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ON ECONOMIC ENTROPY

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ABSTRACT

§ 1. Deduction of the Natural Formulae of Index Numbers

In the theory of information, the "Entropy H" is defined as

$$(1) \qquad H = -\sum_s p_s \log p_s \qquad [\ \sum_s = \sum_{s=1}^{n}\ , \quad s = 1 \sim n\ ]$$

where p_s is a probability that an event Es happens $(s = 1 \sim n)$.

In economics, the corresponding entropy is the demand entropy, which is defined as follows:

$$(2) \qquad H_t^{(n)} = -\sum_s \alpha_t^s \log \alpha_t^s$$

where

$$\alpha_t^s = \frac{p_t^s\, q_t^s}{v_t} \qquad\qquad v_t = \sum_s p_t^s\, q_t^s$$

p_t^s is the market price of good "s" and

q_t^s is its demanded quantity.

Rewriting $H_t^{(n)}$ with v_t, p_t^s and q_t^s's, we have

$$(3) \qquad H_t^{(n)} = \log v_t - \log \pi_t - \log \rho_t$$

where

$$(4) \qquad \log \pi_t = \sum_s \alpha_t^s \log p_t^s\ , \quad \log \rho_t = \sum_s \alpha_t^s \log q_t^s$$

It can easily be proved that when n is sufficiently large, $H_t^{(n)}$ is practically constant over time. Therefore, we get

$$(5) \quad 0 = H_j^{(n)} - H_i^{(n)} = (\log v_j - \log v_i) - (\log \pi_j - \log \pi_i) - \log \rho_j - \log \rho_i)$$

Thus, we have our "Natural Formulae" of index numbers, namely

$$(6) \qquad V_{ij} = \frac{v_j}{v_i}\ , \quad P_{ij} = \frac{\pi_j}{\pi_i}\ , \quad Q_{ij} = \frac{\rho_j}{\rho_i}\ , \quad V_{ij} = P_{ij}\, Q_{ij}$$

§ 2. Deduction of welfare functions

From expression (3) in the previous section, we have

$$(7) \qquad c_t \log \rho_t = - c_t H_t^{(n)} + c_t \log \bar{v}_t \qquad [\bar{v}_t = \frac{v_t}{\pi_t}]$$

We put $u_t = c_t \log \rho_t$, $w_t = - c_t H_t^{(n)}$, and we have welfare functions:

$$(8) \qquad u_t = w_t + c_t \log \bar{v}_t \qquad \text{(for the society as a whole)}$$

Similarly, we get wellbeing functions:

$$(9) \qquad u_t^i = w_t^i + c_t^i \log \bar{v}_t^i \qquad \text{(for the i-th family)}$$

We can construct u_t from its constituents u_t^i's $(i = 1 \sim m)$, making use of the relation:

$$(10) \qquad H_t = \sum_i \beta_t^i H_t^i , \qquad \beta_t^i = \frac{v_t^i}{v_t}, \quad (i = 1 \sim m) .$$

§ 3. Economic Entropy

Our basic assumption concerning the nature of the demand entropy function is now modified to make our model more realistic. We have previously assumed that the principal part of the demand entropy function is equal to "$c_t \log v_t$". This implies the marginal utility of money is $\frac{c_t}{v_t}$ which is an oversimplified and unrealistic restriction. In order to correct this we propose to define a function of the type

$$E(h_t) = \frac{c_t}{1 + e^{-b_t h_t}}$$

as economic entropy where $h_t = \bar{v}_t - a_t$ and a_t, b_t and c_t are Gestalt constants. This economic entropy should replace the principal part of welfare function. A similar argument applies to the individual wellbeing function for the i-th family, $(i = 1 \sim m)$.

