_i,\sigma_{f(i)}) \geq \lambda F_i(\sigma_i,\sigma_{f(i)})+(1-\lambda)F_i(\bar{\sigma}_i,\sigma_{f(i)})$$

is completely established, where as always the continuous function F_i indicates the minimun of payoff function A_i over the product set $\Sigma_{e(i)}$. Thus, F_i is concave with respect to $\sigma_i \in \Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$.

On the other hand, because the payoff function A_i of player $i \in N$, is convex with respect to variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)}) \in \Sigma_i x \Sigma_{f(i)}$, then for each real number $\lambda \in [0,1]$, one has

$$A_i(\sigma_i,\lambda\sigma_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)},\sigma_{f(i)}) \leq \lambda A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)})+(1-\lambda)A_i(\sigma_i,\bar{\sigma}_{e(i)},\sigma_{f(i)})$$

for any pair of points $\sigma_{e(i)}$ and $\bar{\sigma}_{e(i)}$ in the product space $\Sigma_{e(i)}$. Therefore, by taking the maximun in this latter expression over the strategy set Σ_i, the following inequality results immediately

$$G_i(\sigma\lambda_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)},\sigma_{f(i)}) \leq \lambda\ G_i(\sigma_{e(i)},\sigma_{f(i)})+(1-\lambda)\ G_i(\bar{\sigma}_{e(i)},\sigma_{f(i)}),$$

where the continuous function G_i indicates as always the maximun over Σ_i of payoff function A_i. Thus, the concavity of G_i with respect to variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ is assured.

Finally, the latter condition together with the observation that for any joint strategy $\sigma \in \Sigma$ and any player $i \in N$ the maximun $V_i(\sigma_{f(i)})$ and minimax value

$V^i(\sigma_{f(i)})$ of the $\sigma_{f(i)}$-associated game are always related by: $V_i(\sigma_{f(i)}) \leq V^i(\sigma_{f(i)})$, which guarantees the fulfillment of the last requirement of theorem II.9 when applied to the game $\Gamma_{\underline{e}}$.

Thus, that theorem assures the existence of an $\underline{e}$-simple stable point $\bar{\sigma}\epsilon\Sigma$ for the game $\Gamma_{\underline{e}}$, which is determined by

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all player $i\epsilon N$.

Now consider for an arbitrary player $i\epsilon N$ the $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_i,\Sigma_{e(i)};A_i \}$ where the strategy sets Σ_i and $\Sigma_{e(i)}$ are non-empty, compact and convex in euclidean spaces and the payoff function A_i is continuous in the variable $(\sigma_i,\sigma_{e(i)})\epsilon\Sigma_i x\Sigma_{e(i)}$, concave in $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ and convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_i\epsilon\Sigma_i$. Then, by theorem I.1 the game $\Gamma_i(\bar{\sigma}_{f(i)})$ has a value, that is,

$$\max_{s_i\epsilon\Sigma_i} \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} \max_{s_i\epsilon\Sigma_i} A_i(s_i,s_{e(i)},\sigma_{f(i)}) ,$$

which implies the validity of the following assertion:

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\epsilon N$. Therefore the point $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}$-simple saddle point of the game $\Gamma_{\underline{e}}$. (Q.E.D.)

The latter requirement can be seen as an attack and defense property in a modified sense with respect to the concept of e-simple saddle point, which can be interpreted as follows: for any established behavior there is another one which is optimal for each player and his antagonistic coalition in the associated game determined by the old choices of his corresponding indifferent coalition.

Again, as what has been pointed out after theorems II.7 and II.9, we recall the uselessness of this result when the simple structure function is determined by having empty indifferent coalition sets for each player.

We observe that under the established conditions this result can be independently obtained without any reference to theorem II.9. The natural way is determined by considering for each player $i \in N$ and each point σ in the product space Σ the set

$$\varphi_i(\sigma) = \{\tau \in \Sigma : A_i(\tau_i, \tau_{e(i)}, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)})\}$$

that is, the set of all the saddle points in the $\sigma_{f(i)}$-associated game, is a non-empty set. Actually, to apply the fixed point technique it is necessary to obtain the convexity of the intersection of such sets. But this property arises as an immediate consequence of theorem I.4 applied to $\sigma_{f(i)}$-associated game, for each player $i \in N$ which guarantees the convexity of set $\varphi_i(\sigma)$ of saddle points. Thus, we now can proceed in the usual manner, obtaining the desired result.

The above theorem contributes another characterization of e-simple saddle points which in a simple analysis neither includes the result expressed in theorem II.3, nor is included in the old existence theorem. One should observe that they are two very closed formulations, even though complementary, in the sense that the latter conditions in the respective theorems are determined by very different requirements.

On the other hand, the result expressed in theorem II.3 can be applied to a wider class of games than the corresponding kind of games considered by theorem II.9, because it remains valid even in the special case where all the indifferent coalitions are empty. This fact contributes more generality to the first characterization.

Using the previous theorem, one can immediately formulate the corresponding existence theorem for the mixed extension of finite n-person games, which is given as follows:

<u>THEOREM II.12</u>: Let $\Gamma_{\underline{e}} = \{ \Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n \}$ be an n-person game with simple structure function $\underline{e}$, such that the expectation function E_i of player $i \in N$ is linear in $x_{e(i)} \in X_{e(i)}$ for fixed $(s_i, x_{f(i)}) \in X_i x X_{f(i)}$ and for any x in the product space X there is another $y \in X$ which satisfies

$$\max_{s_i \in \Sigma_i} E_i(s_i, y_{e(i)}, x_{f(i)}) = E_i(y_i, y_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(y_i, s_{e(i)}, x_{f(i)})$$

for all the players $i \in N$. Then, the mixed extension $\tilde{\Gamma} = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; E_1,\ldots,E_n \}$ has an $\underline{e}$-simple saddle point.

<u>PROOF</u>: From the linearity with respect to the variable $x_{e(i)} \in X_{e(i)}$ for fixed $(x_i, x_{f(i)}) \in X_i x X_{f(i)}$ of the expectation function E_i of player $i \in N$, this function is convex too. On the other hand, by the multilinearity, E_i always is concave in the variable $x_i \in X_i$ for fixed $(x_{e(i)}, x_{f(i)}) \in X_{e(i)} x X_{f(i)}$. Then, since the latter requirement, of the previous theorem applied to the mixed extension game $\tilde{\Gamma}_{\underline{e}}$, is completely satisfied, the existence of an $\underline{e}$-simple saddle point of $\tilde{\Gamma}_{\underline{e}}$ is guaranteed. (Q.E.D.)

We do note that the above theorem coincides with theorem II.10. Indeed, the latter conditions of both theorems indicate the validity of the minimax theorem in all the associated two-person games, since the payoffs are bilinear in the variable of the player and his antagonistic coalition.

For instance, a kind of finite n-person game, for which the previous result is available, arises when the simple structure function $\underline{e}$ is determined by $\underline{e}(i) = (g(i), N-g(i)\cup\{i\})$ for each player $i \in N$, where $g(i)$ is a subset of players with no more than one element. Thus, if $j \in g(i)$, then $g(j) = g(g(i)) = \{i\}$. Furthermore, the payoff functions are related by

$$A_i(\sigma_i, \sigma_{g(i)}, \sigma_{N-(g(i)\cup\{i\})}) = -c_i A_{g(i)}(\sigma_{g(i)}, \sigma_i, \sigma_{N-(g(i)\cup\{i\})}) + d_i$$

where c_i is any positive real number and d_i any real number, for all the players $i \in N$ with $g(i) \neq \phi$.

Indeed, for any given element x in the product space X, let $y_i \in \Sigma_i$ be one maximun strategy of player $i \in N$ with $g(i) = \{j\}$ and let $y_{g(i)} \in X_{g(i)}$ be a minimax strategy of player $j \in N$ both in the $x_{f(i)}$-associated two-person game. Evidently, such elements always exist. Then, the joint strategy $y \in X$ formed by having y_i as the i-th coordinate for player $i \in N$ with $g(i) \neq \phi$ and any strategy y_i for player $i \in N$ with $g(i) \neq \phi$, satisfies the modified attack and defense properties on the point $x \in X$.

Therefore, for the mixed extension of such a finite n-person game the above theorem, which guarantees the existence of an $\underline{e}$-simple saddle point is available.

We note that the existence of an $\underline{e}$-simple saddle point for the games under consideration, has been obtained in the remarks after the theorem II.4, by observing that the attack and defense properties described in this theorem are completely assured.

Therefore, this constitutes an example where both formulations of the attack and defense properties corresponding to the theorems II.4 and II.12 are satisfied.

CHAPTER III *

III.1. e-Simple Points by Fixed Point Procedure.

The results expressed in the previous chapter, leads us to ask about the corresponding mathematical extensions of those general results for games where the strategy sets have a more complicated topological structure.

In this chapter we treat three different general extensions which apply when the strategy sets are compact and convex set in a real topological linear spaces. As immediate consequences of these new treatments we will obtain existence theorems for mixed extensions of the continuous games which will be introduced after some considerations.

The first extension, which is examined in this section, is based on the generalization of Kakutani's fixed point theorem due to Fan [2] and Glicksberg [6]. Another generalization of those results considered in the following section employs a result concerning the intersection of sets having convex sections introduced in [9]; which is an improvement of a recent very useful theorem given by Fan in [5]. This treatment contains as special cases all the results obtained by using the fixed point procedure.

Finally, in the last section we deal with another generalization which is based on the idea of Nikaido-Isoda introduced in [16] which is to resolve the existence of equilibrium points.

The results obtained in the second section neither include as special cases nor are contained in the results of the third extension.

Before formulating the treatment for e-simple points, the following important concepts and facts should be recalled.

A linear topological space or topological vector space* is a vector space Σ over the field of real numbers R together with a topology such that the addition and multiplication functions

$$+: \Sigma \times \Sigma \to \Sigma \qquad \cdot: R \times \Sigma \to \Sigma$$

are continuous functions.

If in a linear topological space there exists a fundamental base of convex neighborhoods for the point 0, it is said to be locally convex. In such a space we can always choose a symmetric neighborhood U, that is $U = -U$, belonging to the fundamental base.

It is well known that every finite dimentional linear topological space is locally convex. Furthermore, if a Hansdorff linear topological space has finite dimension, then, its topology is euclidean.

A typical example of a linear topological space which is non-locally convex is $\ell_{1/2}$ determined by all the infinite sequences $\sigma = (\sigma_1, \ldots, \sigma_n, \ldots)$ of real numbers σ_i such that

$$\sum_{i=1}^{\infty} |\sigma_i|^{1/2} < \infty .$$

We now extend the notion of upper-semicontinuity for multivalued function defined on linear topological spaces.

Let Σ be a non-empty, compact Hausdorff space, then a multivalued function

$$\varphi : \Sigma \to \Sigma$$

is said to be closed or upper-semicontinuous if for each pair of convergent directed systems

$$\sigma(k) \to \sigma \quad \text{and} \quad \tau(k) \to \tau$$

in the space Σ, such that for every k in the directed set D : $\tau(n) \epsilon \varphi(\sigma(n))$, then $\tau \epsilon \varphi(\sigma)$.

*We do not use the adjective real because we only deal with such spaces.

This definition, in terms of the graph of the multivalued function φ

$$G_\varphi = \{(\sigma,\tau) \in \Sigma \times \Sigma : \tau \in \varphi(\sigma)\} ,$$

is equivalent to determining the closeness of this set in the product space $\Sigma \times \Sigma$.

Using these concepts, one can now furmulate the following fixed point theorem due to Fan [2] and Glicksberg [6], which is the fundamental tool for the subsequent discussion in this section.

THEOREM III.1: Let Σ be a non-empty compact convex set in a locally convex linear Hausdorff space. If the multivalued function

$$\varphi : \Sigma \to \Sigma$$

is such that, for all $\sigma \in \Sigma$, the set $\varphi(\sigma)$ is non-empty and convex, then there exists a fixed point $\bar{\sigma} \in \Sigma$; that is, $\bar{\sigma} \in \varphi(\bar{\sigma})$.

PROOF: Let U be a closed symmetric neighborhood of 0 , then by the upper-semicontinuity of the multivalued function φ , it can be seen that the graph

$$G_{\varphi_U} = \{(\sigma,\tau) \in \Sigma \times \Sigma : \tau \in (\varphi(\sigma) + U) \cap \Sigma\}$$

of the multivalued function

$$\varphi_U = (\varphi + U) \cap \Sigma$$

is a closed set. Furthermore, the set

$$G_{\varphi_U \cap I} = \{(\sigma,\tau) \in \Sigma \times \Sigma : \tau \in (\varphi(\sigma) + U) \cap \Sigma \cap \{\sigma\}\} = G_{\varphi_U} \cap \Delta$$

is also closed, since the graph

$$\Delta = \{(\sigma,\sigma) \in \Sigma \times \Sigma\}$$

of the identity function I , is closed.

Therefore, the projection of set $G_{\varphi_U \cap I}$ on the space Σ :

$$F_U = \{\sigma \in \Sigma : \sigma \in (\varphi(\sigma) \cap U) \cap \Sigma\}$$

is a closed set.

By virtue of the compactness of Σ , there exist a finite number of points $\sigma_1, \ldots, \sigma_n$, such that ,

$$\Sigma \subseteq \bigcup_{i=1}^{n} (\{\sigma_i\} + U) \quad .$$

Let H be the convex hull of the finite set of points $\{\sigma_1, \ldots, \sigma_n\}$, which is a compact set. Since the relative topology is euclidean, we can apply the Kakutani fixed point theorem to the upper-semicontinuous multivalued function

$$\varphi_U^H : H \rightarrow H$$

defined by the non-empty convex set

$$\varphi_U^H(\sigma) = (\varphi(\sigma) + U) \cap H \; .$$

Therefore, there exists a fixed point

$$\tilde{\sigma} \in \varphi((\tilde{\sigma}) + U) \cap H \; ,$$

which implies that the set F_U is non-empty.

From the compactness of the set Σ , for any two arbitrary closed symmetric neighborhoods of 0 , U and V one has

$$F_U \cap F_V \supseteq F_{U \cap V} \; ,$$

that is, non-void intersection, and so the intersection set

$$\bigcap_U F_U$$

is non-empty. Thus, any element $\bar{\sigma}$ of this intersection is a fixed point $\bar{\sigma} \in \varphi(\bar{\sigma})$ of the multivalued function φ . (Q.E.D.).

Now, in order to obtain a systematic exposition, we recall a very simple result which is formulated below:

LEMMA III.2: Let

$$A: \Sigma \times \Sigma \to R \quad \text{and} \quad B: \Sigma \to R$$

be two continuous real functions, where Σ is a compact Hausdorff space, such that for each $\sigma\epsilon\Sigma$ there is a $\tau\epsilon\Sigma$ with $A(\tau,\sigma) = B(\sigma)$. Then, the multivalued function

$$\varphi: \Sigma \to \Sigma$$

defined by

$$\varphi(\sigma) = \{\tau\epsilon\Sigma : A(\tau,\sigma) = B(\sigma)\}$$

is upper-semicontinuous.

PROOF: Consider two arbitrary convergent directed systems

$$\sigma(k) \to \sigma \quad \text{and} \quad \tau(k) \to \tau$$

in the space Σ, such that for every k in the directed set D: $\tau(k)\epsilon\, \varphi(\sigma(k))$, and therefore:

$$A(\tau(k), \sigma(k)) = B(\sigma(k)) .$$

From the continuity of the functions A and B, the following composition directed systems converge:

$$B(\sigma(k)) \to B(\sigma)$$

and

$$A(\tau(k),\sigma(k)) \to A(\tau,\sigma) .$$

This implies

$$A(\tau,\sigma) = B(\sigma) .$$

Thus, the point τ is an element of set $\varphi(\sigma)$, and the upper-semicontinuity of multivalued function φ has been shown. (Q.E.D.).

By using the results just considered, we now formulate a first existence theorem which concerns e-positive simple equilibrium points.

We recall that the cartesian product of linear topological spaces is, with addition and scalar multiplication defined coordinate-wise, and with product topology, a linear topological space. Moreover, the product of two locally convex linear topological spaces, is locally convex too.

<u>THEOREM III.3:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\dots,\Sigma_n;\ A_1,\dots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, and his payoff function A_i continuous in the product variable $\sigma\epsilon\Sigma$ and concave with respect to the variable $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{N-\{i\}}\epsilon\Sigma_{N-\{i\}}$.

Then, the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-positive very simple equilibrium point.

<u>PROOF:</u> Consider the non-empty, convex and compact set Σ in the locally convex linear Hausdorff product space. For an arbitrary point $\sigma\epsilon\Sigma$ and a player $i\epsilon N$ let us consider the following non-empty set

$$\varphi_i(\sigma) = \{\tau\epsilon\Sigma:\ A_i(\tau_i,\sigma_{N-\{i\}}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{N-\{i\}})\} .$$

This is convex by virtue of the concavity of the payoff function A_i in the variable $\sigma_i\epsilon\Sigma_i$.

Now, the multivalued function φ_i determined by $\varphi_i(\sigma)$ for every $\sigma\epsilon\Sigma$, is seen from the previous lemma to be upper-semicontinuous. Since the function's maximum is continuous with respect to the variable $\sigma\epsilon\Sigma$.

Define the multivalued function

$$\varphi = \bigcap_{i\epsilon N} \varphi_i : \ \Sigma \to \Sigma ,$$

whose graph G_φ is the non-empty intersection over $i\epsilon N$ of the closed graphs G_{φ_i}. φ is therefore upper-semicontinuous.

Now, since for any $\sigma\in\Sigma$ the non-empty set $\varphi(\sigma) \subseteq \Sigma$ is convex, we can apply the Fan-Glicksberg fixed point theorem to the multivalued function φ. Thus, the existence of a point $\bar{\sigma}\in \varphi(\bar{\sigma})$ is guaranteed.

At such a point, we have

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{N-\{i\}})$$

for each player $i\in N$, that is, the joint strategy $\bar{\sigma}$ is an $\underline{e}$-positive simple equilibrium point of game $\Gamma_{\underline{e}}$. (Q.E.D.).

Some special consequences of this result can be obtained which are related to mixed extensions of games defined on topological spaces. Before formulating them the following concepts and facts should be recalled[2]: a normed space is a linear topological space with respect to the strong topology, that is, the topology defined by the natural distance. Furthermore, the vector space forming this Banach space, can become a linear topological space by assigning some other interesting topologies. In general, let Σ be a vector space and let $\bar{\Sigma}$ be the space of all linear real functions on Σ. Then, a linear subspace T of $\bar{\Sigma}$ is said to be total if $f(\sigma) = 0$ for all $f\in T$ implies $\sigma = 0$. Every total linear subspace T of the space $\bar{\Sigma}$ of linear functionals on the vector space Σ, determines the T-topology of Σ, which is obtained by having the fundamental base of convex neighborhoods of point 0, defined by

$$N(\epsilon, F) = \{\sigma\in\Sigma: \ |f(\sigma)| < \epsilon \quad \text{for all} \quad f\in F\} ,$$

where F is a finite subset of T and is a real number, $\epsilon > 0$. The T-topology of the vector space Σ with total linear subspace T of $\bar{\Sigma}$ is locally convex. Σ is then a locally convex linear topological space. Of all the possible special cases, there are two very interesting ones. The first one arises when the total

[2]See reference [1].

linear subspace T coincides with the dual space Σ^*, that is, the set of all continuous linear real functions defined on the Banach space Σ. The total property of Σ^* is guaranteed by the distinguished points property which in turn is an immediate consequence of Hahn-Banach theorem. In this instance, the T-topology of Σ is called the <u>weak topology</u>. Finally, the remaining important case is obtained when the Banach space Σ coincides with the dual space Δ^* of all continuous linear functionals on the linear topological space Δ, and the total subspace T is the natural embedding of Δ in Δ^{**}, that is, if it is determined by

$$T = \{ f^*_\delta \in \Sigma^* : \delta \in \Delta ,\ f^*_\delta(\sigma) = \sigma(\delta) \quad \text{for all} \quad \sigma \in \Sigma \} .$$

In such an instance, the T-topology of Σ is usually called the <u>weak*-or w*-topology of Σ</u>. Therefore, if the space Σ is Hausdorff and compact, then the dual space $C^*(\Sigma)$ of the Banach space $C(\Sigma)$ of all continuous real functions on Σ is a locally convex, real-linear Hausdorff space, with respect to the w*-topology.

Given a compact Hausdorff space Σ, let Y be the space of all the regular countable additive measures on Σ. Then, by the Riesz representation theorem the space Y can be represented by $C^*(\Sigma)$ such that the corresponding elements $y \in Y$ and $f^* \in C^*(\Sigma)$ satisfy:

$$E(f^*) = f^*(f) = \int_\Sigma f(\sigma)\, dy$$

for $f \in C(\Sigma)$. Furthermore, the set $X \subseteq Y$ of regular countable additive measures on Σ with total measure one is compact and convex in $C^*(\Sigma)$ with respect to the w*-topology.

Having these facts, we can now extend the concept of mixed extension to a more general class of games.

Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be a n-person game where for player $i\in N$ the pure strategy set Σ_i is a compact Hausdorff space, and the payoff function A_i is continuous on Σ. Then, the mixed strategy set $x_i \in X_i = \tilde{\Sigma}_i$ is defined by the set of regular countable additive measures on Σ_i with total measure one, and the <u>mixed extention game</u> $\tilde{\Gamma} = \{\tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; E_1,\ldots,E_n\}$ by having for $i\in N$ as payoff function the continous expectation

$$E_i(x_1,\ldots,x_n) = \int_{\Sigma} A_i(\sigma_1,\ldots,\sigma_n)\, d(x_1\times\ldots\times x_n) ,$$

which is the restriction on $X = \times_{i\in N} \tilde{\Sigma}_i$ of a multilinear function. This kind of n-person game plays an analogous rule in this chapter to that played by the mixed extensions of finite games in the previous chapter.

Using the preceding theorem, one immediately obtains the following result due to Glicksberg [6], which is related to the existence of $\underline{e}$-positive simple equilibrium points for mixed extensions of the games just considered.

<u>THEOREM III.4:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\in N$ is a compact Hausdorff space, and the payoff function A_i is continous in $\sigma\in\Sigma$. Then, the mixed extension game $\tilde{\Gamma}_{\underline{e}} = \{\tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; E_1,\ldots,E_n\}$ with simple structure function $\underline{e}$, has an $\underline{e}$-positive simple equilibrium point.

<u>PROOF:</u> Consider for player $i\in N$, the set $\tilde{\Sigma}_i$ which is compact and convex in the locally convex linear Hausdorff space $C^*(\Sigma_i)$, and the expectation function E_i which is continous in the variable $x\in X$. E_i is concave in the variable

$x_i \epsilon X_i$ for fixed $x_{N-\{i\}} \epsilon X_{N-\{i\}} = \underset{j \epsilon N-\{i\}}{\times} X_j$, since it is the restriction on X of a multivalued function, all the requirements of theorem III, 3 applied to the mixed extension $\tilde{\Gamma}_e$ are satisfied. Thus, $\tilde{\Gamma}_e$ has an e-positive simple equilibrium point. (Q.E.D.)

This result contains as a special case the following minimax theorem due to Fan [2] and Glicksberg [6], which is obtained by imposing the zero-sum condition on the two-person game.

THEOREM III.5: Let $\Gamma = \{\Sigma_1, \Sigma_2; A\}$ be a zero-sum two-person game, such that the strategy sets Σ_1 and Σ_2 are compact Hausdorff spaces, and the payoff function A is continuous on $\Sigma_1 \times \Sigma_2$. Then, the following equality is satisfied:

$$\max_{x_1 \epsilon \tilde{\Sigma}_1} \min_{x_2 \epsilon \tilde{\Sigma}_2} \int_{\Sigma_1 x \Sigma_2} A \, d(x_1 \times x_2) = \min_{x_2 \epsilon \tilde{\Sigma}_2} \max_{x_1 \epsilon \tilde{\Sigma}_1} \int_{\Sigma_1 x \Sigma_2} A \, d(x_1 \times x_2) ,$$

that is, the mixed extension $\tilde{\Gamma} = \{\tilde{\Sigma}_1, \tilde{\Sigma}_2; E\}$ has a saddle point.

We, now examine for the games under consideration, the analogous treatment for the dual concept of e-negative simple equilibrium point. A general result is described below:

THEOREM III.6: Let $\Gamma_e = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be an n-person game with simple structure function e , such that the strategy set Σ_i of player $i \epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, and his payoff function A_i is continous in the product variable $\sigma \epsilon \Sigma$, and convex with respect to the variable $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$

for fixed $(\sigma_i, \sigma_{f(i)}) \in \Sigma_i \times \Sigma_{f(i)}$. Then, if for each joint strategy $\sigma\epsilon\Sigma$ there is another one $\tau\epsilon\Sigma$ such that for all $i\epsilon N$

$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}) ,$$

the game $\Gamma_{\underline{e}}$ has an e-negative simple equilibrium point.

PROOF: Consider the non-empty, convex and compact set Σ in the locally convex linear Hausdorff product space. For any point σ in the product space Σ and any player $i\epsilon N$, let us consider the following non-empty set

$$\varphi_i(\sigma) = \{\tau\epsilon\Sigma: A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} (\sigma_i, s_{e(i)}, \sigma_{f(i)})\} ,$$

this set is convex, since the payoff function A_i of player $i\epsilon N$ is convex with respect to the product variable $\sigma_{e(i)}\epsilon \Sigma_{e(i)}$.

From here, we can define the multivalued function φ_i which is given by the set $\varphi_i(\sigma)$ for every $\sigma\epsilon\Sigma$. By lemma II.2 this multivalued function φ_i is upper-semicontinuous, because the function's minimum is continuous with respect to the variable $\sigma\epsilon\Sigma$. Therefore the multivalued function

$$\varphi = \bigcap_{i\epsilon N} \varphi_i : \Sigma \to \Sigma ,$$

whose graph G_φ is the non-empty intersection over $i\epsilon N$ of the closed graphs $G_{\varphi i}$, is upper-semicontinuous also. Furthermore, for any joint strategy $\sigma\epsilon\Sigma$ the non-empty set $\varphi(\sigma) \subseteq \Sigma$ is convex.

Now, by a direct application of the Fan-Glicksberg fixed point theorem to φ, the existence of a fixed point $\bar{\sigma}\epsilon\ \varphi(\bar{\sigma})$ is assured.

For this fixed point $\bar{\sigma}\epsilon\Sigma$, the following equality holds

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\epsilon N$. Thus, the point $\bar{\sigma}\epsilon\Sigma$ is an e-negative simple equilibrium point of game Γ_e . (Q.E.D.)

This existence theorem involves a question of linearity on the payoff functions. This draws a correspondence between the kind of games just considered, and those illustrated after theorem I.10 for mixed extension of finite games. We will now extend the result formulated in lemma I.11. These results will be a usefull tool deciding the question of linearity of the expectation functions.

LEMMA III.7: Let $\Sigma_1,\ldots,\Sigma_n$ be non-empty compact Hausdorff space and let A be a continuous real function defined on the product space $\Sigma = \Sigma_1\times\ldots\times\Sigma_n$. Then, the expectation function E is the restriction of a linear function, with respect to the product variable $x\epsilon X = \tilde{\Sigma}_1\times\ldots\times\tilde{\Sigma}_n$ of the respective set $\tilde{\Sigma}_i$ of mixed strategy over Σ_i , with $i\epsilon N = \{1,\ldots, n\}$ if and only if, the function A is expressible as

$$A(\sigma_1,\ldots,\sigma_n) = a_1(\sigma_1) +\ldots+ a_n(\sigma_n) ,$$

where a_i is a function depending only on the variable $\sigma_i\epsilon\Sigma_i$, with $i\epsilon N$.

PROOF: The sufficiency can be seen immediately.

Now, let us consider the converse which we will prove by induction on the number n .

For $n = 1$ the assertion is trivial. Let $n=2$, then the expectation function E is the restriction of a linear function. Thus, for any pair of elements $x = (x_1,x_2)$ and $y = (y_1,y_2)$ belonging to the product set $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$, one has

$$E(\lambda x + (1-\lambda)y) = \lambda E(x) + (1-\lambda) E(y) .$$

On the other hand, because the expectation function is the restriction of a multilinear function, we have:

$$E(\lambda x + (1-\lambda)y) = \lambda^2 E(x) + (1-\lambda)^2 E(y) + (1-\lambda)\lambda[E(x_1,y_2) + E(x_2,y_1)].$$

From these two equalities, we get:

$$E(x) + E(y) = E(x_1,y_2) + E(y_1,x_2)$$

for all x and y of $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$. Now, by choosing y as any fixed point in $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$, this equality guarantees the form

$$E(x_1,x_2) = e_1(x_1) + e_2(x_2) ,$$

for the expectation function, where the functions e_1 and e_2 are restrictions on $\tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ of linears functions:

$$e_i(x_i) = \int_{\Sigma_i} a_i(\sigma_i)\, dx_i \qquad (i: 1,2) .$$

By replacing in the last equality, all the functions, we obtain

$$\int_{\Sigma_1 x \Sigma_2} [A(\sigma_1,\sigma_2) - a_1(\sigma_1) - a_2(\sigma_2)]\, d(x_1 \times x_2) = 0$$

for all $x_1 \epsilon \tilde{\Sigma}_1$ and $x_2 \epsilon \tilde{\Sigma}_2$. This implies that the function A has the following form

$$A(\sigma_1,\sigma_2) = a_1(\sigma_1) + a_2(\sigma_2) .$$

This proves the validity of the assertion for $n = 2$.

Now let n be arbitrary. Then, to the expectation function $E(x_1,\ldots x_{n-1},\sigma_n)$ with fixed $\sigma_n \epsilon \Sigma_n$, which is obtained by taking the mixed strategy $\bar{x}_{n,\sigma_n} \epsilon \tilde{\Sigma}_n$ given by

$$\bar{x}_{n,\sigma_n}(f) = f(\sigma_n)$$

for all $f \in C(\Sigma_n)$, one can apply the hypothesis of induction. Thus, the function A has the shape: $A(\sigma_1,\ldots,\sigma_n) = b_1(\sigma_1,\sigma_2) + \ldots + b_n(\sigma_1,\sigma_n)$.

Replacing this function in the expression of expectation E, we get

$$E(x_1,\ldots,x_n) = f_1(x_1,x_n) + \ldots + f_n(x_{n-1},x_n)$$

where the bilinear function f_i indicates the corresponding expectation of b_i with $i \in N$. Now one can easily see that the assertion of the case for $n = 2$ just examined, applied to function f_i with $i \in N$ gives

$$b_i(\sigma_i,\sigma_n) = a_i(\sigma_i) + c_i(\sigma_n) .$$

Then, by calling $\alpha_n(\sigma_n) = \sum_{i=1}^{n-1} c_i(\sigma_n)$, we have that the function A is expressible as

$$A(\sigma_1,\ldots,\sigma_n) = \sum_{i \in N} a_i(\sigma_i) ,$$

thus the statement for an arbitrary n is demonstrated . (Q.E.D.)

Having this result, we now can formulate an existence theorem for mixed extension games of the games under consideration.

<u>THEOREM III.8:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \in N$ is a compact Hausdorff space, and the payoff function A_i is the sum of the continuous function a_i, a_i^j with $j \in e(i)$ in $\sigma \in \Sigma$:

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) = a_i(\sigma_i,\sigma_{f(i)}) + \sum_{j \in e(i)} a_i^j(\sigma_i,\sigma_j,\sigma_{f(i)})$$

Then, if for each $x \in X$ there is a joint pure strategy $\sigma \in \Sigma$ such that

$$E_i(x_i,\sigma_{e(i)},x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} E_i(x_i,u_{e(i)},x_{f(i)})$$

for all $i \in N$, the mixed extension $\tilde{\Gamma}_{\underline{e}} = \{\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_n; E_1, \ldots, E_n\}$ with simple structure function $\underline{e}$, has an $\underline{e}$-negative simple equilibrium point.

PROOF: Consider for player $i \in N$ the compact convex set $\tilde{\Sigma}_i$ in the locally convex Hausdorff linear topological space $C^*(\Sigma_i)$. The expectation function E_i is continuous in the variable $x \in X$. Furthermore, by the form of the payoff function A_i, the lemma expresses that E_i is convex with respect to the variable $x_{e(i)} \in X_{e(i)} = \underset{j \in e(i)}{\times} \tilde{\Sigma}_j$ for fixed $(x_i, x_{f(i)}) \in X_i \times X_{f(i)}$.

On the other hand, the last condition of the previous theorem applies to the mixed extension game $\tilde{\Gamma}_{\underline{e}}$. Thus, there exists an $\underline{e}$-negative simple equilibrium point of $\tilde{\Gamma}_e$. (Q.E.D.)

By using the same fixed point procedure, one can characterize the $\underline{e}$-simple saddle points for the games under consideration. A first formulation of this is given in the following theorem.

THEOREM III.9: Let $\Gamma_{\underline{e}} = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, and his payoff function A_i is continuous in the product variable $\sigma \in \Sigma$, concave in $\sigma_i \in \Sigma_i$ for fixed $(\sigma_{e(i)}, \sigma_{f(i)}) \in \Sigma_{e(i)} \times \Sigma_{f(i)}$ and convex in $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_i, \sigma_{f(i)}) \in \Sigma_i \times \Sigma_{f(i)}$. If for each joint strategy $\sigma \in \Sigma$ there is another $\tau \in \Sigma$ such that

$$A_i(\tau_i, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)})$$

and

$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$

for all $i \in N$, then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-simple saddle point.

PROOF: Consider the non-empty, convex and compact set Σ in the locally convex product linear Hausdorff space. For any point $\sigma \in \Sigma$ and any player $i \in N$, we define the following non-empty sets

$$\varphi_i^m(\sigma) = \{\tau \in \Sigma: A_i(\tau_i, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)})\}$$

$$\varphi_{m,i}(\sigma) = \{\tau \in \Sigma: A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})\} .$$

Because the payoff function A_i is concave with respect to $\sigma_i \in \Sigma_i$ and is convex in the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$, then both sets are convex, and therefore, their intersection

$$\varphi_i(\sigma) = \varphi_i^m(\sigma) \cap \varphi_{m,i}(\sigma) ,$$

which is also non-empty is convex.

These three sets determine the multivalued functions $\varphi_{m,i}$ φ_i^m and φ_i which are related by:

$$\varphi_i = \varphi_i \cap \varphi_{m,i} .$$

By lemma III.2 the multivalued functions φ_i^m and $\varphi_{m,i}$ are both upper-semicontinuous, since the payoff function together with their minimum and maximin functions are continuous in the product variable $\sigma \in \Sigma$. Hence, the function φ_i is upper-continuous, too, since its graph is the non-empty intersection of the graphs $G^m_{\varphi_i}$ and $G_{\varphi_{m,i}}$. Finally by the same reason the multivalued function

$$\varphi = \bigcap_{i\in N} \varphi_i = \bigcap_{i\in N} (\varphi^m_i \cap \varphi_{m,i}) : \Sigma \to \Sigma$$

is upper-semicontinuous, with the non-empty convex set $\varphi(\sigma) \subseteq \Sigma$.

By applying the Fan-Glicksberg fixed point theorem to φ , the existence of a fixed point $\bar{\sigma} \in \varphi(\bar{\sigma})$ is assured.

At this fixed point $\bar{\sigma}\in\Sigma$, we have:

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

$$= \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N$. Thus, this point is an e-simple saddle point of game Γ_e . (Q.E.D.)

As a special consequence of this result we obtain the following existence theorem for e-simple saddle points for mixed extension games.

THEOREM III.10: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function e , such that the strategy set Σ_i of player $i\in N$ is a compact Hausdorff space, and the payoff function A_i is the sum of continuous functions a_i, a_i^j , in $\sigma\in\Sigma$ with $j\in e(i)$:

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) = a_i(\sigma_i,\sigma_{f(i)}) + \sum_{j\in e(i)} a_i^j(\sigma_i,\sigma_j,\sigma_{f(i)}) .$$

If for each $x\in X$ there is a joint pure strategy $\sigma\in\Sigma$ such that

$$E_i(\sigma_i,x_{e(i)},x_{f(i)}) = \max_{u_i\in X_i} E_i(u_i,x_{e(i)},x_{f(i)})$$

and

$$E_i(x_i,\sigma_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} E_i(x_i,u_{e(i)},x_{f(i)})$$

for all $i\in N$, then the mixed extension $\tilde{\Gamma}_e$ has an e-simple saddle point.

PROOF: Again, consider for player $i \epsilon N$ the compact convex set $\tilde{\Sigma}_i$ in the locally convex linear Hausdorff space $C^*(\Sigma_i)$. The expectation function E_i is continuous in the product variable $x \epsilon X$. On the other hand, by lemma II.7, the expectation function E_i is concave in $x_i \epsilon X_i$ and convex with respect to $x_{e(i)} \epsilon X_{e(i)}$. Finally, the last condition assures the fulfillment of the last requirement of previous theorem applied to the mixed extension game $\tilde{\Gamma}_{\underline{e}}$. Therefore the existence of an $\underline{e}$-simple saddle point of $\tilde{\Gamma}_{\underline{e}}$ is guaranteed. (Q.E.D.)

In a fashion similar to the preceding chapter, from the theorems III.3 and III.6, which respectively characterize the concepts of $\underline{e}$-positive and $\underline{e}$-negative simple equilibrium points, one can derive existence theorems for $\underline{e}_m$, $\underline{e}^m$-simple stable points. A first formulation is given as follows:

THEOREM III.11: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, and his payoff function A_i is continuous in the product variable $\sigma \epsilon \Sigma$, and F_i is concave with respect to $\sigma_i \epsilon \Sigma_i$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$. Then, the game $\Gamma_{\underline{e}}$ has an $\underline{e}_m$-simple stable point.

PROOF: Consider the n-person game $\Gamma' = \{\Sigma_1,\ldots,\Sigma_n; F_1,\ldots,F_n\}$, which completely satisfies all the requirements of theorem III.3, since the payoff function F_i of player $i \epsilon N$ is continuous with respect to $\sigma \epsilon \Sigma$. Therefore, we have the existence of a very simple equilibrium point $\bar{\sigma} \epsilon \Sigma$ of game Γ'. Such a point is an $\underline{e}_m$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

A special result related to mixed extensions, directly follows from this theorem.

<u>THEOREM III.12:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;\ A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is a compact Hausdorff space, and the payoff function A_i is continuous in the variable $\sigma\epsilon\Sigma$. Then, the mixed extension game $\tilde{\Gamma}_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;\ E_1,\ldots,E_n\}$ has an $\underline{e}_m$-simple stable point.

<u>PROOF:</u> For player $i\epsilon N$, since the payoff function E_i is a restriction of a multivalued function, the function

$$F_i(x_i,x_{f(i)}) = \min_{u_{e(i)}\epsilon X_{e(i)}} E_i(x_i,u_{e(i)},x_{f(i)})$$

is concave with respect to $x_i\epsilon\tilde{\Sigma}_i$ for fixed $x_{f(i)}\epsilon X_{f(i)}$. Thus, all the requirements of the previous theorem applied to the mixed extension $\tilde{\Gamma}_{\underline{e}}$ are satisfied. And so the game $\tilde{\Gamma}_{\underline{e}}$ has an $\underline{e}_m$-simple stable point. (Q.E.D.)

The characterization of $\underline{e}^m$-simple stable points is formulated in the following theorem.

<u>THEOREM III.13:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;\ A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, and his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, and G_i is convex with respect to $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. Then, if for each joint strategy $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all $i\epsilon N$, the game $\Gamma_{\underline{e}}$ has an $\underline{e}^m$-simple stable point.

PROOF: Consider the n-person game $\Gamma'' = \{\Sigma_1,\dots,\Sigma_n;\ G_i,\dots,G_n\}$, which has all its payoff functions continuous. Therefore, it satisfies all the requirement asked by theorem III.6. Hence, we have the existence of an $\underline{e}$-negative equilibrium point of game $\Gamma''_{\underline{e}}$. This point is an $\underline{e}^m$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

An immediate consequence of this theorem is obtained for mixed extension games:

THEOREM III.14: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\dots,\Sigma_n;\ A_1,\dots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \in N$ is a compact Hausdorff space, and the payoff function A_i is the sum of the continuous functions a_i, a_i^j in $\sigma \in \Sigma$, with $j \in e(i)$:

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) = a_i(\sigma_i,\sigma_{f(i)}) + \sum_{j\in e(i)} a_i^j(\sigma_i,\sigma_j,\sigma_{f(i)}) .$$

If for each $x \in X$ there is a joint strategy $y \in X$ such that

$$\max_{u_i\in X_i} E_i(u_i, y_{e(i)}, x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} \ \max_{u_i \in X_i} E_i(u_i,u_{e(i)},x_{f(i)})$$

for all $i \in N$, then the mixed extension game $\tilde{\Gamma}_{\underline{e}}$ has an $\underline{e}^m$-simple stable point.

PROOF: For player $i \in N$, because the expectation function E_i is the restriction of a multivalued function, and by the form of payoff function A_i, the lemma III.7 assures that the function

$$G_i(x_{e(i)},x_{f(i)}) = \max_{u_i\in X_i} E_i(u_i,x_{e(i)},x_{f(i)})$$

is convex with respect to $x_{e(i)} \in X_{e(i)}$ for fixed $x_{f(i)} \in X_{f(i)}$. Then, the preceding theorem guarantees the existence of an $\underline{e}^m$-simple stable point of mixed extension $\tilde{\Gamma}_{\underline{e}}$. (Q.E.D.)

The $\underline{e}$-simple stable points are considered in the following existence theorem.

THEOREM III.15: Let $\Gamma_e = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, and $F_i(\sigma_i,\sigma_{f(i)})$ is concave with respect to $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$ and $G_i(\sigma_{e(i)},\sigma_{f(i)})$ is convex with respect to $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. If for each joint strategy $\sigma\epsilon\Sigma$ there is another one $\tau\epsilon\Sigma$ such that

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all $i\epsilon N$, then the game Γ_e has an $\underline{e}$-simple stable point.

