Issued Under Office of Naval Research
Contract No. Nonr- 1858(16)

THE EQUIVALENCE IN TWO DIMENSIONS OF THE STRONG AND WEAK AXIOMS OF REVEALED PREFERENCE

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Econometric Research Program
Research Paper No. 1
17 April 1962

The research described in this paper was supported by the Office of Naval Research.

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1. Introduction

Any points u, x in the positive orthants B, C of real Euclidean spaces of dimension n are to define a balance and a composition. They have scalar product $u'x$; and the composition x is said to be within, on or over the balance u according as $u'x \leq , = , \text{ or } > 1$.

An expenditure figure is defined by a balance u together with a composition on it. Thus $(x \mid u)$ can denote an expenditure figure, where it is to be understood that $u'x = 1$. Any collection of expenditure figures defines an expenditure configuration.

Two expenditure figures $(x_o \mid u_o)$, $(x_1 \mid u_1)$ are said to be linked, in that order, if $u_o'x_1 \leq 1$ and $x_o \neq x_1$, that is, if the composition belonging to the second is within the balance, but different from the composition, belonging to the first. It is taken as determining an order of descent from the first to the second. A sequence of $k + 1$ figures $(x_r \mid u_r)$ $(r = 0, \ldots, k)$ in a configuration may be denoted by $(x_o, \ldots, x_k \mid u_o, \ldots, u_k)$. If each figure in it is linked with its successor, it defines a chain, of k links, on a k-chain, and may be denoted by $(x_o, \ldots, x_k \downarrow u_o, \ldots, u_k)$. If, moreover, the last figure is linked to the first, it becomes a cycle of $k + 1$ links, or a $k + 1$-cycle, denoted by $(x_o, \ldots, x_k \uparrow\downarrow u_o, \ldots, u_k)$. The figures are then in a cyclic order, with the first following the last, with every figure linked with its successor, taken around the cycle.

An hypothesis which can be applied to a given expenditure configuration is that no cycles can be formed from any of its figures. This corresponds to the Strong Axiom of Revealed Preference of Houthakker,[1] and is the condition for the existence of an irreflexive, transitive order relation in which every progression along a chain can be consistently interpreted as a

[1]H. S. Houthakker, "Revealed preference and the utility function," Economica 17 (1950), pp. 159-174.

descent, to be considered as a descent in preference. Then the Weak Axiom of Samuelson[2] just asks that no 2-cycles can be formed; so it is a part of the Strong Axiom. While the Strong Axiom thus implies the Weak Axiom, it is not true that conversely the Weak Axiom implies the Strong Axiom, unless $n = 2$. In this and only this case, the Strong Axiom can be deduced from the Weak, and the two axioms are equivalent.

While this fact has been remarked upon in the literature, appears to be well-known, and indeed has a certain obviousness, no formal proof is to be found there.

In application to a continuously defined expenditure system, that is, a function $x = f(u)$ with the balance conditions $u'x = 1$, an analytical proof can be given, by methods of the calculus. But in more general applications, to an arbitrary expenditure configuration, it has to be an algebraical proof, such as will now be given.

Concerning the inequivalence of the Axioms in more than two dimensions, Gale[3] gives a counter-example to prove it, contrary to various speculations, which he recounts. Uzawa[4] has proposed certain "smoothness" conditions on an expenditure system under which there is equivalence. Nevertheless, it can be proved, on the lines indicated in Afriat,[5] that an expenditure system can be differentiable, satisfy the Weak Axiom, but violate the Strong Axiom, in the following extreme fashion; more than just the existence of cycles, a cycle exists through every pair of points.

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[2] P. A. Samuelson, "Consumption theory in terms of revealed preference," Economica 28 (1948), pp. 243-53.

[3] D. Gale, "A note on revealed preference," Economica (1960), pp. 348-354.

[4] H. Uzawa, "Preference and rational choice in the theory of consumption," Mathematical Methods in the Social Sciences, Stanford Mathematical Studies in the Social Sciences V (Stanford, 1959).

[5] S. N. Afriat, "Preference scales and expenditure systems," Econometrica (forthcoming).

2. Separation

Two balances $u, v \in B$, such that $v = u\alpha$ $(\alpha > 0)$, are said to be parallel. The relation they have, which is to be indicated by $u \parallel v$, is reflexive, symmetric, and transitive, and thus an equivalence. If $\alpha < 1$, then v is said to be an expansion of u, and u a contraction of v; so if $u \parallel v$, and u is an expansion of v, then $u'x < v'x$ for every composition x. Let $\langle u \rangle$ denote the set of balances parallel to u, so that

$$u \parallel v \equiv \langle u \rangle = \langle v \rangle .$$

If u, v are not parallel, let this be indicated by $u \nparallel v$.