ON ECONOMIC ENTROPY

Kazuo Mizutani

1. Deduction of the Natural Formulae of Index Numbers[1]

In The Mathematical Theory of Communication Professor Claude E. Shannon defines[2] "Entropy H" as

$$(1) \qquad H = - \sum_{s} p_s \log p_s \qquad [\ \sum_{s} = \sum_{s=1}^{n}\ ,\ s = 1 \sim n\ ]$$

where p_s is a probability that an event E_s happens $(s = 1 \sim n)$. There are "n" mutually exclusive events $\{E_s\}$, therefore, we have

$$(2) \qquad \sum_{s} p_s = 1$$

In economics, this "Entropy H" may be an index of the randomness or irrationality of our economic behavior.

Now, we shall examine relations between such an entropy and some economic functions. In the first place, we shall take up my "Natural Formulae" for making index numbers of prices and quantities.[3]

Suppose, an individual has an income

$$(3) \qquad v_t = \sum_{s} p_t^{\,s}\, q_t^{\,s}$$

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1 It was Professor Oskar Morgenstern who first suggested to me the possible benefits of studying information theory while I was staying at Princeton University as a Visiting Fellow in 1954.

2 Cf. C. E. Shannon and W. Weaver: The Mathematical Theory of Communication, Urbana, 1949, pp. 18-20.

3 Kazuo Mizutani: "New Formulae for Making Index Numbers of Prices and Quantities", which is coming out shortly.

The original form was published in "Bulletin of the International Statistical Institute, Vol. XXXVIII. Part II, Tokyo, 1961, and also in "Fundamental Laws of Elasticity and Semi-elasticity, and a New Formula for Index-Numbers of Prices and Quantities (in "Momoyama-Gakuin Keizaigaku Ronshu, with English summary, February, 1961).

where

v_t : his income

p_t^s : the market price of good "s"

q_t^s : the quantity of good "s" demanded.[4]

Taking the semi-elasticity[5] of both sides of (3), we obtain

$$(4) \qquad \frac{d \log_e v_t}{dt} = \sum_s \alpha_t^s \frac{d \log_e p_t^s q_t^s}{dt}$$

where

$$(5) \qquad \alpha_t^s = \frac{v_t^s}{v_t} = \frac{p_t^s q_t^s}{v_t}$$: the share of good "s" in his

total expenditure.

By integrating both sides of (4) with respect to "t" from t = i to j, we have[6]

$$(6) \qquad V_{ij} = P_{ij} Q_{ij}$$

where

$$(7) \qquad V_{ij} = \frac{v_j}{v_i}, \quad P_{ij} = \frac{\pi_j}{\pi_i}, \quad Q_{ij} = \frac{\rho_j}{\rho_i}$$

$$(8) \qquad \log \pi_t = \sum_s \alpha_t^s \log p_t^s$$

$$(9) \qquad \log \rho_t = \sum_s \alpha_t^s \log q_t^s$$

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4 The superscript "s" refers to good (or service) "s", while the subscript "t" refers to year "t". Throughout this article "goods" signifies "goods and services".

5 Let $y > 0$ be a real-valued continuously differentiable function of x defined over a positive interval $[x_1, x_2]$, then $\frac{d \log_e y}{dt} = \frac{1}{y} \cdot \frac{dy}{dt}$ shall be called the semi-elasticity of y with respect to x in the interval (x_1, x_2).

6 On condition that $R = \sum_{t=1}^{T} \sum_s \{(\alpha_\tau^s - \alpha_{\tau+1}^s) \log v_{\xi_\tau}^s\}$ be practically equal to zero. The whole interval $[i,j]$ of t is sub-divided into T subintervals $\{I_\tau^s\}$, $(\tau=1 \sim T)$.

Here π_t is the price-level in the year "t" and ρ_t is the quantity-level in the year "t". Therefore, P_{ij} is the index number of prices of the j^{th} year relative to the i^{th} year.

Similarly, Q_{ij} is the index number of quantities, and V_{ij} is the index number of values.

Now, we shall examine the relation between these "Natural Formulae" (6), (7), (8), (9) and so-called "Entropy H."