PROOF: Again, consider the non-empty, convex and compact set Σ in the locally convex linear product Hausdorff space. For any point $\sigma\epsilon\Sigma$ and any player $i\epsilon N$, let us define the sets

$$\varphi_i^m(\sigma) = \{\tau\epsilon\Sigma: F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)})\}$$

and

$$\varphi_{m,i}(\sigma) = \{\tau\epsilon\Sigma: G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_i,\sigma_{f(i)})\}$$

which, by the continuity of payoff functions A_i are non-empty. On the other hand, because the function F_i is concave in the variable $\sigma_i\epsilon\Sigma_i$ for fixed

$\sigma_{f(i)} \epsilon \Sigma_{f(i)}$ the set $\varphi_i^m(\sigma)$ is convex, and because the function G_i is convex in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$, the set $\varphi_{m,i}$ is also convex. Moreover, the intersection $\varphi_i(\sigma) = \varphi_i^m(\sigma) \cap \varphi_{m,i}(\sigma)$ is non-empty and convex too.

These sets define the multivalued functions φ_i^m, $\varphi_{m,i}$ and φ_i. By lemma II.2 the first of these are upper-semicontinuous. Thus, the graph φ_i which is the non-empty intersection of graphs φ_i^m and $\varphi_{m,i}$ is closed, that is, the multivalued function φ_i is also upper-semicontinuous. By the same reason, the multivalued function defined by

$$\varphi = \bigcap_{i \epsilon N} \varphi_i = \bigcap_{i \epsilon N} (\varphi_i^m \cap \varphi_{m,i}) : \Sigma \to \Sigma$$

with the convex set $\varphi(\sigma)$ is also upper-semicontinuous.

Then, the Fan-Glicksberg fixed point theorem applied to φ, gives the existance of a fixed point $\bar{\sigma} \epsilon \varphi(\bar{\sigma})$.

On such a point $\bar{\sigma} \epsilon \Sigma$, we have:

$$F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)}) = \max_{s_i \epsilon \Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all $i \epsilon N$, that is, it is an e-simple stable point of game Γ_e. (Q.E.D.)

From this result, we immediately derive the next existence theorem for mixed extension games.

THEOREM III.16: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function e, such that the strategy set Σ_i of player $i\epsilon N$ is a compact Hausdorff space, and the payoff function A_i is the sum of the continuous functions a_i, a_i^j in $\sigma\epsilon\Sigma$, with $j\epsilon\, e(i)$:

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) = a_i(\sigma_i,\sigma_{f(i)}) + \sum_{j\epsilon e(i)} a_i^j(\sigma_i,\sigma_j,\sigma_{f(i)}) .$$

If for each $x\epsilon X$ there is a joint strategy $y\epsilon X$ such that

$$\min_{u_{e(i)}\epsilon X_{e(i)}} E_i(y_i,u_{e(i)},x_{f(i)}) = \max_{u_i\,\epsilon X_i}\ \min_{u_{e(i)}\epsilon X_i} E_i(u_i,u_{e(i)},x_{f(i)})$$

and

$$\max_{u_i\epsilon X_i} E_i(u_i,y_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\epsilon X_{e(i)}}\ \max_{u_i\epsilon X_i} E_i(u_i,u_{e(i)},x_{f(i)})$$

for all $i\epsilon N$, then the mixed extension game $\tilde{\Gamma}_e$ has an e-simple stable point.

PROOF: Because the expectation function E_i of player $i\epsilon N$ is the restriction of a multivalued function, and by the form of payoff function A_i, lemma III.7 assures that the function

$$F_i(x_i,x_{f(i)}) = \min_{u_{e(i)}\epsilon X_{e(i)}} E_i(x_i,\ u_{e(i)},\ x_{f(i)})$$

is convex with respect to $x_{e(i)}\epsilon X_{e(i)}$ for fixed $x_{f(i)}\epsilon\, X_{f(i)}$. Thus, all the requirements of preceding theorem for mixed extension $\tilde{\Gamma}_e$ are satisfied. And so the existence of a e-simple stable point of $\tilde{\Gamma}_e$ is guaranteed. (Q.E.D.)

Indeed, such a point is an e-simple saddle point, since in all the associated two-person games the minimax holds true, by virtue of the bilinearity of payoff functions.

Another very simple application of theorem III.15 gives a different characterization from that of theorem II.9 for $\underline{e}$-simple saddle points.

THEOREM III.17: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space, and his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, concave in $\sigma_i\epsilon\Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\Sigma_{e(i)}\times\Sigma_{f(i)}$ and convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)})\epsilon\Sigma_i\times\Sigma_{f(i)}$. If for each joint strategy $\sigma\epsilon\Sigma$ there is another one $\tau\epsilon\Sigma$ such that

$$\max_{s_i\epsilon\Sigma_i} A_i(s_i,\tau_{e(i)},\sigma_{f(i)}) = A_i(\tau_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\tau_i,s_{e(i)},\sigma_{f(i)})$$

for all $i\epsilon N$, then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-simple saddle point.

PROOF: By virtue of the concavity of payoff function A_i of player $i\epsilon N$ in the variable $\sigma_i\epsilon\Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\,\Sigma_{e(i)}\times\Sigma_{f(i)}$, the function $F_i(\sigma_i,\sigma_{f(i)})$ is concave in $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. On the other hand, by the convexity in the variable $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)})\epsilon\Sigma_i\times\Sigma_{f(i)}$, the function $G_i(\sigma_{e(i)},\sigma_{f(i)})$ is also convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$.

The last condition assures that for any joint strategy $\sigma\epsilon\Sigma$ the following inequality must be an equality

$$\max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) \leq \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

and therefore, the latter requirement of theorem III.15 is satisfied.

This theorem then guarantees the existence of a e-simple stable point $\bar{\sigma}\epsilon\Sigma$ of the game $\Gamma_{\underline{e}}$, at which one has

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\epsilon N$. Furthermore, by the above relation, at such a point

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\epsilon N$, which shows that the point $\bar{\sigma}\epsilon\Sigma$ is an e-simple saddle point. (Q.E.D.)

This characterization is an extension of theorem II. 11, which can be obtained directly by using the fixed point technique. We point out in a similar way that this procedure envolves the maximin result given in theorem III.5.

Finally, the following formulation results as an immediate consequence of this theorem which is theorem III.16 itself, since the minimax property is satisfied by the bilinearity of payoff functions.

THEOREM III.18: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is a compact Hausdorff space, and the payoff function A_i is the sum of the continuous functions a_i, a_i^j in $\sigma\epsilon\Sigma$, with $j\epsilon\ e(i)$:

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) = a_i(\sigma_i,\sigma_{f(i)}) + \sum_{j\epsilon\ e(i)} a_i^j(\sigma_i,\sigma_j,\sigma_{f(i)}) .$$

If for each $x\epsilon X$ there is a joint strategy $y\epsilon X$ such that

$$\max_{u_i \in X_i} E_i(u_i, y_{e(i)}, x_{f(i)}) = E_i(y_i, y_{e(i)}, x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} E_i(y_i, u_{e(i)}, x_{f(i)})$$

for all $i \in N$, then the mixed extension game $\Gamma_{\underline{e}}$ has an $\underline{e}$-simple saddle point.

Finally, we note that the remaining results of this third chapter are useless when the simple structure function has $f(i) = \emptyset$ for every $i \in N$, since this last condition is the thesis of each theorem. This situation is similar to what has been seen in the previous chapter.

III.2. e-Simple Points by Intersection of Sets with Convex Cylinders Procedure.

Now, we are concerned with extending the general results obtained in the preceding section which have been derived from the fixed point procedure. The new generalization presented in this section is based on the procedure due to Fan in [5], where equilibrium points were considered. This procedure uses a very useful result by the same author [4], which concerns the intersection of sets with convex cylinders. For our purposes, we need a stronger result which has been introduced in [9], for attacking the generalizations of this section. However, the procedure remains that due to Fan.

The more fundamental result is expressed in the following theorem due to Fan [3], which is observed as a generalized form of the Knaster-Kuratowski-Mazurkiewicz's theorem. The assertion of this result is expressed as follows: If the $n+1$ closed subsets $\Sigma_1,\dots,\Sigma_n$ of the n-dimentional simplex Σ, a euclidean space, with the vertices $\sigma_0,\dots,\sigma_n$ satisfy the condition that for each subset $\{i_1,\dots,i_r\}$ with $1 \leq r \leq n+1$ of the set $\{0,\dots,n\}$, the face of Σ determined by the vertices $\sigma_{i_1},\dots,\sigma_{i_r}$ is contained in the set $\Sigma_{i_1} \cup \dots \cup \Sigma_{i_r}$. Then, the intersection

$$\bigcap_{i=1}^{n} \Sigma_i$$

is non-empty.

THEOREM III.19: Let Σ be an arbitrary set in a linear space Δ. For each point $\sigma \in \Sigma$, let $S(\sigma) \subseteq \Delta$, such that the convex hull of any finite numbers of members $\sigma_1,\dots,\sigma_n$ of Σ is contained in $\bigcup_{i=1}^{n} S(\sigma_i)$. If for some $\sigma \in \Sigma$ the set $S(\sigma)$ is compact, then the intersection

$$\bigcap_{\sigma \in \Delta} S(\sigma)$$

is non-empty.

PROOF: First of all, we will show that for any finite number m of members $\sigma_1,\dots,\sigma_m$ of set Σ, the intersection $\bigcap_{i=1}^{m} S(\sigma_i)$ is non-empty. Let $\sigma_1,\dots,\sigma_m$ be m arbitrary points of Σ and $M = \{1,\dots,m\}$. Let T be an $(m-1)$-simplex in an m-dimensional euclidean space, whose corresponding vertices are $t_1,\dots,t_m$. Define the continuous function

$$\varphi : T \to \Delta$$

given by

$$\varphi\left(\sum_{i=1}^{m} \alpha_i t_i\right) = \sum_{i=1}^{m} \alpha_i \sigma_i$$

for any convex combination of the vertices of T, that is, for $i \in M$, $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. This function determines the m closed subsets $K_i = \varphi^{-1} S(\sigma_i) \subseteq T$, with $i \in M$. Therefore, since for any finite subset $N \subseteq M$ and any convex combination we have

$$\sum_{i \in M} \beta_i \sigma_i \subseteq \bigcup_{i \in M} S(\sigma_i),$$

we have in general

$$\sum_{i \in M} \beta_i t_i \in \varphi^{-1}\left(\sum_{i \in M} \beta_i \sigma_i\right) \subseteq \bigcup_{i \in M} K_i \subseteq T.$$

Thus, the simplex spanned by the vertices t_i with $i \in M$ is contained in the union of set K_i with $i \in M$. Therefore the requirements of the Knaster-Kuratowski-Mazukiewicz's theorem applied to the simplex T, whose induced topology is euclidean, together with the sets K_i are satisfied. So, the intersection $\bigcap_{i \in M} K_i$ is non-empty. This fact implies also that the intersection

$$\bigcap_{i \in M} S_i(\sigma_i)$$

is non-empty. Now, by using the existence of a compact set $F(\sigma)$ for at least one $\sigma\epsilon\Delta$, the assertion that the intersection

$$\bigcap_{\sigma\epsilon\Delta} S(\sigma)$$

is non-empty, results immediately. (Q.E.D.)

Using this result one can now formulate the following theorem introduced in [9], which is a straightforward extension of the result given in theorem III, 21 considered in [4] by Fan.

THEOREM III.20: Let $\Sigma_1,\dots, \Sigma_n$ be compact convex sets each in a linear Hausdorff space and for each $i\epsilon N = \{1,\dots,n\}$ let $h(i)$ be a subset of N. Given n subsets $S_1,\dots,S_n$ of the product space $\Sigma = \underset{i\epsilon N}{X} \Sigma_i$, such that for each $i\epsilon N$ and each $\sigma\epsilon\Sigma$ the cylinder

$$S_i(\sigma) = \{\tau\epsilon\Sigma: (\tau_{h(i)},\sigma_{N-h(i)}) \in S_i\}$$

is convex and the cylinder

$$S^i(\sigma) = \{\tau\epsilon\Sigma: (\sigma_{h(i)},\tau_{N-h(i)}) \in S_i\}$$

is open. If for each $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$(\tau_{h(i)},\sigma_{N-h(i)}) \in S_i$$

for all $i\epsilon N$, then the intersection

$$\bigcap_{i\epsilon N} S_i$$

is non-empty.

PROOF: First of all, let us consider some simple facts. Suppose that the set $h(i)$ corresponding to $i\epsilon N$ is empty, then for an arbitrary point $\sigma\epsilon\Sigma$, the cylinders have the following forms:

$$S_i(\sigma) = \{\tau\epsilon\Sigma: \sigma\epsilon S_i\} = \begin{cases} \Sigma & \text{if } \sigma\epsilon S_i \\ \emptyset & \text{if } \sigma\notin S_i \end{cases}$$

and

$$S^i(\sigma) = \{\tau\epsilon\Sigma: \ \tau\epsilon S_i\} = S_i \ .$$

Thus, under the latter condition one has for all $\sigma\epsilon\Sigma$

$$S_i(\sigma) = S^i(\sigma) = S_i = \Sigma \ .$$

On the other hand, if $h(i)$ is the same set N, then for an arbitrary point $\sigma\epsilon\Sigma$, the cylinders are expressed by

$$S_i(\sigma) = \{\tau\epsilon\Sigma: \ \tau \epsilon S_i\} = S_i$$

and

$$S^i(\sigma) = \{\tau\epsilon\Sigma: \ \sigma\epsilon S_i\} = \begin{cases} \Sigma & \text{if } \sigma\epsilon S_i \\ \emptyset & \text{if } \sigma\notin S_i \end{cases} .$$

Consider the set $P \subseteq N$ of all $i\epsilon N$ with $h(i)$ empty, and the set $Q \subseteq N$ of all $i\epsilon N$ for which the set $h(i)$ coincides with N and finally $R = N-(P\cup Q)$ the set of $i\epsilon N$ with both sets $h(i)$ and $N-h(i)$ non-empty.

Let us define for each point $\sigma\epsilon\Sigma$ the set $A(\sigma)$ in the product space Σ, given by the complement of the intersection of cylinders $S^i(\sigma)$:

$$A(\sigma) = c \ [\bigcap_{i\epsilon N} S^i(\sigma)]$$

which by the properties of the cylinders, is compact.

Now, by the last condition, for $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that $(\tau_{h(i)}, \sigma_{N-h(i)}) \epsilon S_i$ for $i\epsilon R$; $\sigma\epsilon S_i = \Sigma$ for $i\epsilon P$ and $\tau\epsilon S_i$ for $i\epsilon Q$. Thus, $S^i(\sigma) = \Sigma$ for $i\epsilon Q$ and therefore obviously $\tau\epsilon S^i(\sigma)$ for $i\epsilon N$. This implies that the intersection

$$\bigcap_{\sigma\epsilon\Sigma} A(\sigma)$$

is empty.

Then, a direct application of the contrapositive result of Lemma III.19 to the sets $A(\sigma)$ with $\sigma\epsilon\Sigma$ determines the existence of a point

$$\tau = \sum_{j=1}^{m} \alpha_j \sigma(j)$$

a convex combination of m points $\sigma(1),\ldots,\sigma(m)$, which does not belong to the union set $\bigcup_{j=1}^{m} A(\sigma(j))$.

Hence, for each $j \in \{1,\ldots,m\}$ and each $i \in N$, one has $\tau \in S^i(\sigma(j))$, that is, $(\sigma_{h(i)}(j) , \tau_{N-h(i)}) \in S_i$ for $i \in R$; $\tau \in S^i(\sigma(j)) = \Sigma$ for $i \in P$ and $\tau \in S^i(\sigma(j)) = \Sigma$ for $i \in Q$. Then, for $i \in Q$ $\sigma(j) \in S_i$ for all $j \in \{1,\ldots,m\}$.

By virtue of these relations, of the following assertion holds true: for each $i \in N$ and each $j \in \{1,\ldots,m\}$: $\sigma(j) \in S_i(\tau)$; and consequently from the convexity of cylinders $S_i(\sigma)$:

$$\tau = \sum_{j=1}^{m} \alpha_j \sigma(j) \in S_i(\tau)$$

for all $i \in N$. This implies that

$$\tau \in \bigcap_{i \in N} S_i ,$$

which proves the theorem. (Q.E.D.)

A special case of the above theorem which constitutes a very useful result has been established by Fan in [4]. This arises when the set $h(i) = \{i\}$ for all $i \in N$. The precise formulation of this is given in the following theorem.

<u>THEOREM III.21:</u> Let $\Sigma_1,\ldots,\Sigma_n$ be compact convex sets each in a linear Hausdorff space. Given n subsets $S_1,\ldots,S_n$ of Σ , such that for each point $\sigma \in \Sigma$ and each $i \in N = \{1,\ldots,n\}$ the cylinder

$$S_i(\sigma) = \{\tau \in \Sigma: (\tau_i, \sigma_{N-\{i\}}) \in S_i\}$$

is non-empty and convex, and the cylinder

$$S^i(\sigma) = \{\tau \in \Sigma: (\sigma_i, \tau_{N-\{i\}}) \in S_i \}$$

is open. Then, the intersection

$$\bigcap_{i \in N} S_i$$

is non-empty.

<u>PROOF:</u> Since for each point $\sigma\epsilon\Sigma$ and $i\epsilon N$, in the cylinder $S_i(\sigma)$ we can choose an $\tau(i)\epsilon S_i(\sigma)$ for each $i\epsilon N$. Thus, for any point $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that $(\tau_i, \sigma_{N-\{i\}})\epsilon S_i$ for all $i\epsilon N$. Thus, for any point $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that $(\tau_i, \sigma_{N-\{i\}})\epsilon S_i$ for all $i\epsilon N$, namely, $\tau = (\tau_1(1), \ldots, \tau_n(n))$. Consequently, all the requirements of the preceding theorem applied to the sets $S_1, \ldots, S_n$ with $h(i) = \{i\}$ for all $i\epsilon N$, are satisfied and we have the desired result. (Q.E.D.)

Once, having the results just considered in this section, we are able to extend the most important theorems related to games given in the previous section. Before going into details of these new formulations, the following important concepts should be recalled: a real function A defined on a Hausdorff space Σ is said to be <u>lower semicontinuous</u> on Σ, if for each real number λ, the set

$$\{\sigma\epsilon\Sigma: \ A(\sigma) > \lambda\}$$

is open. Analogously, A is called <u>uppersemicontinuous</u> if the set

$$\{\sigma\epsilon\Sigma: \ A(\sigma) < \lambda\}$$

is open. A real function on Σ is continuous if and only if it is both lower and upper-semicontinuous.

Directly from the definitions, one obtains that if all A_r with $r\epsilon R$ are lower semicontinuous real functions defined on Σ, then, the function

$$G(\sigma) = \sup_{r\epsilon R} A_r(\sigma)$$

is also lower semicontinuous. Similarly, if the functions A_r with $r\epsilon R$ are upper semicontinuous, then the function

$$F(\sigma) = \inf_{r\epsilon R} A_r(\sigma)$$

is also upper-semicontinuous. Furthermore, an upper-semicontinuous function on a compact set attains its supreme value and for a lower semicontinuous there exists the minimum. Analogously, one can reformulate the corresponding results for upper and lower semicontinuous functions. Indeed by noting that a function is upper semicontinuous if and only if its negative is lower semicontinuous, the analogous results are immediate.

A real function A defined on a convex set Σ of a linear topological space is said to be <u>quasi-concave</u> on Σ, if for any real number λ the set

$$\{\sigma\epsilon\Sigma: \ A(\sigma) > \lambda\}$$

is convex. Similarly, it is called <u>quasi-convex</u> on Σ if the set

$$\{\sigma\epsilon\Sigma: \ A(\sigma) < \lambda\}$$

is convex. Obviously, a convex function is quasi-convex and a concave function is quasi-concave.

Using these concepts, one can now formulate the following theorem.

<u>THEOREM III.22:</u> Let $\Sigma_1,\ldots,\Sigma_n$ be non-empty, conpact and convex sets each in a linear Hausdorff space, and let $A_1,\ldots,A_n$ be real functions defined on the product space $\Sigma = \overset{n}{\underset{i=1}{X}} \Sigma_i$, such that, for each $i\epsilon N = \{1,\ldots,n\}$ and fixed $\sigma_{h(i)}\epsilon\,\Sigma_{h(i)}$, the function $A_i(\ \cdot\ ,\ \cdot\ )$ is lower semicontinuous in the variable $\sigma_{N-h(i)}\epsilon\,\Sigma_{N-h(i)}$ and it is quasi-concave in the variable $\sigma_{h(i)}\epsilon\,\Sigma_{h(i)}$ for fixed $\sigma_{N-h(i)}\epsilon\Sigma_{N-h(i)}$. If, given the vector $\lambda = (\lambda_1,\ldots,\lambda_n)$, for each $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$A_i(\tau_{h(i)}, \sigma_{N-h(i)}) > \lambda_i$$

for all $i\epsilon N$, then, there exists a point $\bar{\sigma}\epsilon\Sigma$ such that

$$A_i(\bar{\sigma}_{h(i)},\bar{\sigma}_{N-h(i)}) > \lambda_i$$

for all $i\epsilon N$.

PROOF: For $i \in N$, consider the set

$$S_i = \{\sigma \epsilon \Sigma: \; A_i(\sigma_{h(i)}, \sigma_{N-h(i)}) > \lambda_i\} .$$

Then, on the one hand, the cylinders

$$S_i(\sigma) = \{\tau \epsilon \Sigma: \; A_i(\tau_{h(i)}, \sigma_{N-h(i)}) > \lambda_i\}$$

are convex, since the functions A_i are quasi-concave in the variable $\sigma_{h(i)}$. On the other hand, the cylinders

$$S^i(\sigma) = \{\tau \epsilon \Sigma: \; A_i(\sigma_{h(i)}, \tau_{N-h(i)}) > \lambda_i\}$$

are open because the functions A_i are lower semicontinuous in the variable $\sigma_{N-h(i)}$. Furthermore, for each point σ in the product space Σ there is another $\tau \epsilon \Sigma$ such that

$$(\tau_{h(i)}, \sigma_{N-h(i)}) \in S_i$$

for all $i \epsilon N$. Then, theorem III.20 assures the existence of a point $\bar{\sigma} \epsilon \Sigma$, a member of all the sets S_i. Such a point satisfies the theorem. (Q.E.D.)

As a consequence of this result, we, now formulate following Fan [4] the following general minimax theorem due to Sion [17].

THEOREM III.23: Let $\Gamma = \{\Sigma_1, \Sigma_2; A\}$ be a zero-sum two-person game, such that the strategy sets are non-empty, compact and convex in a linear Hausdorff space, and the payoff function A is lower semicontinuous and quasi-convex with respect to the variable $\sigma_2 \epsilon \Sigma_2$ for fixed $\sigma_1 \epsilon \Sigma_1$, and it is upper semicontinuous and quasi-convex in the variable $\sigma_1 \epsilon \Sigma_1$ for fixed $\sigma_2 \epsilon \Sigma_2$. Then

$$\max_{s_1 \epsilon \Sigma_1} \; \min_{s_2 \epsilon \Sigma_2} A(s_1, s_2) = \min_{s_2 \epsilon \Sigma_2} \; \max_{s_1 \epsilon \Sigma_1} A(s_1, s_2) .$$

PROOF: Given a real number $\delta > 0$ consider the vector

$$\lambda_\delta = \left(\min_{s_2 \in \Sigma_2} \max_{s_1 \in \Sigma_1} A(s_1,s_2) - \delta , \ - \max_{s_1 \in \Sigma_1} \min_{s_2 \in \Sigma_2} A(s_1,s_2) - \delta \right) ,$$

which is well defined. Indeed, because the payoff function A is upper semicontinuous in the variable $\sigma_1 \in \Sigma_1$, and it is lower semicontinuous in the variable $\sigma_2 \in \Sigma_2$, then on the one hand the values

$$\max_{s_1 \in \Sigma_1} A(s_1,\sigma_2) \quad \text{and} \quad \min_{s_2 \in \Sigma_2} A(\sigma_1,s_2)$$

are attained. On the other hand, by what has been indicated for the maximin and minimum function of a family of lower and upper semicontinuous function, then the functions whose expressions have been just considered are lower and upper semicontinuous respectively. Thus, the maximin and minimax values exist.

Then, by taking $h(i) = \{i\}$ for $i \in N = \{1,2\}$ with $A_1 = A$ and $A_2 = -A$, we see that all the requirements of previous theorem are satisfied for λ_δ, and therefore the existence of a point $\sigma_\delta = (\sigma_1^\delta , \sigma_2^\delta)$ such that

$$A(\sigma_1^\delta,\sigma_2) > \min_{s_2 \in \Sigma_2} \max_{s_1 \in \Sigma_1} A(s_1,s_2) - \delta$$

$$< \max_{s_1 \in \Sigma_1} \min_{s_2 \in \Sigma_2} A(s_1,s_2) + \delta ,$$

is assured. From here, one has the inequality

$$\min_{s_2 \in \Sigma_2} \max_{s_1 \in \Sigma_1} A(s_1,s_2) \leq \max_{s_1 \in \Sigma_1} \min_{s_2 \in \Sigma_2} A(s_1,s_2)$$

where, only the equality sign must hold since the converse inequality is always true. (Q.E.D.)

Unfortunately, we now are not able to extend this theorem for n-person games, with the generality expressed in it.

The following theorem concerning e-simple positive equilibrium points, is a result due to Fan [5], which is a generalization of theorem III.3.

> THEOREM III.24: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function e, such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a linear Hausdorff space and his payoff function A_i is continuous in the product variable $\sigma\in\Sigma$. Then, the game Γ_e has an e-positive simple equilibrium point.

PROOF: For each player $i\in N$ and a number $\delta > 0$, consider the set

$$S_{\delta,i} = \{\sigma\in\Sigma: A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) > \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)}\sigma_{f(i)})-\delta\}.$$

Define $h(i) = \{i\}$. Then, because the payoff function of player A_i is a continuous function on the product space, the function $\max_{s_i\in\Sigma_i} A_i(s_i, \cdot \;, \; \cdot)$ is also a continuous function with respect to the product variable $\sigma\in\Sigma$. Thus, the cylinder

$$S^i_\delta(\sigma) = \{\tau\in\Sigma: A_i(\sigma_i,\tau_{e(i)},\tau_{f(i)}) > \max_{s_i\in\Sigma_i} A_i(s_i,\tau_{e(i)},\tau_{f(i)}) - \delta\}$$

is a non-empty and open set in the product space. On the other hand, because the payoff function A_i of player $i\in N$ is a quasi-concave function in the variable $\sigma_i\in\Sigma_i$, the cylinder

$$S_{\delta,i}(\sigma) = \{\tau\in\Sigma: A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) > \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) - \delta\}$$

is convex. Therefore, by a direct application of Theorem III.21 to the sets S_i, since all the requirements are satisfied, gives that the intersection

$$\bigcap_{i\epsilon N} S_{\delta,i} \subseteq \Sigma \quad \text{for any } \delta > 0$$

is non-empty.

Now for each real number $\delta > 0$ let the non-empty set $\bar{S}_{\delta,i}$ be the closure of set $S_{\delta,i}$ in the product space Σ. Then, the sets $S_\delta = \bigcap_{i\epsilon N} \bar{S}_{\delta,i}$ have the finite intersection property, and therefore since Σ is compact, there exists a point $\bar{\sigma}$ such that

$$\bar{\sigma} \in \bigcap_{\delta>0} \bigcap_{i\epsilon N} \bar{S}_{\delta,i} .$$

Such a point satisfies

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\epsilon N$. Hence, it is an $\underline{e}$-positive simple equilibrium point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

The most important reason, for which we could not extend in a simple way the above result with a generality comparable with that expressed in theorem III.23, is due to the fact that the sections $S^i_\delta(\sigma)$ should be open, which cannot be obtained immediately if the payoff functions are lower or upper semicontinuous.

An immediate consequence of this result is concerning the existence of $\underline{e}_m$-simple stable points as an extension of theorem III.11 which is formulated as follows:

<u>THEOREM III.25:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n;\ A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space and his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, and F_i is quasi-concave with respect to $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. Then, the game $\Gamma_{\underline{e}}$ has an $\underline{e}_m$-simple stable point.

PROOF: Consider the n-person game $\Gamma' = \{\Sigma_1,\ldots,\Sigma_n; F_1,\ldots,F_n\}$, which satisfies all the conditions of the previous theorem, since the payoff function F_i of player $i\epsilon N$ is a continuous function on the product space. Then, a very simple equilibrium point of game Γ' exists. Such a point is an $\underline{e}_m$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

In a similar manner, we now formulate a result concerning the existance of $\underline{e}$-negative simple equilibrium points, which will be seen to be an extension of theorem III.6.

THEOREM III.26: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space and his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, and quasi-convex with respect to the variable $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)})\epsilon\ \Sigma_i \times \Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$A_i(\sigma_i, \tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\epsilon N$, then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-negative equilibrium point.

PROOF: For each player $i\epsilon N$ and real number $\delta > 0$, consider the set

$$S_{\delta,i} = \{\sigma\epsilon\Sigma:\ A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta\ \} .$$

Let $h(i) = e(i)$. Then, the section

$$S^i_\delta(\sigma) = \{\tau\epsilon\Sigma:\ A_i(\tau_i,\sigma_{e(i)},\tau_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\tau_i,s_{e(i)},\tau_{f(i)}) + \delta\ \}$$

is open in the product space Σ , since the payoff function A_i of player $i \in N$ is continuous in the product variable $\sigma \in \Sigma$, which implies also the continuity of the function $\min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\cdot, s_{e(i)}, \cdot)$ with respect to $\sigma \in \Sigma$. On the other hand, the section

$$S_{\delta,i}(\sigma) = \{\tau \in \Sigma: A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}) + \delta\}$$

is convex, by virtue of the quasi-convexity of payoff function A_i with respect to the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$. Finally, by the last condition, for any point $\sigma \in \Sigma$ there is a joint strategy $\tau \in \Sigma$ such that

$$(\tau_{e(i)}, \sigma_{N-e(i)}) \in S_{\delta,i}$$

for all $i \in N$. Thus, all the requirements of theorem III.20 are satisfied. Then, the intersection

$$\bigcap_{i \in N} S_{\delta,i} \qquad \text{for any } \delta > 0$$

is non-empty.

Now, defining for each $\delta > 0$ the set $\bar{S}_{\delta,i}$ as the closure in the product space Σ of set $S_{\delta,i}$, we have for the family of sets $S_\delta = \bigcap_{i \in N} S_{\delta,i}$ the finite intersection property. Hence, because Σ is compact, there exists a point $\bar{\sigma} \in \Sigma$ belonging to the following intersection

$$\bar{\sigma} \in \bigcap_{\delta > 0} \bigcap_{i \in N} \bar{S}_{\delta,i} .$$

At this point, one has

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all $i \in N$. Then, such a point is an $\underline{e}$-negative simple equilibrium point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

We point out that the last condition in this theorem is analogous to the corresponding requirement of theorem III.6.

From here, we can characterize immediately the $\underline{e}^m$-simple stable points.

<u>THEOREM III.27:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space and his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, and g_i is quasi-convex in the variable $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. If for each real number $\delta > 0$ and for each point $\sigma\epsilon\Sigma$ there is a joint strategy $\tau\epsilon\Sigma$ such that

$$G_i(\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\epsilon N$, then the game $\Gamma_{\underline{e}}$ has an $\underline{e}^m$-simple stable point.

<u>PROOF:</u> Consider the n-person game $\Gamma'' = \{\Sigma_1,\ldots k\Sigma_n; g_1,\ldots g_n\}$, which has all the payoff functions continuous. Thus, all the requirements of the previous theorem applied to game Γ'' are satisfied. The, there exists an $\underline{e}$-negative simple equilibrium point for game Γ''. Such a point is an $\underline{e}^m$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

This result generalizes the corresponding theorem III.13.

By using the same technique, we now extend theorem III.15 regarding $\underline{e}$-simple stable points.

<u>THEOREM III.28:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space,

his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$ and G_i quasi-convex with respect to $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$F_i(\tau_i,\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) - \delta$$

and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)} + \delta$$

for all $i\epsilon N$, then, the game $\Gamma_{\underline{e}}$ has an <u>e</u>-simple stable point.

<u>PROOF:</u> For each player $i\epsilon N$ and a real number $\delta > 0$, consider the sets

$$S_{F,\delta,i} = \{\sigma\epsilon\Sigma \colon F_i(\sigma_i,\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) - \delta\ \}$$

$$S_{G,\delta,i} = \{\sigma\epsilon\Sigma \colon G_i(\sigma_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta\ ,$$

and the intersection

$$S_{\delta,i} = S_{F,\delta,i} \cap S_{G,\delta,i}(\sigma)$$

$$= \{\tau\epsilon\Sigma \colon F_i(\tau_i,\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} F_i(\delta_i,\sigma_{f(i)}) - \delta\}$$

$$\cap\ \{\tau\epsilon\Sigma \colon G_i(\tau_{e(i)},\sigma_{f(i)}) \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta\}$$

is convex. Finally, by the last condition, for any point $\sigma\epsilon\Sigma$ there is a joint strategy $\tau\epsilon\Sigma$ such that

$$(\tau_{\{i\}\cup e(i)},\sigma_{f(i)}) \in S_{\delta,i}$$

for all $i \in N$. Thus, all the requirements of theorem III.20 applied to the sets $S_{\delta,i}$ are completely satisfied, and therefore, the non-emptyness of the intersection

$$\bigcap_{i \in N} S_{\delta,i} \qquad \text{for any } \delta > 0$$

is assured.

From here, by defining for each $\delta > 0$ the set $\bar{S}_{\delta,i}$ as the closure in the product space Σ of set $S_{\delta,i}$, one can easily show the finite intersection property for the family of sets $S_{\delta} = \bigcap_{i \in N} \bar{S}_{\delta,i}$. Hence, by the compactness of Σ, there exists a point $\bar{\sigma} \in \Sigma$ belonging to the intersection

$$\bar{\sigma} \in \bigcap_{\delta > 0} \bigcap_{i \in N} \bar{S}_{\delta,i} .$$

On this point we have

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$$

$$= \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all $i \in N$, that is, it is an e-simple stable point of game Γ_e. (Q.E.D.)

With this result, we are able to obtain the following characterization of e-simple saddle points, which is a generalization of theorem III.17.

THEOREM III.29: Let $\Gamma_e = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be an n-person game with simple structure function e, such that the strategy set Σ_i of player $i \in N$ in non-empty, compact and convex in a linear Hausdorff space, his payoff function A_i is continuous in the product variable $\sigma \in \Sigma$, quasi-concave with respect to $\sigma_i \in \Sigma_i$ for fixed $(\sigma_{e(i)}, \sigma_{f(i)}) \in \Sigma_{e(i)} \times \Sigma_{f(i)}$ and quasi-convex with respect to $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_i, \sigma_{f(i)}) \in \Sigma_i \times \Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma \in \Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$A_i(\tau_i,\tau_{e(i)},\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} A_i(s_i,\tau_{e(i)},\sigma_{f(i)}) - \delta/2$$

$$< \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\tau_i,s_{e(i)}\sigma_{f(i)}) + \delta/2$$

for all $i\epsilon N$, then, the game $\Gamma_{\underline{e}}$ has an e-simple saddle point.

PROOF: First of all, we will show that under the conditions of payoff function, the functions F_i and G_i are quasi-concave and quasi-convex respectively. Assume that the function

$$F_i(\sigma_i,\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

were not quasi-concave with respect to $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. Then, there would be $\bar{\sigma}_{f(i)}\epsilon\Sigma_{f(i)}$, and a real number λ such that the set

$$F_\lambda = \{\tau_i\epsilon\Sigma_i : F_i(\tau_i,\bar{\sigma}_{f(i)}) > \lambda\}$$

is not convex. Hence, there exist two points τ_i^1 on τ_i^2 belonging to F_λ and a real number $\mu\epsilon[0,1]$ such that

$$F_i(\lambda\tau_i^1 + (1-\lambda)\tau_i^2, \bar{\sigma}_{f(i)}) \leq \lambda .$$

On the other hand for τ_i^1 and τ_i^2 we have

$$A_i(\tau_i^1,s_{e(i)},\bar{\sigma}_{f(i)}) > \quad \text{and} \quad A_i(\tau_i^2, s_{e(i)},\bar{\sigma}_{f(i)}) > \lambda$$

for all $s_{e(i)}\epsilon\Sigma_{e(i)}$, and in particular for the point $\bar{s}_{e(i)}\epsilon\Sigma_{e(i)}$ for which

$$F_i(\lambda\tau_i^1 + (1-\lambda)\tau_i^2,\sigma_{f(i)}) = A_i(\lambda\tau_i^1 + (1-\lambda)\tau_i^2,\bar{s}_{e(i)},\bar{\sigma}_{f(i)}) \leq \lambda$$

which is impossible by virtue of the quasi-concavity of payoff function A_i in the variable $\sigma_i \epsilon \Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)}) \epsilon \Sigma_{e(i)} \times \Sigma_{f(i)}$. Then, the function F_i is quasi-concave in $\sigma_i \epsilon \Sigma_i$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$.

In a similar way, assume that the function

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i \epsilon \Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

were not quasi-convex in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$. Then for some $\bar{\sigma}_{f(i)} \epsilon \Sigma_{f(i)}$ and a λ the set

$$G_\lambda = \{\tau_{e(i)} \epsilon \Sigma_{e(i)}: \ G_i(\tau_{e(i)},\sigma_{f(i)}) < \lambda \}$$

would be not convex. Then, for some two points $\tau^1_{e(i)}$ and $\tau^2_{e(i)}$ in G_λ and some $\mu \epsilon [0,1]$

$$G_i(\mu\tau^1_{e(i)} + (1-\mu)\,\tau^2_{e(i)},\bar{\sigma}_{f(i)}) \geq \lambda \ ,$$

in particular for a $\bar{s}_i \in \Sigma_i$ where the minimum is attained:

$$G_i(\tau^1_{e(i)},\bar{\sigma}_{f(i)}) = A_i(\bar{s}_i, \mu\tau^1_{e(i)} + (1-\mu)\,\tau^2_{e(i)},\bar{\sigma}_{f(i)}) \geq \lambda \, .$$

But, for the points $\tau^1_{e(i)}$ and $\tau^2_{e(i)}$ we have

$$A(s_i,\tau^1_{e(i)},\bar{\sigma}_{f(i)}) < \lambda \quad \text{and} \quad A(s_i,\tau^2_{e(i)},\bar{\sigma}_{f(i)}) < \lambda$$

for all $s_i \epsilon \Sigma_i$, which is a contradiction, since the payoff function A_i is quasi-convex in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)}) \epsilon \Sigma_i \times \Sigma_{f(i)}$. Then, the function G_i is quasi-convex in the variable $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$. Thus, the game $\underline{\Gamma_e}$ satisfies the first conditions of previous theorem.

Now, we will show that the last requirement is also verified. From the latter condition, for a given $\delta > 0$ and any $\sigma \epsilon \Sigma$ there is an $\tau \epsilon \Sigma$:

$$A_i(\tau_i,\tau_{e(i)},\sigma_{f(i)}) > G_i(\tau_{e(i)},\sigma_{f(i)}) - \delta/2 \geq \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

$$< F_i(\tau_i,\sigma_{f(i)}) + \delta/2 \leq \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)}) + \delta/2$$

for all $i\in N$, and therefore, from

$$\max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)}) \leq \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

we obtain

$$F_i(\tau_i,\sigma_{f(i)}) > \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)}) - \delta$$

and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta ,$$

for all $i\in N$. Thus, the last condition of theorem III.28 is also satisfied. Then, the existence of an e-simple stable point $\bar{\sigma}\in\Sigma$ of game $\Gamma_{\underline{e}}$ is quaranteed:

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N$.

On the other hand, by taking the strategy $\bar{\sigma}_{f(i)}\in\Sigma_{f(i)}$, we have for each player $i\in N$ the $\bar{\sigma}_{f(i)}$ - associated zero-sum two-person game. This game satisfies all the requirements of theorem III.23, since the payoff function is quasi-concave in $\sigma_i\in\Sigma_i$ and quasi-convex with respect to $\sigma_{e(i)}\in\Sigma_{e(i)}$. Thus, for each player $i\in N$ in the $\bar{\sigma}_{f(i)}$-associated game the minimax theorem holds, that is

$$\max_{s_i\in\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)}) ,$$

and therefore, we have

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)})$$
$$= G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$$

for all $i \in N$. Thus, such a point is an e-simple saddle point of game Γ_e . (Q.E.D.)

Having the previous results, we can now extend the results of theorem III.20. Thus, with this new simple extension, we are able to get more generality of some already considered results and moreover to formulate a further characterization of e-simple saddle points.

THEOREM III.30: Let $\Sigma_1, \ldots, \Sigma_n$ be compact convex sets each in a linear Hausdorff space and for each $i \in N = \{1, \ldots, n\}$ let $h(i)$ and $h'(i)$ be two subsets of N . Given $2n$ subsets $S_1, \ldots, S_n$ and $T_1, \ldots, T_n$ of product space $\Sigma = \underset{i \in N}{X} \Sigma_i$, such that for each $i \in N$ and each $\sigma \in \Sigma$ the cylinders

$$S_i(\sigma) = \{\tau \in \Sigma: (\tau_{h(i)}, \sigma_{N-h(i)}) \in S_i\}$$

and

$$T_i(\sigma) = \{\tau \in \Sigma: (\tau_{h'(i)}, \sigma_{N-h'(i)}) \in T_i\}$$

are convex and the cylinders

$$S^i(\sigma) = \{\tau \in \Sigma: (\sigma_{h(i)}, \tau_{N-h(i)}) \in S_i\}$$

and

$$T^i(\sigma) = \{\tau \in \Sigma: (\sigma_{h'(i)}, \tau_{N-h'(i)}) \in T_i\}$$

are open. If for each $\sigma \in \Sigma$ there is another point τ in the product space Σ , such that

$$(\tau_{h(i)}, \sigma_{N-h(i)}) \in S_i \quad \text{and} \quad (\tau_{h'(i)}, \sigma_{N-h'(i)}) \in T_i$$

for all $i \in N$, then the intersection

$$\bigcap_{i \in N} (S_i \cap T_i)$$

is non empty.