A balance w is said to separate a pair of balances u, v, or the relation $\langle u \mid w \mid v \rangle$ is defined to hold, if $w \in \langle u \rangle + \langle v \rangle$, in other words if $w = u\alpha + v\beta$ when $\alpha, \beta > 0$. This relation of separation of two balances by a third has certain properties which are now going to be shown.

Obviously,

$$\langle u \mid v \mid u \rangle \Longleftrightarrow u \parallel v .$$

Also

$$\langle t \mid u \mid v \rangle \wedge \langle u \mid v \mid w \rangle \Longrightarrow \langle t \mid u,v \mid w \rangle ,$$

where

$$\langle t \mid u,v \mid w \rangle \text{ means } \langle t \mid u \mid w \rangle \text{ and } \langle t \mid v \mid w \rangle .$$

For if

$$u = t\alpha + v\beta , \; v = u\gamma + w\delta \quad (\alpha, \beta, \gamma, \delta > 0) ,$$

then

$$u = t\alpha + (u\gamma + w\delta)\beta$$

so that

$$u(1 - \gamma\beta) = t\alpha + w\delta\beta .$$

But the vectors have positive elements, and the result of combining t, w with positive coefficient $\alpha, \delta\beta$ must be a vector with positive elements. Hence, $1 - \gamma\beta > 0$; and there follows a relation of the form $u = t\mu + w\upsilon$; that is $< t \mid u \mid w >$. Similarly $< t \mid v \mid w >$.

By induction

$$< u \mid v_o \mid v_1 > \wedge < v_o \mid v_1 \mid v_n > \wedge \ldots \wedge < v_{m-1} \mid v_m 'w >$$
$$\Longrightarrow < u \mid v_o \ldots v_m \mid w > .$$

The distinctive property of $n = 2$ is that, given any three balances no two of which are parallel, one separates the other two. For they must be linearly independent, so that

$$u_o\alpha_o + u_1\alpha_1 + u_2\alpha_2 = 0$$

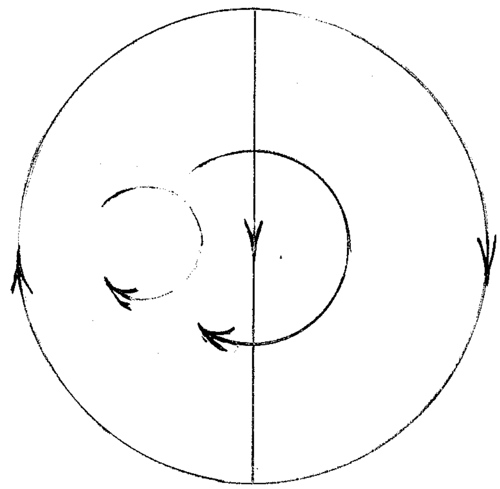
for some α's , not all zero. Two plainly cannot be zero, since the $u_r \neq 0$. Were one α zero, two of the balances must be parallel, contrary to hypothesis. Hence all the α's must be non-zero. They cannot then all be of the same sign, since if they are all positive or negative, $u_o\alpha_o + u_1\alpha_1 + u_2\alpha_2$ would be positive or negative, and not zero, since the u's all have positive or negative. Hence only two, say α_1, α_2 , will be of the same sign. This implies a relation of the form

$$u_o = u_1\beta_1 + u_2\beta_2 \quad (\beta_1, \beta_2 > 0) .$$

That is, $< u_1 \mid u_o \mid u_2 >$.

3. Proof of equivalence

An irreducible cycle is defined as a cycle in which there are no cross-links, in other words, in which the only links are those from one figure to its successor. Given any cycle, there can be constructed from it an irreducible subcycle. For if there is a cross-link, it becomes a link between consecutive figures in a subcycle; and so forth with that subcycle, until an irreducible cycle is obtained. Hence, given an expenditure configuration in



two dimensions, with a cycle, but no 2-cycles, in other words, which satisfies the Weak Axiom but, if possible, not the Strong Axiom, there must exist an irreducible cycle of more than two links. Let $(x_o \ldots x_k \uparrow\downarrow u_o \ldots u_k)$, with $k \geq 2$, be such a cycle. Then

$$\begin{array}{l} u_o'x_o = 1 \, , \; u_o'x_1 \leq 1 \, , \; u_o'x_2 > 1 \;\ldots\ldots\; u_o'x_k > 0 \\ u_1'x_o > 1 \, , \; u_1'x_1 = 1 \, , \; u_1'x_2 \leq 1 \;\ldots\ldots\; u_1'x_k > 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_k'x_o \leq 1 \, , \; u_k'x_1 > 1 \, , \; u_k'x_2 > 1 \;\ldots\ldots\; u_k'x_k = 1 \, . \end{array}$$

Assuming $n = 2$, a contradiction will be deduced from these conditions in the following manner.