First, we note that α_t^s has the properties of a probability:

(i) $0 \leqq \alpha_t^s \leqq 1$

(ii) $\sum_s \alpha_t^s = 1$

For this reason, the demand entropy for all goods may well be defined as:

(10) $$H_t^{(n)} = - \sum_s \alpha_t^s \log \alpha_t^s$$ [7]

Substituting (5) for α_t^s in (10), we have:

(11) $$H_t^{(n)} = \log v_t - \log \pi_t - \log \rho_t \quad .$$

Consequently, if we can assume[8] that $H_t^{(n)}$ remains constant throughout the whole interval of time "t" , then we have

(12) $$0 = H_j^{(n)} - H_i^{(n)} = (\log v_j - \log v_i) - (\log \pi_j - \log \pi_i) - (\log \rho_j - \log \rho_i)$$

This is nothing but the relation shown in (6), (7), (8) and (9). Thus we see that we can deduce our Natural Formulae from the demand entropy for all goods (10), that is (11), on condition that $H_t^{(n)}$ be constant over time.

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7 The superscript (n) of H_t signifies the number of the different kinds of goods.

8 The proof is given on pages 4 to 5.

One might expect that as the economy develops and the number of different kinds of goods increases and consumption goods become diversified in their forms, in their qualities and in many other different respects, that the stochastic character of consumer's choice becomes more conspicuous. But if on the other hand, one is willingly to concede that as the markets of different consumption goods become better organized, that the fluctuations of prices and quantities demanded will become smaller and smaller, this latter assumption is sufficient to prove the invariance of $H_t^{(n)}$.

In order to see this, we shall assume as follows:

(i) $\{ X_s = \log_e \alpha_t^s \}$ is an infinite sequence of stochastic variables, with given probability distributions[9]

(ii) The X_s's are uniformly bounded, and they each have finite means and variances such as

$$\mu_s = E(X_s) , \quad \sigma_s^2 = Var (X_s)$$

(iii) The sum $S_n = \sum_s X_s$ has also finite mean and variance

$$m_n = \sum_s \mu_s, \quad s_n^2 = \sum_s \sigma_s^2 + 2 \sum_{r,s} Cov(X_r,X_s)$$

the last sum extending over each of the $\binom{n}{2}$ pairs (X_r,X_s) with $r < s$

(iv)
$$\lim_{n\to\infty} \frac{s_n}{n} = 0$$

Then, invoking the theorem on the law of large numbers, we have the constancy over time,[10] namely, for every $\epsilon > 0$ and $\eta > 0$, there exists $n_o (\epsilon,\eta)$ such that

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[9] Cf. W. Feller: An Introduction to Probability Theory and Its Applications, Vol. I, sec. ed., New York, 1957, pp. 238 and 246. The X_s's are not necessarily independent of each other.

[10] I would like to express my sincere thanks to Professor William Feller, to Professor Shizuo Kakutani and to Professor Hideya Gamo for their valuable comments upon my proof of the constancy of the demand entropy over time.

(13) $$P\{\,|H_t^{(n)} - H^{(n)}| > \epsilon\} < \eta \quad [\text{for any } n>n_o \text{ and for any } t]$$

where

(10) $$H_t^{(n)} = -\sum_s \alpha_t^s \log \alpha_t^s \qquad (s = 1\sim n)$$ [11]

and $H^{(n)}$ is the mathematical expectation of $H_t^{(n)}$.

§ 2. Deduction of Welfare Function

Last year, I published an article on "A New Welfare Function" in Japanese.[12]

(1) This welfare function was meant to be measurable in the sense of cardinal numbers.[13]

(2) This welfare function for the whole economy consists of the individual wellbeing functions of the constituent families which have properties similar to the whole welfare function.

(3) This social welfare function u_t must satisfy the following conditions:

(a) In equilibrium, it must satisfy the law of equi-marginal utilities, namely:

(14) $$u_t^s = \frac{\partial u_t}{\partial q_t^s} = \lambda_t\, p_t^s \qquad (s = 1\sim n)$$

where

λ_t is the social marginal utility of money, all these variables are functions of "t".

p_t^s is the market price of good"s".

q_t^s is its quantity demanded and supplied.