PROOF: We recall that if $h(i)$ is empty, then under the last condition the sections are

$$S_i(\sigma) = S_i = \Sigma \quad \text{and} \quad S^i(\sigma) = S_i = \Sigma .$$

If $h(i) = N$ then the sections are given by

$$S_i(\sigma) = S_i \quad \text{and} \quad S^i(\sigma) = \begin{cases} \Sigma & \text{if } \sigma \in S_i \\ \emptyset & \text{if } \sigma \notin S_i \end{cases} .$$

Analogously for the corresponding cases of $h'(i)$.

Let $P \subseteq N$ be the set of all $i \in N$ with $h(i)$ empty and let $P' \subseteq N$ be the set of $i \in N$ for which $h'(i)$ is empty. Similarly, let $Q \subseteq N$ and $Q' \subseteq N$ be the sets of $i \in N$ for which $h(i)$ and $h'(i)$ coincide with N, respectively and finally

$$R = N - (P \cup Q) \quad \text{and} \quad R' = N - (P' \cup Q') .$$

Define the set $A(\sigma)$ in the product space, for every $\sigma \in \Sigma$, to be given by the complement of the intersection of sets $S^i(\sigma) \cap T^i(T)$:

$$A(\sigma) = c \left[\bigcap_{i \in N} (S^i(\sigma) \cap T^i(\sigma)) \right],$$

which by the closeness of the sections, is compact. On the other hand, by the last condition, for a $\sigma \in \Sigma$ there is a point in the product space $\tau \in \Sigma$ such that

$$(\tau_{h(i)}, \sigma_{N-h(i)}) \in S_i$$

for all $i \in R$,

$$(\tau_{h'(i)}, \sigma_{N-h'(i)}) \in T_i$$

for all $i \in R'$, $\sigma \in S_i = \Sigma$ for $i \in P$ and $\sigma \in T_i = \Sigma$ for $i \in P'$; $\tau \in S_i$ for $i \in Q$ and $\tau \in T_i$ for $i \in Q'$. This condition implies the non-emptyness of the intersection

$$\bigcap_{\sigma \in \Sigma} A(\sigma) .$$

Now, as an immediate consequence of the contrapositive result of Lemma III.19 applied to the sets $A(\sigma)$ with $\sigma \in \Sigma$, one has guaranteed the existence of a point

$$\tau = \sum_{j=1} \alpha_j \sigma(j)$$

convex combination of m points $\sigma(1),\dots,\sigma(m)$, which does not belong to the union set

$$\bigcup_{j=1}^{m} A(\sigma(j)).$$

Then, for each $j\in\{1,\dots,m\}$ and each $i\in N$, $\tau\in S^i(\sigma(j)) \cap T^i(\sigma(j))$. Hence, $(\sigma_{h(i)}(j), \tau_{N-h(i)}) \in S_i$ for all $i\in R$, $(\sigma_{h'(i)}(j), \tau_{N-h'(i)}) \in T_i$ for all $i\in R'$; $\tau\in S^i(\sigma(j)) = \Sigma$ for every $i\in P$ and $\tau\in T^i(\sigma(j)) = \Sigma$ for every $i\in P'$; $\tau\in S^i(\sigma(j)) = \Sigma$ for $i\in Q$ and $\tau\in S^i(\sigma(j)) = \Sigma$ for $i\in Q'$. From the last condition one deduces, that for $i\in Q$: $\sigma(j)\in S_i$ and for $i\in Q'$: $\sigma(j)\in T_i$ for $j\in\{1,\dots,m\}$. Then, by all these relations we have for each $i\in N$ and each $j\in\{1,\dots,m\}$ the condition

$$\sigma(j) \in S_i(\tau) \cap T_i(\tau)$$

holds, and therefore by virtue of the convexity of sections $S_i(\sigma)$ and $T_i(\sigma)$

$$\tau = \sum_{j=1}^{m} \alpha_j \sigma(j) \in S_i(\tau) \cap T_i(\tau)$$

for all $i\in N$. From here, we obtain that the point $\tau\in\Sigma$ is a member of the intersection $\bigcap_{i\in N} (S_i \cap T_i)$. This implies the validity of our theorem. (Q.E.D.)

We regard theorem III.20 as a particular case of the above result, which occurrs when for each $i\in N$ the sets $h(i)$ and $h'(i)$ are equal and the sets S_i and T_i coincide.

By applying the result expressed in the preceding theorem, we now in the following formulation extend the theorem III.22.

<u>THEOREM III.31:</u> Let $\Sigma_1,\dots,\Sigma_n$ be non-empty, compact and convex sets each in a linear Hausdorff space, and let $A_1,\dots,A_n$; $B_1,\dots,B_n$ be 2n

real functions defined on the product space $\Sigma = \underset{i\in N}{X} \Sigma_i$, such that, for each $i\epsilon N = \{1,\ldots,n\}$ and given subsets $h(i)$ and $h'(i)$ of N and fixed $\sigma_{h(i)}\epsilon\Sigma_{h(i)}$ the function A_i is lower semicontinuous in the variable $\sigma_{N-h(i)}\epsilon\Sigma_{N-h(i)}$ and it is quasi-concave in the variable $\sigma_{h(i)}\epsilon\Sigma_{h(i)}$ for fixed $\sigma_{N-h(i)}\epsilon\Sigma_{N-h(i)}$. For each $i\epsilon N$ and each $\sigma_{h'(i)}\epsilon\Sigma_{h'(i)}$, the function B_i is lower semicontinuous in the variable $\sigma_{N-h'(i)}\epsilon\Sigma_{N-h'(i)}$ and it is quasi-concave in the variable $\sigma_{h'(i)}\epsilon\Sigma_{h'(i)}$ for fixed $\sigma_{N-h'(i)}\epsilon\Sigma_{N-h'(i)}$. If, given the vectors $\lambda = (\lambda_1,\ldots\lambda_n)$ and $\lambda^1 = \lambda^1_1,\ldots,\lambda^1_n)$, for each $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$A_i(\tau_{h(i)},\sigma_{N-h(i)}) > \lambda_i \quad \text{and} \quad B_i(\tau_{h'(i)},\sigma_{N-h'(i)}) > \lambda^1_i$$

for all $i\epsilon N$, then, there exists a point $\bar{\sigma}\epsilon\Sigma$ such that

$$A_i(\bar{\sigma}_{h(i)},\bar{\sigma}_{N-h(i)}) > \lambda_i \quad \text{and} \quad B_i(\bar{\sigma}_{h'(i)},\bar{\sigma}_{N-h'(i)}) > \lambda^1_i$$

for all $i\epsilon N$.

<u>PROOF:</u> For $i\epsilon N$, define the sets

$$S_i = \{\sigma\epsilon\Sigma: \ A_i(\sigma_{h(i)},\sigma_{N-h(i)}) > \lambda_i \}$$

and

$$T_i = \{\sigma\epsilon\Sigma: \ B_i(\sigma_{h'(i)},\sigma_{N-h'(i)}) > \lambda^1_i \} \quad .$$

From here on the one hand, the cylinders

$$S_i(\sigma) = \{\tau\epsilon\Sigma: A_i(\tau_{h(i)},\sigma_{N-h(i)}) > \lambda_i \}$$

and

$$T_i(\sigma) = \{\tau\epsilon\Sigma: B_i(\tau_{h'(i)},\sigma_{N-h'(i)}) > \lambda^1_i \}$$

are both convex, since the functions A_i and B_i are quasi-concave in the variables $\sigma_{h(i)}\epsilon\Sigma_{h(i)}$ and $\sigma_{h'(i)}\epsilon\Sigma_{h'(i)}$, respectively. On the other hand, the cylinders

$$S^i(\sigma) = \{\tau\epsilon\Sigma:\ A_i(\sigma_{h(i)},\tau_{N-h(i)}) > \lambda_i\}$$

and

$$T^i(\sigma) = \{\tau\epsilon\Sigma:\ B_i(\sigma_{h'(i)},\tau_{N-h'(i)}) > \lambda_i^1\}$$

are open because the functions A_i and B_i are lower-semicontinuous with respect to the variables $\sigma_{N-h(i)}\epsilon\Sigma_{N-h(i)}$ and $\sigma_{N-h'(i)}\epsilon\Sigma_{N-h'(i)}$, respectively. Moveover, by the last condition for each point $\sigma\epsilon\Sigma$ there is a point τ in the product space Σ such that

$$(\tau_{h(i)},\sigma_{N-h(i)})\epsilon\ S_i \quad \text{and} \quad (\tau_{h'(i)},\sigma_{N-h'(i)}) \ \epsilon\ T_i$$

for all $i\epsilon N$. Thus, all the requirements of previous theorem applied to the sets S_i and T_i, are satisfied, and therefore the existence of a point $\bar{\sigma}\epsilon\Sigma$ belonging to all the sets $S_i \cap T_i$ is guaranteed. Such a point fulfills the theorem. (Q.E.D.)

This result permits us to extend a particular case of Sion's minimax result given in theorem III.23.

<u>THEOREM III.32:</u> Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n;\ A_1,\ldots,A_n\}$ be an n-person game, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space, his payoff function A_i is continuous in the product variable, quasi-concave in the variable $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{N-\{i\}}\epsilon\Sigma_{N-\{i\}}$, and quasi-convex with respect to the variable $\sigma_{N-\{i\}}\epsilon\Sigma_{N-\{i\}}$ for fixed $\sigma_i\epsilon\Sigma_i$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$A_i(\tau_i,\sigma_{N-\{i\}}) > \min_{s_{N-\{i\}}\epsilon\Sigma_{N-\{i\}}} \ \max_{s_1\epsilon\Sigma_1} A_i(s_i,s_{N-\{i\}}) - \delta$$

and

$$A_i(\sigma_i,\tau_{N-\{i\}}) < \max_{s_i\epsilon\Sigma_i} \ \min_{s_{N-\{i\}}\epsilon\Sigma_{N-\{i\}}} A_i(s_i,s_{N-\{i\}}) + \delta$$

for all $i \in N$. Then, there exists a point $\bar{\sigma} \in \Sigma$ such that

$$A_i(\sigma_i, \sigma_{N-\{i\}}) = \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} \max_{s_i \in \Sigma_i} A_i(s_i, s_{N-\{i\}})$$

$$= \max_{s_i \in \Sigma_i} \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(s_i, s_{N-\{i\}})$$

<u>PROOF:</u> Given a real number $\delta > 0$, consider for player $i \in N$, the following two vectors

$$\lambda_\delta = (\min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} \max_{s_i \in \Sigma_i} A_i(s_i, s_{N-\{i\}}) - \delta$$

and

$$\lambda^1_\delta = (\max_{s_i \in \Sigma_i} \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(s_i, s_{N-\{i\}}) + \delta) .$$

Now, let $h(i) = \{i\}$ and $h'(i) = N-\{i\}$ with $i \in N$ and the functions $A_i = A_i$ and $B_i = -A_i$. Then, all the requirements of theorem III.31 applied to the functions A_i and B_i are completely satisfied. Thus, there exists a point $\sigma^\delta \in \Sigma$ such that

$$A_i(\sigma^\delta_i, \sigma^\delta_{N-\{i\}}) > \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} \max_{s_i \in \Sigma_i} A_i(s_i, s_{N-\{i\}}) - \delta$$

$$< \max_{s_i \in \Sigma_i} \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(s_i, s_{N-\{i\}}) + \delta$$

for all $i \in N$. Now let $\bar{\sigma}$ be a cluster point of the directed system σ^δ with $\delta \to 0$. Then by virtue of the continuity of payoff functions for the joint strategy $\bar{\sigma} \in \Sigma$ we have

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{N-\{i\}}) = \max_{s_i \in \Sigma_i} \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} A_i(s_i, s_{N-\{i\}})$$

$$= \min_{s_{N-\{i\}} \in \Sigma_{N-\{i\}}} \max_{s_i \in \Sigma_i} A_i(s_i, s_{N-\{i\}})$$

for all $i \in N$. (Q.E.D.)

We do note that unfortunately, such a point, characterized in the above theorem, is not necessarily an e-simple saddle point of game Γ_e.

Now a further application of the general result expressed in theorem III.30 of is related with another examination of e-simple saddle points which is obtained in the following simple way:

THEOREM III.33: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n,A_1,\ldots,A_n\}$ be an n-person game with simple structure function e, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space and his payoff function A_i is continuous in the product variable $\sigma\epsilon\Sigma$, quasiconcave in the variable $\sigma_i\epsilon\Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\Sigma_{e(i)}\times\Sigma_{f(i)}$ and quasi-convex with respect to the variable $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $(\sigma_i\sigma_{f(i)})\epsilon\ \Sigma_i\times\Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) - \delta$$

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\epsilon N$, then, the game Γ_e has an e-simple saddle point.

PROOF: For each player $i\epsilon N$ and a real number $\delta > 0$, consider the following two sets

$$S_{\delta,i} = \{\sigma\epsilon\Sigma\colon A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) - \delta\}$$

and

$$T_{\delta,i} = \{\sigma\epsilon\Sigma\colon A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta\}.$$

Let $h(i) = \{i\}$ and $h'(i) = e(i)$ for all $i \in N$. Then, for a joint strategy $\sigma \in \Sigma$ both sections

$$S^i_\delta(\sigma) = \{\tau \in \Sigma: A_i(\sigma_i, \tau_{e(i)}, \tau_{f(i)}) > \max_{s_i \in \Sigma_i} A_i(s_i, \tau_{e(i)}, \tau_{f(i)}) - \delta \}$$

and

$$T^i_\delta(\sigma) = \{\tau \in \Sigma: A_i(\tau_i, \sigma_{e(i)}, \tau_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\tau_i, s_{e(i)}, \tau_{f(i)}) + \delta \}$$

are open, because the payoff function A_i of player $i \in N$ is continuous in the product variable $\sigma \in \Sigma$. This implies also, the continuity of the functions

$$\max_{s_i \in \Sigma_i} A_i(s_i, \cdot \quad \cdot) \quad \text{and} \quad \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\cdot \quad s_{e(i)}, \cdot)$$

with respect to $\sigma \in \Sigma$. On the other hand, both sections

$$S_{\delta,i}(\sigma) = \{\tau \in \Sigma: A_i(\tau_i, \sigma_{e(i)}, \sigma_{f(i)}) > \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)}) - \delta \}$$

and

$$T_{\delta,i}(\sigma) = \{\tau \in \Sigma: A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}) + \delta \}$$

are convex by virtue of the quasi-concavity of payoff function A_i with respect to $\sigma_i \in \Sigma_i$ and the quasi-convexity with respect to the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$. Finally, by the last condition, for any joint strategy $\sigma \in \Sigma$ there is a point τ in the product space Σ such that

$$(\tau_i, \sigma_{N-\{i\}}) \in S_{\delta,i} \quad \text{and} \quad (\tau_{e(i)}, \sigma_{N-\{i\}}) \in T_{\delta,i}$$

for all $i \in N$. Thus, all the conditions of theorem III.30 applied to the sets $S_{\delta,i}$ and $T_{\delta,i}$ are satisfied, and therefore the intersection

$$\bigcap_{i \in N} (S_{\delta,i} \cap T_{\delta,i}) \qquad \text{for any } \delta > 0$$

is non-empty.

Now, by defining for each $\delta > 0$ the set $\bar{S}_{\delta,i} \cap \bar{T}_{\delta,i}$ as the intersection of the closures in the product space Σ of sets $S_{\delta,i}$ and $T_{\delta,i}$ it can be easily seen that the family of sets $R_R = \bigcap_{i\in N} (\bar{S}_{\delta,i} \cap \bar{T}_{\delta,i})$ has the finite intersection property. Hence, because Σ is compact, there exists a point $\bar{\sigma}\in\Sigma$ belonging in the following intersection

$$\bar{\sigma} \in \bigcap_{\delta>0} \bigcap_{i\in N} (\bar{S}_{\delta,i} \cap \bar{T}_{\delta,i}) \ .$$

Such a point satisfies:

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

$$= \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i,\delta_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N$, that is, it is an e-simple saddle point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

This later characterization of e-simple saddle points has the advantage of the characterization given in theorem III.29, i.e., that it is valid even if the simple structure function has all the sets $f(i)$ empty. In this case theorem III.29 has not any value. This situation appears for almost all the other theorems which are concerned with e-simple points.

Although the results obtained in this paragraph using the intersection technique extend all the corresponding theorems of previous sections it is remarkable to note that it is not known whether or not this method includes in a natural way the fixed point procedure; i.e. whether or not the Fan-Glicksberg fixed point theorem can be obtained as a consequence of theorem III.19 and its corollaries. The similarity of these facts has also been pointed out by Fan in [3] .

III.3 e-Simple Points by Maximum Function Procedure

In this section we will deal with the treatment of e-simple points for games defined on compact sets in linear topological spaces. This further extension is built on the base of a result due to Nikaido and Isoda [16], concerning maximum of functions, which has been used to examine the existence of equilibrium points. This new treatment does not generalize completely that which we just examined in the previous section, since its applications can be performed only for games whose payoff functions are convex and concave, but not quasi-convex and quasi-concave. However, the new procedure will allow us to extend the examination to games, which do not have the Hausdorff property for their sets. Thus, even though there is certain similarity between the results expressed in this section and those just examined by the "intersection of sets with convex sections" procedure. Nevertheless, neither do the results obtained here include the other, nor are they included in them.

The fundamental tool of this section, on which our examination is built is the following theorem due to Nikaido-Isoda [16], which uses the Brouwer's fixed point theorem. This last theorem, which is a special case of Kakutani's fixed point, is concerned with simple functions and assures the validity of the assertion: If a function $A: \Sigma \to \Sigma$ on an non-empty, compact and convex set Σ in an euclidean space into itself, is continuous, then, there exists a fixed point: $\bar{\sigma} = A(\bar{\sigma})$.

THEOREM III.35: Let φ be a real function defined on the product space $\Sigma \times \Sigma$, where the set Σ is non-empty, compact and convex in a linear topological space, such that for each $\tau \epsilon \Sigma$ the functions $\varphi(\tau,\sigma)$ and $\varphi(\sigma,\sigma)$ are continuous in the variable $\sigma \epsilon \Sigma$ and the function $\varphi(\sigma,\tau)$

is concave in $\sigma\epsilon\Sigma$. Then, there exists a point $\bar{\sigma}\epsilon\Sigma$ such that

$$\varphi(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi(s,\bar{\sigma}) \quad .$$

PROOF: Suppose that there is not a point having the property just mentioned. Then, for each point $\sigma\epsilon\Sigma$ there is another point $\tau\epsilon\Sigma$ such that

$$\varphi(\sigma,\sigma) < \varphi(\tau,\sigma)$$

Define the following set

$$\theta_\tau = \{\sigma\epsilon\Sigma: \ \varphi(\sigma,\sigma) < \varphi(\tau,\sigma)\}$$

included in the product space Σ . By the continuity of the functions $\varphi(\tau,\sigma)$ and $\varphi(\sigma,\sigma)$ in the variable $\sigma\epsilon\Sigma$ for fixed $\sigma\epsilon\Sigma$, there exist a finite number of points $\tau_1,\ldots,\tau_n \in \Sigma$ such that $\bigcup_{i=1}^{n} \theta_{\tau_i} = \Sigma$.

Consider the functions

$$\rho_i(\sigma) = \max\,[\varphi(\tau_i,\sigma) - \varphi(\sigma,\sigma),\ 0\ ]$$

for i: 1,...,n . From here, we have immediately that

$$\rho(\sigma) = \sum_{i=1}^{n} \rho_i(\sigma) > 0 \qquad \text{for all } \sigma\epsilon\Sigma \ .$$

Now, consider the following continuous function $\psi : \Sigma \rightarrow \Sigma$ given by

$$\psi(\sigma) = \sum_{i=1}^{n} \frac{\rho_i(\sigma)}{\rho(\sigma)} \tau_i \ ,$$

which is well defined, since the product space Σ is convex.

The convex hull of points $\tau_1,\ldots,\tau_n$ in the product space Σ is homeomorphic to a simplex in a euclidean space. Thus, the application of Brower's fixed point theorem to the continuous function ψ guarantees the existence of a fixed point $\bar{\sigma}\epsilon\Sigma$:

$$\bar{\sigma} = \sum_{i=1}^{n} \frac{\rho_i(\bar{\sigma})}{\rho(\bar{\sigma})} \tau_i \ .$$

By replacing this value in the function $\varphi(\sigma,\tau)$, which is concave in the variable $\tau\epsilon\Sigma$ for fixed $\sigma\epsilon\Sigma$, we obtain

$$\varphi(\bar{\sigma},\bar{\sigma}) \geq \sum_{i=1}^{n} \frac{\rho_i(\bar{\sigma})}{\rho(\bar{\sigma})} \varphi(\tau_i,\bar{\sigma}) ,$$

which is impossible, since by the definition of the function ρ_i: $\varphi(\tau_i,\bar{\sigma}) > \varphi(\bar{\sigma},\bar{\sigma})$ for all $i: 1,\ldots,n$ with $\rho_i(\bar{\sigma}) > 0$. Then,

$$\varphi(\bar{\sigma},\bar{\sigma}) \geq \sum_{i=1}^{n} \frac{\rho_i(\bar{\sigma})}{\rho(\bar{\sigma})} \varphi(\tau_i,\bar{\sigma}) > \sum_{i=1}^{n} \frac{\rho_i(\bar{\sigma})}{\rho(\bar{\sigma})} \varphi(\bar{\sigma},\bar{\sigma}) = \varphi(\bar{\sigma},\bar{\sigma}) .$$

Thus, the existence of a point $\bar{\sigma}\epsilon\Sigma$ such that

$$\varphi(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi(s,\bar{\sigma})$$

has been guaranteed. Q.E.D.

We point out that this result cannot be immediately extended for quasi-concave functions. In fact, under this new condition we cannot obtain the necessary inequality

$$\varphi_1(\bar{\sigma},\bar{\sigma}) \geq \sum_{i=1}^{n} \frac{\rho_i(\bar{\sigma})}{\rho(\sigma)} \varphi(\tau_i,\bar{\sigma}) .$$

Having this result, we now will show how we can use it to examine the $\underline{e}$-positive simple equilibrium points.

<u>LEMMA III.36:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$. Then, a joint strategy $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}$-positive simple equilibrium point of the game $\Gamma_{\underline{e}}$, if and only if

$$\varphi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi_1(s,\bar{\sigma}) ,$$

where the function φ_1 is defined by

$$\varphi_1(\sigma,\tau) = \sum_{i\epsilon N} A_i(\sigma_i,\tau_{N-\{i\}}) .$$

PROOF: Let the point $\bar{\sigma}\in\Sigma$ be an $\underline{e}$-positive simple equilibrium point of the game $\Gamma_{\underline{e}}$. Thus,

$$A_i(\bar{\sigma},\bar{\sigma}_{N-\{i\}}) = \max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{N-\{i\}})$$

for all $i\in N$, and therefore, the following inequalities hold true:

$$\varphi_1(\bar{\sigma},\bar{\sigma}) = \sum_{i\in N} \max_{s_i\in\Sigma_i} A_i(s_i,\bar{\sigma}_{N-\{i\}}) \geq \max_{s\in\Sigma} \varphi_1(s,\bar{\sigma}) ,$$

where on the second inequality sign only the equality sign must hold. Hence,

$$\varphi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \varphi_1(s,\bar{\sigma})$$

which proves the necessity of the assertion. Now, we examine the sufficiency. Let us assume that the point $\bar{\sigma}\in\Sigma$ satisfies

$$\varphi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \varphi_1(s,\bar{\sigma}) ,$$

and suppose that it is not an $\underline{e}$-positive simple equilibrium point of game $\Gamma_{\underline{e}}$. Thus, there exists a point τ in the product space Σ and a non-empty set $I\subseteq N$ such that

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{N-\{i\}}) < A_i(\tau_i,\bar{\sigma}_{N-\{i\}})$$

for all $i\in I$. Define the strategy $\bar{\tau}\in\Sigma$ by

$$\bar{\tau} = \begin{cases} \tau_i & \text{if } i\in I \\ \bar{\sigma}_i & \text{if } i\notin I \end{cases} ,$$

then, according to the assumptions, one has

$$\varphi_1(\bar{\sigma},\bar{\sigma}) = \sum_{i\in N} A_i(\bar{\sigma}_i,\bar{\sigma}_{N-\{i\}}) < \sum_{i\in N} A_i(\bar{\tau}_i,\bar{\sigma}_{N-\{i\}}) = \varphi_1(\bar{\tau},\bar{\sigma})$$

which is impossible. The contradiction, thus obtained, assures the validity of the theorem. (Q.E.D.)

Now, by using these facts, the following characterization of $\underline{e}$-positive simple equilibrium points due to Nikaido-Isoda [16] is a simple matter.

THEOREM III.37: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$ such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear topological space, his payoff function A_i is concave in the variable $\sigma_i \epsilon \Sigma_i$ for fixed $\sigma_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}$, it is continuous in the variable $\sigma_{N-\{i\}} \epsilon \Sigma_{N-\{i\}}$ for fixed $\sigma_i \epsilon \Sigma_i$ and finally the function

$$\sum_{i\epsilon N} A_i(\sigma_i, \sigma_{N-\{i\}})$$

is continuous with respect to $\sigma\epsilon\Sigma$. Then, the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-positive simple equilibrium point.

PROOF: Consider the function

$$\varphi_1(\sigma,\tau) = \sum_{i\epsilon N} A_i(\sigma_i, \tau_{N-\{i\}})$$

defined on the product space $\Sigma \times \Sigma$. For each point $\tau\epsilon\Sigma$, the functions $\varphi_1(\tau,\sigma)$ and $\varphi_1(\sigma,\sigma)$ are both continuous in the variable $\sigma\epsilon\Sigma$, since they are the sums of continuous functions. On the other hand, the function $\varphi_1(\sigma,\tau)$ is concave in the variable $\sigma\epsilon\Sigma$ for fixed $\tau\epsilon\Sigma$, since it is the sum of concave functions. Therefore, the conditions of theorem III.35 are thus satisfied for the function φ_1. This guarantees the existence of a point $\bar{\sigma}\epsilon\Sigma$ with the property

$$\varphi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi_1(s,\bar{\sigma}) .$$

From here and the lemma just considered, it follows that the point $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}$-positive simple equilibrium point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

A very special case appears when the payoff function of player $i \in N$ is continuous in the product variable. It is important to note that the continuity requirements on the payoff functions have some unusual and delicate forms.

As an immediate consequence of this result we now have the following existence theorem.

THEOREM III.38: Let $\Gamma_{\underline{e}} = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a linear topological space, his corresponding function $F_i(\sigma_i, \sigma_{f(i)})$ is concave in the variable $\sigma_i \in \Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ and continuous in $\sigma_{N-\{i\}} \in \Sigma_{N-\{i\}}$ for fixed $\sigma_i \in \Sigma_i$. Then, if the function

$$\sum_{i \in N} F_i\,(\sigma_i, \sigma_{f(i)})$$

is continuous in $\sigma \in \Sigma$, the game $\Gamma_{\underline{e}}$ has an $\underline{e}_m$-simple stable point.

PROOF: Consider the n-person game $\Gamma' = \{\Sigma_1, \ldots, \Sigma_n; F_1, \ldots, F_n\}$, which completely satisfies all the requirements of the previous theorem. Thus, that game Γ' has an $\underline{e}$-positive simple equilibrium point. Such a point is an $\underline{e}_m$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

A further application of theorem III.37 arises for zero-sum two-person games, whose formulation is given in the following theorem of Nikaido [15]:

THEOREM III.39: Let $\Gamma = \{\Sigma_1, \Sigma_2; A\}$ be a zero-sum two-person game, such that the strategy sets are non-empty, compact and convex in a linear topological space, and the payoff function A is continuous in each variable separately, concave in $\sigma_1 \in \Sigma_1$ and convex in $\sigma_2 \in \Sigma_2$. Then, the game Γ has a saddle point.

PROOF: Since the payoff of the second person is $-A$, then, we have

$$A_1(\sigma_1,\sigma_2) + A_2(\sigma_1,\sigma_2) \equiv 0 .$$

Furthermore, all the other requirements of theorem III.37 for game Γ are completely satisfied. Thus, the existence of a saddle point of game Γ is guaranteed. (Q.E.D.)

By considering a similar examination, we will obtain the existence of $\underline{e}$-negative simple equilibrium points as a simple consequence of the following lemma.

LEMMA III.40: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that for each real number $\delta > 0$ and each $\sigma\epsilon\Sigma$ there is a joint strategy $\tau\epsilon\Sigma$ such that

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\epsilon I$ for which

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) > \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

for all $i\epsilon N-I$.

Then, a joint strategy $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}$-negative simple equilibrium point of game $\Gamma_{\underline{e}}$, if and only if

$$\varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi_2(s,\bar{\sigma})$$

where the function φ_2 is defined by

$$\varphi_2(\sigma,\tau) = \sum_{i\epsilon N} [-A_i(\tau_i,\sigma_{e(i)},\tau_{f(i)})] .$$

PROOF: First we examine the necessity. Let the point $\bar{\sigma}\epsilon\Sigma$ be an e-negative simple equilibrium point of game Γ_e. Thus, it satisfies

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\epsilon N$, and therefore, we have

$$\varphi_2(\bar{\sigma},\bar{\sigma}) = \sum_{i\epsilon N} \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})] \geq \max_{s\epsilon\Sigma} \varphi_2(s,\bar{\sigma}) ,$$

where only the equality sign must hold. Hence, it follows immediately that

$$\varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi_2(s,\bar{\sigma}) .$$

This shows the necessity. Now, we are going to show the sufficiency. Assume the point $\bar{\sigma}\epsilon\Sigma$ satisfies

$$\varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi_2(s,\bar{\sigma})$$

and suppose that it is not an e-negative simple equilibrium point of game Γ_e. From here it follows immediately that there exist a non-empty greater subset $I\subseteq N$ and a real number $\delta > 0$ such that for all $i\epsilon I$

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) > \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)}) + \delta.$$

But then by the hypothesis, for $\delta > 0$ and $\bar{\sigma}\epsilon\Sigma$ there is another joint strategy $\bar{\tau}\epsilon\Sigma$ such that

$$A_i(\bar{\sigma}_i,\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)}) > \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)}) + \delta$$

for all $i\epsilon I$ and

$$A_i(\bar{\sigma}_i,\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N-I$, and therefore, we have

$$\varphi_2(\bar{\tau},\bar{\sigma}) = \sum_{i\in N} [-A_i(\bar{\sigma}_i,\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)})] > \sum_{i\in N} [-A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})] = \varphi_2(\bar{\sigma},\bar{\sigma}) ,$$

which is impossible, according to the definition of point $\bar{\sigma}\in\Sigma$. The contradiction proves the sufficiency. (Q.E.D.)

Having this simple result, the proof of existence of e-negative simple point arises immediately.

THEOREM III.41: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function e, such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a linear topological space, his payoff function A_i is convex in the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $(\sigma_i,\sigma_{f(i)})\in\Sigma_i\times\Sigma_{f(i)}$, it is continuous in the variable $(\sigma_i,\sigma_{f(i)})\in\Sigma_i\times\Sigma_{f(i)}$ for fixed $\sigma_{e(i)}\in\Sigma_{e(i)}$, and finally the function

$$\sum_{i\in N} A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)})$$

is continuous in $\sigma\in\Sigma$. If for each real number $\delta > 0$ and each $\sigma\in\Sigma$ there is a joint strategy $\tau\in\Sigma$ such that

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\in I$ for which

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) > \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

for all $i \in N-I$. Then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-negative simple equilibrium point. (Q.E.D.)

PROOF: Consider the function

$$\varphi_2(\sigma,\tau) = \sum_{i\in N} [-A_i(\tau_i,\sigma_{e(i)},\tau_{f(i)})]$$

defined on the product space $\Sigma \times \Sigma$. By virtue of the continuity of the payoff function A_i of player $i \in N$, on the one hand, with respect to the variable $(\sigma_i,\sigma_{f(i)}) \in \Sigma_i \times \Sigma_{f(i)}$, and on the other hand, of the sum of them with respect to the variable $\sigma \in \Sigma$, then, it follows that for each point $\tau \in \Sigma$, the functions $\varphi_2(\tau,\sigma)$ and $\varphi_2(\sigma,\sigma)$ are continuous in $\sigma \in \Sigma$. The convexity of the payoff function A_i in the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ assures that the function $\varphi_2(\sigma,\tau)$ is concave in $\sigma \in \Sigma$ for fixed $\tau \in \Sigma$. Thus, all the conditions expressed in Theorem III.35 are completely satisfied by the function φ_2 , and therefore the existence of a point $\bar{\sigma} \in \Sigma$ with

$$\varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \varphi_2(s,\bar{\sigma}) \qquad \text{is guaranteed.}$$

Now, by using the result included in the previous lemma since the last condition is satisfied, we obtain that such a point is an $\underline{e}$-negative simple equilibrium point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

THEOREM III.42: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a linear topological space, his corresponding function $G_i(\sigma_{e(i)},\sigma_{f(i)})$ is convex in the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$, and continuous with respect to $\sigma_{f(i)} \in \Sigma_{f(i)}$ for fixed $\sigma_{e(i)} \in \Sigma_{e(i)}$, and finally

$$\sum_{i \in N} G_i(\sigma_{e(i)}, \sigma_{f(i)})$$

is continuous in $\sigma \in \Sigma$. If for each real number $\delta > 0$ and each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)}) + \delta ,$$

for all $i \in I$ for which

$$G_i(\sigma_{e(i)}, \sigma_{f(i)}) > \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)})$$

for all $i \in N-I$.

Then the game $\Gamma_{\underline{e}}$ has an $\underline{e}^m$-simple stable point.

<u>PROOF:</u> Consider the n-person game $\Gamma'' = \{\Sigma_1, \ldots, \Sigma_n; G_1, \ldots, G_n\}$. This game Γ'' satisfies all the conditions imposed by the preceding theorem. Thus we know of the existence of an $\underline{e}$-negative simple equilibrium point for game Γ''. Such a point is an $\underline{e}^m$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

The corresponding characterization of $\underline{e}$-simple stable points will arise as a simple consequence of the following lemma.

<u>LEMMA III.43:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that for each real number $\delta > 0$ and each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that

$$F_i(\tau_i, \sigma_{f(i)}) > \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}) - \delta$$

for all $i \in I$ for which

$$F_i(\sigma_i,\sigma_{f(i)}) < \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)}) ,$$

and

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

for all $i\in N-I$, and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\in J$ for which

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) > \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

and finally

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all $i\in N-J$. Then, a joint strategy $\bar{\sigma}\in\Sigma$ is an $\underline{e}$-simple stable point of game $\Gamma_{\underline{e}}$, if and only if

$$\varphi_1^F(\bar{\sigma},\bar{\sigma}) + \varphi_2^G(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} [\varphi_1(s,\bar{\sigma}) + \varphi_2(s,\bar{\sigma})]$$

where the functions are given by

$$\varphi_1^F(\sigma,\tau) = \sum_{i\in N} F_i(\sigma_i,\tau_{f(i)})$$

and

$$\varphi_2^G(\sigma,\tau) = \sum_{i\in N} [-G_i(\sigma_{e(i)},\sigma_{f(i)})] .$$

PROOF: First of all, we examine the necessity. For this reason, consider the point $\bar{\sigma}\in\Sigma$ as an $\underline{e}$-simple stable point of game $\Gamma_{\underline{e}}$. Thus we have

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i \epsilon N$, and therefore

$$\varphi_1^F(\bar{\sigma},\bar{\sigma}) + \varphi_2^G(\bar{\sigma},\bar{\sigma}) = \sum_{i\epsilon N} F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) + \sum_{i\epsilon N} [-G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})]$$

$$= \sum_{i\epsilon N} \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)}) + \sum_{i\epsilon N} \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-G_i(s_{e(i)},\bar{\sigma}_{f(i)})]$$

$$\geq \max_{s\epsilon\Sigma} \varphi_1^F(s,\bar{\sigma}) + \max_{s\epsilon\Sigma} \varphi_2^G(s,\bar{\sigma}) \geq \max_{s\epsilon\Sigma} [\varphi_1^F(s,\bar{\sigma}) + \varphi_2^G(s,\bar{\sigma})] ,$$

where only the equality sign must hold. Hence,

$$\varphi_1^F(\bar{\sigma},\bar{\sigma}) + \varphi_2^G(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1^F(s,\bar{\sigma}) + \varphi_2^G(s,\bar{\sigma})] .$$

Conversely, let us assume that the point $\sigma\epsilon\Sigma$ satisfies

$$\varphi_1^F(\bar{\sigma},\bar{\sigma}) + \varphi_2^G(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1^F(s,\bar{\sigma}) + \varphi_2^G(s,\bar{\sigma})] ,$$

and suppose that it is not an $\underline{e}$-simple stable point of game $\Gamma_{\underline{e}}$. Then there exists a real number $\delta > 0$ and larger subsets $I \subseteq N$ and $J \subseteq N$ such that

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) < \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)}) - \delta$$

for all $i\epsilon I$,

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) > \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)}) + \delta$$

for all $i\epsilon J$ where I or J are non-empty. But, by hypothesis, for $\delta > 0$ and $\bar{\sigma}\epsilon\Sigma$ there is a joint strategy $\bar{\tau}\epsilon\Sigma$ such that

$$F_i(\bar{\tau}_i,\bar{\sigma}_{f(i)}) > \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)}) - \delta$$

for all $i\epsilon I$

$$F_i(\bar{\tau}_i,\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)})$$

for all $i\epsilon N-I$,

$$G_i(\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)}) < \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)}) + \delta$$

for all $i\in J$ and

$$G_i(\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N-J$. From these relations and the above inequalities, in each instance we have

$$\varphi_1^F(\bar{\tau},\bar{\sigma}) + \varphi_2^G(\bar{\tau},\bar{\sigma}) = \sum_{i\in N} F_i(\bar{\tau}_i,\bar{\sigma}_{f(i)}) + \sum_{i\in N} [-G_i(\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)})]$$

$$> \sum_{i\in N} F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) + \sum_{i\in N} [-G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})]$$

$$= \varphi_1^F(\bar{\sigma},\bar{\sigma}) + \varphi_2^G(\bar{\sigma},\bar{\sigma})$$

which is impossible, according to the definition of the point $\bar{\sigma}\in\Sigma$. The contradiction proves the sufficiency. (Q.E.D.)

Once established by this result, the examination of e-simple stable points is straightforward. An existence theorem introduced also in [10], will be reconsidered as follows:

THEOREM III.44: Let $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function e, such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a linear topological space for each $\sigma_{f(i)}\in\Sigma_{f(i)}$, his corresponding function $F_i(\sigma_i,\sigma_{f(i)})$ is concave in $\sigma_i\in\Sigma_i$ and $G_i(\sigma_{e(i)},\sigma_{f(i)})$ is convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$; for each $(\sigma_i,\sigma_{e(i)})\in\Sigma_i\times\Sigma_{e(i)}$ both functions $F_i(\sigma_i,\sigma_{f(i)})$ and $G_i(\sigma_{e(i)},\sigma_{f(i)})$ are continuous in $\sigma_{f(i)}\in\Sigma_{f(i)}$, and finally both functions

$$\sum_{i\in N} F_i(\sigma_i,\sigma_{f(i)}) \quad \text{and} \quad \sum_{i\in N} G_i(\sigma_{e(i)},\sigma_{f(i)})$$

are continuous in $\sigma\epsilon\Sigma$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$F_i(\tau_i,\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) - \delta$$

for all $i\epsilon I$ for which

$$F_i(\sigma_i,\sigma_{f(i)}) < \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) ;$$

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

for all $i\epsilon N-I$, and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\epsilon J$ for which

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) > \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

and finally

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all the remaining $i\epsilon N-J$. Then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-simple stable point.

PROOF: Consider the functions

$$\varphi_1^F(\sigma,\tau) = \sum_{i\epsilon N} F_i(\sigma_i,\tau_{f(i)})$$

and

$$\varphi_2^G(\sigma,\tau) = \sum_{i\epsilon N} [-G_i(\sigma_{e(i)},\tau_{f(i)})]$$

defined on the product space. By virtue of the continuity of functions F_i and G_i of player $i\epsilon N$, with respect to the variables $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$, and on the other hand, by the continuity of their sums with respect to the variable $\sigma\epsilon\Sigma$, then, it follows immediately, that for each point $\tau\epsilon\Sigma$ the functions $\varphi_1^F(\tau,\sigma) + \varphi_2^G(\tau,\sigma)$ and $\varphi_1^F(\sigma,\sigma) + \varphi_2^G(\sigma,\sigma)$ are both continuous in $\sigma\epsilon\Sigma$. The concavity of the function F_i in the variable $\sigma_i\epsilon\Sigma_i$ and the convexity of the function G_i in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$, assure that the function $\varphi_1^F(\sigma,\tau) + \varphi_2^G(\sigma,\tau)$ is convex in the variable $\sigma\epsilon\Sigma$ for fixed $\tau\epsilon\Sigma$. Thus, all the requirements given in theorem III.35 for the function $\varphi_1^F + \varphi_2^G$ are satisfied, and therefore the existence of a point $\bar{\sigma}\epsilon\Sigma$:

$$\varphi_1^F(\bar{\sigma},\bar{\sigma}) + \varphi_2^G(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1^F(s,\bar{\sigma}) + \varphi_2^G(s,\bar{\sigma})]$$

is shown. Now, by the last condition, the preceding lemma is also satisfied. Then such a point $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}$-simple stable point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

As an application of this result, we will now characterize the $\underline{e}$-simple saddle points in the next theorem, where we assume stronger conditions of continuity.