Consider any triple of consecutive figures, with order according to the cyclic order, say $(u_o u_1 u_2 \mid x_o x_1 x_2)$. It will be shown that $< u_o \mid u_1 \mid u_2 >$, and similarly $< u_1 \mid u_2 \mid u_3 >$, ..., $< u_{k-1} \mid u_k \mid u_o >$; from which it will follow that $< u_o \mid u_1 \ldots u_k \mid u_o >$, and then that $u_o \parallel u_1 \parallel \ldots \parallel u_k$. Also the contrary of this conclusion is deduced. So a contradiction is obtained, showing that the hypothesis of a cycle with no 2-cycles is impossible, and thus that the Weak Axiom implies the Strong

Axiom.

No consecutive balances such as u_o, u_1, u_2, are parallel. It only has to be shown that $u_o \nparallel u_1, u_2$; and then, similarly, $u_1 \nparallel u_2, u_3$, and so on round the cycle.

Thus, $u_o \parallel u_1$ is impossible. For $u_o'x_2 > 1$, $u_1'x_2 \leq 1$, which implies $u_1'x_2 < u_o'x_2$, would then establish u_1 as an expansion of u_o . But also $u_o'x_1 \leq 1$ and $u_1'x_1 = 1$ imply $u_1'x_1 \geq u_o'x_1$, which contradicts that u_1 is an expansion of u_o .

Further, $u_o \nparallel u_2$. For otherwise there would be a similar contradiction in $u_o'x_1 < u_2'x_1$ and $u_o'x_2 > u_2'x_2$.

Now, since no pair of u_o, u_1, u_2 are parallel, one must separate the other two; and the same holds for every consecutive triple around the cycle.

But $< u_1 \mid u_o \mid u_2 >$ is impossible. For suppose $u_o = u_1\alpha + u_2\beta$. Then from $u_o'x_2 > 1$, $u_1'x_2 \leq 1$ and $u_2'x_2 = 1$ follows

$$1 < u_o'x_2 = u_1'x_2\alpha + u_2'x_2\beta \leq \alpha + \beta ,$$

and hence

$$1 < \alpha + \beta .$$

Also from $u_o'x_1 \leq 1$, $u_1'x_1 = 1$ and $u_2'x_1 > 1$ follows

$$1 \geq u_o'x_1 = u_1'x_1\alpha + u_2'x_1\beta > \alpha + \beta ,$$

and hence

$$1 > \alpha + \beta .$$

So there is a contradiction.

Also $< u_o \mid u_2 \mid u_1 >$ is impossible. For suppose $u_2 = u_o\alpha + u_1\beta$. Then $u_2'x_1 > 1$, $u_o'x_1 \leq 1$ and $u_1'x_1 = 1$ imply

$$1 < u_2'x_1 = u_o'x_1\alpha + u_1'x_1\beta \leq \alpha + \beta ,$$

so that

$$1 < \alpha + \beta .$$

And $u_2'x_3 \leq 1$, $u_1'x_3 > 1$ and $u_o'x_3 \geq 1$ imply

$$1 \geq u_2'x_3 = u_o'x_3\alpha + u_1'x_3\beta > \alpha + \beta ,$$

giving

$$1 > \alpha + \beta ,$$

so there is again a contradiction.

The only possibility which remains is $< u_o \mid u_1 \mid u_2 >$, which is now proved, since $n = 2$. Similarly, $< u_1 \mid u_2 \mid u_3 >$, ..., $< u_{k-1} \mid u_k \mid u_o >$. But these separation conditions together imply

$$< u_o \mid u_1 \ldots u_k \mid u_o > ,$$

which is impossible, unless $u_o \parallel u_1 \parallel \ldots \parallel u_k$. But it has already been shown that $u_o \nparallel u_1 \nparallel \ldots$. Hence there is a contradiction, so the original hypothesis, of a cycle without 2-cycles, is false.