(b) The equilibrium must be stable, therefore, the Hessian of order k :

(15) $$D_k = |\,u_t^{h\ell}| \qquad (h,k,\ell = 1 \sim n)$$

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[11] The superscript (n) of H_t and H signifies the number of the different kinds of goods.

[12] Kazuo Mizutani: "A New Welfare Function" (in the Commemorative Issue of ACADEMIA for the Celebration of the Enlargement of Department of Economics, the Catholic University of Nagoya, March, 1964.)

[13] Throughout this article "measurable" means measurable in the sense of cardinal numbers.

should satisfy the condition:

$$(16) \qquad (-1)^k D_k > 0 \qquad (k = 1 \sim n)$$

where

$$(17) \qquad u_t^{h\ell} = \frac{\partial^2 u_t}{\partial q_t^\ell \, \partial q_t^h} \quad (h,\ell = 1 \sim n) \quad .$$

Consequently,

$$(18) \qquad u_t^{hh} < 0 \qquad (h = 1 \sim n)$$

(c) u_t must take into account the following factors:

(i) Luxury grade of the people

(ii) Leisure ratio of the people

(iii) Grade of culture of the people

(iv) Housing conditions of the people

(v) Standard of living of the people

Among those functions, which satisfy these conditions (1), (2), and $(3)_{(a,b,c)}$, one of the simplest may be

$$(19) \qquad u_t = w_t + c_t \log \frac{v_t}{\pi_t} \quad .$$

Here w_t is a negative definite quadratic form in a broader sense,[14] where $\tilde{q}_t^s$ is a stochastic variable, whose distribution (density) function is of the normal-type with mode at the equilibrium quantity q_t^s, $(s=1\sim n)$.

The parameter "c_t" is a function of parameters $\theta_1(t), \theta_2(t), \ldots, \theta_6(t)$, which take into account those factors such as Luxury grade etc. Since "c_t" is independent of p_t^s and q_t^s, we shall call this c_t a "Gestalt constant".

[14] Because, w_t contains stochastic variables, I have annexed the adjectival phrase "in a broader sense" to it.

Such a welfare function as u_t is interpreted, information-theoretically, as the quantity-level component of the demand entropy H_t .

Because

(11) $$H_t = \log v_t - \log \pi_t - \log \rho_t$$

the quantity-level component of H_t is:

(21) $$\log \rho_t = - H_t + \log \frac{v_t}{\pi_t}$$

Consequently, putting $w_t = - c_t H_t$, we have

(22) $$c_t \log \rho_t = w_t + c_t \log \frac{v_t}{\pi_t} = u_t$$

Thus, our welfare function for the whole society is interpreted, information-theoretically, as a function indicating the quantity-level with the Gestalt constant c_t as a weight to take into account those factors such as Luxury grade etc.

The partial derivative of u_t with respect to q_t^s , that is the marginal utility of good "s" in period t , is, in equilibrium:

(23) $$u_t^s = \frac{\partial u_t}{\partial q_t^s} = p_t^s c_t \frac{1 - \epsilon_s}{v_t} , \quad [\epsilon_s = \log p_t^s - \log \pi_t]$$ [15]

But, since $\log \pi_t$ is the arithmetic average of $\log p_t^s$, $\epsilon_s = \log p_t^s - \log \pi_t$ is very small in the normal case as compared with v_t . Consequently, we have in the normal case $\frac{1 - \epsilon_s}{v_t} \doteqdot \frac{1}{v_t}$, and $\frac{c_t}{v_t}$ is common to all goods. Hence, we obtain

(14) $$u_t^s = \lambda_t p_t^s \qquad [\lambda_t = \frac{c_t}{v_t}]$$

as is required by the condition (3,a). Here λ_t is taken to be the social marginal utility of money.

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15 In equilibrium, we have $\frac{\partial w_t}{\partial q_t^s} = 0$, because $\tilde{q}_t^r = q_t^r$, (r =1~n).

It is easy to see that u_t as shown in (19) satisfies all the other conditions (1), (2) and $(3)_{(a,b,c)}$.[16]

With regard to the relation between this welfare function of the whole society, we must examine the structure of the social income.