THEOREM III.45: Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear topological space and his payoff function is continuous in the product variable $\sigma\epsilon\Sigma$, concave in $\sigma_i\epsilon\Sigma_i$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\Sigma_{e(i)} \times \Sigma_{f(i)}$ and convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed τ $(\sigma_i,\sigma_{f(i)})\epsilon\Sigma_i \times \Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is another $\tau\epsilon\Sigma$ such that

$$F_i(\tau_i,\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) - \delta$$

for all $i\epsilon I$ for which

$$F_i(\sigma_i,\sigma_{f(i)}) < \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) ,$$

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

for all the remaining $i \in N-I$,

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)}) + \delta$$

for all $i \in J$ for which

$$G_i(\sigma_{e(i)}, \sigma_{f(i)}) > \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)})$$

and finally

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)})$$

for all the remaining $i \in N-J$. Then the game $\Gamma_{\underline{e}}$ has an e-simple saddle point.

PROOF: For player $i \in N$, since his payoff function A_i is concave in $\sigma_i \in \Sigma_i$, then, also his corresponding function $F_i(\sigma_i, \sigma_{f(i)})$ is concave with respect to the variable $\sigma_i \in \Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. On the other hand, by the convexity of A_i in $\sigma_{e(i)} \in \Sigma_{e(i)}$, the convexity of function $G_i(\sigma_{e(i)}, \sigma_{f(i)})$ with respect to $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ holds true. The conditions of continuity required by the previous theorem, are obviously completely satisfied. Finally for the last condition, the validity of the last requirement of the above theorem holds true. Thus, the existence of an e-simple stable point $\bar{\sigma} \in \Sigma$ of game $\Gamma_{\underline{e}}$ is demonstrated:

$$F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)}) = \max_{s_i \in \Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all $i \in N$.

Now, for player $i \in N$, consider the $\bar{\sigma}_{f(i)}$-associated zero-sum two-person game. This game, since the payoff function A_i is concave in $\sigma_i \in \Sigma_i$ and convex in $\sigma_{e(i)} \in \Sigma_{e(i)}$, has the minimax property, in virtue of theorem III.39. Then, it

implies
$$\max_{s_i \in \Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)}) ,$$
and therefore, one has
$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)})$$
$$= G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$$
for all $i \in N$. Such a point is an $\underline{e}$-simple saddle point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

A further characterization of these points will follow immediately after the following result, which indeed is essentially the lemma III.43.

<u>LEMMA III.46:</u> Let $\Gamma_{\underline{e}} = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that for each number $\delta > 0$ and each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that
$$A_i(\tau_i, \sigma_{e(i)}, \sigma_{f(i)}) > \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)}) - \delta$$
for all $i \in I$ for which
$$A_i(\sigma_i, \sigma_{e(i)}, \sigma_{f(i)}) < \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)}) ;$$
$$A_i(\tau_i, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)})$$
for the remaining $i \in N-I$, and
$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}) + \delta$$
for all $i \in J$ for which
$$A_i(\sigma_i, \sigma_{e(i)}, \sigma_{f(i)}) > \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$
and finally
$$A_i(\sigma_i, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$
for all the remaining $i \in N-J$.

Then a joint strategy $\bar{\sigma}\epsilon\Sigma$ is an e-simple saddle point of game Γ_e , if and only if

$$\varphi_1(\bar{\sigma},\bar{\sigma}) + \varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1(s,\bar{\sigma}) + \varphi_2(s,\bar{\sigma})]$$

PROOF: First we will show the necessity. Let the point $\bar{\sigma}\epsilon\Sigma$ be an e-simple saddle point of game Γ_e . Then

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

$$= \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\epsilon N$, and therefore

$$\varphi_1(\bar{\sigma},\bar{\sigma}) + \varphi_2(\bar{\sigma},\bar{\sigma}) = \sum_{i\epsilon N} A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) + \sum_{i\epsilon N} [-A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})]$$

$$= \sum_{i\epsilon N} \max_{s_i\epsilon\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) + \sum_{i\epsilon N} \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)})]$$

$$\geq \max_{s\epsilon\Sigma} [\varphi_1(s,\bar{\sigma}) + \varphi_2(s,\bar{\sigma})] ,$$

where only the equality sign must hold, that is

$$\varphi_1(\bar{\sigma},\bar{\sigma}) + \varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1(s,\bar{\sigma}) + \varphi_2(s,\bar{\sigma})] .$$

Conversely, let us assume that the point $\bar{\sigma}\epsilon\Sigma$ satisfies

$$\varphi_1(\bar{\sigma},\bar{\sigma}) + \varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1(s,\bar{\sigma}) + \varphi_2(s,\bar{\sigma})]$$

and suppose that such a point is not an e-simple saddle point of game Γ_e . Then there exists a real number $\delta > 0$ and larger subsets $I \subseteq N$ and $J \subseteq N$ such that

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) < \max_{s_i\epsilon\Sigma_i} A_i(s_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) - \delta$$

for all $i\epsilon I$,

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) > \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) + \delta$$

for all $i \in J$ with I or J non-empty. But by hypothesis, for $\delta > 0$ and the point $\bar{\sigma} \in \Sigma$ there is another joint strategy $\bar{\tau} \in \Sigma$ such that

$$A_i(\bar{\tau}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) > \max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) - \delta$$

for all $i \in I$

$$A_i(\bar{\tau}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$$

for all $i \in N-I$,

$$A_i(\bar{\sigma}_i, \bar{\tau}_{e(i)}, \bar{\sigma}_{f(i)}) < \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) + \delta$$

for all $i \in J$ and finally

$$A_i(\bar{\sigma}_i, \bar{\tau}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for the remaining $i \in N-J$. From all of these strict inequalities, we have in each instance

$$\begin{aligned}\varphi_1(\bar{\tau},\bar{\sigma}) + \varphi_2(\bar{\tau},\bar{\sigma}) &= \sum_{i \in N} A_i(\bar{\tau}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) + \sum_{i \in N} [-A_i(\bar{\sigma}_i, \bar{\tau}_{e(i)}, \bar{\sigma}_{f(i)})] \\ &> \sum_{i \in N} A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) + \sum_{i \in N} [-A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})] \\ &= \varphi_1(\bar{\sigma},\bar{\sigma}) + \varphi_2(\bar{\sigma},\bar{\sigma})\end{aligned}$$

which is impossible, according to the definition of joint strategy $\bar{\sigma} \in \Sigma$. The contradiction proves the assertion. (Q.E.D.)

Having this simple result a further examination of e-simple saddle points is a very simple task.

<u>THEOREM III.47</u>: Let $\Gamma = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an n-person game with simple structure function $\underline{e}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear topological space, for each $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$ his payoff function $A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)})$ is concave in $\sigma_i\epsilon\Sigma_i$ for fixed $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ and is convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_i\epsilon\Sigma_i$; for each $\sigma_i\epsilon\Sigma_i$ it is continuous in $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\Sigma_{e(i)}\times\Sigma_{f(i)}$ and for each $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ it is continuous in $(\sigma_i,\sigma_{f(i)})\epsilon\Sigma_i\times\Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) > \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) - \delta$$

for all $i\epsilon I$ for which

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) < \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) \;;$$

$$A_i(\tau_i,\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)})$$

for the remaining $i\epsilon N-I$, and

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\epsilon J$ for which

$$A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)}) > \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

and finally

$$A_i(\sigma_i,\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)},\sigma_{f(i)})$$

for the remaining $i\epsilon N-J$. Then the game $\Gamma_{\underline{e}}$ has an $\underline{e}$-simple saddle point.

PROOF: Consider the functions $\varphi_1(\sigma,\tau)$ and $\varphi_2(\sigma,\tau)$ defined on the product space. On the one hand from the continuity of payoff function A_i of player $i\epsilon N$, with respect to the variable $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$, for fixed $(\sigma_i,\sigma_{e(i)})\epsilon\Sigma_i \times \Sigma_{e(i)}$, the function $\varphi_1(\tau,\sigma) + \varphi_2(\tau,\sigma)$ is continuous with respect to $\sigma\epsilon\Sigma$ for fixed $\tau\epsilon\Sigma$. On the other hand, the function $\varphi_1(\sigma,\sigma) + \varphi_2(\sigma,\sigma)$ is continuous, because it is zero identically. Finally by the concavity and convexity properties of payoff function A_i, we have that the function $\varphi_1(\sigma,\tau) + \varphi_2(\sigma,\tau)$ is convex in the variable $\sigma\epsilon\Sigma$ for fixed $\tau\epsilon\Sigma$. Thus, all the requirements given in theorem III.35 for the function $\varphi_1 + \varphi_2$ are completely satisfied, and therefore the existence of a point $\bar{\sigma}\epsilon\Sigma$:

$$\varphi_1(\bar{\sigma},\bar{\sigma}) + \varphi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} [\varphi_1(s,\bar{\sigma}) + \varphi_2(s,\bar{\sigma})]$$

is given. Now, by the above lemma, since the condition on it for game $\Gamma_{\underline{e}}$ is also satisfied, such a point is an e-simple saddle point of game $\Gamma_{\underline{e}}$. (Q.E.D.)

We note that the Nikaido's result given in theorem III.39, arises also immediately as a particular case of the previous existence theorem.

Finally, we recall that as has been pointed out in other sections, some of the theorems considered in this section have no value in the case when the simple structure function has an empty indifferent coalition for all the players.

CHAPTER IV

IV.1 E-Equilibrium Points

This chapter is devoted to an extension of the concepts formulated in the first and second chapters for general games. Technically, the first part is a straightforward extension of those results.

The introduction of e-simple equilibrium points has been founded on the concepts of simple structure functions and associated zero-sum games. The associated game of a player represents the real situation he sees in the game, since he is considered embedded in it. From this point of view, the situation of the players depend strongly upon the structure function, that is, on the composition of the respective indifferent and antagonistic coalitions. These technical concepts have been introduced using the important intuitive concepts of enemy and indifferent player. The very important concept of friendness player, is only considered when the first player in the associated game is assumed normal, since he is trying to obtain greater winnings, we can consider heuristically that he is friend of himself.

Now, one can try to extend the concept of friendness in a natural manner, similar to what has been done for indifferent and antagonistic coalitions. In this way a structure for the game will be imposed.

Let us consider an n-person game $\Gamma = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ with the usual properties, that is, where the strategy set of player $i \in N$ is a non-empty compact set in a euclidean space and his own payoff function is a continuous function in the product space. Furthermore, the real situation involving the representation of game Γ allows only non-cooperative behavior among the players.

For a given player $i \in N$ of the game Γ, we now can see the set of players N, divided into three disjoint groups of players, such that

$$N = d(i) \cup e(i) \cup f(i) \ .$$

The new set of players given by $d(i)$, possesses the role of the player $i \in N$ himself in the old description, that is the player $i \in N$ is now replaced by the set of players $d(i)$. In the old descriptions the first player was considered normal in the associated game, now the players in $d(i)$ represent this. They are the players in the game Γ, who are trying to help player $i \in N$ with respect to his own position. By this reason, it seems natural to consider the players in the set $d(i)$ as the friends of player $i \in N$. On the other hand, the remaining sets $e(i)$ and $f(i)$ are regarded as in the old description, to be the antagonistic and indifferent players, respectively. In this new approach the condition $N - \{i\} = e(i) \cup f(i)$ is not necessarily satisfied.

Formally, a function

$$E: N \to \underline{P}_N \times \underline{P}_N \times \underline{P}_N$$

which for each player $i \in N = \{1, \ldots, n\}$ of the n-person game Γ assigns three disjoint subsets of players given by $E(i) = (d(i), e(i), f(i))$, such that $d(i) \cup e(i) \cup f(i) = N$, is called a <u>structure function</u> of the game Γ. Furthermore, the set of players $d(i)$ is said to be the <u>friend coalition</u> of player $i \in N$.

Again, an n-person game with an associated structure function E, which is represented by $\Gamma_E = (\Gamma, E)$, will be for simplicity, also called a game.

We note that the concept of simple structure function, now appears as a special case of this new concept just introduced. In fact, if the friend coalition of each player $i \in N$ is formed by himself alone, that is: $d(i) = \{i\}$, then the simple structure function is obtained.

It is interesting to point out, that due to the flexibility and the symmetry of this new structure function, one can easily handle many "pathological" situations, which arise when a player does not belong to his corresponding friend coalition. On such circumstances, he is a member of his indifferent coalition, or even more peculiar he is a member of his own antagonistic coalition. Of course, these situations generally are not considered in the literature, but it is interesting to observe that they could still have real representations.

Having already introduced the new structure function and in accordance with the above, we again consider the player $i \in N$ embedded in game Γ, in a two-person conflict depending on the choice of the indifferent coalition $f(i)$, where his own friend coalition has the role of the first player and his antagonistic coalition has the place of the second player.

In a formal manner, for the player $i \in N = \{1,\dots,n\}$ in the game Γ_E and a joint strategy $\sigma_{f(i)} \in \Sigma_{f(i)}$ of the indifferent coalition $f(i)$, we define the $\sigma_{f(i)}$- associated zero-sum two-person game with respect to the game Γ_E with structure function E, by

$$\Gamma_i(\sigma_{f(i)}) = \{ \Sigma_{d(i)}, \Sigma_{e(i)}; A_i(\sigma_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) \}$$

With the introduction of the new associated game, now, we are able to introduce new concepts regarding games Γ_E with an associated structure function. In fact, such an introduction will be a natural consequence of the assigned character of the constituent parts in the associated two-person games. Of course, the role assigned to the indifferent coalition $f(i)$ of the player $i \in N$, will remain constant throughout the discussion, similar to the way it was done in the previous considerations.

First of all, we will extend the concept of positive simple equilibrium point. This will be done by assigning to the player formed by the coalition $e(i)$, the indifferent character in the associated game. Furthermore, the first player (formed by the friend coalition $d(i)$) is considered as having a normal player, thus it will try to maximize the position of the corresponding player $i \in N$.

Indeed, the situation described by the above assignations, can be seen more naturally by considering the indifferent coalition of player $i \in N$ to be formed by $e(i) \cup f(i)$.

A point $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ belonging to the product space Σ is called an <u>E-positive equilibrium point</u> of the n-person game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with structure function E, if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_{d(i)}, \Sigma_{e(i)}; A_i \}$ of player $i \in N$:

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) .$$

This concept was introduced in [11] . From an intuitive viewpoint, an E-positive equilibrium point is a rule of behavior which is such that if some members of the friend coalition of a player change from it, then the profit of this player will be decreased, assuming that the players not in the friend coalition abide by it.

Before formulating an existence theorem for E-positive equilibrium points, the following concepts should be reformulated. A real function A defined on a convex set Σ in a euclidean space is called <u>quasi-concave</u> on Σ, if for any real number λ the set

$$\{ \sigma \in \Sigma : A(\sigma) > \lambda \}$$

is convex. Similarly, it is said to be <u>quasi-convex</u> on Σ if the set

$$\{ \sigma \in \Sigma : A(\sigma) < \lambda \}$$

is convex. Equivalently, one can define such a function by replacing the strict inequality sign by the non-strict inequality sign in the definition of the level sets. Again, we note that a concave function is quasi-concave. Indeed, let τ and $\bar{\tau}$ be two points such that

$$A(\tau) > \lambda \quad \text{and} \quad A(\bar{\tau}) > \lambda$$

for a real number λ , then because the function is convex, for any real number μ in the closed unit interval, we have

$$A(\mu\tau+(1-\lambda)\bar{\tau}) \geq \mu A(\tau)+(1-\mu)A(\bar{\tau}) > \lambda$$

which implies the quasi-concavity of the function. Similarly, a convex function is quasi-convex.

An important property of such kinds of functions which will be used later is formulated in the following result.(*)

<u>LEMMA IV.1:</u> Let Σ_1 and Σ_2 be non-empty, compact and convex sets each in a euclidean space. Let A and B be a continuous real function defined on the product space $\Sigma_1 x \Sigma_2$, such that for each $\sigma_2 \in \Sigma_2$, the function $A(\sigma_1,\sigma_2)$ is quasi-concave and the function $B(\sigma_1,\sigma_2)$ is quasi-convex with respect to the variable $\sigma_1 \in \Sigma_1$. Then, the functions

$$\min_{s_2 \in \Sigma_2} A(\sigma_1, s_2) \quad \text{and} \quad \max_{s_2 \in \Sigma_2} A(\sigma_1, s_2)$$

are respectively quasi-concave and quasi-convex in the variable $\sigma_1 \in \Sigma_1$.

(*) These definitions and results expressed have already been used in the more advanced approach given in Chapter III.

PROOF: Let $\bar{\sigma}_1$ and $\tilde{\sigma}_1$ be two points in Σ_1 for which the function $\min_{s_2 \in \Sigma_2} A(\cdot, s_2)$ takes values greater than a given real number λ. Suppose that for a real number $\mu \in [0,1]$

$$\min_{s_2 \in \Sigma_2} A(\mu\bar{\sigma}_1 + (1-\mu)\tilde{\sigma}_1, s_2) \leq \lambda$$

Then, for at the point $\tau_2 \in \Sigma_2$, at which this minimun is reached, on one hand we would have

$$A(\mu\bar{\sigma}_1 + (1-\mu)\tilde{\sigma}_1, \tau_2) \leq \lambda$$

On the other hand, because on the points $\bar{\sigma}_1$ and $\tilde{\sigma}_1$

$$A(\bar{\sigma}_1, s_2) > \lambda \quad \text{and} \quad A(\tilde{\sigma}_1, s_2) > \lambda$$

for all $s_2 \in \Sigma_2$, it follows that

$$A(\bar{\sigma}_1, \tau_2) > \lambda \quad \text{and} \quad A(\tilde{\sigma}_1, \tau_2) > \lambda \quad .$$

This is impossible since the function A is quasi-concave with respect to the variable $\sigma_1 \in \Sigma_1$ at the point $\tau_2 \in \Sigma_2$. Then the function $\min_{s_2 \in \Sigma_2} A(\cdot, s_2)$ is quasi-concave in the variable $\sigma_1 \in \Sigma_1$.

Similarly, let the function $\max_{s_2 \in \Sigma_2} B(\cdot, s_2)$ not be quasi-convex with respect to $\sigma_2 \in \Sigma_2$. Then, for some real number λ, the set

$$\{ B_\lambda = \{ \sigma_1 \in \Sigma_1 : \max_{s_2 \in \Sigma_2} B(\sigma_1, s_2) < \lambda \}$$

would be non-convex, and therefore for some two elements $\bar{\tau}_1$ and $\tilde{\tau}_1$ in B_λ and some real number μ in the closed unit interval, we would have

$$\max_{s_2\in\Sigma_2} B(\mu\bar{\tau}_2+(1-\mu)\tilde{\tau}_1, s_2) \geq \lambda$$

From this, for the point $\tau_2\in\Sigma_2$ where this maximun is reached, it follows that

$$B(\mu\bar{\tau}_1+(1-\mu)\tilde{\tau}_1, \tau_2) \geq \lambda$$

This is impossible, since $B(\bar{\tau}_1, \tau_2)$ and $B(\tilde{\tau}_1, \tau_2)$ are less than λ and the function B is quasi-convex in the variable $\sigma_1\in\Sigma_1$. Hence, the function $\max_{s_2\in\Sigma_2} B(\cdot, s_2)$ is quasi-convex in $\sigma_1\in\Sigma_1$. (Q.E.D.)

Another very simple and useful tool for all the theorems considered in the subsequent discussion is given in the following formulation, which could follow immediately from Lemma III.2.

<u>LEMMA IV.2</u>: Let Σ be non-empty, compact set in a euclidean space. Let A be a continuous real function defined on the product space $\Sigma x\Sigma$. Then the multivalued functions

$$\varphi^m, \varphi_m : \Sigma \to \Sigma$$

defined by

$$\varphi^m(\sigma) = \{ \tau\in\Sigma : A(\tau,\sigma) = \max_{s\in\Sigma} A(s,\sigma) \}$$

and

$$\varphi_m(\sigma) = \{ \tau\in\Sigma : A(\tau,\sigma) = \min_{s\in\Sigma} A(s,\sigma) \}$$

are upper-semicontinuous.

PROOF: Consider two arbitrary converging sequences

$$\sigma(k) \to \sigma \quad \text{and} \quad \tau(k) \to \tau$$

in the space Σ, such that for every positive integer k : $\tau(k) \epsilon \varphi^m(\sigma(k))$, that is

$$A(\tau(k),\sigma(k)) = \max_{s\epsilon\Sigma} A(s,\sigma(k)) .$$

Now, by virtue of the continuity of the function A in the variable product, the function maximun is also continuous, and therefore the convergence of the following sequences of real numbers is guaranteed:

$$A(\tau(k),\sigma(k)) \to A(\tau,\sigma)$$

and

$$\max_{s\epsilon\Sigma} A(s,\sigma(k)) \to \max_{s\epsilon\Sigma} A(s,\sigma) .$$

This implies equality between the limiting values:

$$A(\tau,\sigma) = \max_{s\epsilon\Sigma} A(s,\sigma) ,$$

which means that the point τ is an element of the set $\varphi^m(\sigma)$. Hence, the multivalued function φ^m is upper-semicontinuous. By taking $B = -A$ in the previous discussion, the upper-semicontinuity of multivalued function φ_m can be proved. (Q.E.D.)

Having these results, we now are going to formulate an existence theorem for E-positive equilibrium points.

THEOREM IV.3: Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ be an n-person game with structure function E, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i continuous in the variable $\sigma\epsilon\Sigma$; and quasi-concave with respect to $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\Sigma_{e(i)}x\Sigma_{f(i)}$. If for each joint strategy $\sigma\epsilon\Sigma$ there is another point $\tau\epsilon\Sigma$ such that for all $i\epsilon N$

$$A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) ,$$

then the game Γ_E has an E-positive equilibrium point.

<u>PROOF</u>: Consider for a point σ in the non-empty, compact and convex product space the non-empty convex set

$$\varphi_i(\sigma) = \{ \tau\epsilon\Sigma: A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) \} .$$

Indeed, given two arbitrary elements τ and $\bar{\tau}$ of the set $\varphi_i(\sigma)$, we have from the definition:

$$A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = A_i(\bar{\tau}_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) ,$$

which implies

$$A_i(\lambda\tau_{d(i)}+(1-\lambda)\bar{\tau}_{d(i)},\sigma_{e(i)},\sigma_{f(i)})=\lambda A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)})+(1-\lambda)A_i(\bar{\tau}_{d(i)},\sigma_{e(i)},\sigma_{f(i)})$$

$$= \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) ,$$

for all real numbers λ in the unit closed interval, since the payoff function A_i is quasi-concave in $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$. This means that the point $\lambda\tau+(1-\lambda)\bar{\tau}$ is an element of $\varphi_i(\sigma)$, since all the strategy sets are convex. Hence, the set $\varphi_i(\sigma)$ is convex.

Let us consider the multivalued function

$$\psi: \Sigma \to \Sigma$$

defined by the convex set

$$\psi(\sigma) = \bigcap_{i \in N} \varphi_i(\sigma)$$

for each point σ in the product space Σ . The set $\psi(\sigma)$ is non-empty by the last condition. Furthermore, by Lemma IV.2, the multivalued function $\varphi_i(\sigma)$, for each player $i \in N$, is upper-semicontinuous and therefore, because the graph of the multivalued function ψ is the intersection over $i \in N$ of the graphs of φ_i:

$$G_\varphi = \bigcap_{i \in N} G_{\varphi_i}$$

we obtain the upper-semicontinuity of φ .

Thus, as usual, we are able to apply the Kakutani's Fixed Point Theorem to the multivalued function ψ , obtaining a fixed point $\bar{\sigma} \in \Sigma : \bar{\sigma} \in \psi(\bar{\sigma})$, on which the payoff function of player $i \in N$ is:

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) .$$

Then, this point is an E-positive equilibrium point of game Γ_E . (Q.E.D.)

The last condition of this theorem, which is regarded as the defense property of the game with respect to the E-positive equilibrium concept could be interpreted as follows: for any accepted behavior among the players there exists another behavior such that if all the players which are not in the friend coalition of a player abide by the first one, then, the second one maximizes his own position.

As an application of the result described previously, we formulate the following existence theorem for mixed extensions of finite n-person games.

<u>THEOREM IV.4</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be a finite n-person game with structure function E, such that the payoff function of player $i \in N$ is of the form:

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j \in d(i)} a_i^j(\sigma_j,\sigma_{e(i)},\sigma_{f(i)}) .$$

If for each point $x \in X = \times_{j \in N} \tilde{\Sigma}_j$ there is another point $y \in X$ such that for all $i \in N$:

$$E_i(y_{d(i)},x_{e(i)},x_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)},x_{e(i)},x_{f(i)}) ,$$

then, the mixed extension $\tilde{\Gamma}_E = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;E_1,\ldots,E_n \}$ has an E-positive equilibrium point.

<u>PROOF</u>: Consider for player $i \in N$, the expectation function E_i, which by the form of the payoff function A_i, has the following form

$$E_i(x_{d(i)},x_{e(i)},x_{f(i)}) = \sum_{j \in e(i)} e_i^j(x_j,x_{e(i)},x_{f(i)})$$

where e_i^j denotes the expectation function of the function a_i^j. Because each function e_i^j is linear with respect to the variable $x_j \in X_j$, Lemma I.11 then says that the expectation function E_i is linear in the variable $x_{d(i)} \in X_{d(i)}$. Thus, the requirement on the form of payoff functions of theorem IV.3 is satisfied by the mixed extension game Γ_E with structure function E. Furthermore, the last condition is also verified. Thus, the existence of an E-positive equilibrium point of mixed extension $\tilde{\Gamma}$ is obtained. (Q.E.D.)

An example, for which that condition holds true is illustrated by the game Γ_E, with structure function E given by the partition $\underline{P} = (P_1,\dots,P_r)$ of the players set N, that is, for all $i \in P \in \underline{P}$ then $d(i) = P$ and $e(i) \cup f(i) = N-P$, and the payoff functions are determined by

$$A_i(\sigma_{d(i)}, \sigma_{e(i)\cup f(i)}) = \sum_{j \in d(i)} a_i^j(\sigma_j, \sigma_{e(i)\cup f(i)})$$

where for every joint mixed strategy $x_{e(i)\cup f(i)}$ and for each $j \in d(i) = P$ there is a $y_j \in \tilde{\Sigma}_j$ for which

$$e_i^j(y_j, x_{e(i)\cup f(i)}) = \max_{s_j \in \Sigma_j} e_i^j(s_j, x_{e(i)\cup f(i)})$$

for all $j \in d(i) = P$, where e_i^j denotes the expectation of the function a_i^j. A special case is obtained when for each $P \in \underline{P}$ all the functions a_i^j with $i,j \in P$ coincide. Thus, in such situations there is some E-positive equilibrium point.

The dual concept of E-positive equilibrium point can be derived, by regarding on the one hand, the second player, in the associated game of each player, determined by the choice of his indifferent coalition, as normal. Thus, the antagonistic coalition is acting to hurt this corresponding player. On the other hand, assigning to each friend coalition an apathetic behavior with respect to the position of its corresponding player.

On the base of this observation, formally, given an n-person game $\Gamma_E = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ with structure function E, a joint strategy $\bar{\sigma} \in \Sigma$ is said to be an <u>E-negative equilibrium point</u> of the game Γ_E, if for each $\bar{\sigma}_{f(i)}$- associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)}, \Sigma_{e(i)}; A_i\}$ of the player $i \in N$:

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_{d(i)}, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for every player $i \in N$.

This concept is a simple extension of the e-negative simple equilibrium point which also is found in [11] . This latter concept is derived when the structure function E of the game Γ is determined by having all the friend coalition identified with the corresponding player.

Heuristically speaking, an E-negative equilibrium point is a rule of behavior which is such that if some of the players belonging to the antagonistic coalition of a player change from it, then, the position of this player will be increased, if his friend and indifferent coalitions abide by it.

We note that, apparently, one may think that this concept essentially coincides with the e-negative equilibrium point by associating the following simple structure function $\underline{e}(i) = (e(i), d(i) \cup f(i))$ for every player $i \in N$. But, unfortunately, we do not obtain such a connection, since we can be in a pathological situation, that is, some players could be a member of his own antagonistic coalition.

A general characterization of this kind of points is formulated in the following result, which is essentially theorem II.1.

<u>THEOREM IV.5</u>: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be a finite n-person game with structure function E , such that the strategy set Σ_i of player $i \in N = \{1, \ldots, n\}$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i is continuous in the variable $\sigma \in \Sigma$, and quasi-convex with respect to $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_{d(i)}, \sigma_{f(i)}) \in \Sigma_{d(i)} \times \Sigma_{f(i)}$. If for each joint strategy $\sigma \in \Sigma$ there is another point $\tau \in \Sigma$ such that for all $i \in N$:

$$A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)})$$

then, the game Γ_E has an E-negative equilibrium point.

<u>PROOF</u>: For a point σ in the non-empty, compact convex product space Σ , let

$$\varphi_i(\sigma) = \{ \tau\in\Sigma: A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{d(i)}\in\Sigma_{d(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)})\}$$

be a non-empty set in Σ , corresponding to player $i\in N$. It is convex, since the payoff function A_i is quasi-convex with respect to the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$.

Define the multivalued function

$$\psi: \Sigma \to \Sigma$$

as the convex set

$$\psi(\sigma)= \bigcap_{i\in N} \varphi_i(\sigma)$$

for each point σ in the product space Σ . The set $\psi(\sigma)$ is non-empty by virtue of the last condition. By the continuity of the payoff functions, then, Lemma IV.2, assures the upper-semicontinuity of the multivalued function ψ .

Thus, Kakutani's Fixed Point Theorem applied to ψ , assures the existence of a fixed point $\bar{\sigma}\in\Sigma:\bar{\sigma}\in\psi(\bar{\sigma})$. Such a point is an E-positive equilibrium point of the game Γ_E . (Q.E.D.)

The attack property of the game Γ_E with respect to the E-negative equilibrium point is expressed in the last condition of the previous theorem, and has the following intuitive meaning: for any accepted joint behavior between the players there is another actuation such that if all the players which are not members of the antagonistic coalition of a player abide by the first one, the second one minimizes his own position.

As an immediate consequence of this result regarding E-negative equilibrium points, we derive the following existence theorem for mixed extensions of finite games.

THEOREM IV.6: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be a finite n-person game with structure function E, such that the payoff function of player $i \in N$ is of the form:

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j \in e(i)} a_i^j(\sigma_{d(i)},\sigma_j,\sigma_{f(i)}) \ .$$

If for each point $x \in X = \underset{j \in N}{\times} \widetilde{\Sigma}_j$ there is another point $y \in \Sigma$ such that for all $i \in N$

$$E_i(x_{d(i)},y_{e(i)},x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_{d(i)},s_{e(i)},x_{f(i)}) \ ,$$

then, the mixed extension $\widetilde{\Gamma}_E = \{\widetilde{\Sigma}_1,\ldots,\widetilde{\Sigma}_n;E_1,\ldots,E_n\}$ has an E-negative equilibrium point.

PROOF: By way of the form of the payoff function A_i of player $i \in N$ in the game Γ_E, Lemma I.11 then assures that the expectation function E_i is a linear function with respect to the variable $x_{e(i)} \in X_{e(i)}$. Thus, all the conditions required in the previous theorem, are satisfied by the mixed extension $\widetilde{\Gamma}_E$ and so the game $\widetilde{\Gamma}_E$ has an E-negative equilibrium point. (Q.E.D.)

We note that the E-negative and E-positive equilibrium points are connected by a simple relation. In fact, consider for the game $\Gamma_E = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ with structure function E the following associated n-person game: $\Gamma^*_{\bar{E}} = \{ \Sigma_1,\dots,\Sigma_n;-A_1,\dots,-A_n \}$ where the new structure function $\bar{E}$ is defined by $\bar{E}(i) = (e(i),d(i),f(i))$ for all player $i \in N$. Thus, from the definition, we have that the E-positive equilibrium points of game Γ_E coincide with the $\bar{E}$-negative equilibrium points of game $\Gamma^*_{\bar{E}}$ and the E-negative equilibrium points of game Γ_E are the $\bar{E}$-positive equilibrium points of game $\Gamma^*_{\bar{E}}$.

Both previous concepts, together, determine an extension of e-simple saddle point. This is achieved by assigning the normal roles to the respective friend and antagonistic coalitions of every player in the associated game determined by the actions of his respective indifferent coalition.

Formally, given an n-person game $\Gamma_E = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ with the structure function E, a joint strategy $\bar{\sigma} \in \Sigma$ is said to be an E-neutral equilibrium point or E-saddle point of game Γ_E, if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\sigma_{f(i)}) = \{\Sigma_{d(i)},\Sigma_{e(i)};A_i\}$ of the player $i \in N$:

$$\max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{d(i)} \in \Sigma_{d(i)}} A_i(\bar{\sigma}_{d(i)},s_{e(i)},\bar{\sigma}_{f(i)}) .$$

In other words, if it is an E-positive and E-negative equilibrium point.

An E-saddle point is a rule of behavior which for each player $i \in N$ is a saddle point of the resulting game, if all the players of the indifferent coalition abide by it. In other words, it is optimal for each friend coalition and antagonistic coalition with respect to the position of the respective player, given the actions of the indifferent coalitions.

A first result concerning these points for the games under consideration (which is a simple extension of Theorem II.3) is formulated as follows:

THEOREM IV.7: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with structure function E, such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i is continuous in the variable $\sigma\in\Sigma$, quasi-concave in the variable $\sigma_{d(i)}\in\Sigma_{d(i)}$ for fixed $(\sigma_{d(i)},\sigma_{f(i)})\in\Sigma_{d(i)}\times\Sigma_{f(i)}$, and quasi-convex in the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $(\sigma_{d(i)},\sigma_{f(i)})\in\Sigma_{d(i)}\times\Sigma_{f(i)}$. If for each joint strategy $\sigma\in\Sigma$ there is a point $\tau\in\Sigma$ such that

$$A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)}) ,$$

then, the game Γ_E has an E-saddle point.

PROOF: For a point σ in the product space define the non-empty set

$$\varphi_i(\sigma) = \{ \tau\in\Sigma: A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) \}$$

$$\cap \{ \tau\in\Sigma: A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)}) \}$$

for player $i\in N$. It is convex, since the payoff function A_i is quasi-concave in the variable $\sigma_{d(i)}\in\Sigma_{d(i)}$ and quasi-convex with respect to $\sigma_{e(i)}\in\Sigma_{e(i)}$.

Thus, by the latter condition on the payoff functions, for any $\sigma\in\Sigma$ the set

$$\psi(\sigma) = \bigcap_{i\in N} \varphi_i(\sigma)$$

is non-empty and convex, since all the strategy sets are convex. This then determines a multivalued function

$$\psi: \Sigma \to \Sigma$$

which, by virtue of the continuity of payoff function, and Lemma IV.2, is upper-semicontinuous. Hence, the Kakutani Fixed Point assures a point $\bar{\sigma}\in\Sigma: \bar{\sigma}\in\psi(\bar{\sigma})$. Such a point is an E-saddle point to the game Γ_E. (Q.E.D.)

Again, the latter condition on the previous theorem has the character of the attack and defense property for the game Γ_E with structure function E, with respect to the E-saddle point concept. This can be seen in the following heuristic point of view for any accepted joint action between the players in the game Γ_E, there is another one which is such that, if all the players of the indifferent and antagonistic coalition of a player abide by the first one, the second one maximizes his own position and minimizes it in the resulting game of the actions of his friend and indifferent coalition.

As a special case, we obtain the following result regarding mixed extension of finite games.

<u>THEOREM IV.8</u>: Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be a finite n-person game with structure function E, such that the payoff function of the player $i\in N$ is of the form:

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} \sum_{k\in e(i)} a_i^{j,k}(\sigma_j,\sigma_k,\sigma_{f(i)}).$$

If for each point $x \in X = \underset{j \in N}{\times} \tilde{\Sigma}_j$, there is another point $y \in X$ such that for all $i \in N$

$$E_i(y_{d(i)}, x_{e(i)}, x_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, x_{e(i)}, x_{f(i)})$$

and

$$E_i(x_{d(i)}, y_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_{d(i)}, s_{e(i)}, x_{f(i)}) ,$$

then, the mixed extension $\tilde{\Gamma}_E = \{ \tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_n; E_1, \ldots, E_n \}$ has an E-saddle point.

PROOF: By the form of the payoff function A_i of palyer $i \in N$, Lemma I.11 assumes that the expectation function E_i is bilinear with respect to the variable $(x_{d(i)}, x_{e(i)}) \in X_{d(i)} x X_{e(i)}$ for fixed $x_{f(i)} \in X_{f(i)}$. Thus, all the requirements of the preceding theorem applied to the mixed extension $\tilde{\Gamma}_E$ are completely satisfied. Then, the game $\tilde{\Gamma}_E$ has an E-saddle point. (Q.E.D.)

As what has been commented in the first two chapters about the equivalence and interchangeability properties of e-simple points, one can easily observe that the E-positive, E-negative equilibrium and E-saddle point do not satisfy generally such properties. Nevertheless, in the special case where the structure function has all the indifferent coalitions empty, then for E-saddle points the same properties hold true. Indeed, these facts are illustrated in the following results.

THEOREM IV.9: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E such that all the indifferent coalitions f(i) are empty, the strategy set Σ_i of player $i \in N$ is a non-empty, compact and convex set in a euclidean space, and his payoff function A_i is continuous with respect to the variable $\sigma \in \Sigma$. Then, all the E-saddle points of game Γ_E are equivalents, that is, if $\bar{\sigma}$, $\tilde{\sigma} \in \Sigma$ are two E-saddle points of game Γ_E , then

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}) = A_i(\tilde{\sigma}_{d(i)}, \tilde{\sigma}_{e(i)})$$

for all the players $i \in N$.

PROOF: For each player $i \in N$, consider the associated game $\Gamma_i = \{ \Sigma_{d(i)}, \Sigma_{e(i)}; A_i \}$. Because the payoff function A_i is continuous, the requirements of theorem I.3 are satisfied for the points $\bar{\sigma}$ and $\tilde{\sigma}$. Thus, the payoff function of player $i \in N$ coincide on them. (Q.E.D.)

For the typical kind of games considered in Theorem IV.7 we have the following result which is a simple extension of Theorem I.4.

THEOREM IV.10: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E such that the indifferent coalitions f(i) are empty, the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex set in a euclidean space, his payoff function A_i is continuous with respect to the variable $\sigma \in \Sigma$, quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ for fixed $\sigma_{e(i)} \in \Sigma_{e(i)}$ and quasi-convex in $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{d(i)} \in \Sigma_{d(i)}$. If, for each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that

$$A_i(\tau_{d(i)}, \sigma_{e(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)})$$

and

$$A_i(\sigma_{d(i)}, \tau_{e(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_{d(i)}, s_{e(i)})$$

for all $i \in N$, then the set of E-saddle points of game Γ_E is non-empty, compact and convex.

PROOF: By theorem IV.7, the set of E-saddle points of game Γ_E is non-empty. Now, we are going to show the compactness of this set. Consider an arbitrary sequence $\sigma(k) \to \sigma$ of E-saddle points for the game Γ_E , that is, for any positive integer k:

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_{d(i)}(k), s_{e(i)}) = A_i(\sigma_{d(i)}(k), \sigma_{e(i)}(k)) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)}(k))$$

for all the players $i \in N$. By the continuity of payoff functions, the three sequences having their general terms formed by the latter expressions given in the equalities, converge to the values obtained by replacing the point $(\sigma_{d(i)}(k), \sigma_{e(i)}(k))$ with $(\sigma_{d(i)}, \sigma_{e(i)})$. Thus, for the limit points, we have

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_{d(i)}, s_{e(i)}) = A_i(\sigma_{d(i)}, \sigma_{e(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)}) .$$

for every $i \in N$. Then, the set of E-saddle points of game Γ_E is compact.

For the convexity, let $\bar{\sigma}$ and $\tilde{\sigma}$ be two arbitrary E-saddle points of game Γ_E . For the player $i \in N$, consider the associated two-person game $\Gamma_i = \{ \Sigma_{d(i)}, \Sigma_{e(i)}; A_i \}$ where the points

$$(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}) \qquad \text{and} \qquad (\tilde{\sigma}_{d(i)}, \tilde{\sigma}_{e(i)})$$

are saddle points. Then, from the definition it follows that

$$V_i = A_i(\bar{\sigma}_{d(i)}, \tilde{\sigma}_{e(i)}) = A_i(\tilde{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}) = A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}) ,$$

since they are equivalents. From here, the two points $(\bar{\sigma}_{d(i)}, \tilde{\sigma}_{e(i)})$ and $(\bar{\sigma}_{d(i)}, \tilde{\sigma}_{e(i)})$ are saddle points of game Γ_i . Now, consider the point

$$(\bar{\sigma}_{d(i)}, \lambda\tilde{\sigma}_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)}) ,$$

for a real number $\lambda\in[0,1]$. On this point, the payoff function A_i takes the value V_i, since it is quasi-convex in the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$.

Now, we will show, that these points are saddle points of game Γ_i. Consider an arbitrary point $\sigma_{d(i)}\in\Sigma_{d(i)}$. By the definition of minimax strategy for the points $\tilde{\sigma}_{e(i)}$ and $\bar{\sigma}_{e(i)}$ we have for this point $\sigma_{d(i)}\in\Sigma_{d(i)}$ that

$$A_i(\sigma_{d(i)},\tilde{\sigma}_{e(i)}) \leq V_i \quad \text{and} \quad A_i(\sigma_{d(i)},\bar{\sigma}_{e(i)}) \leq V_i .$$

Now consider, the set

$$A_{w_i} = \{ \sigma_{e(i)}\in\Sigma_{e(i)}: A_i(\sigma_{d(i)},\sigma_{e(i)}) \leq w_i \}$$

where the value w is given by

$$w_i = \max \{A_i(\sigma_{d(i)},\tilde{\sigma}_{e(i)}) , A_i(\sigma_{d(i)},\bar{\sigma}_{e(i)}) \} .$$

This set must be convex, by virtue of the quasi-convexity of the payoff function A_i. Thus, because $\lambda\tilde{\sigma}_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)}\in A_{w_i}$, we have

$$A_i(\sigma_{d(i)}, \lambda\tilde{\sigma}_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)}) \leq A_i(\bar{\sigma}_{d(i)}, \lambda\tilde{\sigma}_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)}) = V_i ,$$

and therefore, the point $(\sigma_{d(i)}, \lambda\tilde{\sigma}_{e(i)}+(1-\lambda)\bar{\sigma}_{e(i)})$ is a saddle point of Γ_i.