Suppose, the society S is composed of F families, and that the social income v_t is distributed among F families in period "t" as follows:

$$(24) \qquad v_t = \sum_i v_t^i \,, \quad \beta_t^i = \frac{v_t^i}{v_t} \,, \ (i=1\sim F), \quad \sum_i = \sum_{i=1}^{F}$$

then the relation between the social budget equation and those of the individual families are as follows:

$$(25) \qquad v_t = \sum_s p_t^s q_t^s \,, \quad v_t^i = \sum_s p_t^s q_t^s \,, \quad q_t^s = \sum_i q_t^{is}$$

$$\alpha_t^s = \frac{v_t^s}{v_t} = \frac{p_t^s q_t^s}{v_t} \,, \quad \alpha_t^{is} = \frac{v_t^{is}}{v_t^i} = \frac{p_t^s q_t^s}{v_t^i}$$

We shall call the demand entropy of the constituent individual family the elementary demand entropy, and we shall denote it by H_t^i. Then it is easy to see that

$$(26) \qquad H_t = \sum_i \beta_t^i H_t^i$$

$$(27) \qquad H_t^i = - \sum_s \alpha_t^{is} \log_e \alpha_t^{is} = \log_e v_t^i - \log_e \pi_t^i - \log \rho_t^i$$

From this, we can easily deduce the individual wellbeing function u_t^i for the i^{th} family, which shall be:

$$(28) \qquad u_t^i = w_t^i + c_t^i \log \frac{v_t^i}{\pi_t^i} \,, \quad [\log \pi_t^i = \sum_s \alpha_t^{is} \log p_t^s]$$

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16 In this connection, the following fact should also be noted. From the relation $w_t = - c_t H_t$, we can easily see that the fact that w_t includes stochastic variables $\tilde{q}_t^s$, $(s=1\sim n)$ clearly verifies the legitimacy of our interpretation of H_t as an index of the randomness or the irrationality of human economic behaviors.

Hence, the marginal utility of good "s" to this family is, in equilibrium,[17]

(29) $$u_t^{is} = \frac{\partial u_t^i}{\partial q_t^{is}} = \lambda_t^i p_t^s \qquad [\lambda_t^i = \frac{c_t^i}{v_t^i}]$$

where λ_t^i is the marginal utility of money to the i^{th} family. Then, for the marginal utility $\tilde{u}_t^i$ of the typical good of all n goods to the i^{th} family, the following relation holds:

(30) $$\tilde{u}_t^i = \lambda_t^i \pi_t^i \qquad [\log_e \tilde{u}_t^i = \sum_s \alpha_t^{is} \log_e u_t^{is}]$$

Therefore, it may be natural to suppose that the marginal utility $\tilde{u}_t$ of the typical good to the representative family M of the society S should be

(31) $$\tilde{u}_t = \Lambda_t \pi_t = \frac{c_t}{v_t''}$$

where

(32) $$\log \tilde{u}_t = \sum_i \beta_t^i \log_e \tilde{u}_t^i , \ \log_e \Lambda = \sum_i \beta_t^i \log \lambda_t^i ,$$

$$\log \pi_t = \sum_i \sum_s \beta_t^i \alpha_t^{is} \log p_t^s$$

that is, Λ_t is the marginal utility of money for M, which is representative for the society S as a whole.

π_t is the price-level of the social economy S.

c_t is the Gestalt constant of the society S, that is:

$$\log c_t = \sum_i \beta_t^i \log c_t^i$$

v_t'' is the real income of M namely, $v_t'' = \frac{\tilde{v}_t}{\pi_t}$, where $\tilde{v}_t$ is the monetary income of M, that is,

$$\log \tilde{v}_t = \sum_i \beta_t^i \log v_t^i$$

17 Here, we have neglected the term $\frac{\log_e p_t^s - \log_e \pi_t^i}{v_t^i}$ as on page 7 .

From (31), we can deduce that the social welfare function u_t of the society S is

$$(33) \qquad u_t = \int \frac{c_t}{v''_t} dv''_t = a_t + c_t \log v''_t$$

Since the real social income of the society is $v'_t = \frac{v_t}{\pi_t}$, the principal part of u_t, namely $c_t \log_e v''_t$, is evaluated by the definite integration:

$$(34) \qquad \int_1^{v'_t} \frac{c_t}{v''_t} dv'' = c_t \log_e v'_t = c_t \log_e \frac{v_t}{\pi_t}$$

Comparing (33) with (19) [=(22)], we see $a_t = w_t$.
Thus we get (19) [=(22)] from the individual wellbeing function of the constituent families.[18]

[18] Note that we have made use of the relation

$$H_t = \sum_i \beta^i_t H^i_t$$

in order to get u_t of S from the constituent functions u^i_t.

§ 3. Logistic Entropy

In this section we introduce a refinement into our model. We have up to now assumed that the marginal utility of money to the individual family was reciprocally proportional to its income. It would be preferable if we could replace this assumption by a more realistic assumption.

In speaking of ordinary objective economic transactions, it may be sufficient to speak of an entropy like that which is used in the theory of communication.[19] In such cases utility considerations have no direct applicability.

However, where psychological evaluation is involved we need a correspondingly more complex definition of entropy. In situations characterized by diminishing marginal returns such as is usually assumed for utility, it seems to me that the natural entropy should be replaced by the corresponding logistic entropy.

By logistic entropy, we mean

$$(35) \quad E(h_t) = \frac{c_t}{1 + e^{-h_t}}$$

where

$$h_t = b_t (\bar{v}_t - a_t)$$

$$\bar{v}_t = \frac{v_t}{\pi_t}$$

and $a_t, b_t, c_t > 0$ are the Gestalt constants.

If we assume that the main part of u_t is of this type, then we may express u_t as follows:

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[19] Take, for example, the case of index numbers. My natural index numbers are derived from the ordinary entropy concept which is as it should be since they characterize objective transactions, not the psychic value of such transactions.

$$(36) \qquad u_t = w_t + E(h_t)$$

where w_t is a negative definite quadratic form as shown in (19).[20]

In equilibrium $\frac{\partial w_t}{\partial q_t^s} = 0$, $(s = 1 \sim n)$.

Moreover, when t, p_t and v_t are given constants, our economic rational behavior to maximize our utility u_t under the constraint

$$v_t = \sum_s p_t^s q_t^s$$

is interpreted as if we are maximizing u_t under the constraint

$$(37) \qquad \bar{v}_t = \sum_s \bar{p}_t^s q_t^s$$

for given constants t, $\bar{p}_t^s$, $\bar{v}_t$, where $\bar{p}_t^s = \frac{p_t^s}{\pi_t}$, $\bar{v}_t = \frac{v_t}{\pi_t}$. 21

Therefore, the marginal utility of good s in year t is given by

$$(38) \qquad u_t^s = \frac{\partial u_t}{\partial q_t^s} = \lambda_t \bar{p}_t^s \qquad \text{(in equilibrium)}$$

where

$$\lambda_t = \frac{b_t c_t}{2\gamma_t}$$

$$\gamma_t = 1 + \cosh(h_t) = 1 + \frac{e^{h_t} + e^{-h_t}}{2}$$

Hence, we have:

$$(39) \qquad u_t^{rs} = \frac{\partial^2 u_t}{\partial q_t^r \, \partial q_t^s} = 2\, b_{rs} + \mu_t \bar{p}_t^r \bar{p}_t^s$$

(in general)

where

$$\mu_t = -\frac{(b_t)^2 c_t \sigma_t}{2(\gamma_t)^2} ,$$

20 cf. (19) on page 6.