Finally, we will prove that the point

$$\sigma_{\mu,\lambda} = (\mu\bar{\sigma}_{d(i)}+(1-\mu)\tilde{\sigma}_{d(i)}, \lambda\bar{\sigma}_{e(i)}+(1-\lambda)\tilde{\sigma}_{e(i)})$$

with $\mu\in[0,1]$, also it is a saddle point of game Γ_i. On the one hand, on it we have

$$V_i = A_i(\sigma_{\mu,\lambda}) \geq A_i(\sigma_{d(i)}, \sigma\bar{\lambda}_{e(i)} + (1-\lambda)\tilde{\sigma}_{e(i)})$$

for all $\sigma_{d(i)} \in \Sigma_{d(i)}$, since the payoff function A_i is quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$. On the other hand, for an arbitrary $\sigma_{e(i)} \in \Sigma_{e(i)}$, since the points $\bar{\sigma}_{d(i)}$ and $\tilde{\sigma}_{d(i)}$ are maximin strategies,

$$A_i(\bar{\sigma}_{d(i)}, \sigma_{e(i)}) \geq V_i \quad \text{and} \quad A_i(\tilde{\sigma}_{d(i)}, \sigma_{e(i)}) \geq V_i .$$

Then, by taking the value w_i as

$$w_i = \min \{ A_i(\bar{\sigma}_{d(i)}, \sigma_{e(i)}) \ , \ A_i(\tilde{\sigma}_{d(i)}, \sigma_{e(i)}) \}$$

one has that the set

$$A_{w_i} = \{ \sigma_{d(i)} \in \Sigma_{d(i)} : A_i(\sigma_{d(i)}, \sigma_{e(i)}) \geq w_i \}$$

is convex, since A_i is quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$. This, because $\mu\bar{\sigma}_{d(i)} + (1-\mu)\tilde{\sigma}_{d(i)} \in A_{w_i}$ then ,

$$A_i(\mu\bar{\sigma}_{d(i)} + (1-\mu)\tilde{\sigma}_{d(i)}, \sigma_{e(i)}) \geq A(\sigma_{\mu,\lambda}) = V_i,$$

for all $\sigma_{e(i)} \in \Sigma_{e(i)}$.

Then, the point $\sigma_{\mu,\lambda} \in \Sigma$ is saddle point of Γ_i , which implies that the set of saddle points of Γ_i is convex. In particular for all the points of this type

$$\sigma_{\lambda,\lambda} = (\lambda\bar{\sigma}_{d(i)}+(1-\lambda)\tilde{\sigma}_{e(i)}, \ \lambda\bar{\sigma}_{e(i)}+(1-\lambda)\tilde{\sigma}_{e(i)})$$

$$= \lambda\bar{\sigma} + (1-\lambda)\tilde{\sigma}\in\Sigma \quad .$$

Hence, this point is a saddle point of the game Γ_i for each player $i\in N$, that is, the set of E-saddle points of game Γ_E is convex. (Q.E.D.)

Again, as what has been pointed out for very simple saddle points of n-person game in general the interchangeability among E-saddle points does not hold true. The reason for this is essentially the same as for those points. As a special case of the above theorem, one can formulate the corresponding property for mixed extension of finite games.

All the results introduced until now have, from a technical point of view, the same structure, which is the application of Kakutani's fixed point theorem to an appropriate multivalued function. The difference between them arises only in the different definitions of these multivalued functions. Then, it seems natural to think about an uniform result which involves all the previous theorems only as particular cases. The advantage of such a result is to have a simple tool for subsequent applications. This is formulated in the following theorem.

<u>THEOREM IV.11</u>: Let $\Gamma_E^B = \{ \Sigma_1,\ldots,\Sigma_n;B_1,\ldots,B_n \}$ and $\Gamma_E^C = \{ \Sigma_1,\ldots,\Sigma_n;C_1,\ldots,C_n \}$ be two n-person games with the same structure function E, such that the set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space, and his payoff functions B_i and C_i are continuous in the product variable $\sigma\in\Sigma$, where B_i is quasi-concave in the variable $\sigma_{d(i)}\in\Sigma_{d(i)}$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\in\Sigma_{e(i)}x\Sigma_{f(i)}$ and C_i

is quasi-convex in the variable $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $(\sigma_{d(i)}, \sigma_{f(i)}) \in \Sigma_{d(i)} \times \Sigma_{f(i)}$. If, for each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that

$$B_i(\tau_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} B_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)})$$

and

$$C_i(\sigma_{d(i)}, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} C_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)})$$

for all $i \in N$. Then, there exists a joint strategy $\bar{\sigma} \in \Sigma$, which is an E-positive equilibrium point of game Γ_E^B and an E-negative equilibrium point of game Γ_E^C .

PROOF: Consider for a point σ in the non-empty, compact and convex product space Σ the non-empty set

$$\varphi_i(\sigma) = \{ \tau \in \Sigma : B_i(\tau_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} B_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) \}$$

$$\cap \{ \tau \in \Sigma : C_i(\sigma_{d(i)}, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} C_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)}) \}$$

which is convex, since the strategy sets are convex, the payoff function B_i is quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ and the payoff function C_i is quasi-convex in $\sigma_{e(i)} \in \Sigma_{e(i)}$.

Let us consider the multivalued function

$$\psi : \quad \Sigma \to \Sigma$$

defined by the convex set

$$\psi(\sigma) = \bigcap_{i \in N} \varphi_i(\sigma)$$

for a fixed point $\sigma \in \Sigma$. The set $\psi(\sigma)$ is non-empty by virtue of the latter requirement on the payoff functions B_i and C_i . Furthermore, by virtue of the continuity of the payoff functions B_i and C_i, Lemma IV.2 assures the upper-semicontinuity of the multivalued function ψ . Thus, the Kakutani Fixed Point Theorem guarantees a fixed point $\bar{\sigma} \in \Sigma : \bar{\sigma} \in \psi(\bar{\sigma})$. Such a point is E-positive equilibrium point of game Γ_E^B and an E-negative equilibrium point of game Γ_E^C . (Q.E.D.)

Having this useful result, we note that the existence theorem for E-negative equilibrium points for the games expressed in theorem IV.3, arises as an immediate consequence of this theorem. In fact putting $B_i = A_i$ and $C_i = K_i$ for every player $i \in N$ where K_i indicates an arbitrary constant, one can see that this latter result coincides with Theorem IV.3. Furthermore, by introducing $B_i = K_i$ and $C_i = A_i$ where K_i is constant, this theorem coincides with theorem IV.5 which guarantees the existence of E-negative equilibrium points. Finally, by considering in the above theorem $B_i = C_i = A_i$, we obtain theorem IV.7.

Other simple applications of this result will be realized in the next paragraph.

Of course, one could easily associate an intuitive meaning with the above result by considering the simultaneous realization of both games.

IV.2. E-Stable Points

The second section of second chapter where the e-simple stable points were introduced, motivates us now to assign new roles to the associated two-person games of the players. The principal reason for the introduction of these different roles, is because, in the associated two-person games the maximin and minimax value do not generally coincide. Thus, it seems natural to consider the normality of the players formed by the friend and antagonistic coalitions referred to the maximin and minimax strategies.

Given a game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n ; A_1, \ldots, A_n \}$ with the structure function E, and with the usual properties, consider for a joint strategy $\sigma_{f(i)} \in \Sigma_{f(i)}$ of the indifferent coalition of player $i \in N$, the $\sigma_{f(i)}$-associated two-person game $\Gamma_i(\sigma_{f(i)}) = \{ \Sigma_{d(i)}, \Sigma_{e(i)} ; A_i \}$, then, the maximun position guaranteed to player $i \in N$ with whole safety, is the maximin value of the game $\Gamma_i(\sigma_{f(i)})$, which, by simplicity, is again referred to as:

$$V_i(\sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} \; \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(s_{d(i)}, s_{e(i)}, \sigma_{f(i)}) \; .$$

This value is obtained by an appropriate maximin strategy of his friend coalition, which is analytically characterized by

$$F_i(\sigma_{d(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} F_i(s_{d(i)}, \sigma_{f(i)})$$

where, for reasons of simplicity, again we keep the old notation for the function

$$F_i(\sigma_{d(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)}) \; .$$

Now, if the role of the friend coalition is normal, with respect to the above sense, and the antagonistic coalition is looked at as hurting this player (that is, it either attacks him or this player sees his antagonistic coalition as a group of players in whom he cannot have any confidence), then, the following concept introduced in [8] results immediately.

A joint strategy $\bar{\sigma}\in\Sigma$ is said to be an E-maximin stable point or for short an E_m-stable point of the game $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ with structure function E, if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_{d(i)},\Sigma_{e(i)};A_i \}$ for player $i\in N$, the joint strategy $\bar{\sigma}_{d(i)}\in\Sigma_{d(i)}$ is maximin, that is

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\bar{\sigma}_{f(i)}) .$$

Intuitively, an E_m-stable point is a rule of behavior which on the one hand assures at least the amount $V_i(\bar{\sigma}_{f(i)})$ to each player, independently of the behavior of his antagonistic coalition and on the other hand such that the value $V_i(\bar{\sigma}_{f(i)})$ is the maximun safety value which the mentioned player is able to obtain by an appropriate behavior of his friend coalition, if in each instance all the players of his indifferent coalition abide by it.

Once, the E_m-stable point $\bar{\sigma}\in\Sigma$ has been established, the outcome of player $i\in N$ satisfies:

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) \geq V_i(\bar{\sigma}_{f(i)}) .$$

Of course, the concept of $\underline{e}_m$-stable point is a special case of this new one. Furthermore, it is interesting to observe that also the concept of E-positive equilibrium point arises as a special case of E_m-stable point. Indeed, consider a structure function $\bar{E}$ determined by

$$\bar{E}(i) = (d(i), \phi, f(i))$$

for all $i \in N$. Then we have

$$F_i(\sigma_{d(i)}, \sigma_{f(i)}) = A_i(\sigma_{d(i)}, \sigma_{f(i)})$$

which indicates that the E-positive equilibrium point is an E_m-stable point. Of course, this fact is obvious according to the heuristic viewpoint.

Another important connection between both of these concepts is related as follows: a point $\bar{\sigma} \in \Sigma$ is an E_m-stable point of game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ if and only if it is an E-positive equilibrium point of the n-person game

$$\Gamma'_E = \{ \Sigma_1, \ldots, \Sigma_n; F_1, \ldots, F_n \}$$

obtained by replacing in the game Γ_E the payoff function A_i by the minimun function F_i . Indeed, they are equivalent for any game $\Gamma_{\bar{E}}$ with structure function $\bar{E}$ such that

$$d(i) = \bar{d}(i) \qquad \text{and} \qquad f(i) \subseteq \bar{f}(i) \cup \bar{e}(i)$$

for every player $i \in N$.

Using this connection, since the E-positive equilibrium points have a characterization given in theorem IV.3, we can easily formulate the existence of the new points, which is given in the following result generalizing theorem II.5.

<u>THEOREM IV.12</u>: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a euclidean space and his payoff function A_i is continuous in the product variable $\sigma \in \Sigma$ and the function $F_i(\sigma_{d(i)}, \sigma_{f(i)})$ of player $i \in N$ is quasi-concave with respect to the variable $\sigma_{d(i)} \in \Sigma_{d(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. If for each $\sigma \in \Sigma$ there is a $\tau \in \Sigma$ such that

$$F_i(\tau_{d(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\sigma_{f(i)})$$

for all $i\in N$, then the game Γ_E has an E_m-stable point.

PROOF: Consider the game $\Gamma'_E = \{ \Sigma_1,\ldots,\Sigma_n;F_1,\ldots,F_n \}$. This game completely satisfies all the conditions involved in Theorem IV.3, since the function F_i is continuous with respect to the product variable $\sigma\in\Sigma$. Thus, there exists an E-positive equilibrium point of game Γ_E , which is an E_m-stable point of the original game Γ_E . (Q.E.D.)

We note that the above result also is an extension of Theorem IV.3, by what has been said above.

The latter requirement is seen as the defense property for game Γ_E with respect to the notion of E_m-stable point. Given an arbitrary situation there exists another behavior such that if all the players of the indifferent coalition of a player abide by the first one, the second one is maximin for his friend coalition in the resulting game. This is a suitable interpretation of that condition.

An immediate consequence of this theorem is obtained for finite games.

THEOREM IV.13: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be a finite n-person game with structure function E such that the payoff function of player $i\in N$ is of the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} a_i^j(\sigma_j,\sigma_{e(i)},\sigma_{f(i)}) .$$

If for each point $x\in X = \times_{j\in N} \tilde{\Sigma}_j$ there is another point $y\in X$ such that

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(y_{d(i)},s_{e(i)},x_{f(i)}) = \max_{u_{d(i)}\in X_{d(i)}} \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(u_{d(i)},s_{e(i)},x_{f(i)})$$

for all $i\in N$, then, the mixed extension $\tilde{\Gamma}_E = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;E_1,\ldots,E_n \}$ has an E_m-stable point.

PROOF: For the player $i\in N$, by virtue of lemma I.11, the continuous expectation function E_i is linear in the variable $x_{d(i)}\in X_{d(i)}$. Thus, the function

$$F_i(x_{d(i)},x_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(x_{d(i)},s_{e(i)},x_{f(i)})$$

is concave in $x_{d(i)}\in X_{d(i)}$. Furthermore, the last requirement of theorem IV.12 is satisfied by the mixed extension $\tilde{\Gamma}_E$. And so the game $\tilde{\Gamma}_E$ has an E_m-stable point in mixed strategy. (Q.E.D.)

The natural dual notion of E_m-stable points is easily obtained by substituting the dual roles in the associated games assigned to each player. Thus, if the antagonistic coalition in is assumed to have a normal behavior, then it is natural to consider that it will act in accordance with a minimax strategy. On the other hand, the friend coalition is seen as an apathetic player, or rather, without any specified task.

These considerations hold to the following formal notion treated also in [8].

Consider an n-person game $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ having the structure function E , then, a joint strategy $\bar{\sigma}\in\Sigma$ is said to be an <u>E-minimax stable point</u> or for short <u>E^m-stable point</u> of game Γ_E if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_{d(i)},\Sigma_{e(i)};A_i\}$ of player $i\in N$ the joint strategy $\bar{\sigma}_{e(i)}\in\Sigma_{e(i)}$ is a minimax strategy, that is

$$\max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},s_{e(i)},\sigma_{f(i)})=V_i(\bar{\sigma}_{f(i)})$$

or equivalently:

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N$, where G_i indicates the maximun function over $\Sigma_{e(i)}$ of the payoff function A_i .

A possible interpretation of an E^m-stable point is as a rule of behavior which assures each antagonistic coalition that its corresponding player cannot safely obtain more than the amount $V^i(\bar{\sigma}_{f(i)})$ independent of the behavior of his friend coalition. With the property also that this value is the maximun value that the antagonistic coalition will be able safely to limit player $i\in N$ to against the corresponding behavior of his friend coalition, if in each instance all the players of his indifferent coalition abide by it.

Once, such an E^m-stable point $\bar{\sigma}\in\Sigma$ has been established, the outcome of the player $i\in N$ with respect to $\bar{\sigma}\in\Sigma$ satisfies

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) \leq V^i(\bar{\sigma}_{f(i)}) .$$

An interesting case arises when every friend coalition $d(i)$ is constituted by only the player $i\in N$ himself. Here an E^m-stable point is an $\underline{e}^m$-simple stable point.

Another case of an E^m-stable point appears if the friend coalition of each player is empty, that is, if for player $i\in N$:

$$E(i) = (\phi, e(i), f(i)) .$$

In this situation, the definition of E^m-stable point coincides with the E-negative equilibrium point, since

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) = A_i(\sigma_{e(i)},\sigma_{f(i)}) ,$$

for each $i\in N$.

On the other hand, there is another important connection between such concepts, namely: a point $\bar{\sigma}\in\Sigma$ is an E^m-stable point of game $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ if and only if it is an E-negative equilibrium point of the n-person game

$$\Gamma_E'' = \{ \Sigma_1,\ldots,\Sigma_n; G_1,\ldots, G_n \}$$

obtained by substituting in the game Γ_E the payoff function A_i by the maximun function G_i . Furthermore, they are equivalent for any game $\Gamma_{\bar{E}}$ with structure function $\bar{E}$ such that

$$e(i) = \bar{e}(i) \qquad \text{and} \qquad f(i) \subseteq \bar{f}(i) \cup \bar{d}(i) ,$$

for every player $i\in N$.

For this relation, then by using theorem IV.5, one has the following existence theorem which is an extension of theorem II.7.

<u>THEOREM IV.14</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with structure function E such that the set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space and his payoff function A_i is continuous in the variable $\sigma\in\Sigma$ and the function $G_i(\sigma_{e(i)},\sigma_{f(i)})$ of player $i\in N$ is quasi-convex with respect to the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$. If for each $\sigma\in\Sigma$ there is a $\tau\in\Sigma$ such that

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all $i\in N$, then the game Γ_E has an E_m-stable point.

PROOF: Consider the game $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n; G_1,\ldots, G_n \}$. This game fulfills every requirement of theorem IV.5, since the function G_i is continuous in the product variable. Then, the existence of an E-negative equilibrium point for Γ_E is guaranteed. This point is an E^m-stable point of game Γ_E. (Q.E.D.)

The last condition of the above theorem can be expressed as: for any established behavior among the players, there is another one which, if all the players of the indifferent coalition of a player abide by the first one, the second one is minimax for his antagonistic coalition in the resulting game. This property is observed as the attack property of the game Γ_E with respect to the notion of E^m-stable point.

We point out that the previous characterization can be derived in another simple way. Indeed, given a game $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ with structure function E, the zero-sum two-person game

$$\bar{\Gamma}_i(\sigma_{f(i)}) = \{ \Sigma_{e(i)},\Sigma_{d(i)};-A_i \}$$

has its maximin and minimax values

$$\bar{V}_i(\sigma_{f(i)}) \quad \text{and} \quad \bar{V}^i(\sigma_{f(i)})$$

connected with the respective values of the associated game $\Gamma_i(\sigma_{f(i)})$ by the expressions:

$$\bar{V}_i(\sigma_{f(i)}) = -V^i(\sigma_{f(i)}) \qquad \text{and} \qquad \bar{V}^i(\sigma_{f(i)}) = -V_i(\sigma_{f(i)}) \quad .$$

Then, it follows from these equalities that an E^m-stable point of the game Γ_E is an $\bar{E}^m$-stable point of the game

$$\Gamma_{\bar{E}} = \{ \Sigma_1,\ldots,\Sigma_n;-A_1,\ldots,-A_n \}$$

where the structure function $\bar{E}$ is given by

$$\bar{d}(i) = e(i) \quad , \quad \bar{e}(i) = d(i) \quad , \quad \bar{f}(i) = f(i)$$

for every $i \in N$, and conversely.

These statements, in other words, describe the dual roles of the E_m-stable and E^m-stable points for n-person games. This duality is analogous to that between maximin and minimax in zero-sum two-person games.

As an immediate consequence of the above theorem, the following result is considered:

<u>THEOREM IV.15</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be a finite n-person game with structure function E , such that the payoff function of player $i \in N$ is of the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j \in e(i)} a_i^j(\sigma_{d(i)},\sigma_j,\sigma_{f(i)}) \ .$$

If for each point $x \in X = \times_{j \in N} \tilde{\Sigma}_j$ there is another point $y \in X$ such that

$$\max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)},y_{e(i)},x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)},u_{e(i)},x_{f(i)})$$

for every $i \in N$, then the mixed extension $\tilde{\Gamma}_E = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;E_1,\ldots,E_n \}$ has an E^m-stable point.

<u>PROOF</u>: For player $i \in N$, Lemma I.11 gives us that the continuous expectation function E_i is linear in the variable $x_{e(i)} \in X_{e(i)}$. Hence, the function G_i is convex in $x_{e(i)} \in X_{e(i)}$. Thus, the conditions of previous theorem for mixed extension $\tilde{\Gamma}_E$ are fulfilled. And so $\tilde{\Gamma}_E$ has an E^m-stable point. (Q.E.D.)

Having developed the two previous notions which have been obtained by assigning different roles to the players in the associated games, it seems natural to introduce another point by considering that both players (that is, the friend and antagonistic coalition) act in a normal way. This assumption leads immediately to the following formal definition also considered in [11].

Given an n-person game $\Gamma_E = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ with structure function E, then a joint strategy $\bar{\sigma} \in \Sigma$ is called an E-stable point of game Γ_E if for each $\bar{\sigma}_{f(i)}$-associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)}, \Sigma_{e(i)}; A_i\}$ of player $i \in N$ the projection $\bar{\sigma}_{d(i)} \in \Sigma_{d(i)}$ is a maximin strategy of the first player and the projection $\bar{\sigma}_{e(i)} \in \Sigma_{e(i)}$ is a minimax strategy of the second player. In other words,

$$F_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} F_i(s_{d(i)}, \bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)})$$

for every player $i \in N$.

An E-stable point is a rule of behavior which is maximin for each friend coalition and minimax for the corresponding antagonistic coalition in the resulting game, with respect to the respective position of the players, if in each instance all the players of the indifferent coalition abide by it.

Once an E-stable point $\bar{\sigma} \in \Sigma$ has been reached, the outcome of player $i \in N$ satisfies:

$$V_i(\bar{\sigma}_{f(i)}) \leq A_i(\bar{\sigma}_{f(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \leq V^i(\bar{\sigma}_{f(i)}) \quad .$$

A characterization of these kind of points is immediately obtained by using theorem IV.11 applied to the games Γ_E' and Γ_E''.

THEOREM IV.16: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a euclidean space and his payoff function A_i is continuous in the variable $\sigma \in \Sigma$; the function F_i quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ and the function G_i quasi-convex in $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. If for each $\sigma \in \Sigma$ there is $\tau \in \Sigma$ such that for each $i \in N$

$$F_i(\tau_{d(i)}, \sigma_{f(i)}) = V_i(\sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = V^i(\sigma_{f(i)}) \quad ,$$

for all $i \in N$, then the game Γ_E has an E-stable point.

PROOF: Consider the games $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; F_1, \ldots, F_n \}$ and $\Gamma_E'' = \{ \Sigma_1, \ldots, \Sigma_n; G_1, \ldots, G_n \}$ derived from the original game Γ_E . From the continuity of the payoff functions A_i , the functions F_i and G_i are seen to be continuous in $\sigma \in \Sigma$. Therefore all the requirements of theorem IV.11 are satisfied by Γ_E' and Γ_E'' . Hence, the existence of an E-stable point for game Γ_E is guaranteed. (Q.E.D.)

The last condition is observed to be the attack and defense property with respect to the concept of E-stable point.

From this result, we immediately derive the following theorem for finite games.

THEOREM IV.17: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E such that the payoff function of player $i \in N$ is of the form:

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} \sum_{k\in e(i)} a_i^{jk}(\sigma_j,\sigma_k,\sigma_{f(i)}) .$$

If for each point $x\in X = \times_{j\in N} \tilde{\Sigma}_j$, there is another point $y\in X$ such that

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(y_{d(i)},s_{e(i)},x_{f(i)}) = \max_{u_{d(i)}\in X_{d(i)}} \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(u_{d(i)},s_{e(i)},x_{f(i)})$$

and

$$\max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},y_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} \max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},u_{e(i)},x_{f(i)})$$

for all $i\in N$, then, the mixed extension $\tilde{\Gamma}_E = \{ \tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n;E_1,\ldots,E_n \}$ has an E-stable point.

PROOF: For player $i\in N$, in virtue of Lemma I.11, the continuous expectation function E_i of player $i\in N$, is bilinear in the variable $(x_{d(i)},x_{e(i)})\in X_{d(i)}\times X_{e(i)}$. The function F_i is concave in $x_{d(i)}\in X_{d(i)}$ and the function G_i is convex in $x_{e(i)}\in X_{e(i)}$. By the latter requirements on these functions, all the conditions of the previous theorem for mixed extension $\tilde{\Gamma}_E$ are satisfied. And so, $\tilde{\Gamma}_E$ has an E-stable point. (Q.E.D.)

Indeed, such an E-stable point is an E-saddle point, because in each associated two-person game the minimax theorem holds true, since it satisfies all the requirements of theorem IV.7.

An important special case of these E-stable points $\bar{\sigma}\in\Sigma$ occurs, when for each player $i\in N$ the equalities

$$V_i(\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = V^i(\bar{\sigma}_{f(i)})$$

hold. In this instance, since in the game $\Gamma_i(\bar{\sigma}_{f(i)})$ the minimax theorem holds, the point $\bar{\sigma}\epsilon\Sigma$ is an E-saddle point.

This simple observation leads to a second characterization of E-saddle points in games, which appears as an application of theorem IV.16.

THEOREM IV.18: Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a euclidean space, and his payoff function is continuous in the variable $\sigma\epsilon\Sigma$, quasi-concave in $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ for fixed $(\sigma_{e(i)},\sigma_{f(i)})\epsilon\Sigma_{e(i)}x\Sigma_{f(i)}$ and quasi-convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $(\sigma_{d(i)},\sigma_{f(i)})\epsilon\Sigma_{d(i)}x\Sigma_{f(i)}$. If for each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$A_i(\tau_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\tau_{e(i)},\sigma_{f(i)})$$

and

$$= \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\tau_{d(i)},s_{e(i)},\sigma_{f(i)})$$

for all $i\epsilon N$, then, the game Γ_E has an E-saddle point.

PROOF: By virtue of the quasi-concavity and quasi-convexity of payoff function A_i of player $i\epsilon N$, Lemma IV.1 assures the quasi-concavity of function F_i with respect to $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ and the quasi-convexity of function G_i in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. Thus, since the last condition implies the latter requirement of theorem IV.16 this then determines the existence of an E-stable point $\bar{\sigma}\epsilon\Sigma$ of game Γ_E. At such a point we have

$$V_i(\bar{\sigma}_{f(i)}) \leq A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) \leq V^i(\bar{\sigma}_{f(i)})$$

for every $i \in N$.

On the other hand, applying theorem IV.7 to the associated two-person game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{ \Sigma_{d(i)},\Sigma_{e(i)};A_i \}$ considering both players normal, the existence of minimax theorem is guaranteed, that is: $V_i(\bar{\sigma}_{f(i)}) = V^i(\bar{\sigma}_{f(i)})$. Therefore, according to the definition of maximun and minimax strategies, we have

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)})$$
$$= G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

for every player $i \in N$. Hence the point $\bar{\sigma} \in \Sigma$ is an E-saddle point of the game Γ_E. (Q.E.D.)

We are now in a similar position to Chapter II, since theorems IV.7 and IV.18 are two complementary characterizations for E-saddle points.

The previous theorem can be proved in a different fashion by assigning to each point $\sigma \in \Sigma$ and each player $i \in N$ the set

$$\varphi_i(\sigma) = \{\tau \in \Sigma : A_i(\tau_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)},\tau_{e(i)},\sigma_{f(i)})$$

$$= \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\tau_{d(i)},s_{e(i)},\sigma_{f(i)}) \}$$

that is to say, all the saddle points of the $\sigma_{f(i)}$-associated game. Then, we need only the convexity of this set to apply the fixed point technique. But, this condition is guaranteed by theorem IV.10. Thus, by using the usual procedure we obtain the desired result.

A simple consequence of this result is formulated in the following characterization for mixed extension games.

THEOREM IV.19: Let $\Gamma_E = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ be a finite n-person game with structure function E such that the payoff function of the player $i \in N$ is of the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} \sum_{k\in e(i)} a_i^{jk} (\sigma_j,\sigma_k,\sigma_{f(i)}) .$$

If for each point $x \in X = \underset{j\in N}{\times} \tilde{\Sigma}_j$, there is another point $y \in X$ such that for all $i \in N$

$$E_i(y_{d(i)},y_{e(i)},x_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},y_{e(i)},x_{f(i)})$$

$$= \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(y_{d(i)},s_{e(i)},x_{f(i)})$$

then, the mixed extension $\tilde{\Gamma}_E = \{ \tilde{\Sigma}_1,\dots,\tilde{\Sigma}_n;E_1,\dots,E_n \}$ has an E-saddle point.

PROOF: The expectation function E_i of player $i \in N$ is bilinear in the variable $(x_{d(i)},x_{e(i)}) \in X_{d(i)} \times X_{e(i)}$ for fixed $x_{f(i)} \in X_{f(i)}$, since the requirements of lemma I.11 are satisfied. Thus, for the latter condition, theorem IV.18 guarantees an E-saddle point for the mixed extension Γ_E. (Q.E.D.)

We note that, indeed, theorem IV.19 and theorem IV.17 coincide, since the minimax theorem holds in the associated games.

In conclusion we note that the formulations of the existence theorems for E-stable points are completely useless in the case where the structure function has all the friend coalitions void. Moreover, in such an instance also the second characterization of E-saddle points determined in theorem IV.18 is also without usefulness.

IV.3. Comparison of the Concepts for Different Structure Functions

The results derived in the previous sections are concerned with the existence of those concepts, which arose as natural extensions of the notions introduced in the two first chapters.

No reference to the relationship between the concepts for different structures assigned to a game, has been yet considered. This will be partially attempted in the present paragraph.

THEOREM IV.20: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n \}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\in N$ is non-empty and compact in a euclidean space and the payoff function A_i is continuous. If a structure function $\bar{E}$ satisfies: $d(i)\supseteq\bar{d}(i)$ for all players $i\in N$. Then the set $P(\Gamma_E)$ of E-positive equilibrium points of Γ_E is contained in the set $P(\Gamma_{\bar{E}})$ of $\Gamma_{\bar{E}}$.

PROOF: For any point σ in the product space Σ

$$\max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) \geq \max_{s_{\bar{d}(i)}\in\Sigma_{\bar{d}(i)}} A_i(s_{\bar{d}(i)},\sigma_{\bar{e}(i)},\sigma_{\bar{f}(i)})$$

for all players $i\in N$, since $d(i)\supseteq\bar{d}(i)$. Thus, for an E-positive equilibrium point $\bar{\sigma}\in\Sigma$ of game Γ_E with structure function E, that is,

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N$, we have

$$A_i(\bar{\sigma}_{\bar{d}(i)},\bar{\sigma}_{\bar{e}(i)},\bar{\sigma}_{\bar{f}(i)}) \geq \max_{s_{\bar{d}(i)}\in\Sigma_{\bar{d}(i)}} A_i(s_{\bar{d}(i)},\bar{\sigma}_{\bar{e}(i)},\bar{\sigma}_{\bar{f}(i)})$$

for every player $i \in N$, since

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_{\bar{d}(i)},\bar{\sigma}_{\bar{e}(i)},\bar{\sigma}_{\bar{f}(i)}) \quad .$$

In the last inequality only the strict equality sign must hold, since, in the second term the maximun over the set $\Sigma_{\bar{d}(i)}$ appears. Hence, the point $\bar{\sigma}\in\Sigma$ is an $\bar{E}$-positive equilibrium point of game $\Gamma_{\bar{E}}$, that is to say: $P(\Gamma_E) \subseteq P(\Gamma_{\bar{E}})$. (Q.E.D.)

From here, we can derive, from a heuristic viewpoint, that, after an E-positive equilibrium point has been reached in a game, the friends of a given player can be seen as an indifferent player. Thus, the afore mentioned behavior does not lose its new character in the new structure.

Furthermore, without any reference to this result, theorem IV.3 applied to the structure function E also contributes a characterization for $\bar{E}$-positive equilibrium points such that $d(i) \supseteq \bar{d}(i)$. Indeed, by the same arguments in the proof just considered, under such conditions, Γ_E satisfied all the requirements of theorem IV.3.

An analogous statement is derived for the dual concept of E-negative equilibrium points.

<u>THEOREM IV.21</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with structure function E, such that the strategy set Σ_i of player $i \in N$ is non-empty and compact in a euclidean space and the payoff function A_i is continuous. If a structure function $\bar{E}$ satisfies: $e(i) \supseteq \bar{e}(i)$ for all the players $i \in N$, then the set $Q(\Gamma_E)$ of E-negative equilibrium point of Γ_E is contained in the set $Q(\Gamma_{\bar{E}})$ of the game $\Gamma_{\bar{E}}$.

PROOF: For an arbitrary point $\sigma \in \Sigma$ we have

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)}) \leq \min_{s_{\bar{e}(i)} \in \Sigma_{\bar{e}(i)}} A_i(\sigma_{\bar{d}(i)}, s_{\bar{e}(i)}, \sigma_{\bar{f}(i)})$$

for all players $i \in N$, since $e(i) \supseteq \bar{e}(i)$. In particular, this holds for an E-negative equilibrium point $\bar{\sigma} \in \Sigma$ of the game Γ_E. Therefore, from that inequality and the definition of this point we get

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \leq \min_{s_{\bar{e}(i)} \in \Sigma_{\bar{e}(i)}} A_i(\bar{\sigma}_{\bar{d}(i)}, s_{\bar{e}(i)}, \bar{\sigma}_{\bar{f}(i)})$$

for all players $i \in N$. In all these inequalities only the strict equality sign must hold since in the latter terms the minimun over the set $\Sigma_{e(i)}$ appears. Thus, we have $Q(\Gamma_E) \subseteq Q(\Gamma_{\bar{E}})$. (Q.E.D.)

Any antagonistic player can be seen as an indifferent with respect to the corresponding player, after an E-negative equilibrium point has been established. Thus, it is invariant under this change in the structure. This is a possible intuitive interpretation of the above result. Moreover, under the requirements of theorem IV.5 for structure function E, we obtain the characterization of E-negative equilibrium points.

Considering both of the previous results together, we get:

THEOREM IV.22: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E, such that the strategy set Σ_i of player $i \in N$ is non-empty and compact in a euclidean space and the payoff function A_i is continuous. If a structure function $\bar{E}$ satisfies: $d(i) \supseteq \bar{d}(i)$ and $e(i) \supseteq \bar{e}(i)$ for all the players $i \in N$, then the set $R(\Gamma_E)$ of E-saddle points of Γ_E is contained in the set $R(\Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

PROOF: Directly from the definitions, we have that

$$R(\Gamma_E) = P(\Gamma_E) \cap Q(\Gamma_E) .$$

Hence, by both previous lemmas,

$$P(\Gamma_E) \subseteq P(\Gamma_{\bar{E}}) \quad \text{and} \quad Q(\Gamma_E) \subseteq Q(\Gamma_{\bar{E}})$$

which implies $R(\Gamma_E) \subseteq R(\Gamma_{\bar{E}})$. (Q.E.D.)

In particular, an E-saddle point of game Γ_E with all the indifferent coalitions empty, it is an E-saddle point for game $\Gamma_{\bar{E}}$ with the structure function $\bar{E}$ such that $d(i) \supseteq \bar{d}(i)$ and $e(i) \supseteq \bar{e}(i)$ for every player $i \in N$.

After reaching such a point any antagonists and friends of a given player can be seen as indifferents. Thus, the point remains invariant under the new structure. This is a possible interpretation of the above lemma.

By this connection of the concept of E-saddle point between different structure we note that theorem IV.7 contributes a direct characterization for $\bar{E}$-saddle points where the structure function $\bar{E}$ is related as above.

The situation for stable points is a little more complicated, and therefore we formulate some auxiliary results.

LEMMA IV.23: Let A be a continuous function on the product space $\Sigma = \times_{i \in N} \Sigma_i$, where for each $i \in N$ the non-empty set Σ_i is compact in a euclidean space. Let

$$\underline{P} = (d,e,f) \quad \text{and} \quad \underline{\bar{P}} = (\bar{d},\bar{e},\bar{f})$$

be two partitions of the set N , such that

$$d \supseteq \bar{d} \ , \quad e \subseteq \bar{e} \ , \quad f \supseteq \bar{f} \ .$$

Then for each point $\sigma\in\Sigma$,

$$\max_{s_d\in\Sigma_d}\ \min_{s_e\in\Sigma_e} A(s_d,s_e,\sigma_f) \geq \max_{s_{\bar{d}}\in\Sigma_{\bar{d}}}\ \min_{s_{\bar{e}}\in\Sigma_{\bar{e}}} A(s_{\bar{d}},s_{\bar{e}},\sigma_{\bar{f}})$$

and

$$\min_{s_e\in\Sigma_e}\ \max_{s_d\in\Sigma_d} A(s_d,s_e,\sigma_f) \geq \min_{s_{\bar{e}}\in\Sigma_{\bar{e}}}\ \max_{s_{\bar{d}}\in\Sigma_{\bar{d}}} A(s_{\bar{d}},s_{\bar{e}},\sigma_{\bar{f}})$$

<u>PROOF</u>: Given a $\sigma\in\Sigma$, then for any point $\tau_d\in\Sigma_d$, because $\bar{e}\supseteq f-\bar{f}$, we have

$$\min_{s_e\in\Sigma_e} A(\tau_d,s_e,\sigma_f) \geq \min_{s_{e\cup(f-\bar{f})}\in\Sigma_{e\cup(f-\bar{f})}} A(\tau_d,s_{e\cup(f-\bar{f})},\sigma_{\bar{f}}) ,$$

and therefore

$$\max_{s_d\in\Sigma_d}\ \min_{s_e\in\Sigma_e} A(s_d,s_e,\sigma_f) \geq \max_{s_d\in\Sigma_d}\ \min_{s_{e\cup(f-\bar{f})}\in\Sigma_{e\cup(f-\bar{f})}} A(s_d,s_{e\cup(f-\bar{f})},\sigma_{\bar{f}}) .$$

On the other hand, since $\bar{e} = e\cup(d-\bar{d})\cup(f-\bar{f})$ for each $\tau_d\in\Sigma_d$, we have

$$\min_{s_{e\cup(f-\bar{f})}\in\Sigma_{e\cup(f-\bar{f})}} A(\tau_d,s_{e\cup(f-\bar{f})},\sigma_{\bar{f}}) \geq \min_{s_{\bar{e}}\in\Sigma_{\bar{e}}} A(\tau_{\bar{d}},s_{\bar{e}},\sigma_{\bar{f}})$$

which implies the following inequality

$$\max_{s_d\in\Sigma_d}\ \min_{s_{e\cup(f-\bar{f})}\in\Sigma_{e\cup(f-\bar{f})}} A(s_d,s_{e\cup(f-\bar{f})},\sigma_{\bar{f}}) \geq \max_{s_{\bar{d}}\in\Sigma_{\bar{d}}}\ \min_{s_{\bar{e}}\in\Sigma_{\bar{e}}} A(s_{\bar{d}},s_{\bar{e}},\sigma_{\bar{f}})$$

From these relations we deduce the validity of the first assertion.

Now, in a similar way we prove the second inequality. Given a point $\sigma\in\Sigma$ then for each $\tau_{e\cup(d-\bar{d})}\in\Sigma_{e\cup(d-\bar{d})}$ the following relation holds

$$\max_{s_d\in\Sigma_d} A(s_d,\tau_e,\sigma_f) \geq \max_{s_{\bar{d}}\in\Sigma_{\bar{d}}} A(s_{\bar{d}},\tau_{e\cup(f-\bar{f})},\sigma_f) ,$$

which implies the inequality

$$\min_{s_e\in\Sigma_e}\max_{s_d\in\Sigma_d} A(s_d,s_e,\sigma_f) \geq \min_{s_{e\cup(d-\bar{d})}\in\Sigma_{e\cup(d-\bar{d})}}\max_{s_{\bar{d}}\in\Sigma_{\bar{d}}} A(s_{\bar{d}},s_{\bar{e}\cup(f-\bar{f})},\sigma_f) .$$

On the other hand,

$$\min_{s_{e\cup(d-\bar{d})}\in\Sigma_{e\cup(d-\bar{d})}}\max_{s_{\bar{d}}\in\Sigma_{\bar{d}}} A(s_{\bar{d}},s_{e\cup(d-\bar{d})},\sigma_f) \geq \min_{s_{\bar{e}}\in\Sigma_{\bar{e}}}\max_{s_{\bar{d}}\in\Sigma_{\bar{d}}} A(s_{\bar{d}},s_{\bar{e}},\sigma_{\bar{f}}) .$$

Thus, by composing both relations the second assertion is guaranteed. (Q.E.D.)

LEMMA IV.24: Let A be a continuous function on the product space $\Sigma = \times_{i\in N}\Sigma_i$ where for each $i\in N$ the non-empty set Σ_i is compact in a euclidean space. Let

$$\underline{P} = (d,e,f) \quad\text{and}\quad \bar{\underline{P}} = (\bar{d},\bar{e},\bar{f})$$

be two partitions of the set N, such that

$$d\subseteq\bar{d} , \quad e\supseteq\bar{e} , \quad f\supseteq\bar{f} .$$

Then, for each point $\sigma\in\Sigma$

$$\max_{s_d\in\Sigma_d}\min_{s_e\in\Sigma_e} A(s_d,s_e,\sigma_f) \leq \max_{s_{\bar{d}}\in\Sigma_{\bar{d}}}\min_{s_{\bar{e}}\in\Sigma_{\bar{e}}} A(s_{\bar{d}},s_{\bar{e}},\sigma_{\bar{f}})$$

and

$$\min_{s_e \in \Sigma_e} \max_{s_d \in \Sigma_d} A(s_d, s_e, \sigma_f) \leq \min_{s_{\bar{e}} \in \Sigma_{\bar{e}}} \max_{s_{\bar{d}} \in \Sigma_{\bar{d}}} A(s_{\bar{d}}, s_{\bar{e}}, \sigma_{\bar{f}}) \; .$$

PROOF: First we will prove the first relation. Given an arbitrary point $\sigma \in \Sigma$ for a $\tau_{d \cup (e-\bar{e})} \in \Sigma_{d \cup (e-\bar{e})}$, we have that

$$\min_{s_e \in \Sigma_e} A(\tau_d, s_e, \sigma_f) \leq \min_{s_{\bar{e}} \in \Sigma_{\bar{e}}} A(\tau_{d \cup (e-\bar{e})}, s_{\bar{e}}, \sigma_f)$$

since $\bar{d} \supseteq e-\bar{e}$. From this, we obtain

$$\max_{s_d \in \Sigma_d} \min_{s_e \in \Sigma_e} A(s_d, s_e, \sigma_f) \leq \max_{s_{d \cup (e-\bar{e})} \in \Sigma_{d \cup (e-\bar{e})}} \min_{s_{\bar{e}} \in \Sigma_{\bar{e}}} A(s_{d \cup (e-\bar{e})}, s_{\bar{e}}, \sigma_f) \; .$$

On the other hand, because $\bar{d} \supseteq f-\bar{f}$, the following is always true:

$$\max_{s_{d \cup (e-\bar{e})} \in \Sigma_{d \cup (e-\bar{e})}} \min_{s_{\bar{e}} \in \Sigma_{\bar{e}}} A(s_{d \cup (e-\bar{e})}, s_{\bar{e}}, \sigma_f) \leq \max_{s_{\bar{d}} \in \Sigma_{\bar{d}}} \min_{s_{\bar{e}} \in \Sigma_{\bar{e}}} A(s_{\bar{d}}, s_{\bar{e}}, \sigma_{\bar{f}}) \; ,$$

and therefore the validity of the first assertion is proven.

Now, we prove the second relation. For an arbitrary point $\sigma \in \Sigma$ and a given $\tau_e \in \Sigma_e$,

$$\max_{s_d \in \Sigma_d} A(s_d, \tau_e, \sigma_f) \leq \max_{s_{d \cup (f-\bar{f})} \in \Sigma_{d \cup (f-\bar{f})}} A(s_{d \cup (f-\bar{f})}, \tau_e, \sigma_{\bar{f}}) \; ,$$

and therefore,

$$\min_{s_e \in \Sigma_e} \max_{s_d \in \Sigma_d} A(s_d, s_e, \sigma_f) \leq \min_{s_e \in \Sigma_e} \max_{s_{d \cup (f-\bar{f})} \in \Sigma_{d \cup (f-\bar{f})}} A(s_{d \cup (f-\bar{f})}, s_e, \sigma_{\bar{f}}) \; .$$

On the other hand, because $\bar{d} = d \cup (e-\bar{e}) \cup (f-\bar{f})$, for $\tau_e \epsilon \Sigma_e$, we have:

$$\max_{s_{d \cup (f-\bar{f})} \epsilon \Sigma_{d \cup (f-\bar{f})}} A(s_{d \cup (f-\bar{f})}, \tau_e, \sigma_f) \quad \max_{s_{\bar{d}} \epsilon \Sigma_{\bar{d}}} A(s_{\bar{d}}, \tau_e, \sigma_{\bar{f}}) \quad ,$$

which implies

$$\min_{s_e \epsilon \Sigma_e} \max_{s_{d \cup (f-\bar{f})} \epsilon \Sigma_{d \cup (f-\bar{f})}} A(s_{d \cup (f-\bar{f})}, s_e, \sigma_f) \quad \min_{s_{\bar{e}} \epsilon \Sigma_{\bar{e}}} \max_{s_{\bar{d}} \epsilon \Sigma_{\bar{d}}} A(s_{\bar{d}}, s_{\bar{e}}, \sigma_{\bar{f}}) \quad .$$

Thus, by combining both relations the second assertion is proved. (Q.E.D.)

As an immediate consequence of this, we get the following simple statement.

<u>THEOREM IV.25</u>: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E , such that the strategy set Σ_i of player $i \epsilon N$ is non-empty and compact in a euclidean space and the payoff function A_i is continuous. If a structure function $\bar{E}$ satisfies:

$$d(i) \supseteq \bar{d}(i) \quad , \quad e(i) = \bar{e}(i) \quad , \quad f(i) \subseteq \bar{f}(i)$$

for all the players $i \epsilon N$, then the set $S(\Gamma_E)$ of E_m-stable points of game Γ_E is contained in the set $S(\Gamma_{\bar{E}})$ of the game $\Gamma_{\bar{E}}$.

<u>PROOF</u>: Let $\bar{\sigma} \epsilon \Sigma$ be an E_m-stable point of game Γ_E , that is:

$$F_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{f(i)}) = V_i(\bar{\sigma}_{f(i)})$$

for every player $i \epsilon N$. By lemma IV.24 applied with

$$d = \bar{d}(i) \supseteq d(i) = d \quad , \quad e = \bar{e} = e(i) \; , \quad f = \bar{f}(i) \subseteq f(i) = \bar{f}$$

for player $i \epsilon N$, and at point $\bar{\sigma} \epsilon \Sigma$ we have

$$V_i(\bar{\sigma}_{f(i)}) \geq V_i(\bar{\sigma}_{\bar{f}(i)}) \quad .$$

But

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_{\bar{d}(i)},\bar{\sigma}_{\bar{f}(i)}) \quad ,$$

which implies

$$F_i(\bar{\sigma}_{\bar{d}(i)},\bar{\sigma}_{\bar{f}(i)}) \geq V_i(\bar{\sigma}_{\bar{f}(i)})$$

for all players $i \in N$. In these inequalities only the strict equality sign must hold since in the second terms appears the maximin of the game $\Gamma_i(\bar{\sigma}_{\bar{f}(i)})$. Hence, we have $S(\Gamma_E) \subseteq S(\Gamma_{\bar{E}})$. (Q.E.D.)

We can easily perform an analogous interpretation similar to what has been done for the previous results.

In a similar way, we derive for E^m-stable points the following assertion by comparing different structures.

<u>THEOREM IV.26</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with structure function E , such that the strategy set Σ_i of player $i \in N$ is non-empty and compact in a euclidean space and the payoff function A_i is continuous. If a structure function $\bar{E}$ satisfies:

$$d(i) = \bar{d}(i) \quad , \quad e(i) \supseteq \bar{e}(i) \quad , \quad f(i) \subseteq \bar{f}(i) \; ,$$

for all the players $i \in N$, then the set $T(\Gamma_E)$ of E^m-stable points of game Γ_E is contained in the set $T(\Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

<u>PROOF</u>: Let $\bar{\sigma} \in \Sigma$ be an E^m-stable point of game Γ_E , that is:

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = V^i(\bar{\sigma}_{f(i)})$$

for all players $i \in N$. Now, by applying lemma IV.23 with

$$d = \bar{d} = d(i) \quad , \quad e = \bar{e}(i) \subseteq e(i) = \bar{e} \quad , \quad f = \bar{f}(i) \supseteq f(i) = \bar{f}$$

for player $i \in N$, we have

$$V^i(\bar{\sigma}_{f(i)}) \leq V^i(\bar{\sigma}_{\bar{f}(i)}) \quad .$$

On the other hand,

$$G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{\bar{e}(i)}, \bar{\sigma}_{\bar{f}(i)})$$

which implies

$$G_i(\bar{\sigma}_{\bar{e}(i)}, \sigma_{\bar{f}(i)}) \leq V^i(\sigma_{\bar{f}(i)})$$

for all players $i \in N$. Here only the equality sign must hold, since in the second term there appears the minimax value of the game $\Gamma_i(\bar{\sigma}_{\bar{f}(i)})$. Hence $T(\Gamma_E) \subseteq T(\Gamma_{\bar{E}})$. (Q.E.D.)

Unfortunately, we are not able to obtain a similar result for E-stable points. The reason for this is due to the formation between the antagonist, friend and indifferent coalitions given in the preceding results, about stable points.

CHAPTER V

V.1 E-Composed Points

In the previous chapter the original notions of the first chapters have been extended in a natural manner. The examination just considered was essentially motivated by the introduction of the structure function, which involves the three basic concepts of friend, antagonistic and indifferent players. Of course, one could try to extend all those results by seeking to incorporate a new notion similar to these, with respect to a player. For the moment, we do not have any new concepts, which would play analogous roles as these considered. For this reason, the results treated in the preceding chapter, in reference with our systematic exposition, are tabled. Nevertheless, there are some other possible ways of extending them.

In this chapter we are concerned with two different approaches to generalizations. These approaches have completely distinguished outlines and therefore we will consider them separately.

Until now, we have assumed that all the players have the same characterization. Indeed, when the associated games, determined by the situation of the indifferent players was considered, the friend and antagonistic coalition of every player were assumed to behave, with respect to our associated game, only in one specified way. In other words, the assumptions on the behavior of the friend and antagonistic coalitions, have been the same for all the players.

Of course, such a restriction could be weakened in many ways. This examination will be the subject matter of this paragraph, which constitutes one of the two approaches considered in this chapter.

As before we consider the players of the original game, embedded in the corresponding two-person game determined by the situations of the indifferent coalitions. Thus, these situations at the instant of the analyzing are considered fixed. Actually, the corresponding friend and antagonistic coalitions of different players can be seen as behaving with non-analogous roles.

Among all the possibilities of associated behaviors in the coalitions, there are some quite simple ones, which will be examined as follows. These are simple illustrations of the general case, which will be introduced in a more natural manner after this formulation.

As a first special case, let us consider that the set of players N of game Γ is divided into two disjoint sets, N_1 and N_2. For all the players $i \in N$ in the set N_1, their assigned antagonistic coalition $e(i)$, in the associated game, is supposed to be indifferent with respect of their position, and their friend coalitions are seen as normal players. On the other hand, for every player $i \in N_2$, his own friend coalition is considered a normal player without any reference to his antagonistic coalition.

Formally, given a game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with structure function E, and a partition $\{N_1, N_2\}$ of the set N, a joint strategy $\bar{\sigma} \in \Sigma$ is said to an <u>$N_1^+ N_{2,s}^+$ E-composed point</u> or concisely an <u>E-composed point</u> with respect to N_1^+, $N_{2,s}^+$ of game Γ_E, if

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$$

for all the players $i \in N_1$ and

$$F_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} F_i(s_{d(i)}, \bar{\sigma}_{f(i)})$$

for every player $i \in N_2$.

Of course, this concept is a straightforward extension of the concept of E-positive equilibrium point which is derived if the set N_2 is empty, and of the notion of E_m-stable point, obtained when N_1 is void.

In a similar fashion to what has been derived in theorems IV.20 and IV.25, it follows immediately that given a game Γ with structure function E, and a partition $\{N_1, N_2\}$ of the set of players. If a structure function $\bar{E}$ satisfies: $d(i) \supseteq \bar{d}(i)$ for all the players $i \in N_1$ and

$$d(i) \supseteq \bar{d}(i) \quad , \quad e(i) = \bar{e}(i) \quad , \quad f(i) \subseteq \bar{f}(i)$$

for all players $i \in N_{2,s}$, then the set of $N_1^+ N_{2,s}^+$ E-composed points of game Γ_E : $P(N_1^+ N_{2,s}^+, \Gamma_E)$ is contained in the set $P(N_1^+ N_{2,s}^+, \Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

The existence of such a point will be derived in the same way as the other concepts of this paragraph, as a direct application of the following general theorem, which extends theorem IV.11.

<u>THEOREM V.1</u>: Let $\Gamma^j_{E_j} = \{ \Sigma_1, \ldots, \Sigma_n; B^j_1, \ldots, B^j_n \}$ $(j \in P = \{1, \ldots, p\})$ be n-person games defined on the same strategy sets with the structure functions E_j, respectively such that the set Σ_i of player $i \in N$ is non-empty, compact and convex in a euclidean space, for all $j \in P$, his payoff function B^j_i in game $\Gamma^j_{E_j}$ is continuous in the product variable and quasi-concave in the variable $\sigma^j_{d(i)} \in \Sigma_{f^j(i)}$ for fixed $\sigma_{e^j(i) \cup f^j(i)} \in \Sigma_{d^j(i) \cup f^j(i)}$. If for each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that

$$B^j_i(\tau_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) = \max_{s_{d^j(i)} \in \Sigma_{d^j(i)}} B^j_i(s_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)})$$

for all $j \in P$ and all the players $i \in N$, then there exists a joint strategy $\bar{\sigma} \in \Sigma$, which is simultaneously for all $j \in M$, an E_j-positive equilibrium point of game $\Gamma^j_{E_j}$.

PROOF: For an arbitrary joint strategy σ in the product space consider the set

$$\varphi_i^j(\sigma) = \{ \tau\epsilon\Sigma : B_i^j(\tau_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) = \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) \}$$

which is convex by virtue of the quasi-concavity of the payoff function B_i^j in $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$ and by the convexity of the strategy sets.

Let us define the multivalued function

$$\psi : \Sigma \rightarrow \Sigma$$

as the set

$$\psi(\sigma) = \bigcap_{i\epsilon N} \bigcap_{j\epsilon M} \varphi_i^j(\sigma)$$

for all the points $\sigma\epsilon\Sigma$. The set $\psi(\sigma)$ is non-empty by virtue of the latter condition. On the other hand, because the payoff functions are all continuous, and so from the lemma IV.2 we establish the upper-semicontinuity of the multivalued function ψ. Then, the Kakutani fixed point theorem guarantees a point $\bar{\sigma}\epsilon\Sigma : \bar{\sigma}\epsilon\psi(\bar{\sigma})$. Such a point is an E_j-positive equilibrium point of the game $\Gamma_{E_j}^j$, for all $j\epsilon P$. (Q.E.D.)

Having this general result, the existence of the previous E-composed points is derived as an immediate consequence.

THEOREM V.2: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a euclidean space, his payoff function A_i is continuous with respect to the product variable $\sigma\epsilon\Sigma$, A_i is quasi-concave in the variable $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ for fixed $\sigma_{e(i)\cup f(i)}\epsilon\Sigma_{e(i)\cup f(i)}$. If for a partition $\{N_1, N_2\}$ of the set of players N and each joint strategy $\sigma \in \Sigma$ there is a point $\tau \in \Sigma$ such that

$$A_i(\tau_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)})$$

for all the players $i \in N_1$ and

$$F_i(\tau_{d(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} F_i(s_{d(i)}, \sigma_{f(i)})$$

for all players $i \in N_2$. Then the game Γ_E has a $N_1^+ N_{2,s}^+$ E-composed point.

PROOF: Given the n-person game Γ_E with structure function E , consider the following associated games

$$\Gamma_E^B = \{\Sigma_1, \ldots, \Sigma_n; B_1, \ldots, B_n\}, \qquad \Gamma_E^C = \{\Sigma_1, \ldots, \Sigma_n; C_1, \ldots, C_n\}$$

whose structure function coincides with E .

The payoff functions are given by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i \in N_1 \\ K_i & \text{if } i \in N_{2,s} \end{cases}$$

and

$$C_i(\sigma) = \begin{cases} K_i & \text{if } i \in N_1 \\ F_i(\sigma_{d(i)}, \sigma_{f(i)}) & \text{if } i \in N_{2,s} \end{cases}$$

for player $i \in N$, where K_i indicates an arbitrary constant.

Thus, all the payoff functions B_i of game Γ_E^B are quasi-concave in the variable $\sigma_{d(i)} \in \Sigma_{d(i)}$ for fixed $\sigma_{e(i) \cup f(i)} \in \Sigma_{e(i) \cup f(i)}$. Furthermore, from the quasi-concavity of A_i, lemma IV.1, and the definition of structure function E, we see that the payoff function C_i is quasi-concave with respect to $\sigma_{d(i)} \in \Sigma_{d(i)}$.

Hence, the games Γ_E^B and Γ_E^C completely satisfy all the requirements of theorem V.1 and therefore, there exists a joint strategy $\bar{\sigma}\in\Sigma$ which is simultaneously an E-positive equilibrium point for both the game Γ_E^B and the game Γ_E^C. Clearly, such a point is an E-composed point with respect to N_1^+, $N_{2,s}^+$. (Q.E.D.)

For example, if for every player $i\in N$ $d(i) = \{i\}$, then for any partition $\{N_1,N_2\}$ a game Γ_E satisfying the requirements of continuity and quasi-convexity of the previous theorem, always has an N_1^+ N_{2s}^+ E-composed point.

From this we easily derive the characterization for finite games.

<u>THEOREM V.3</u>: Let $\Gamma_E = \{ \Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n \}$ be a finite n-person game with structure function E such that the payoff function of player $i\in N$ has the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} a_i^j(\sigma_j,\sigma_{e(i)},\sigma_{f(i)}) .$$

If for a partition $\{N_1,N_2\}$ of the set of players N, and for each point $x\in X = \underset{j\in N}{\times} \tilde{\Sigma}_j$, there is another point $y\in X$ such that

$$E_i(y_{d(i)},x_{e(i)},x_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},x_{e(i)},x_{f(i)})$$

for all $i\in N_1$ and

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(y_{d(i)},s_{e(i)},x_{f(i)}) = \max_{u_{d(i)}\in X_{d(i)}} \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(u_{d(i)},s_{e(i)},x_{f(i)})$$

for all $i\in N_2$, then, the mixed extension $\tilde{\Gamma}_E$ has an $N_1^+N_{2,s}^+$ E-composed point.

<u>PROOF</u>: Again, by lemma I.11, the expectation function E_i of player $i\in N$ is linear in the variable $x_{d(i)}\in X_{d(i)}$, which implies the concavity of the functions minimin in such a variable. Thus, all the requirements of the previous theorem apply to mixed extension $\tilde{\Gamma}_E$. Hence, the game $\tilde{\Gamma}_E$ has an $N_1^+N_{2,s}^+$ E-composed point. (Q.E.D.)

In a certain sense, specified as follows, the latter concept has a dual description, which can be easily obtained, by changing the roles of the players in the associated games. Indeed, given an n-person game Γ, whose set of players N is seen to be divided into two sets N_1 and N_2. On the one hand, the players $i \in N$ in the first set N_1 have the property that their corresponding antagonistic coalitions, in the associated game determined by the behavior of their respective indifferent players, act in a manner so as to hurt the first player, that is, the friend coalition. The first player is not trying to defend his respective player $i \in N$, but only is assumed to be apathetic with respect to him. On the other hand, for all the players in the complement of $N_1 : N_2$, their antagonistic coalitions are assumed to behave normally which is in accordance with a minimax strategy. Here, again the friend coalition has no specified role with respect to the position of his representative player.

These considerations can be described by the following formulation: given an n-person game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with structure function E, and a partition $\{N_1, N_2\}$ of the set of players N, a joint strategy $\bar{\sigma} \in \Sigma$ is said to be an <u>$N_1^- N_{2,s}^-$ E-composed point</u> or shortly an <u>E-composed point with</u> respect to $N_1^- N_{2,s}^-$ of game Γ_E, if

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_{d(i)}, s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all the players $i \in N_1$ and

$$G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)})$$

for all the players $i \in N_2$.

In particular we get the notion of E-negative equilibrium point when the set N_2 is empty. Furthermore, in the other extreme, when the set N_1 is void, the notion of E^m-stable point is derived.

By the respective duality between the concepts of E-positive and E-negative equilibrium points and the E_m and E^m-stable points, the following simple connection with E-composed points is immediately derivable.

Given an n-person game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with structure function E , consider the game $\Gamma'_{\bar{E}} = \{ \Sigma_1, \ldots, \Sigma_n; -A_1, \ldots, -A_n \}$ obtained from the original Γ_E by substituting for payoff function its respective negative, whose structure function $\bar{E}$ is given by:

$$\bar{d}(i) = e(i) \quad , \quad \bar{e}(i) = d(i) \quad , \quad \bar{f}(i) = f(i) \quad .$$

Then, from the simple relations between the minimax and maximin expressions it follows that for an arbitrary partition $\{ N_1, N_2 \}$ of N , a joint strategy is an $N_1^+ N_{2,s}^+$ E-composed point of game $\Gamma_{\bar{E}}$, if and only if it is an $N_1^- N_{2,s}^-$ $\bar{E}$-composed point of associated game $\Gamma_{\bar{E}}$.

Also from theorems IV.21 and IV.26, one derives the following immediate comparison: Given a game Γ with structure function E and a partition $\{N_1, N_2\}$ of the set of players N ; if a structure function $\bar{E}$ fulfills $\bar{e}(i) \supseteq e(i)$ for all the players $i \in N_1$ and

$$d(i) = \bar{d}(i) \quad , \quad e(i) \supseteq \bar{e}(i) \quad , \quad f(i) \subseteq \bar{f}(i)$$

for all the players $i \in N_2$, then the set of $N_1^- N_{2,s}^-$ E-composed points of game Γ_E . $Q(N_1^-, N_{2,s}^-, \Gamma_E)$ is contained in the set $Q(N_1^-, N_{2,s}^-, \Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

As an immediate consequence of the general result formulated in theorem IV.1, we now derive a characterization of the points just introduced.

THEOREM V.4: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i is continuous with respect to the product variable $\sigma\in\Sigma$ and A_i is quasi-convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{d(i)\cup f(i)}\in\Sigma_{d(i)\cup f(i)}$. If for a partition $\{N_1,N_2\}$ of the set of players N and for each joint strategy $\sigma\in\Sigma$ there is a point $\tau\in\Sigma$ such that

$$A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)})$$

for all the players $i\in N_1$ and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all the players $i\in N_2$, then, the game Γ_E has a $\bar{N}_1\bar{N}_{2,s}$ E-composed point.

PROOF: For the game Γ_E with structure function E, consider the following n-person games derived from Γ_E:

$$\Gamma_E^B = \{ \Sigma_1,\ldots,\Sigma_n;B_1,\ldots,B_n \}, \quad \Gamma_E^C = \{ \Sigma_1,\ldots,\Sigma_n;C_1,\ldots,C_n \}$$

whose structure function $\bar{E}$ is given by

$$\bar{d}(i) = e(i) \quad , \quad \bar{e}(i) = d(i) \quad , \quad \bar{f}(i) = f(i)$$

for player $i\in N$, and their payoff functions are defined as

$$B_i(\sigma) = \begin{cases} -A_i(\sigma) & \text{if } i\in N_1 \\ K_i & \text{if } i\in N_2 \end{cases}$$

and

$$C_i(\sigma) = \begin{cases} K_i & \text{if } i\in N_1 \\ -\,G_i(\sigma_{e(i)},\sigma_{f(i)}) & \text{if } i\in N_2 \end{cases}$$

where K_i is an arbitrary constant.

Because the payoff function A_i is quasi-convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$, then lemma IV.1 says that the functions B_i and C_i are quasi-concave in the variable $\sigma_{e(i)}\in\Sigma_{e(i)}$.

From this and the conditions satisfied by the original game Γ_E, the associated games Γ_E^B and Γ_E^C fulfill all the requirements asked by theorem V.1. Thus, the existence of a joint strategy $\bar{\sigma}\in\Sigma$ which is simultaneously E-positive equilibrium point of both games Γ_E^B and Γ_E^C is guaranteed. Such a point is an E-composed point with respect to $N_1^-, N_{2,s}^-$. (Q.E.D.)

From here, one immediately derives the following characterization regarding finite games.

<u>THEOREM V.5</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n \}$ be a finite n-person game with structure function E such that the payoff function of player $i\in N$ has the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in e(i)} a_i^j(\sigma_{d(i)},\sigma_j,\sigma_{f(i)}) \quad .$$

If for a partition $\{N_1,N_2\}$ of the set of players N, and for each point $x\in X = \underset{j\in N}{\times} \tilde{\Sigma}_j$, there is another point $y\in X$ such that

$$E_i(x_{d(i)},y_{e(i)},x_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(x_{d(i)},s_{e(i)},x_{f(i)})$$

for all $i\in N_1$ and

$$\max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)}, y_{e(i)}, x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} \max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)}, u_{e(i)}, x_{f(i)})$$

for all $i\in N_2$. Then, the mixed extension $\tilde{\Gamma}_E$ has an $N_1^- N_{2,s}^-$ E-composed point.

PROOF: The linearity of the expectation function E_i in the variable $x_{e(i)}\in X_{e(i)}$ is guaranteed by lemma I.11. On the other hand, the function's maximum is also convex in that variable. Thus, theorem V.4 for mixed extension game $\tilde{\Gamma}_E$ is verified, and therefore the validity of our assertion is proven. (Q.E.D.)

Finally, before we go into detail in the examination of the general case, we consider the following description, which is derived by assuming that the players forming the two sets N_1 and N_2 of the partition of set of players N, have other different characteristics associated with them. Thus, the antagonistic and friend coalitions, having the respective roles of first and second player in the situation determined by the acts of the indifferent coalition, are considered normal with respect to the position of their represented player, if this player is a member of N_1 and they behave according to a saddle point, whereas if he belongs to N_2 then the approach of maximin and minimax behavior is assumed.

This assumption leads immediately to the following formal definition given an n-person game $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ with structure function E, and a partition $\{N_1,N_2\}$ of the players set N, a joint strategy $\bar{\sigma}\in\Sigma$ is called an <u>$N_1^= N_{2,s}^=$ E-composed point</u> or an E-composed point with respect to $N_1^= N_{2,s}^=$, of game Γ_E, if it is an $N_1^+ N_{2,s}^+$ and $N_1^- N_{2,s}^-$ E-composed point of game Γ_E, that is, if

$$\max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_{d(i)},s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\in N_1$,

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\bar{\sigma}_{f(i)})$$

and

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\in N_2$.

This concept extends in a natural way to the notion of E-saddle point and E-stable points introduced in the previous chapter.

From the general result expressed in theorem V.1, we now derive the following result which characterizes the latter notion.

<u>THEOREM V.6</u>: Let $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space, and his payoff function A_i is continuous in the product variable $\sigma\in\Sigma$; with a partition $\{N_1,N_2\}$ of the set of players N , such that for all $i\in N_1$ the payoff function A_i is quasi-concave in $\sigma_{d(i)}\in\Sigma_{d(i)}$ for fixed $\sigma_{e(i)\cup f(i)}\in\Sigma_{e(i)\cup f(i)}$ and quasi-convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{d(i)\cup f(i)}\in\Sigma_{d(i)\cup f(i)}$; and for all $i\in N_2$ the function F_i is quasi-concave in the variable $\sigma_{d(i)}\in\Sigma_{d(i)}$ and the function G_i is quasi-convex with respect to $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$. If for each joint strategy $\sigma\in\Sigma$ there is another point $\tau\in\Sigma$ such that

$$A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)})$$

and

$$A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)})$$

for all the players $i \in N_1$ and

$$F_i(\tau_{d(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} F_i(s_{d(i)}, \sigma_{f(i)})$$

and

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)})$$

for all the players $i \in N_2$, then, the game Γ_E has an $\bar{\bar{N}}_1 \bar{\bar{N}}_{2,s}$ E-composed point.

<u>PROOF</u>: From the original game Γ_E , we now derive the four following games:

$$\Gamma_E^B = \{ \Sigma_1, \ldots, \Sigma_n; B_1, \ldots, B_n \} , \quad \Gamma_{\bar{E}}^C = \{ \Sigma_1, \ldots, \Sigma_n; C_1, \ldots, C_n \}$$

$$\Gamma_E^D = \{ \Sigma_1, \ldots, \Sigma_n; D_1, \ldots, D_n \} \text{ and } \Gamma_{\bar{E}}^E = \{ \Sigma_1, \ldots, \Sigma_n; E_1, \ldots, E_n \}$$

whose respective structure functions are E and $\bar{E}$, where $\bar{E}$ is defined by

$$\bar{d}(i) = e(i) , \quad \bar{e}(i) = d(i) , \quad \bar{f}(i) = f(i)$$

for each player $i \in N$. His payoff functions are given by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i \in N_1 \\ K_i & \text{if } i \in N_2 \end{cases}$$

$$C_i(\sigma) = \begin{cases} -A_i(\sigma) & \text{if } i \in N_1 \\ K_i & \text{if } i \in N_2 \end{cases}$$

$$D_i(\sigma) = \begin{cases} K_i & \text{if } i \in N_1 \\ F_i(\sigma_{d(i)}, \sigma_{f(i)}) & \text{if } i \in N_2 \end{cases}$$

and

$$E_i(\sigma) = \begin{cases} K_i & \text{if } i\epsilon N_1 \\ -C_i(\sigma_{e(i)},\sigma_{f(i)}) & \text{if } i\epsilon N_2 \end{cases}$$

where K_i indicates an arbitrary constant.

From the quasi-concavity and quasi-convexity of the payoff function A_i, we see that the payoff functions B_i and C_i are quasi-concave in $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ and $\sigma_{\bar{d}(i)}\epsilon\Sigma_{\bar{d}(i)}$ respectively. Furthermore, the payoff functions D_i and E_i are also quasi-concave. Thus, this last condition implies that all the requirements of theorem V.1 applied to the four games just considered, are satisfied, and therefore there exists a point which is simultaneously an E-positive equilibrium point of the above four games. Such a point is an $\bar{\bar{N}}_1\bar{\bar{N}}_{2,s}$ E-composed point of the original game Γ_E. (Q.E.D.)

We point out that in the previous theorem the quasi-concavity and quasi-convexity properties of the payoff functions of players belonging to the set N_2 were not assumed but only these properties for the maximun and minimun functions. If such conditions are assumed, then we also obtain the existence of a point $\bar{\sigma}\epsilon\Sigma$ which in this case is an E-saddle point. Indeed, applying theorem IV.7 for all the players i in the set N_2, to the associated game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)},\Sigma_{e(i)};A_i\}$ we have by the minimax theorem, that

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) .$$

This implies that the point $\bar{\sigma}\epsilon\Sigma$ is an E-saddle point of game Γ_E.

As an immediate consequence of this result we obtain the following theorem regarding finite games.

<u>THEOREM V.7</u>: Let $\Gamma_E = \{ \Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n \}$ be a finite n-person game with structure function E such that the payoff function of player $i \in N$ is of the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} \sum_{k\in e(i)} a_i^{jk} (\sigma_j,\sigma_k,\sigma_{f(i)}) \quad .$$

If for a partition $\{N_1,N_2\}$ of players set N , and for each point $x \in X = \underset{j\in N}{\times} \tilde{\Sigma}_j$, there is another point $y \in X$ such that

$$E_i(y_{d(i)},x_{e(i)},x_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},x_{e(i)},x_{f(i)})$$

and

$$E_i(x_{d(i)},y_{e(i)},x_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(x_{d(i)},s_{e(i)},x_{f(i)})$$

for all the players $i\in N_1$, and

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(y_{d(i)},s_{e(i)},x_{f(i)}) = \max_{u_{d(i)}\in X_{d(i)}} \min_{s_{e(i)}\in\Sigma_{e(i)}} E_i(u_{d(i)},s_{e(i)},x_{f(i)})$$

and

$$\max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},y_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} \max_{s_{d(i)}\in\Sigma_{d(i)}} E_i(s_{d(i)},u_{e(i)},x_{f(i)})$$

for all the players $i\in N_2$, then, the mixed $\tilde{\Gamma}_E$ has an $\bar{\bar{N_1}}\bar{\bar{N}}_{2,s}$ E-composed point.

<u>PROOF</u>: The bilinearity of the expectation function E_i with respect to the variables $x_{d(i)}\in X_{d(i)}$ and $x_{e(i)}\in X_{e(i)}$ for fixed $x_{f(i)}\in X_{f(i)}$ follows from lemma I.11 and the form of payoff function of player $i\in N$. The maximun function F_i is concave in $x_{d(i)}\in X_{d(i)}$ and the minimun function G_i is convex in $x_{e(i)}\in X_{e(i)}$. Thus, theorem V.8 determines the existence of an $\bar{\bar{N_1}}\bar{\bar{N}}_{2,s}$ E-composed point of mixed extension $\tilde{\Gamma}_E$. (Q.E.D.)

We note that indeed, this point is an E-saddle point, since the minimax theorem is satisfied by all the associated two-person games.

Having all these results, it is natural to extend all of them into a general notion which is constituted by all the previous particular notions. This notion will be derived by assuming that the players in the game act in different ways according to some prefixed concepts. We actually have six different general notions, so we suppose for the general approach, that the players set is partitioned into six sets, namely: N_1, N_2, N_3, N_4, N_5 and N_6. Thus, for all the players belonging to the first set N_1 the antagonistic coalition is assumed to be indifferent with respect to the position of the respective player and the friend coalition as a normal player. For all the players in the set N_2, the friend coalition is supposed to be a normal player without any reference to the antagonistic coalition. The antagonistic coalitions for players in N_3 is considered normal while the corresponding friend coalition is apathetic. For players in N_4 the role of the friend coalition is not considered. Finally, for players in N_5 both coalitions are assumed to play rational roles with respect to saddle points, and for those in N_6 both players are considered rational with respect to the maximin and minimax principle.

From here, in a precise manner we have the following formulation: given an n-person game $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ with structure function E, and a partition $\underline{N} = \{N_1, N_2, N_3, N_4, N_5, N_6\}$ of the set of players N, a joint strategy $\bar{\sigma} \in \Sigma$ is said to be an <u>$N_1^+ N_{2,s}^+ N_3^- N_{4,s}^- N_5^= N_{6,s}^=$ E-composed point</u> or an <u>E-composed point</u> with respect to $N_1^+ N_{2,s}^+ N_3^- N_{4,s}^- N_5^= N_{6,s}^=$ of game Γ_E, if

$$A_i(\bar{\sigma}_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$$

for all the players $i \in N_1 \cup N_5$

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\bar{\sigma}_{f(i)})$$

for all the players $i \in N_2 \cup N_6$

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_{d(i)},s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i \in N_3 \cup N_5$ and finally

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i \in N_4 \cup N_6$.

This concept covers all previous concepts treated until now.

A comparison among such points arises immediately by considering their relation to the points introduced in the previous chapter. Indeed, given a game Γ with structure function E and a partition $\underline{N} = \{N_1,N_2,N_3,N_4,N_5,N_6\}$ of the set of players N , if a structure function $\bar{E}$ satisfies: $d(i) \supseteq \bar{d}(i)$ for all the players $i \in N_1 \cup N_5$, $e(i) \supseteq \bar{e}(i)$ for all $i \in N_3 \cup N_6$,

$$d(i) \supseteq \bar{d}(i) \quad , \quad e(i) = \bar{e}(i) \ , \ f(i) \subseteq \bar{f}(i)$$

for all the players $i \in N_2$ and finally

$$d(i) = \bar{d}(i) \quad , \quad e(i) \supseteq \bar{e}(i) \quad , \ f(i) \subseteq \bar{f}(i)$$

for all $i \in N_4$, then the set of $N_1^{+}N_{2,s}^{+}N_3^{-}N_{4,s}^{-}N_5^{=}N_{6,s}^{=}$ E-composed points of game Γ_E : $R(N_1^{+}N_{2,s}^{+}N_3^{-}N_{4,s}^{-}N_5^{=}N_{6,s}^{=},\Gamma_E)$ is contained in the set $R(N_1^{+}N_{2,s}^{+}N_3^{-}N_{4,s}^{-}N_5^{=}N_{6,s}^{=},\Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

A characterization of this kind of points arises immediately as a consequence of theorem V.1, and is formulated in the following result.

THEOREM V.8: Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space, his payoff function A_i is continuous in the product variable $\sigma\in\Sigma$. Given a partition $\{N_1,N_2,N_3,N_4,N_5,N_6\}$ of the set of players N, such that for all $i\in N_1\cup N_5$ the payoff function A_i is quasi-concave in $\sigma_{d(i)}\in\Sigma_{d(i)}$ for fixed $\sigma_{e(i)\cup f(i)}\in\Sigma_{e(i)\cup f(i)}$, and quasi-convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{d(i)\cup f(i)}\in\Sigma_{d(i)\cup f(i)}$ for all the players $i\in N_3\cup N_5$. Also for all $i\in N_2\cup N_6$ the function F_i is quasi-concave in $\sigma_{d(i)}\in\Sigma_{d(i)}$ and for all $i\in N_4\cup N_6$ and the function G_i is quasi-convex with respect to $\sigma_{e(i)}\in\Sigma_{e(i)}$, for fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$. If for each joint strategy $\sigma\in\Sigma$ there is another point $\tau\in\Sigma$ such that

$$A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)})$$

for all $i\in N_1\cup M_5$,

$$A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)})$$

for all $i\in N_3\cup M_5$,

$$F_i(\tau_{d(i)},\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\sigma_{f(i)})$$

for all $i\in N_2\cup(N_5-M_5)\cup N_6$ and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)})$$

for all $i\in N_4\cup(N_5-M_6)\cup N_6$, where M_5 is an arbitrary subset of N_5. Then, the game Γ_E has an $N_1N_{2,s}N_3N_{4,s}N_5N_{6,s}$ E-composed point.

PROOF: Having already the game Γ_E with structure function E, consider the following n-person games:

$$\Gamma_E^B = \{ \Sigma_1,\dots,\Sigma_n; B_1,\dots,B_n \} \quad , \qquad \Gamma_{\bar{E}}^C = \{ \Sigma_1,\dots,\Sigma_n; C_1,\dots,C_n \}$$

$$\Gamma_E^D = \{ \Sigma_1,\dots,\Sigma_n; D_1,\dots,D_n \} \quad \text{and} \qquad \Gamma_{\bar{E}}^E = \{ \Sigma_1,\dots,\Sigma_n; E_1,\dots,E_n \}$$

whose respective structure functions are E and $\bar{E}$, where $\bar{E}$ is defined by

$$\bar{d}(i) = e(i) \quad , \quad \bar{e}(i) = d(i) \quad , \quad \bar{f}(i) = f(i)$$

for every player $i \in N$. Furthermore, the payoff function of player $i \in N$ is given by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i \in N_1 \cup M_5 \\ K_i & \text{otherwise} \end{cases}$$

$$C_i(\sigma) = \begin{cases} -A_i(\sigma) & \text{if } i \in N_3 \cup M_5 \\ K_i & \text{otherwise} \end{cases}$$

$$D_i(\sigma) = \begin{cases} F_i(\sigma_{d(i)}, \sigma_{f(i)}) & \text{if } i \in N_2 \cup (N_5 - M_5) \cup N_6 \\ K_i & \text{otherwise} \end{cases}$$

and finally

$$E_i(\sigma) = \begin{cases} -G_i(\sigma_{e(i)}, \sigma_{f(i)}) & \text{if } i \in N_4 \cup (N_5 - M_5) \cup N_6 \\ K_i & \text{otherwise} \end{cases}$$

where K_i indicates an arbitrary constant. From the quasi-convexity of the payoff function A_i with respect to the variable $\sigma_{d(i)} \in \Sigma_{d(i)}$ for player $i \in M_5$, lemma IV.1 gives that all the functions B_i and C_i are also quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ and $\sigma_{\bar{d}(i)} \in \Sigma_{\bar{d}(i)}$ respectively. Hence, because the last condition of theorem V.1

is satisfied by the preceding four games there exists a point $\bar{\sigma}\in\Sigma$ which is simultaneously E-positive equilibrium point of all of these games. Now, consider the players in the set N_5-M_6 . The associated two-person game $\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)},\Sigma_{e(i)};A_i(\sigma_{d(i)},\sigma_{e(i)},\bar{\sigma}_{f(i)})\}$ of a player $i\in N_5$-M_5 fulfills all the requirements of theorem IV.7 applied to $\Gamma_i(\bar{\sigma}_{f(i)})$. Thus, in such a game the minimax theorem holds true, and therefore

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_{d(i)},s_{e(i)},\bar{\sigma}_{f(i)})$$

for all $i\in N_5$-M_5 .

From this, we obtain that the point $\bar{\sigma}\in\Sigma$ is an $N_1^+N_2^+,_s N_3^-N_4^-,_s N_5^=N_6^=,_s$ E-composed point of game Γ_E . (Q.E.D.)

From this, we immediately derive the following existence theorem regarding finite games.

<u>THEOREM V.9:</u> Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ be a finite n-person game with structure function E and let $N = \{N_1,N_2,N_3,N_4,N_5,\phi\}$ be a partition of the players set N , such that the payoff function of player $i\in N_1\cup N_2$ is of the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} a_i^j(\sigma_j,\sigma_{e(i)},\sigma_{f(i)}) \quad ,$$

that of the player $i\in N_3\cup N_4$ is

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in e(i)} a_i^j(\sigma_{d(i)},\sigma_j,\sigma_{f(i)}) \quad ;$$

and that of the player $i\in N_5$ is

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} \sum_{k\in e(i)} a_i^{jk}(\sigma_j,\sigma_k,\sigma_{f(i)}) \quad .$$

If for each joint strategy $x \in X = \times_{j \in N} \tilde{\Sigma}_j$, there is another point $y \in X$ such that

$$E_i(y_{d(i)}, x_{e(i)}, x_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, x_{f(i)}, x_{f(i)}) ;$$

for all $i \in N_1 \cup M_5$

$$E_i(x_{d(i)}, y_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_{d(i)}, s_{e(i)}, x_{f(i)})$$

for all $i \in N_3 \quad M_5$,

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(y_{d(i)}, y_{e(i)}, x_{f(i)}) = \max_{u_{d(i)} \in X_{d(i)}} \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(u_{d(i)}, s_{e(i)}, x_{f(i)})$$

for all $i \in N_2 \cup (N_5 - M_5)$, and finally

$$\max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, y_{e(i)}, x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, u_{e(i)}, x_{f(i)})$$

for all $i \in N_4 \cup (N_5 - M_5)$, where M_5 indicates an arbitrary subset of N_5 , then, the mixed extension $\tilde{\Gamma}_E$ has an $N_1^+ N_2^+{}_{,s} N_3^- N_4^-{}_{,s} N_5^=$ E-composed point.

<u>PROOF</u>: From lemma I.11, the linearity of the expectation function E_i in the variable $x_{d(i)} \in X_{d(i)}$ for all the players $i \in N_1 \cup N_5$ follows as does the linearity of E_i in the variable $x_{e(i)} \in X_{e(i)}$ for all the players $i \in N_2 \quad N_5$. Analogously, we have that the minimun function F_i for $i \in N_2$ is concave in $x_{d(i)} \in X_{d(i)}$ and the maximun function Σ_i for $i \in N_4$ is convex with respect to $x_{e(i)} \in X_{e(i)}$. Thus, from the last condition on the expectation functions all the requirements of theorem V.8 for the mixed extension $\tilde{\Gamma}_E$ are satisfied, and therefore there exists an $N_1^+ N_2^+{}_{,s} N_3^- N_4^-{}_{,s} N_5^=$ E-composed point of game $\tilde{\Gamma}_E$. (Q.E.D.)