21 The proof shall be given in appendix.

$$\sigma_t = \sinh(h_t) = \frac{e^{h_t} - e^{-h_t}}{2}$$

$$2\, b_{rs} = \frac{\partial^2 w_t}{\partial q_t^r\, \partial q_t^s}$$

With regard to the individual wellbeing function for a family of a society S , we have similarly:

(40) $$u_t^i = w_t^i + E^i(h_t^i) \qquad \text{(for the i-th family)}$$

where

(41) $$E^i(h_t^i) = \frac{c_t^i}{1 + e^{-h_t^i}}$$

$$h_t^i = b_t^i\, (\bar{v}_t^i - a_t^i)$$

$$\bar{v}_t^i = \frac{v_t^i}{\pi_t^i}$$

a_t^i , b_t^i , $c_t^i > 0$ are Gestalt constants and w_t^i is a negative definite quadratic form in $(\tilde{q}_t^{is} - q_t^{is})$'s .

Here, we have likewise:

(42) $$u_t^{is} = \frac{\partial\, u_t^i}{\partial q_t^{is}} = \lambda_t^i\, \bar{p}_t^s \quad \text{(in equilibrium)}$$

$$\lambda_t^i = \frac{b_t^i\, c_t^i}{2\, \gamma_t^i}, \quad \gamma_t^i - 1 = \cosh(h_t{}^i) = \frac{e^{h_t^i} + e^{-h_t^i}}{2}$$

and also,

(43) $$\frac{\partial^2 u_t^i}{\partial q_t^{ir}\, \partial q_t^{is}} = 2\, b_{rs}^i + u_t^i\, \bar{p}_t^r\, \bar{p}_t^s \qquad \text{(in general)}$$

$$u_t^i = \frac{(b_t^i)^2\, c_t^i\, \sigma_t^i}{2(\gamma_t^i)^2},$$

$$\sigma_t^i = \sin h(h_t^i) = \frac{e^{h_t^i} - e^{-h_t^i}}{2}, \; 2\, b_{rs}^i = \frac{\partial^2 w_t^i}{\partial q_t^{ir} \partial q_t^{is}}.$$

It is not difficult to construct welfare function for the society S as a whole from all individual wellbeing functions for the families which constitute the society S , making use of the concept of the representative family as stated above.[22]

It is also clear from (38), (39), (42) and (43) that these functions (36) and (40) are more realistic as compared with those given by (19) and (28), which were derived from natural entropy.[23]

I would like to extend my deep gratitude toward Professor Oskar Morgenstern for his kind assistance in preparing this paper.

I would also like to express my sincere thanks to Professor W. Feller, Professor S. Kakutani, Professor H. S. Houthakker and Professor H. Gamo for their valuable comments.

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[22] Cf. (31) on page 9.

[23] Compare (38) with (14), and also compare (42) with (29).

APPENDIX

Under the constraint:

$$(1) \qquad \sum_s p_t^s \, q_t^s = v_t$$

with given market prices $\{ p_t^s \}$, $(s = 1 \sim n)$, and an income v_t for fixed t, we seek to maximize

$$(2) \qquad u_t = u_t \, (q_t^1 \cdot q_t^2 \, , \, \ldots , \, q_t^n \,)$$

In the first place, we form a Lagrangian function

$$(3) \qquad F_t = u_t + \lambda_t \, (\bar{v}_t - \Sigma \, \bar{p}_t^s \, q_t^s \,)$$

$$\bar{v}_t = \frac{v_t}{\pi_t} \, , \qquad \bar{p}_t^s = \frac{p_t^s}{\pi_t}$$

We have, as a part of necessary conditions for the utility u_t to be a maximum,

$$(4) \qquad \frac{\partial F_t}{\partial q_t^s} = \frac{\partial u_t}{\partial q_t^s} + \lambda_t \, (v_t - \sum_s p_t^s \, q_t^s \,) \, . \, [\, \frac{\partial}{\partial q_t^s} \, (\, \frac{1}{\pi_t}) \,] - \frac{\lambda_t}{\pi_t} \, p_t^s = 0$$

However, since by (1)

$$v_t - \sum_s p_t^s \, q_t^s = 0$$

we get from (4)

$$(5) \qquad \frac{\partial u_t}{\partial q_t^s} = \lambda_t \, \frac{p_t^s}{\pi_t} = \lambda_t \, \bar{p}_t^s \, . \qquad \text{Q.E.D.}$$