We do note that, although in this theorem the last set N_6 of the partition was explicitly considered null, the result obtained by changing this would coincide with the just treated formulation. Indeed, for those players in the set N_6 , the minimax theorem for their corresponding two-person associated game holds true, and therefore they can be considered as members of the set N_5 .

V.2 E-Partially Composed Points

In this section, as has been mentioned in the introduction of the previous paragraph, we are concerned with a further approach, based on a different point of view.

Until now, we have considered all the players to have been assigned some behavior in the situation they are assumed embedded in, that is, in the associated two-person game . Of course, as before, we have obtained a more general concept by combining all the available notions. But, one could consider the possibility that some players are not associating with any of those notions, but in another new, specified or not specified manner. Thus, if we possess some new concept, all the previous results might be extended by the incorporation of this one. But, if one does not have such a new notion, which is the case for us at the present moment, the global description of the behavior in the game is given only partially, that is, it will be described explicitly only by the consideration of the joint acts of a subset of players in the whole game. In this way, we will obtain new concepts which we will incorporate into those just considered, partially; that means only for a certain group of players.

First of all, let us consider one of the most simple of these new notions which is formally introduced as follows: given an n-person game $\Gamma_E = \{ \Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n \}$ with structure function E and a subset M contained in the players set N , a joint strategy $\bar{\sigma}\in\Sigma$ is said to be an <u>E-partially positive equilibrium point</u> with respect to $M\subseteq N$ of game Γ_E , if for all the players $i\in M\subseteq N$:

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) .$$

At this point, we do note that although the notion of these points does not involve the whole set N but only the subset M, it does not mean that they are independent of the actions of the players in $N-M$.

Directly from the definitions, we have that a point is an E-positive equilibrium point of game Γ_E if and only if for every subset $M\subseteq N$ it is an E-partially positive equilibrium point with respect both subset M and $N-M$.

In a way similar to that used in the preceding chapter for E-positive equilibrium points, we have that if $M\supseteq\bar{M}$ and for the structure function $\bar{E}$ $d(i)\supseteq\bar{d}(i)$ for all $i\in\bar{M}$, then the set of E-partially positive equilibrium points with respect to M of game Γ_E: $P(M,\Gamma_E)$ is contained in the set $P(\bar{M},\Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

On the other hand, the notion just introduced is a straightforward extension of E-positive equilibrium point which appears when the set M coincides with the set of players N. On the other hand, one can derive this as an E-positive equilibrium point of a modified game. Indeed given a game Γ_E, consider the following associated n-person game

$$\Gamma_E^a = \{ \Sigma_1,\ldots,\Sigma_n; B_1,\ldots,B_n \}$$

where the new payoff function B_i is defined by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i\in M \\ K_i & \text{otherwise} \end{cases}$$

where K_i indicates an arbitrary constant. Then a point is an E-partially positive equilibrium point with respect to M of Γ_E if and only if it is an E-equilibrium point of the associated game Γ_E^a.

A characterization of points is a particularization of the following general result, which could also be useful in obtaining other existence theorems for most general concepts.

THEOREM V.10: Let $\Gamma^j_{E_j} = \{ \Sigma_1,\ldots,\Sigma_n; B^j_1,\ldots,B^j_n \}$ $(j\in P = \{1,\ldots,p\})$ be p n-person games defined on the same strategy sets with the respective structure functions E_j, such that the set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space. Which for any p given subsets $M_j \subseteq N$ with $j\in P$ for game $\Gamma^j_{E_j}$, the payoff function B^j_i of player $i\in M_j$ in the game $\Gamma^j_{E_j}$, is continuous in the product variable $\sigma\in\Sigma$ and quasi-concave with respect to the variable $\sigma_{d^j(i)}\in\Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i)\cup f^j(i)}\in\Sigma_{e^j(i)\cup f^j(i)}$. If for each joint strategy $\sigma\in\Sigma$ there is a point $\tau\in\Sigma$ such that

$$B^j_i(\tau_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) = \max_{s_{d^j(i)}\in\Sigma_{d^j(i)}} B^j_i(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})$$

for all $j\in P$ and all the players $i\in M$. Then there exists a joint strategy $\bar{\sigma}\in\Sigma$, which is simultaneously an E_j-partially positive equilibrium point with respect to M_j of game $\Gamma^j_{M_j}$, for all $j\in P$.

PROOF: Consider for each $j\in P$ the associated game

$$\Gamma^{ja}_{E_j} = \{ \Sigma_1,\ldots,\Sigma_n; C^j_1,\ldots,C^j_n \}$$

obtained from game $\Gamma^j_{E_j}$ by substituting for the payoff function B^j_i of players $i\in N-M_j$ an arbitrary constant $C^j_i = K^j_i$. Thus, for all these new games all the requirements of theorem V.1 are satisfied, and therefore the existence of a joint strategy $\bar{\sigma}\epsilon\Sigma$ which is simultaneously E-positive equilibrium point of games $\Gamma^{ja}_{E_j}$ with $j\epsilon P$ is guaranteed. Such a point is simultaneously an E-partially positive equilibrium point with respect to M_j of game $\Gamma^{j}_{M_j}$, with $j\epsilon P$. (Q.E.D.)

Having this result, we are now going to introduce the most general concept. Given an n-person game $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n;A_1,\ldots,A_n\}$ with structure function E and a partition $\underline{M} = \{M_1,M_2,M_3,M_4,M_5,M_6\}$ of a subset M of the players set N, a joint strategy $\bar{\sigma}\epsilon\Sigma$ is said to be an $M_1^+M_{2,s}^+M_3^-M_{4,s}^-M_5^=M_{6,s}^=$ E-partially composed point of game Γ_E, if

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\bar{\sigma}_{e(i)},\sigma_{f(i)}) ,$$

for all the players $i\epsilon M_1 \cup M_5$,

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_{d(i)},s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\epsilon M_3 \cup M_5$,

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} F_i(s_{d(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\epsilon M_2 \cup M_6$ and finally

$$G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)})$$

for all the players $i\epsilon M_4 \cup M_6$.

This concept generalizes all the previous notions considered. There are several ways of observing this extension, for example, an $N_1^+N_{2,s}^+N_3^-N_{4,s}^-N_5^=N_{5,s}^=$ E-composed point can be observed to be simultaneously an N_1^+ E-partially,

$N^{+}_{2,s}$ E-partially , N^{-}_{3} E-partially , $N^{-}_{4,s}$ E-partially, $N^{=}_{5}$ E-partially, $N^{=}_{6,s}$ E-partially composed point. Of course many other relations could be obtained immediately from the definitions.

The relationships between such points follow immediately from the results of the previous chapter. Thus, for a game Γ with structure function E and two partitions $\{M_1,M_2,M_3,M_4,M_5,M_6\}$ of a subset M and $\{\bar{M}_1,\bar{M}_2,\bar{M}_3,\bar{M}_4,\bar{M}_5,\bar{M}_6\}$ of the subset $\bar{M}$ of players set N , if a structure function E verifies $d(i)\supseteq\bar{d}(i)$ for $i\in\bar{M}_1\cup\bar{M}_5$, $e(i)\supseteq\bar{e}(i)$ for $i\in\bar{M}_3\cup\bar{M}_6$,

$$d(i)\supseteq\bar{d}(i) \quad , \quad e(i)=\bar{e}(i) \quad , \quad f(i)\subseteq\bar{f}(i)$$

for all the players $i\in\bar{M}_2$ and

$$d(i)=\bar{d}(i) \quad , \quad e(i)\supseteq\bar{e}(i) \quad , \quad f(i)\subseteq\bar{f}(i)$$

for all $i\in\bar{M}_4$, then if $\bar{M}_k\subseteq M_k$ for $k=1,\dots,6$, the set of $M^{+}_{1}M^{+}_{2,s}M^{-}_{3}M^{-}_{4,s}M^{=}_{5}M^{=}_{6,s}$ E-partially composed points of game Γ_E: $S(M^{+}_{1}M^{+}_{2,s}M^{-}_{3}M^{-}_{4,s}M^{=}_{5}M^{=}_{6,s},M,\Gamma_E)$ is contained in the set $S(\bar{M}^{+}_{1}\bar{M}^{+}_{2,s}\bar{M}^{-}_{3}\bar{M}^{-}_{4,s}\bar{M}^{=}_{5}\bar{M}^{=}_{6,s},\bar{M},\Gamma_{\bar{E}})$ of game $\Gamma_{\bar{E}}$.

A characterization of these points is formulated in the following result.

<u>THEOREM V.11</u>: Let $\Gamma_E=\{\Sigma_1,\dots,\Sigma_n;A_1,\dots,A_n\}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a euclidean space. Let there also be given a partition $\{M_1,M_2,M_3,M_4,M_5,M_6\}$ of a subset M of the set of players N , such that all the payoff functions A_i of players $i\in M$ are continuous, and for all $i\in M_1\cup M_5$, A_i is quasi-concave in $\sigma_{d(i)}\in\Sigma_{d(i)}$ for fixed $\sigma_{e(i)\cup f(i)}\in\Sigma_{e(i)\cup f(i)}$, and quasi-convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$ for fixed $\sigma_{d(i)\cup f(i)}\in\Sigma_{d(i)\cup f(i)}$ for all the players

$i \in M_2 \cup M_5$. Also for all $i \in M_2 \cup M_6$ the function F_i is quasi-concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ and for all $i \in M_4 \cup M_6$ the function Σ_i is quasi-convex with respect to $\sigma_{e(i)} \in \Sigma_{e(i)}$, for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. If for each joint strategy $\sigma \in \Sigma$ there is another point $\tau \in \Sigma$ such that

$$A_i(\tau_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)})$$

for all $i \in M_1 \cup L_5$,

$$A_i(\sigma_{d(i)}, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)})$$

for all $i \in M_3 \cup L_5$,

$$F_i(\tau_{d(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} F_i(s_{d(i)}, \sigma_{f(i)})$$

for all $i \in M_2 \cup (M_5 - L_5) \cup M_6$ and

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)})$$

for all $i \in M_4 \cup (M_5 - L_5) \cup M_6$, where L_5 is an arbitrary subset of M_5 . Then the game Γ_E has an $M_1^+ M_{2,s}^+ M_3^- M_{4,s}^- M_5^= M_{6,s}^=$ E-partially composed point.

<u>PROOF</u>: Given the game Γ_E with structure function E , consider the following associated n-person game $\Gamma_E^a = \{ \Sigma_1, \ldots, \Sigma_n ; B_1, \ldots, B_n \}$ with structure function E and the payoff functions defined by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i \in M \\ K_i & \text{otherwise} \end{cases}$$

where K_i indicates an arbitrary constant. On the other hand, let us consider the partition $\{M_1 \cup (N-M), M_2, M_3, M_4, M_5, M_6\}$ of the set of players N . Thus, the associated game Γ_E^a satisfies all the requirements of theorem V.8 for the partition just considered, and therefore this game Γ_E^a has an $(M_1 \cup (N-M))^+ M_{2,s}^+ M_3^- M_{4,s}^- M_5^= M_{6,s}^=$ E-composed point. Such a point is an $M_1^+ M_{2,s}^+ M_3^- M_{4,s}^- M_5^= M_{6,s}^=$ E-partially composed point of game Γ_E. (Q.E.D.)

Directly from this result, we now derive a general theorem regarding mixed extensions of finite games.

<u>THEOREM V.12</u>: Let $\Gamma_E = \{ \Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n \}$ be a finite n-person game with structure function E and let $\underline{M} = \{M_1, M_2, M_3, M_4, M_5, \phi\}$ be a partition of a subset M of the set of players N such that the payoff function of player $i \in M_1 \cup M_2$ is of the form

$$A_i(\sigma_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \sum_{j \in d(i)} a_i^j(\sigma_j, \sigma_{e(i)}, \sigma_{f(i)}) ,$$

that of the player $i \in M_3 \cup M_4$ is

$$A_i(\sigma_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \sum_{j \in e(i)} a_i^j(\sigma_{d(i)}, \sigma_j, \sigma_{f(i)}) ;$$

and that of the player $i \in M_5$ is

$$A_i(\sigma_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \sum_{j \in d(i)} \sum_{k \in e(i)} a_i^{jk}(\sigma_j, \sigma_k, \sigma_{f(i)}) .$$

If for each joint strategy $x \in X = \underset{j \in N}{\times} \tilde{\Sigma}_j$, there is another point $j \in X$ such that

$$E_i(y_{d(i)}, x_{e(i)}, x_{f(i)}) = \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, x_{e(i)}, x_{f(i)})$$

$$E_i(x_{d(i)}, y_{e(i)}, x_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(x_{d(i)}, s_{e(i)}, x_{f(i)})$$

for all $i \in M_3 \cup L_5$,

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(y_{d(i)}, s_{e(i)}, x_{f(i)}) = \max_{u_{d(i)} \in X_{d(i)}} \min_{s_{e(i)} \in \Sigma_{e(i)}} E_i(u_{d(i)}, s_{e(i)}, x_{f(i)})$$

for all $i \in M_2 \cup (M_5 - L_5)$, and finally

$$\max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, y_{e(i)}, x_{f(i)}) = \min_{u_{e(i)} \in X_{e(i)}} \max_{s_{d(i)} \in \Sigma_{d(i)}} E_i(s_{d(i)}, u_{e(i)}, x_{f(i)})$$

for all $i \in M_4 \cup (M_5 - L_5)$, where L_5 is an arbitrary subset of M_5 . Then, the mixed extension $\tilde{\Gamma}_E$ has an $M_1^{+}M_{2,s}^{+}M_3^{-}M_{4,s}^{-}M_5^{=}M_{6,s}^{=}$ E-partially composed point.

PROOF: From the forms of the payoff functions, and lemma I.11, the expectation function E_i for all the players $i \in M_1 \; M_5$ is seen to be linear in the variable $\sigma_{d(i)} X_{d(i)}$; for players $i \in M_2 \cup M_5$ it is linear with respect to $x_{e(i)} \in X_{e(i)}$. On the other hand, the minimun function F_i for $i \in M_2$ is concave in $x_{d(i)} \in X_{d(i)}$ and the maximun function G_i is convex in $x_{e(i)} \in X_{e(i)}$, for $i \in M_4$. Thus, by the last condition on the expectation functions, all the requirements of the previous theorem, for mixed extension game $\tilde{\Gamma}_E$ are met and therefore it has an $M_1^{+}M_{2,s}^{+}M_3^{-}M_{4,s}^{-}M_5^{=}$ E-partially composed point. (Q.E.D.)

We do note that the preceding results involve more generality than can be pretended at first.

CHAPTER VI*

VI.1 E-Points by Fixed Point Procedure

This last chapter is devoted to extending those concepts and characterizations already formulated in the two preceding chapters. Thus, the results obtained here, which are the most general presented, are in a certain sense final.

This chapter is to the results of the previous chapter as the third chapter was to the first two chapters.

The simpler approach, which is considered first uses the fixed point procedure which is essentially of a repetition of the treatment just considered.

The existence theorems of E-points for general games defined on linear topological spaces will be derived from the following result which is a direct extension of theorem V.10.

THEOREM VI.1: Let $\Gamma^j_{E_j} = \{\Sigma_1,\ldots,\Sigma_n; B^j_1,\ldots, B^j_n\}$ $(j\epsilon P = \{1,\ldots,p\})$ be p n-person games defined on the same strategy sets with respective structure functions E_j, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space. For each $j\epsilon P$ let M_j be a subset of N, such that for all the players $i\epsilon M_j$, the payoff function B^j_i is continuous in the product space Σ, and quasi-concave in $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i)\cup f^j(i)} \epsilon \Sigma_{e^j(i)\cup f^j(i)}$. If for each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$B^j_i(\tau_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) = \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B^j_i(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})$$

for all $j\epsilon P$ and all the players $i\epsilon M_j$, then, there exists a joint strategy $\bar{\sigma}\epsilon\Sigma$, which is simultaneously for all $j\epsilon P$ or $\underline{E}_j$-partially positive equilibrium point with respect to M_j of game $\Gamma^j_{E_j}$.

PROOF: Consider for an arbitrary joint strategy $\sigma\epsilon\Sigma$, and a player $i\epsilon M_j$ with $j\epsilon P$ the set

$$\varphi_i^j(\sigma) = \{\tau\epsilon\Sigma:\ B_i^j(\tau_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) = \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})\}$$

which is well defined by virtue of the continuity of payoff function B_i^j. It is convex because B_i^j is quasi-concave with respect to $\sigma_{d^j(i)}\epsilon\Sigma_{d^j(i)}$, and the convexity of the strategy sets Σ_i.

Let us define the multivalued function

$$\psi : \Sigma \to \Sigma$$

as the set

$$\psi(\sigma) = \bigcap_{j\epsilon P}\ \bigcap_{i\epsilon M_j} \varphi_i^j(\sigma)$$

for all the joint strategies $\sigma\epsilon\Sigma$. Indeed, that the set $\psi(\sigma)$ is non-empty, follows from the last condition on the payoff functions. On the other hand, the upper-semicontinuity of the multivalued function ψ is guaranteed by the continuity of the payoff functions and lemma III.2. Thus, the fixed point theorem given in theorem III.1 can be applied to the function ψ, since the product space Σ is locally convex linear Hausdorff space, and therefore there exists a point $\bar{\sigma}\epsilon\Sigma$ such that $\bar{\sigma}\epsilon\psi(\bar{\sigma})$. Such a point is an $\underline{E}_j$-partially positive equilibrium point with respect to M_j of game $\Gamma^j_{\underline{E}_j}$ for all $j\epsilon P$. (Q.E.D.)

As an immediate consequence of this result we derive the following general theorem dealing with E-partially composed points:

THEOREM VI.2: Let $\Gamma_E = \{\Sigma_1,\dots,\Sigma_n;\ A_1,\dots,A_n\}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a locally convex linear Hausdorff space.

Given a partition $\{M_1, M_2, M_3, M_4, M_5, M_6\}$ of a subset M of the players set N such that all the payoff functions A_i of players $i \epsilon M$ are continuous; for all $i \epsilon M_1 \cup M_5$, A_i is quasi-concave in $\sigma_{d(i)} \epsilon \Sigma_{d(i)}$ for fixed $\sigma_{e(i) \cup f(i)} \epsilon \Sigma_{e(i) \cup f(i)}$, and for all the players $i \epsilon M_2 \cup M_5$ quasi-convex in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $\sigma_{d(i) \cup f(i)} \epsilon \Sigma_{d(i) \cup f(i)}$. Also for all $i \epsilon M_2 \cup M_6$ the function F_i is quasi-concave in $\sigma_{d(i)} \epsilon \Sigma_{d(i)}$ and for all $i \epsilon M_4 \cup M_5$ the function G_i is quasi-convex with respect to $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$, for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$. If for each joint strategy $\sigma \epsilon \Sigma$ there is another point $\tau \epsilon \Sigma$ such that

$$A_i(\tau_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \epsilon \Sigma_{d(i)}} A_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)})$$

for all $i \epsilon M_1 \cup L_5$,

$$A_i(\sigma_{d(i)}, \tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} A_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)})$$

for all $i \epsilon M_3 \cup L_5$,

$$F_i(\tau_{d(i)}, \sigma_{f(i)}) = \max_{s_{d(i)} \epsilon \Sigma_{d(i)}} F_i(s_{d(i)}, \sigma_{f(i)})$$

for all $i \epsilon M_2 \cup (M_5 - L_5)\ M_6$ and

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{d(i)} \epsilon \Sigma_{d(i)}} G_i(s_{d(i)}, \sigma_{f(i)})$$

for all $i \epsilon M_4 \cup (M_5 - L_5) \cup M_6$, where L_5 is an arbitrary subset of M_5, then the game Γ_E has an $M_1^+\ M_{2,s}^+\ M_3^-\ M_{4,s}^-\ M_5^=\ M_{6,s}^=$ E-partially composed point.

<u>PROOF:</u> Given the game Γ_E having the structure function E and the subset M, consider the following n-person games

$$\Gamma_E^B = \{\Sigma_1, \ldots, \Sigma_n;\ B_1, \ldots, B_n\}\ ,\ \Gamma_{\bar{E}}^C = \{\Sigma_1, \ldots, \Sigma_n;\ C_1, \ldots, C_n\}$$

$$\Gamma_E^D = \{\Sigma_1, \ldots, \Sigma_n;\ D_1, \ldots, D_n\} \text{ and } \Gamma_{\bar{E}}^E = \{\Sigma_1, \ldots, \Sigma_n;\ E_1, \ldots, E_n\}$$

whose respective structure functions are $\underline{E}$ and $\bar{E}$, where $\bar{E}$ is defined by

$$\bar{d}(i) = e(i)\ ,\quad \bar{e}(i) = d(i)\ ,\quad f(i) = f(i)$$

for every player $i\epsilon N$. The payoff functions are defined by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i\epsilon M_1 \cup L_5 \\ K_i & \text{otherwise} \end{cases}$$

$$C_i(\sigma) = \begin{cases} -A_i(\sigma) & \text{if } i\epsilon M_3 \cup L_5 \\ K_i & \text{otherwise} \end{cases}$$

$$D_i(\sigma) = \begin{cases} F_i(\sigma_{d(i)},\sigma_{f(i)}) & \text{if } i\epsilon M_2 \cup (M_5 \cup L_5) \\ K_i & \text{otherwise} \end{cases}$$

and lastly

$$E_i(\sigma) = \begin{cases} -G_i(\sigma_{e(i)},\sigma_{f(i)}) & \text{if } i\epsilon M_4 \cup (M_5 - L_5) \cup M_6 \\ K_i & \text{otherwise} \end{cases}$$

where, as usual, K_i indicates an arbitrary constant.

The quasi-concavity of payoff function A_i with respect to $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ of a player $i\epsilon L_5$, implies as has been asserted in the proof of theorem III.29, that the minimum function F_i is quasi-concave in the variable $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$. In the same way, because for player $i\epsilon L_5$ A_i is quasi-convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$, then the maximum function G_i is also quasi-convex in this variable.

Hence, all the requirements of theorem VI.1 applies to the previous four games in a global way. Thus, the existence of a point $\bar{\sigma}\epsilon\Sigma$ which is simultaneously an E-positive equilibrium point for each of those games, is assured.

Now, for the players $i\epsilon M_5 - L_5$, let us consider the two-person game

$\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)},\Sigma_{e(i)};\ A_i(\sigma_{d(i)},\sigma_{e(i)},\bar{\sigma}_{f(i)})\ \}$, which fulfills all the requirements of theorem III.9 Thus, the minimax theorem holds for it, which implies:

$$A_i(\bar\sigma_{d(i)},\bar\sigma_{e(i)},\bar\sigma_{f(i)}) = \max_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\bar\sigma_{e(i)},\bar\sigma_{f(i)})$$

$$= \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar\sigma_{d(i)},s_{e(i)},\bar\sigma_{f(i)})$$

for all $i\in M_5-L_5$.

By virtue of these relations, we have that the point $\bar\sigma\in\Sigma$ is an M_1^+ $M_{2,s}^+$ M_3^- $M_{4,s}^-$ $M_5^=$ $M_{6,s}^=$ E-partially compound point of original game Γ_E. (Q.E.D.)

From here, we now obtain the characterization of such points for mixed extension of continuous games.

<u>THEOREM VI.3:</u> Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with structure function E , such that the strategy set Σ_i of player $i\in N$ is a compact Hausdorff space. Given a partition $\{M_1, M_2, M_3, M_4, M_5, \emptyset\}$ of a subset M of players set N , such that all the payoff functions A_i of players $i\in M$ are of the form

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)} a_i^j(\sigma_j,\sigma_{e(i)},\sigma_{f(i)})$$

for player $i\in M_1 \cup M_2$,

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in e(i)} a_i^j(\sigma_{d(i)},\sigma_j,\sigma_{f(i)})$$

for player $i\in M_3 \cup M_4$ and finally

$$A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) = \sum_{j\in d(i)}\sum_{k\in e(i)} a_i^{jk}(\sigma_j,\sigma_k,\sigma_{f(i)})$$

for player $i\in M_5$, where all the involved functions are continuous in their respective variable. If for each joint strategy $x\in X$ there is a point $y\in X$ such that

$$E_i(y_{d(i)},x_{e(i)},x_{f(i)}) = \max_{u_{d(i)}\in X_{d(i)}} E_i(u_{d(i)},x_{e(i)},x_{f(i)})$$

for all $i\in M_1 \cup L_5$,

$$E_i(x_{d(i)},y_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} E_i(x_{d(i)},u_{e(i)},x_{f(i)})$$

for all $i\in M_3 \cup L_5$,

$$\min_{u_{e(i)}\in X_{e(i)}} E_i(y_{d(i)},u_{e(i)},x_{f(i)}) = \max_{u_{d(i)}\in X_{d(i)}} \min_{u_{e(i)}\in X_{e(i)}} E_i(x_{d(i)},u_{e(i)},x_{f(i)})$$

for all $i\in M_2 \cup (M_5-L_5)$ and finally

$$\max_{u_{d(i)}\in X_{d(i)}} E_i(u_{d(i)},y_{e(i)},x_{f(i)}) = \min_{u_{e(i)}\in X_{e(i)}} \max_{u_{d(i)}\in X_{d(i)}} E_i(u_{d(i)},u_{e(i)},x_{f(i)})$$

for all $i\in M_4 \cup (M_5-L_5)$, where L_5 is an arbitrary subset of M_5 , then, the mixed extension $\tilde{\Gamma}_E$ has an $M_1^+ \; M_{2,s}^+ \; M_3^- \; M_{4,s}^- \; M_5^=$ E-partially composed point.

PROOF: From the form of payoff functions and lemma III.7 we see that the expectation functions E_i for player $i\in M_1 \cup M_5$ is linear with respect to $x_{d(i)}\in X_{d(i)}$. By the same reasoning, for player $i\in M_2 \cup M_2$ it is linear in $x_{e(i)}\in X_{e(i)}$. Furthermore, the minimum function F_i and the maximum function G_i are respectively concave in $x_{d(i)}\in X_{d(i)}$ and convex in $x_{e(i)}\in X_{e(i)}$.

Thus, because of the last condition, all the requirements of theorem VI.2 for mixed extension game $\tilde{\Gamma}_E$ are fulfilled, and therefore the existence of an $M_1^+ \; M_{2,s}^+ \; M_3^- \; M_{4,s}^- \; M_5^=$ E-partially composed point is proven. (Q.E.D.)

A further application of the above results is derived below and is related to a certain class of "simultaneous" E-saddle points.

THEOREM VI.4: Let $\Gamma_{E_j}^j = \{\Sigma_1,\ldots,\Sigma_n; B_1^j,\ldots,B_n^j\}$ $(j\in P = \{1,\ldots,p\})$ n-person games defined on the same strategy sets with the respective structure functions E_j such that for all $j\in P$ and all the players

$i \in N$: $f^j(i) = \emptyset$. The strategy set Σ_i of player $i \in N$ is assumed to be non-empty, compact and convex in a locally convex linear Hausdorff space, his payoff functions B_i^j with $j \in P$ are continuous in the product variable $\sigma \in \Sigma$, quasi-concave in $\sigma_{d^j(i)} \in \Sigma_{d^j(i)}$, for fixed $\sigma_{e^j(i)} \in \Sigma_{e^j(i)}$, and it is quasi-convex in $\sigma_{e^j(i)} \in \Sigma_{e^j(i)}$ for fixed $\sigma_{d^j(i)} \in \Sigma_{d^j(i)}$. If for each point $\sigma \in \Sigma$ there is a joint strateby $\tau \in \Sigma$ such that

$$B_i^j(\tau_{d^j(i)}, \sigma_{e^j(i)}) = \max_{s_{d^j(i)} \in \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)})$$

and

$$B_i^j(\sigma_{d^j(i)}, \tau_{e^j(i)}) = \min_{s_{e^j(i)} \in \Sigma_{e^j(i)}} B_i^j(\sigma_{d^j(i)}, s_{e^j(i)})$$

for all $j \in P$ and all the players $i \in N$, then there exists a point $\bar{\sigma} \in \Sigma$ which is simultaneously for all $j \in P$ E_j-saddle point of the game $\Gamma^j_{E_j}$.

<u>PROOF:</u> Consider the following 2p associated games

$$\Gamma^{Cj}_{\bar{E}_j} = \{\Sigma_1, \ldots, \Sigma_n; \ C_1^j, \ldots, C_n^j\}$$

whose strategy sets coincide with the corresponding strategy sets of original games. The structure function $\bar{E}_j$ for $j \in \{1, \ldots, 2p\}$ is given by: $\bar{E}_j = E_j$ if $j \in P$ and by $\bar{d}^j(i) = e^j(i)$, $\bar{e}^j(i) = d^j(i)$ if $j \in \{p+1, \ldots, 2p\}$. Finally the payoff functions are defined by

$$C_i^j(\sigma) = \begin{cases} B_i^j(\sigma) & \text{if } j \in P \\ B_i^j(\sigma) & \text{otherwise.} \end{cases}$$

Hence, all the requirements of theorem VI.1 are fulfilled for these games and therefore the existence of a point $\bar{\sigma}\epsilon\Sigma$ which is $\bar{E}_j$-positive equilibrium point for all these games, is demonstrated. Such a point is simultaneously an E_j-saddle point of the original games $\Gamma^j_{E_j}$. (Q.E.D.)

VI.2: E-points by Intersection of Sets with Convex Cylinders Procedure.

In this section we deal with a second approach for E-points in general, similar to what was done for e-simple points in the third chapter, which is based on the procedure of Fan [5].

These results will contain as particular cases those already derived in the previous paragraph by using a fixed point procedure.

The results of this section, will be derived in an analogous manner to that of the third chapter, as consequence of the following general result, which is a straight forward extension of theorem III.30. It will be proven in an indirect way using that theorem.

THEOREM VI.5: Let $\Sigma_1,\dots,\Sigma_n$ be compact and convex sets each in a linear Hausdorff space and for each $i\epsilon N = \{1,\dots,n\}$ and each $j\epsilon P = \{1,\dots,p\}$, let $h^j(i)$ be subsets of N. Let there also be given pn subsets $S^j_1,\dots,S^j_n$ $j\epsilon P$ of the product space $\Sigma = \underset{i\epsilon N}{\times} \Sigma_i$, such that for each $i\epsilon N$ and each point $\sigma\epsilon\Sigma$ all the cylinders

$$S^j_i(\sigma) = \{\tau\epsilon\Sigma:\ (\tau_{h^j(i)},\sigma_{N-h^j(i)})\epsilon S^j_i\}$$

with $j\epsilon P$ are convex and the cylinders

$$S^{j^i}(\sigma) = \{\tau\epsilon\Sigma:\ (\sigma_{h^j(i)},\tau_{N-h^j(i)})\epsilon S^j_i\}$$

with $j \in P$ are convex and the cylinders

$$S^{j^i}(\sigma) = \{\tau \in \Sigma : (\sigma_{h^j(i)}, \tau_{N-h^j(i)}) \in S_i^j\}$$

are open. If for each $\sigma \in \Sigma$ there is another point $\tau \in \Sigma$ such that

$$(\tau_{h^j(i)}, \sigma_{N-h^j(i)}) \in S_i^j$$

for all $i \in N$ and $j \in P$, then the intersection

$$\bigcap_{j \in P} \bigcap_{i \in P} S_i^j$$

is non-empty.

PROOF: If the set P is formed by only one element, then this theorem coincides with the result expressed in theorem III.20. On the other hand if the set P is composed of two elements, it is theorem III.30 itself. Finally, suppose that the set P has more than two elements; that is $P = \{1,\ldots,p\}$ with $p > 2$. Let us consider two arbitrary different elements of set P, which without loss of generality are assumed to be 1 and 2. Then theorem III.30 applies to the subsets $S_1^1,\ldots,S_n^1$ and $S_1^2,\ldots,S_n^2$, since all the respective requirements are verified, we have that the intersection

$$\bigcap_{i \in N} (S_i^1 \cap S_i^2)$$

is non-empty. Now, define the subset $T_i^1 \subseteq \Sigma$ as the set $S_i^1 \cap S_i^2$. Again consider the subsets $T_1^1,\ldots,T_n^1$ and $S_1^3,\ldots,S_n^3$, which again satisfy all the conditions of theorem III.30, and therefore, the non-voidness of the intersection

$$\bigcap_{i \in N} (T_i^1 \cap S_i^3)$$

is assured. By following in this way, applying the mentioned theorem each time, the non-emptyness of the complete intersection

$$\bigcap_{j \in P} \bigcap_{i \in N} S_i^j$$

is proven since the number of element of P is finite. (Q.E.D.

We note that the number $p.n$ of subsets involved in this theorem is not essential. Indeed, if one has m subsets where

$$(p-1).n < m < p.n$$

for a determined natural number p, then by considering the product space Σ as those remaining $p.n-m$ subsets demanded by theorem, one obtains the desired result only for those m subsets.

As a first application of the above theorem we now derive a general result which generalized theorem VI.2 concerning E-partially compound points formulated as follows:

<u>THEOREM VI.6:</u> Let $\Gamma_E = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game with structure function $\underline{E}$, such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space. Let there also be given a partition $\{M_1,M_2,M_3,M_4,M_5,M_6\}$ of a subset M of players set N such that all the payoff functions A_i of players $i\epsilon M$ are continuous; for all $i\epsilon M_1 \cup M_5$, A_i is quasi-concave in $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ for fixed $\sigma_{e(i)\cup f(i)}\epsilon\Sigma_{e(i)\cup f(i)}$, and for all players $i\epsilon M_2 \cup M_5$ it is quasi-convex in $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$ for fixed $\sigma_{d(i)\cup f(i)}\epsilon\Sigma_{d(i)\cup f(i)}$; also for all $i\epsilon M_2 \cup M_6$ the function F_i is quasi-concave in $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ and for all $i\epsilon M_5 \cup M_6$ the function G_i is quasi-convex with respect to $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$, for fixed $\sigma_{f(i)}\epsilon\Sigma_{f(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) > \max_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) - \delta$$

for all $i\epsilon M_1 \cup L_5$,

Define the sets $h^1(i) = h^3(i) = d(i)$ and $h^2(i) = h^4(i) = e(i)$ for every player $i\epsilon N$. Then the cylinder

$$S^{1i}_{\delta}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma: A_i(\sigma_{d(i)},\tau_{e(i)},\tau_{f(i)}) > \max\limits_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\tau_{e(i)},\tau_{f(i)}) - \delta\} & \text{if } i\epsilon M_1 \cup L_5 \\ \Sigma & \text{otherwise} \end{cases}$$

of player $i\epsilon N$ is open, since the payoff function A_i is continuous in the product variable. By the same reasoning the cylinders

$$S^{2i}_{\delta}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma: A_i(\tau_{d(i)},\sigma_{e(i)},\tau_{f(i)}) < \min\limits_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\tau_{d(i)},s_{e(i)},\tau_{f(i)}) + \delta\} & \text{if } i\epsilon M_3 \cup L_5 \\ \Sigma & \text{otherwise} \end{cases}$$

are also open. Moreover by the continuity of payoff functions, the minimum functions F_i and the maximum functions G_i are also continuous, and therefore the cylinders

$$S^{3i}_{\delta}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma: F_i(\sigma_{d(i)},\tau_{f(i)}) > \max\limits_{s_{d(i)}\epsilon\Sigma_{d(i)}} F_i(s_{d(i)},\tau_{f(i)}) - \delta\} & \text{if } i\epsilon M_2 \cup (M_5 - L_5) \cup M_6 \\ \Sigma & \text{otherwise} \end{cases}$$

and

$$S^{4i}_{\delta}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma: G_i(\sigma_{e(i)},\tau_{f(i)}) < \min\limits_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\tau_{f(i)}) + \delta\} & \text{if } i\epsilon M_4 \cup (M_5 - L_5 (\cup M_6 \\ \Sigma & \text{otherwise} \end{cases}$$

are open too. The quasi-concavity of functions A_i and F_i with respect to the variable $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$, implies that the cylinders

$$S^{1}_{\delta,i}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma: A_i(\tau_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) > \max\limits_{s_{d(i)}\epsilon\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) - \delta\} & \text{if } i\epsilon M_1 \cup L_5 \\ \Sigma & \text{otherwise} \end{cases}$$

and

$$A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\in M_3 \cup L_5$,

$$F_i(\tau_{d(i)},\sigma_{f(i)}) > \max_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\sigma_{f(i)}) - \delta$$

for all $i\in M_2\cup(M_5-L_5)\cup M_6$ and

$$G_i(\tau_{e(i)},\sigma_{f(i)}) < \min_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta$$

for all $i\in M_4 \cup (M_5-L_5) \cup M_6$, where L_5 is an arbitrary subset of M_5 , then the game $\bar{\bar{E}}$ has an $M_1^+ \; M_{2,s}^+ \; M_3^- \; M_{4,s}^- \; M_5^= \; M_{6,s}^=$ $\underline{E}$-partially composed point.

<u>PROOF:</u> For each player $i\in N$ and a real number $\delta > 0$, consider the following sets defined by

$$S^1_{\delta,i} = \begin{cases} \{\sigma\in\Sigma\colon A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) > \max\limits_{s_{d(i)}\in\Sigma_{d(i)}} A_i(s_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) - \delta\} & \text{if } i\in M_1\cup L_5 \\ \Sigma & \text{otherwise} \end{cases}$$

$$S^2_{\delta,i} = \begin{cases} \{\sigma\in\Sigma\colon A_i(\sigma_{d(i)},\sigma_{e(i)},\sigma_{f(i)}) < \min\limits_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)}) + \delta\} & \text{if } i\in M_3\cup L_5 \\ \Sigma & \text{otherwise} \end{cases}$$

$$S^3_{\delta,i} = \begin{cases} \{\sigma\in\Sigma\colon F_i(\sigma_{d(i)},\sigma_{f(i)}) > \max\limits_{s_{d(i)}\in\Sigma_{d(i)}} F_i(s_{d(i)},\sigma_{f(i)}) - \delta\} & \text{if } i\in M_2\cup(M_5-L_5)\cup M_6 \\ \Sigma & \text{otherwise} \end{cases}$$

and finally

$$S^4_{\delta,i} = \begin{cases} \{\sigma\in\Sigma\colon G_i(\sigma_{e(i)},\sigma_{f(i)}) < \min\limits_{s_{e(i)}\in\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) + \delta\} & \text{if } i\in M_4\cup(M_5-L_5)\cup M_6 \\ \Sigma & \text{otherwise.} \end{cases}$$

$$S^3_{\delta,i}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma\colon F_i(\tau_{d(i)},\sigma_{f(i)}) > \max\limits_{s_{d(i)}\epsilon\Sigma_{d(i)}} F_i(s_{d(i)},\sigma_{f(i)}) - \delta\} & \text{if } i\epsilon M_3 \cup (M_5-L_5) \cup M_6 \\ \Sigma & \text{otherwise} \end{cases}$$

are convex. Furthermore, the quasi-convexity of function A_i and G_i in the variable $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$, implies that both of the cylinders

$$S^2_{\delta,i}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma\colon A_i(\sigma_{d(i)},\tau_{e(i)},\sigma_{f(i)}) < \min\limits_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_{d(i)},s_{e(i)},\sigma_{f(i)}) + \delta\} & \text{if } i\epsilon M_3 \cup L_5 \\ \Sigma & \text{otherwise} \end{cases}$$

and

$$S^4_{\delta,i}(\sigma) = \begin{cases} \{\tau\epsilon\Sigma\colon F_i(\tau_{e(i)},\sigma_{f(i)}) < \min\limits_{s_{e(i)}\epsilon\Sigma_{e(i)}} F_i(s_{e(i)},\sigma_{f(i)}) + \delta\} & \text{if } i\epsilon M_4 \cup (M_5-L_5) \cup M_6 \\ \Sigma & \text{otherwise} \end{cases}$$

are convex. Finally, by the latter condition, the last requirement of the previous theorem applies to those subsets already considered. Thus, the intersection

$$\bigcap_{i\epsilon N} (S^1_{\delta,i} \cap S^2_{\delta,i} \cap S^3_{\delta,i} \cap S^4_{\delta,i}) \qquad \text{for any } \delta > 0$$

is non empty. Now, defining

$$(\overline{S^1_{\delta,i}} \cap \overline{S^2_{\delta,i}} \cap \overline{S^3_{\delta,i}} \cap \overline{S^4_{\delta,i}}) \quad \text{for a given } \delta > 0,$$

where S indicates the closure of S, we then have that the family of sets

$$R_\delta = \bigcap_{i\epsilon N} (\overline{S^1_{\delta,i}} \cap \overline{S^2_{\delta,i}} \cap \overline{S^3_{\delta,i}} \cap \overline{S^4_{\delta,i}})$$

satisfies the finite intersection property. Therefore, because the product space Σ is compact, the intersection

$$\bigcap_{\delta>0} \bigcap_{i\epsilon N} (\overline{S^1_{\delta,i}} \cap \overline{S_{\delta,i}} \cap \overline{S_{\delta,i}} \cap \overline{S_{\delta,i}})$$

is non-empty. Hence, there exists a point $\bar{\sigma}\epsilon\Sigma$ which is an $M^+_1\, M^+_{2,s}\, M^-_3\, M_{4,s}\, L^=_5\, [M_6 \cup (M_5-L_5)]^=$ E-partially composed point of game Γ_E.

Now, consider for a player $i\epsilon M_5-L_5$ the two-person associated game

$$\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)}, \Sigma_{e(i)}; A_i(\sigma_{d(i)},\sigma_{e(i)},\bar{\sigma}_{f(i)})\}$$

which is determined by the choice $\bar{\sigma}_{f(i)}\epsilon\Sigma_{f(i)}$ of the respective indifferent coalition. This game satisfies all the requirements of the minimax theorem III.23. Hence, for it, by the definition of functions F_i and G_i we have that the following equality holds:

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

which implies

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)})$$
$$= G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)})$$

for each $i\epsilon M_5-L_5$. Then, the point $\bar{\sigma}\epsilon\Sigma$ is an $M_1^+ \; M_{2,3}^+ \; M_3^- \; M_{4,s}^- \; M_5^= \; M_{6,s}^=$ E-partially composed point of game Γ_E. (Q.E.D.)

We note that the last condition involved in the above theorem is weaker than the respective one in theorem VI.2.

As has been shown in the previous chapter and in the preceding section, applying to the corresponding procedures, the respective particularization of this result has always been a special case of a further general result. Of course in this case we also have a general extension of theorem IV.1 which is formulated as follows:

<u>THEOREM VI.7:</u> Let $\Gamma_{E_j}^j = \{\Sigma_1,\dots,\Sigma_n; B_1^j,\dots,B_n^j\}$ $(j\epsilon P = \{1,\dots,p\})$ be p n-person games defined on the same strategy sets with the respective structure functions E_j such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space. For each $j\epsilon P$

let M_j be a subset of N, such that for all the players $i\epsilon M_j$, the payoff function B_i^j is continuous in the product space Σ and quasi-concave in $\sigma_{d^j(i)}\epsilon\Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i)\cup f^j(i)}\epsilon\Sigma_{e^j(i)\cup f^j(i)}$. If for each real number $\delta > 0$ and each joint strategy $\sigma\epsilon\Sigma$ there is a point $\tau\epsilon\Sigma$ such that

$$B_i^j(\tau_{d^j(i)},\sigma_{e^j(i)}\sigma_{f^j(i)}) > \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) - \delta$$

for all $j\epsilon P$ and all the players $i\epsilon M_j$; then, there exists a joint strategy $\bar{\sigma}\epsilon\Sigma$, which is simultaneously for all $j\epsilon P$ on E_j-partially equilibrium point with respect to M_j of game $\Gamma_{E_j}^j$.

<u>PROOF:</u> For a given real number $\delta > 0$ and a player $i\epsilon N$, consider the sets

$$S_{\delta,i}^j = \begin{cases} \{\sigma\epsilon\Sigma: B_i^j(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) > \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) - \delta\} & \text{if } i\epsilon M_j \\ \Sigma & \text{otherwise} \end{cases}.$$

Let us define $h^j(i) = d^j(i)$ for every player $i\epsilon N$. Then, the continuity of payoff function B_i^j, for any $\sigma\epsilon\Sigma$ implies that the cylinder $S_\delta^{ji}(\sigma)$ is open and its quasi-concavity in the variable $\sigma_{d^j(i)}\epsilon\Sigma_{d^j(i)}$, implies that the cylinder $S_{\delta,i}^j(\sigma)$ is convex. Furthermore, by the last condition, theorem VI.5 is satisfied by sets $S_{\delta,i}^j$, and therefore the non-emptyness of the intersection

$$\bigcap_{j\epsilon P}\ \bigcap_{i\epsilon N} S_{\delta,i}^j \qquad \text{for any } \delta > 0,$$

is assured. Now, defining for each $\delta > 0$ the closure set $\bar{S}_{\delta,i}$, one can easily see that the family $R_\delta = \bigcap_{i\epsilon N}\ \bigcap_{j\epsilon P} \overline{S_{\delta,i}^j}$ has the finite intersection property, which implies the existence of a point $\bar{\sigma}\epsilon\Sigma$, a member of the intersection

$$\bar{\sigma} \epsilon \bigcap_{\delta>0}\ \bigcap_{i\epsilon N}\ \bigcap_{j\epsilon P} \overline{S_{\delta,i}^j},$$

since the product space Σ is compact. Such a point is simultaneously for all $j \epsilon P$ on E_j-partially positive equilibrium point with respect to M_j of game $\Gamma^j_{E_j}$. (Q.E.D.)

A special case of the above theorem is related in the following result concerned with some special E-saddle points, which extends theorem VI.4.

<u>THEOREM VI.8:</u> Let $\Gamma^j_{E_j} = \{\Sigma_1,\ldots,\Sigma_n; B^j_1,\ldots,B^j_n\}$ $(j\epsilon P = \{1,\ldots,p\})$ be p n-person games defined on the same strategy sets with the respective structure functions E_j such that for all $j\epsilon P$ and all the players $i\epsilon N$: $f^j(i) = \emptyset$. Also assumed that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear Hausdorff space, his payoff functions B^j_i with $j\epsilon P$ are continuous in the product variable $\sigma\epsilon\Sigma$, quasi-concave in $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i)} \epsilon \Sigma_{e^j(i)}$ and it is quasi-convex in $\sigma_{e^j(i)} \epsilon \Sigma_{e^j(i)}$ for fixed $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$. If for each real number $\delta > 0$ and each point $\sigma\epsilon\Sigma$ there is a joint strategy $\tau\epsilon\Sigma$ such that

$$B^j_i(\tau_{d^j(i)}, \sigma_{e^j(i)}) > \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B^j_i(s_{d^j(i)}, \sigma_{e^j(i)}) - \delta$$

and

$$B^j_i(\sigma_{d^j(i)}, \tau_{e^j(i)}) < \min_{s_{e^j(i)} \epsilon \Sigma_{e^j(i)}} B^j_i(\sigma_{d^j(i)}, s_{e^j(i)}) + \delta$$

for all $j\epsilon P$ and all the players $i\epsilon N$, then, there exists a point $\bar{\sigma}\epsilon\Sigma$ which is simultaneously, for all $j\epsilon P$, an E_j-saddle point of the game $\Gamma^j_{E_j}$.

PROOF: Consider the following 2p associated games

$$\Gamma_{\bar{E}_j}^{C_j} = \{\Sigma_1,\ldots,\Sigma_n;\ C_1^j,\ldots,C_n^j\}$$

with structure functions $\bar{E}_j$ defined by: $\bar{E}_j = E_j$ if $j\epsilon P$ and by $\bar{d}^j(i) = e^j(i)$, $\bar{e}^j(i) = d^j(i)$ if $j\epsilon\{p+1,\ldots,2p\}$, whose strategy sets are those respective of the original game, and finally the payoff functions are given by

$$C_i^j(\sigma) = \begin{cases} B_i^j(\sigma) & \text{if } j\epsilon P \\ -B_i^j(\sigma) & \text{otherwise.} \end{cases}$$

Then, all the requirements of theorem VI.7 apply to these games and therefore the existence of a point $\bar{\sigma}\epsilon\Sigma$ which is an $\bar{E}_j$-positive equilibrium point for all these games, is guaranteed. Such a point is simultaneous on E_j-saddle points of the original games $\Gamma_{E_j}^j$. (Q.E.D.)

At this point, it is interesting to ask for an extension of the result related in theorem III.32. Of course, in order to realize such an approach we should use a straightforward generalization of theorem III.30, which is formulated as follows:

THEOREM VI.9: Let $\Sigma_1,\ldots,\Sigma_n$ be non-empty, compact and convex sets each in a linear Hausdorff space, and for each $j\epsilon P = \{1,\ldots,p\}$ let $B_1^j,\ldots,B_n^j$ be n real functions defined on the product space $\Sigma = \times_{i\epsilon N} \Sigma_i$ such that for each $i\epsilon N = \{1,\ldots,n\}$ and $j\epsilon P$ let $h^j(i)$ be a subset of set N such that the function $B_i^j(\sigma_{h^j(i)},\sigma_{N-h^j(i)})$ is lower semicontinuous in the variable $\sigma_{N-h^j(i)}\epsilon\Sigma_{N-h^j(i)}$ for fixed $\sigma_{h^j(i)}\epsilon\Sigma_{h^j(i)}$ and it is quasi-concave in the variable $\sigma_{h^j(i)}\epsilon\Sigma_{h^j(i)}$ for fixed $\sigma_{N-h^j(i)}\epsilon\Sigma_{N-h^j(i)}$.

If given p vectors $\lambda^j = (\lambda^j_1, \ldots, \lambda^j_n)$ with $j \epsilon P$, for each $\sigma \epsilon \Sigma$ there is a point $\tau \epsilon \Sigma$ such that

$$B^j_i(\tau_{h^j(i)}, \sigma_{N-h^j(i)}) > \lambda^j_i$$

for all $j \epsilon P$ and $i \epsilon N$, then there exist a point $\bar{\sigma} \epsilon \Sigma$ such that

$$B^j_i(\bar{\sigma}_{h^j(i)}, \bar{\sigma}_{N-h^j(i)}) > \lambda^j_i$$

for all $j \epsilon P$ and $i \epsilon N$.

PROOF: For each $j \epsilon P$ and $i \epsilon N$, let us define the sets

$$S^j_i = \{\sigma \epsilon \Sigma: B^j_i(\sigma_{h^j(i)}, \sigma_{N-h^j(i)} > \lambda^j_i\} .$$

Thus, the quasi-concavity of functions B^j_i with respect to $\sigma_{h^j(i)} \epsilon \Sigma_{h^j(i)}$, implies that the cylinders

$$S^j_i(\sigma) = \{\sigma \epsilon \Sigma: B^j_i(\tau_{h^j(i)}, \sigma_{N-h^j(i)}) > \lambda^j_i\}$$

are convex. On the other hand, the cylinders

$$S^{ji}(\sigma) = \{\sigma \epsilon \Sigma: B^j_i(\sigma_{h^j(i)}, \tau_{N-h^j(i)}) > \lambda^j_i\}$$

are open, since the functions B^j_i are lower semicontinuous in the variable $\sigma_{N-h^j(i)} \epsilon \Sigma_{N-h^j(i)}$. Finally, by the latter condition, for each point $\sigma \epsilon \Sigma$ there is another one $\tau \epsilon \Sigma$ such that

$$(\tau_{h^j(i)}, \sigma_{N-h^j(i)}) \epsilon S^j_i$$

for all $j \epsilon P$ and $i \epsilon N$. Thus, all the requirements of theorem VI.5 apply these sets and therefore there exists a point $\bar{\sigma} \epsilon \Sigma$ belonging to the intersection:

$$\bar{\sigma} \epsilon \bigcap_{j \epsilon P} \bigcap_{i \epsilon N} S^j_i .$$

Such a point satisfies:

$$B_i^j(\bar{\sigma}_{h^j(i)}, \bar{\sigma}_{N-h^j(i)}) > \lambda_i^j$$

for all $j \epsilon P$ and $i \epsilon N$. (Q.E.D.)

Now as an immediate consequence of the previous theorem, we will derive the following result. It is a natural extension of theorem III.32.

<u>THEOREM VI.10:</u> Let $\Gamma_{E_j}^j = \{\Sigma_1, \ldots, \Sigma_n; A_i^j, \ldots, A_n^j\}$ $(j \epsilon P = \{1, \ldots, p\})$ be p n-person games defined on the same strategy sets with the respective structure functions E_j such that for all $j \epsilon P$ and all the players $i \epsilon N$: $f^j(\sigma) = \emptyset$. Assume that the strategy set Σ_i of player $i \epsilon N$ is non-empty, compact and convex in a linear Hausdorff space, and his payoff functions B_i^j are continuous in the product variable $\sigma \epsilon \Sigma$, quasi-concave in $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i)} \epsilon \Sigma_{e^j(i)}$, and quasi-concave in $\sigma_{e^j(i)} \epsilon \Sigma_{e^j(i)}$ for fixed $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$. If for each real number $\delta > 0$ and each point $\sigma \epsilon \Sigma$ there is a joint strategy $\tau \epsilon \Sigma$ such that

$$B_i^j(\tau_{d^j(i)}, \sigma_{e^j(i)}) > \min_{s_{e^j(i)} \epsilon \Sigma_{e^j(i)}} \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)}) - \delta$$

and

$$B_i^j(\sigma_{d^j(i)}, \tau_{e^j(i)}) < \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} \min_{s_{e^j(i)} \epsilon \Sigma_{e^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)}) + \delta$$

for all $j \epsilon P$ and $i \epsilon N$, then, there exists a point $\bar{\sigma} \epsilon \Sigma$ such that

$$B_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{e^j(i)}) = \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} \min_{s_{e^j(i)} \epsilon \Sigma_{e^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)})$$

$$= \min_{s_{d^j(i)} \epsilon \Sigma_{e^j(i)}} \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)})$$

for all $j \epsilon P$ and $i \epsilon N$.

PROOF: Given a real number $\delta > 0$, consider the following $2p$ vectors $\lambda_\delta^j = (\lambda_{\delta 1}^j,\ldots,\lambda_{\delta n}^j)$ and $\lambda_\delta^{1j} = (\lambda_{\delta 1}^{1j},\ldots,\lambda_{\delta n}^{1j})$, with $j\epsilon P$, by taking the components

$$\lambda_{\delta i}^j = \min_{s_{e^j(i)}\epsilon\Sigma_{d^j(i)}} \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)}) - \delta$$

and

$$\lambda_{\delta i}^{1i} = \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} \min_{s_{e^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)}) + \delta .$$

Now, for $j\epsilon\{1,\ldots,2p\}$ and player $i\epsilon N$, define the subset to be $h^j(i) = d^j(i)$ if $j\epsilon P = \}1,\ldots,p\}$ and $h^j(i) = e^j(i)$ if $j\epsilon\{p+1,\ldots,2p\}$, and let the functions be B_i^j if $j\epsilon P$ and $-B_i^j$ if $j\epsilon\{p+1,\ldots,n\}$. Thus, all the requirements of theorem VI.9 apply to these $2pn$ functions. Hence the existence of a point $\sigma^\delta\epsilon\Sigma$ such that

$$B_i^j(\sigma^\delta_{d^j(i)}, \sigma^\delta_{e^j(i)}) > \min_{s_{e^j(i)}\epsilon\Sigma_{e^j(i)}} \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)}) - \delta$$

$$< \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} \min_{s_{e^j(i)}\epsilon\Sigma_{e^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)}) + \delta$$

for all $j\epsilon P$ and all the players $i\epsilon N$, is guaranteed. Now, consider the directed system σ^δ with $\delta\to 0$ and let $\bar\sigma\epsilon\Sigma$ be a cluster point of σ^δ. Then, by virtue of the continuity of the payoff functions for the joint strategy $\bar\sigma\epsilon\Sigma$, we have

$$B_i^j(\bar\sigma_{d^j(i)}, \bar\sigma_{e^j(i)}) = \min_{s_{e^j(i)}\epsilon\Sigma_{e^j(i)}} \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)})$$

$$= \max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} \min_{s_{e^j(i)}\epsilon\Sigma_{e^j(i)}} B_i^j(s_{d^j(i)}, s_{e^j(i)})$$

for all $j\epsilon P$ and $i\epsilon N$. (Q.E.D.)

Again, unfortunately we are not able to derive that such a point is for each $j \in P$ an E_j-saddle point of the game $\Gamma^j_{E_j}$.

Finally, we now formulate a further characterization for E-saddle points which is a natural extension of theorem III.29. It will be obtained as an immediate consequence of the general theorem VI.7.

<u>THEOREM VI.11:</u> Let $\Gamma^j_{E_j} = \{\Sigma_1,\ldots,\Sigma_n; B^j_1,\ldots,B^j_n\}$ $(j \in P = \{1,\ldots,p\})$ be p n-person games defined on the same strategy sets with the respective structure functions E_j such that the strategy set Σ_i of player $i \in N$ is non-empty, compact and convex in a linear Hausdorff space. The payoff functions B^j_i of player $i \in N$ are continuous in the product space and quasi-concave with respect to $\sigma_{d^j(i)} \in \Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i) \cup f^j(i)} \in \Sigma_{e^j(i) \cup f^j(i)}$ and quasi-convex in $\sigma_{e^j(i)} \in \Sigma_{e^j(i)}$ for fixed $\sigma_{d^j(i)} \in \Sigma_{d^j(i)}$. If for each real number $\delta > 0$ and each point $\sigma \in \Sigma$ there is a joint strategy $\tau \in \Sigma$ such that

$$B^j_i(\tau_{d^j(i)},\tau_{e^j(i)}\sigma_{f^j(i)}) > \max_{s_{d^j(i)} \in \Sigma_{d^j(i)}} B^j_i(s_{d^j(i)},\tau_{e^j(i)},\sigma_{f^j(i)}) - \frac{\delta}{2}$$

$$< \min_{s_{e^j(i)} \in \Sigma_{e^j(i)}} B^j_i(\tau_{d^j(i)},s_{e^j(i)},\sigma_{f^j(i)}) + \frac{\delta}{2}$$

for all $j \in P$ and all the players $i \in N$, then, there exists a joint strategy $\bar{\sigma} \in \Sigma$ which is simultaneously for all $j \in P$, an E_j-saddle point of the game $\Gamma^j_{E_j}$.

<u>PROOF:</u> First of all, by the quasi-concavity of payoff functions B_i^j in $\sigma_{d^j(i)} \in \Sigma_{d^j(i)}$, we know that the minimum function F_i^j is also quasi-concave in the same variables. Similarly, the maximum function G_i^j is quasi-convex with respect to $\sigma_{e^j(i)} \in \Sigma_{e^j(i)}$, since B_i^j satisfies that condition too.

Now consider the $2p$ associated n-person games

$$\Gamma_{\bar{E}_j}^{C_j} = \{\Sigma_1,\ldots,\Sigma_n;\ C_1^j,\ldots,C_n^j\}$$

with structure function $\bar{E}_j$ defined by $\bar{E}_j = E_j$ if $j \epsilon P$ and by $\bar{d}^j(i) = e^j(i)$, $\bar{e}^j(i) = d^j(i)$ and $\bar{f}^j(i) = f^j(i)$ if $j \epsilon \{p+1,\ldots,2p\}$, whose strategy sets are those of the original game and finally the payoff functions are given by

$$C_i^j(\sigma) = \begin{cases} F_i^j(\sigma_{d(i)},\sigma_{f(i)}) & \text{if } j \epsilon P \\ -G_i^j(\sigma_{e(i)},\sigma_{f(i)}) & \text{otherwise} \end{cases} .$$

Then the first requirements of theorem VI.7 are satisfied by these games. Now we will show that the last condition also holds. Indeed, we have that for each real number $\delta > 0$ and each point $\sigma \epsilon \Sigma$ there is a joint strategy $\tau \epsilon \Sigma$ such that

$$B_i^j(\tau_{d^j(i)},\tau_{e^j(i)},\sigma_{f^j(i)}) > G_i^j(\tau_{e^j(i)},\sigma_{f^j(i)}) - \frac{\delta}{2} \geq$$

$$\geq \min_{s_{e^j(i)} \epsilon \Sigma_{e^j(i)}} G_i^j(s_{e^j(i)},\sigma_{f^j(i)}) - \frac{\delta}{2}$$

$$> F_i^j(\tau_{d^j(i)},\sigma_{f^j(i)}) + \frac{\delta}{2} \leq \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} F_i^j(s_{d^j(i)},\sigma_{f^j(i)}) + \frac{\delta}{2}$$

for all $j \epsilon P$ and $i \epsilon N$. Hence, because we always have

$$\max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} F_i^j(s_{d^j(i)},\sigma_{f^j(i)}) \leq \min_{s_{e^j(i)} \epsilon \Sigma_{e^j(i)}} G_i^j(s_{e^j(i)},\sigma_{f^j(i)})$$

we have the following inequalities

$$F_i^j(\tau_{d^j(i)}, \sigma_{f^j(i)}) > \max_{s_{d^j(i)} \in \Sigma_{e^j(i)}} G_i^j(s_{e^j(i)}, \sigma_{f^j(i)}) - \delta$$

and

$$G_i^j(\tau_{e^j(i)}, \sigma_{f^j(i)}) < \min_{s_{e^j(i)} \in \Sigma_{e^j(i)}} G_i^j(s_{e^j(i)}, \sigma_{f^j(i)}) + \delta$$

for all $j \in P$ and all the players $i \in N$. Thus, also the last requirement is satisfied and therefore, there exists a point $\bar{\sigma} \in \Sigma$ such that

$$F_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{e^j(i)}) = \max_{s_{d^j(i)} \in \Sigma_{d^j(i)}} F_i^j(s_{d^j(i)}, \bar{\sigma}_{f^j(i)})$$

and

$$G_i^j(\bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) = \min_{s_{e^j(i)} \in \Sigma_{e^j(i)}} G_i(s_{e^j(i)}, \bar{\sigma}_{f^j(i)})$$

for all $j \in P$ and all $i \in N$.

Finally, consider for a given $j \in P$ and $i \in N$ the associated two-person game determined by the joint strategy $\sigma_{f^j(i)} \in \Sigma_{f^j(i)}$, which according with theorem VI.8 has the minimax property, that is

$$F_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{f^j(i)}) = G_i^j(\bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) .$$

Hence, at the point $\bar{\sigma} \in \Sigma$ we have

$$A_i^j(\sigma_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) = F_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{f^j(i)})$$

$$= G_i^j(\bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)})$$

for all $j \in P$ and all the players $i \in N$. (Q.E.D.)

VI.3 E-Points by Maximum Function Procedure

Finally, this paragraph is devoted to the natural extension of the results derived in the latter section of the third chapter. Indeed, as we had in the previous pages, we now are concerned with verifying the maximum function procedure, built on the result due to Nikaido and Isoda [16], is applicable for characterizations of E-points for games defined on compact and convex strategy sets in linear topological spaces without the Hausdorff condition. But again this new treatment is not a complete extension of that which was derived in the previous paragraph, since we are not able to extend theorem III.35 for functions having stronger quasi-concavity property.

The general result will be derived immediately from the following auxiliar result concerning simultaneous E-partially postive equilibrium points where we have weakened the requirements on the existence of the strategies.

LEMMA VI.12: Let $\Gamma^j_{E_j} = \{\Sigma_1,\dots,\Sigma_n; B^j_1,\dots,B^j_n\}$ $(j\epsilon P = \{1,\dots,p\})$ be p n-person games defined on the same strategy sets with the respective structure functions E_j. For each $j\epsilon P$ let M_j be a subset of N. If for each joint strategy $\sigma\epsilon\Sigma$ such that

$$\sum_{j\epsilon P}\sum_{i\epsilon M_j} B^j_i(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) < \sum_{j\epsilon P}\sum_{i\epsilon M_j}\max_{s_{d^j(i)}\epsilon\Sigma_{d^j(i)}} B^j_i(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})$$

there exists a point $\tau\epsilon\Sigma$ such that

$$\sum_{j\epsilon P}\sum_{i\epsilon M_j} B^j_i(\tau_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) > \sum_{j\epsilon P}\sum_{i\epsilon M_j} B^j_i(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) ,$$

then, a joint strategy $\bar\sigma\epsilon\Sigma$ is an E_j-partially positive equilibrium point of game $\Gamma^j_{E_j}$ for every $j\epsilon P$, if and only if

$$\varphi(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \varphi(s,\bar{\sigma})$$

where the function φ is defined by

$$\varphi(\sigma,\tau) = \sum_{j\in P} \sum_{i\in M_j} B_i^j(\sigma_{d^j(i)}, \tau_{e^j(i)}, \tau_{f^j(i)}) .$$

PROOF: Let the point $\bar{\sigma}\in\Sigma$ be an E_j-partially positive equilibrium point of game $\Gamma_{E_j}^j$ for every $j\in P$. Thus, we have

$$B_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) = \max_{s_{d^j(i)}\in\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)})$$

for all $j\in P$ and $i\in M_j$, and therefore, the following inequality holds

$$\varphi(\bar{\sigma},\bar{\sigma}) = \sum_{j\in P} \sum_{i\in M_j} \max_{s_{d^j(i)}\in\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) \geq \max_{s\in\Sigma} \varphi(s,\bar{\sigma}) .$$

But in the second term of this inequality the maximum of the function appears, which implies that

$$\varphi(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \varphi(s,\bar{\sigma})$$

which shows the necessity of the assertion. Now, we will show the sufficiency. Let us suppose that there exists a point $\bar{\sigma}\in\Sigma$ which satisfies

$$\varphi(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \varphi(s,\bar{\sigma})$$

and it is not an E_j-partially positive equilibrium point of the game $\Gamma_{E_j}^j$ for every $j\in P$, then there exists a real number $\delta > 0$ and non-empty sets $J \subseteq P$ and $L_j \subseteq M_j$ for each $j\in J$, such that

$$B_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) < \max_{s_{d^j(i)}\in\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) - \delta$$

for all $j \epsilon J$ and $i \epsilon L_j$. But, then we have

$$\sum_{j\epsilon P} \sum_{i\epsilon M_j} B_i^j(\bar{\sigma}_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)}) < \sum_{j\epsilon P} \sum_{i\epsilon M_j} \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \bar{\sigma}_{e^j(i)}, \bar{\sigma}_{f^j(i)})$$

and therefore by the hypothesis, there exists a joint strategy $\bar{\tau}\epsilon\Sigma$ such that

$$\varphi(\bar{\tau},\bar{\sigma}) > \varphi(\bar{\sigma},\bar{\sigma})$$

which we see is impossible from the definition of the point $\bar{\sigma}\epsilon\Sigma$. (Q.E.D.)

We note that the following condition, which is a natural extension of the requirements used in Chapter III for the corresponding simple concepts, is completely satisfied by the above: for each real number $\delta > 0$ and each point $\sigma\epsilon\Sigma$ there is joint strategy $\tau\epsilon\Sigma$ such that

$$B_i^j(\tau_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) > \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)})$$

for all $j\epsilon P$ and $i\epsilon L_j$ for which

$$B_i^j(\sigma_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) < \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)})$$

and finally

$$B_i^j(\tau_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) = \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)})$$

for all the remaining $i\epsilon M_j - L_j$ with $j\epsilon P$.

Indeed, if for each $j\epsilon P$ the set L_j is empty, the general condition of the above lemma holds trivially, whereas, if for some $j\epsilon P$ the sets L_j are non-empty, then there exists a real number $\delta > 0$ and L_j for $i\epsilon L_j$ such that:

$$B_i^j(\sigma_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) < \max_{s_{d^j(i)} \epsilon \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)}, \sigma_{f^j(i)}) - \delta$$

for all $i\epsilon M_j$ with $j\epsilon P$. Thus, from this condition applied to this δ, there is a $\bar{\tau}\epsilon\Sigma$ for which:

$$\sum_{j\in P}\sum_{i\in M_j} B_i^j(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) < \sum_{j\in P}\sum_{i\in M_j} B_i^j(\tau_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})$$

and therefore, the requirement in the above lemma is satisfied.

Of course, one could formulate many other requirements which completely satisfy that which was adopted above.

With this result, we will now formulate a general characterization of those points.

<u>THEOREM VI.13:</u> Let $\Gamma_{E_j}^j = \{\Sigma_1,\dots,\Sigma_n; B_1^j,\dots,B_n^j\}$ be $(j\in P = \{1,\dots,p\}$ p n-person games defined on the same strategy sets having the respective structure functions E_j, such that the strategy set Σ_i of player $i\in N$ is non-empty, compact and convex in a linear topological space. For each $j\in P$ let M_j be a subset of N, such that for all $j\in P$ and $i\in M_j$ the payoff functions B_i^j are concave in $\sigma_{d^j(i)}\in\Sigma_{d^j(i)}$ for fixed

$\sigma_{e^j(i)\cup f^j(i)}\in\Sigma_{e^j(i)\cup f^j(i)}$, and continuous in the variable

$\sigma_{e^j(i)\cup f^j(i)}\in\Sigma_{e^j(i)\cup f^j(i)}$ for fixed $\sigma_{d^j(i)}\in\Sigma_{d^j(i)}$, and finally the function

$$\sum_{j\in P}\sum_{i\in M_j} B_i^j(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})$$

is continuous in $\sigma\in\Sigma$. If for each joint strategy $\sigma\in\Sigma$ such that

$$\sum_{j\in P}\sum_{i\in M_j} B_i^j(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) < \sum_{j\in P}\sum_{i\in M_j} \max_{s_{d^j(i)}\in\Sigma_{d^j(i)}} B_i^j(s_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)})$$

there exists a point $\tau\in\Sigma$ such that

$$\sum_{j\in P}\sum_{i\in M_j} B_i^j(\tau_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) > \sum_{j\in P}\sum_{i\in M_j} B_i^j(\sigma_{d^j(i)},\sigma_{e^j(i)},\sigma_{f^j(i)}) ,$$

then, there exists a joint strategy $\bar{\sigma}\epsilon\Sigma$ which is simultaneously for all $j\epsilon P$ an E_j-partially positive equilibrium point of game $\Gamma^j_{E_j}$.

PROOF: Consider the function

$$\varphi(\sigma,\tau) = \sum_{j\epsilon P} \sum_{i\epsilon M_j} B^j_i(\sigma_{d^j(i)}, \tau_{e^j(i)}, \tau_{f^j(i)})$$

defined on the product space $\Sigma \times \Sigma$. The continuity of the payoff functions B^j_i with respect to the variable $\sigma_{e^j(i)\cup f^j(i)} \epsilon \Sigma_{e^j(i)\cup f^j(i)}$, for each $\tau\epsilon\Sigma$ implies that the function $\varphi(\tau,\sigma)$ is continuous in $\sigma\epsilon\Sigma$. The other condition, implies that the function $\varphi(\sigma,\sigma)$ is also continuous with respect to $\sigma\epsilon\Sigma$. Finally, the concavity of payoff functions B^j_i in $\sigma_{d^j(i)} \epsilon \Sigma_{d^j(i)}$ implies the concavity in $\sigma\epsilon\Sigma$ of the function $\varphi(\sigma,\tau)$ for fixed $\tau\epsilon\Sigma$. Thus, all the requirements of theorem III.35 are satisfied by the function and therefore, a point $\bar{\sigma}\epsilon\Sigma$ exists such that

$$\varphi(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \varphi(s,\bar{\sigma}) \ .$$

But the latter condition on the payoff functions and lemma VI.12, imply that this point $\bar{\sigma}\epsilon\Sigma$ is simultaneously an E_j-partially positive equilibrium point of the game $\Gamma^j_{E_j}$ for all $j\epsilon P$. (Q.E.D.)

We now derive a characterization for partially compound points directly from this general result; which is formulated as follows:

THEOREM VI.14: Let $\Gamma_E = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be an n-person game with structure function E such that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear topological space. Let there also be given a partition $\{M_1,M_2,M_3,M_4,M_5,M_6\}$ of a subset M of players set N and an arbitrary $L_5 \subseteq M_5$ such that for $i\epsilon M_1 \cup M_5$ the payoff function A_i

and for $i \in M_2 \cup (M_5 - L_5) \cup M_6$ the minimum function F_i are continuous in $\sigma_{e(i) \cup f(i)} \in \Sigma_{e(i) \cup f(i)}$ for fixed $\sigma_{d(i)} \in \Sigma_{d(i)}$ and for $i \in M_1 \cup M_5$ A_i is concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ for fixed $\sigma_{e(i) \cup f(i)} \in \Sigma_{e(i) \cup f(i)}$, for $i \in M_3 \cup M_5$ A_i and for $i \in M_5 \cup (M_5 - L_5) \cup M_6$ the maximum function G_i are continuous is the variable $\sigma_{d(i) \cup f(i)} \in \Sigma_{d(i) \cup f(i)}$ for fixed $\sigma_{e(i)} \in \Sigma_{e(i)}$ and for $i \in M_3 \cup M_5$ A_i is convex in $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{d(i) \cup f(i)} \in \Sigma_{d(i) \cup f(i)}$. Furthermore let the function

$$\theta(\sigma) = \sum_{i \in M_1 \cup L_5} A_i(\sigma_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) + \sum_{i \in M_3 \cup L_5} A_i(\sigma_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)})$$

$$+ \sum_{i \in M_2 \cup (M_5 - L_5) \cup M_6} F_i(\sigma_{d(i)}, \sigma_{f(i)}) + \sum_{i \in M_4 \cup (M_5 - L_5) \cup M_6} G_i(\sigma_{e(i)}, \sigma_{f(i)})$$

be continuous in $\sigma \in \Sigma$. If for each point $\sigma \in \Sigma$ such that

$$\theta(\sigma) < \max_{s_{d(i)} \in \Sigma_{d(i)}} \sum_{i \in M_1 \cup L_5} A_i(s_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) + \max_{s_{e(i)} \in \Sigma_{e(i)}} \sum_{i \in M_3 \cup L_5} [-A_i(\sigma_{d(i)}, s_{e(i)}, \sigma_{f(i)})]$$

$$\max_{s_{d(i)} \in \Sigma_{d(i)}} \sum_{i \in M_2 \cup (M_5 - L_5) \cup M_6} F_i(s_{d(i)}, \sigma_{f(i)}) + \max_{s_{e(i)} \in \Sigma_{e(i)}} \sum_{i \in M_4 \cup (M_5 - L_5) \cup M_6} [-G_i(s_{e(i)}, \sigma_{f(i)})]$$

there exists a joint strategy $\tau \in \Sigma$ such that

$$\sum_{i \in M_1 \cup L_5} A_i(\tau_{d(i)}, \sigma_{e(i)}, \sigma_{f(i)}) + \sum_{i \in M_3 \cup L_5} A_i(\sigma_{d(i)}, \tau_{e(i)}, \sigma_{f(i)}) +$$

$$+ \sum_{i \in M_2 \cup (M_5 - L_5) \cup M_6} F_i(\tau_{d(i)}, \sigma_{f(i)}) + \sum_{i \in M_4 \cup (M_5 - L_5) \cup M_6} G_i(\tau_{e(i)}, \sigma_{f(i)}) > \theta(\sigma),$$

then, the game Γ_E has an $M_1^+ M_{2,s}^+ M_3^- M_{4,s}^- M_5^= M_{6,s}^=$ E-partially compound point.

PROOF: Given the game Γ_E with structure function E , the subset M with the corresponding partition and the set $L_5 \subseteq M_5$, consider the following n-person games

$$\Gamma_E^B = \{\Sigma_1,\ldots,\Sigma_n;\ B_1,\ldots,B_n\}\ ,\quad \Gamma_{\bar{E}}^C = \{\Sigma_1,\ldots,\Sigma_n;\ C_1,\ldots,C_n\}$$

$$\Gamma_E^D = \{\Sigma_1,\ldots,\Sigma_n;\ D_1,\ldots,D_n\} \text{ and } \Gamma_{\bar{E}}^E = \{\Sigma_1,\ldots,\Sigma_n;\ E_1,\ldots,E_n\}$$

whose respective structure functions are E and $\bar{E}$, where $\bar{E}$ is defined by

$$\bar{d}(i) = d(i)\ ,\quad \bar{e}(i) = d(i)\ ,\quad \bar{f}(i) = f(i)$$

for every player $i \in N$. The payoff functions are defined by

$$B_i(\sigma) = \begin{cases} A_i(\sigma) & \text{if } i \in M_1 \cup L_5 \\ K_i & \text{otherwise} \end{cases}$$

$$C_i(\sigma) = \begin{cases} -A_i(\sigma) & \text{if } i \in M_3 \cup M_5 \\ K_i & \text{otherwise} \end{cases}$$

$$D_i(\sigma) = \begin{cases} F_i(\sigma_{d(i)}, \sigma_{f(i)}) & \text{if } i \in M_2 \cup (M_5 - L_5) \cup M_6 \\ K_i & \text{otherwise} \end{cases}$$

and finally

$$E_i(\sigma) = \begin{cases} -G_i(\sigma_{e(i)}, \sigma_{\bar{f}(i)}) & \text{if } i \in M_4 \cup (M_5 - L_5) \cup M_6 \\ K_i & \text{otherwise} \end{cases}$$

where K_i is an arbitrary constant. By virtue of the conditions on the payoff functions, we have on the one hand that $B_i(\sigma)$ and $D_i(\sigma)$ are concave in $\sigma_{d(i)} \in \Sigma_{d(i)}$ and continuous in $\sigma_{e(i) \cup f(i)} \in \Sigma_{e(i) \cup f(i)}$. On the other hand, the functions $C_i(\sigma)$ and $E_i(\sigma)$ are continuous in $\sigma_{\bar{e}(i) \cup \bar{f}(i)} \in \Sigma_{\bar{e}(i) \cup \bar{f}(i)}$ and concave in $\sigma_{\bar{d}(i)} \in \Sigma_{\bar{d}(i)}$. Furthermore the function $\theta(\sigma)$ is continuous in $\sigma \in \Sigma$. Thus, by the last condition all the requirements of the previous theorem are satisfied by the four games introduced above, and therefore a point $\bar{\sigma} \in \Sigma$ exists which is simultaneously an E-positive equilibrium point of the games Γ_E^B , Γ_E^D and also an E-positive equilibrium point of the games Γ_E^C and Γ_E^E .

Consider for player $i\epsilon M_5-L_5$ the associated two-person game

$$\Gamma_i(\bar{\sigma}_{f(i)}) = \{\Sigma_{d(i)},\Sigma_{e(i)};\ A_i(\sigma_{d(i)},\sigma_{e(i)},\bar{\sigma}_{f(i)})\}$$

determined by the joint strategy $\bar{\sigma}_{f(i)}\epsilon\Sigma_{f(i)}$ of the respective indifferent coalition. From the continuity of payoff function A_i separately in the variables $\sigma_{d(i)}\epsilon\Sigma_{d(i)}$ and $\sigma_{e(i)}\epsilon\Sigma_{e(i)}$, and its concavity and convexity it follows that the theorem III.39 applys to it. Thus this game has a saddle point, which implies

$$F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) ,$$

and therefore

$$A_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)},\bar{\sigma}_{f(i)}) = F_i(\bar{\sigma}_{d(i)},\bar{\sigma}_{f(i)}) = G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)} .$$

Hence, the point $\bar{\sigma}\epsilon\Sigma$ is an $M_1^+\ M_{2,s}^+\ \ M_3^-\ \ M_{4,s}^-\ M_5^=\ \ M_{6,s}^=$ E-partially compound point of original game Γ_E . (Q.E.D.)

Another immediate consequence of theorem VI.13 is the following characterization of a special kind of simultaneous E-saddle point.

<u>THEOREM VI.15:</u> Let $\Gamma_{E_j}^j = \{\Sigma_1,\ldots,\Sigma_n;\ B_1^j,\ldots,B_n^j\}$ be $(j\epsilon P = \{1,\ldots,p\})$ p n-person games defined on the same strategy sets having the respective structure functions E_j , such that for all $j\epsilon P$ and all the players $i\epsilon N$ $f^j(i) = \emptyset$. Assume that the strategy set Σ_i of player $i\epsilon N$ is non-empty, compact and convex in a linear topological space, and that the payoff functions B_i^j are continuous and concave in $\sigma_{d^j(i)}\epsilon\Sigma_{d^j(i)}$ for fixed $\sigma_{e^j(i)}\epsilon\Sigma_{e^j(i)}$ and convex in $\sigma_{e^j(i)}\epsilon\Sigma_{e^j(i)}$ for fixed

$\sigma_{d^j(i)} \in \Sigma_{d^j(i)}$, furthermore that the function

$$\theta(\sigma) = 2 \sum_{j\in P} \sum_{i\in N} B_i^j(\sigma_{d^j(i)}, \sigma_{e^j(i)})$$

is continuous in $\sigma\in\Sigma$. If for each joint strategy $\sigma\in\Sigma$ such that

$$\theta(\sigma) < \sum_{j\in P} \sum_{i\in N} \max_{s_{d^j(i)} \in \Sigma_{d^j(i)}} B_i^j(s_{d^j(i)}, \sigma_{e^j(i)})$$

$$+ \sum_{j\in P} \sum_{i\in N} \max_{s_{e^j(i)} \in \Sigma_{e^j(i)}} [-B_i^j(\sigma_{d^j(i)}, s_{e^j(i)})]$$

there exists a point $\tau\in\Sigma$ such that

$$\sum_{j\in P} \sum_{i\in N} B_i^j(\tau_{d^j(i)}, \sigma_{e^j(i)}) + \sum_{j\in P} \sum_{i\in N} B_i^j(\sigma_{d^j(i)}, \tau_{e^j(i)}) > \theta(\sigma) ,$$

then, there exists a point $\bar{\sigma}\in\Sigma$ which is simultaneously E_j-saddle point of the game $\Gamma^j_{E_j}$ for all $j\in P$.

PROOF: Given the games $\Gamma^j_{E_j}$ with $j\in P$, define the $2\ p$ associated n-person games

$$\Gamma^{Cj}_{E_j} = \{\Sigma_1,\ldots,\Sigma_n;\ C_1^j,\ldots,C_n^j\} \quad \text{and} \quad \Gamma^{Dj}_{\bar{E}_j} = \{\Sigma_1,\ldots,\Sigma_n;\ D_1^j,\ldots,D_n^j\}$$

defined on the same strategy set, with structure functions E_j and $\bar{E}_j$ respectively, where $\bar{E}_j$ is given by

$$\bar{d}^j(i) = e^j(i) \quad \text{and} \quad \bar{e}^j(i) = d^j(i) .$$

The payoff functions are defined by

$$C_i^j(\sigma) = B_i^j(\sigma) \quad \text{and} \quad D_i^j(\sigma) = -B_i^j(\sigma) .$$

Thus, by the condition previously expressed, all the requirements of theorem VI.13 are satisfied by those games and therefore there exists a point $\bar{\sigma}\in\Sigma$ which is simultaneously an E_j-positive equilibrium point of the game $\Gamma^{C_j}_{E_j}$ for all $j \epsilon P$, and an $\bar{E}_j$-positive equilibrium point of the game $\Gamma^{D_j}_{\bar{E}_j}$. Such a point is simultaneously an E_j-saddle point of the original games $\Gamma^{j}_{E_j}$ (Q.E.D.)

We note that a similar result regarding simultaneous E-saddle points could be easily obtained by imposing a condition on the minimum and maximum functions and using the Nikaido's minimax theorem. Indeed, such an approach is similar to that adopted in theorem VI.14 with the introduction of set L_5.

Finally we do note that the most general results derived here are those related in theorems VI.7, VI.9, and VI.13. From these and their consequences all the results expressed here can be easily derived as special cases.

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INDEX

INDEX