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UNDER RISK

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The abundance of recent literature on capital budgeting and the variety of places in which this literature has appeared attest to the prominent position research on this subject is being given. Economists and businessmen alike are making heavy "investments" in the study of how businesses should optimally invest. A number of authors have brought tools of mathematical programming to bear on this capital-budgeting problem.¹ But for the most part, these writers as well as authors in the non-programming literature have confined their discussions to a world of certainty. The articles by Cord, Lintner, Näslund, and Weingartner, a chapter in Näslund's thesis, and the Stanford Technical Report by Hillier are notable programming-literature deviations from the well-tread path.

The research discussed in this paper and the work of which it forms a part constitute another exception to the general stream of capital-budgeting literature. The larger work is concerned first with decision-making under risk when the decisions and the risks extend over more than one period of time. Then capital-budgeting under risk is studied as a particular example of such decision-making. This paper limits its concern to a discussion of a particular type of capital-budgeting problem when risk exists.

The environment is intentionally defined as one of risk as opposed to one of uncertainty. The decisions to be analyzed are assumed to be made with knowledge of the probability distributions - be they subjective or objective - of outcomes for the non-deterministic variables involved. On the other hand, an uncertain

¹[3], [5], [6], [9], [11], [13], [15], [16], [18], [20], [21].

environment would be one in which the decision-maker does not possess such probability distributions.

Decisions made by enterprises with regard to capital investment are made in an environment of risk. The decision-makers have an idea of the probabilities with which different values are realized for net returns or net outlays on projects they consider. Neither the precision of these estimates nor their subjective or objective nature is now the issue. The important fact is that actual capital-budgeting decisions are made in a non-deterministic environment. If theory is to come closer to analyzing these decisions and to helping the individuals making them, the theory must cope with the presence of risk.

The capital-budgeting problem, as considered here, is the problem of allocating fixed budget dollars in several time periods among competing investment projects. The discussion is confined to the most stringent (I hesitate to call it the simplest) capital-budgeting situation: pure capital rationing. The firm neither borrows nor lends.² It has for its own use and only its own use fixed amounts of money in each period. Assuming there are T periods for which the firm is making plans, denote the budget ceiling in the t^{th} period by C_t . The firm's problem is then to choose from among the available investment opportunities, extending over its planning horizon, the optimal subset of projects to be undertaken given that the sequence of budget dollars is C_1, C_2, \dots, C_T .

As for the potential investment projects, suppose they are n in number. Each proposal can be represented by a vector, similar to the description of any activity in an activity-analysis problem. The first T elements in the $2T$ -element

²The larger work of which this paper is a part considers the case of capital markets where the firm can borrow and lend.

vector describing the i^{th} project are the period-by-period net returns of cash from that project. The last T elements in the vector are the period-by-period inputs of cash required by it.³

The net cash inflow resulting from project i in period t will be denoted a_{ti} while the cash outlay on project i in period t will be written as c_{ti} . In the present model we assume that gross and net returns from projects are stochastic but that outlays on projects are known with certainty. This asymmetrical treatment of returns and outlays can be justified if a period-analytic view of the firm is adopted.⁴ The firm has the funds for a given period, C_t , at the start of that period. It uses this money for cash outlays during the period, the c_{ti} 's. But only at the end of the period does it receive the gross returns from projects, $c_{ti} + a_{ti}$, and these are stochastic. Hence while the budget constraint quantities are deterministic, the net returns are stochastic. It is also posited that these net returns are not used for future cash outlays within the planning horizon.

For example, the net returns are immediately distributed to the owners of the firm.

³The fact that a project begun in one of the T periods of the firm's planning horizon will yield returns in post-horizon periods creates something of a problem. Some model-builders in capital-budgeting seem to side step the issue, tacitly assuming the firm's planning horizon is long enough to encompass all returns from all projects started during it. (See [3].) Others discount post-horizon flows at some discount rate, usually the company's "cost of capital". (See [21].) Both approaches seem less than acceptable: the latter due to difficulties involved in the concept of the "cost of capital" under capital rationing; the former because it seems hard to envision a firm with a planning horizon long enough so that all post-horizon flows are negligible. I shall attempt to deal with the problem by including in the T^{th} period net-return figure for project i not only the net cash inflow from project i in period T but also the market value of the project in period T .

⁴See [3, p. 321] for an alternative position, namely, that returns and outlays must be treated symmetrically.

The investment proposals considered in the model are discrete indivisible alternatives. A given project is either accepted in full or it is completely rejected. Thus if y_i denotes the i^{th} potential project, y_i is an integer-valued variable. Moreover, if two identical projects are proposed simultaneously, they will be considered to be distinct projects. This means y_i is not only restricted to be integer-valued, it is further required to be a zero-one variable. It has value zero if project i is rejected and it has value one if project i is accepted.⁵

In addition to the zero-one restriction on each project variable, the constraint set of the capital-budgeting problem considered here contains the T period-by-period budget constraints. If physical interdependences - mutual exclusion or contingency relationships - exist, they too find expression in the constraint set of the problem.⁶ In what follows we shall, for simplicity of exposition, assume that such physical interdependences among projects do not exist. That is, while we shall discuss the presence of stochastic interrelationships between projects, we will assume the projects are physically independent of one another.

Describing the objective function for the capital-budgeting problem is more difficult. The objective function clearly depends on the net cash returns from

⁵ Alternatively, all y_i variables could simply have been required to be integral and a further restriction imposed on the subset of y_i variables representing all-or-nothing decisions to force them to be zero or one. The approach taken here is preferable for my purpose because the capital-investment proposals with which I am concerned are basically large ventures. Hence, in most cases the projects will be of the yes-no type. An ability or desire to accept a proposal more than once will be the exception rather than the rule.

⁶ For a further discussion of these interrelationships and their constraint representations, see [2, p. 11 and pp. 32-33].

the projects, the a_{ti} 's. The question is "How?" Some insights into answering the question for the risk model can be gained by considering the objective function appropriate for a world of certainty. Since the primary purpose of this paper is to discuss the problem of capital budgeting under risk, I shall restrict myself to a summary of my work on the certainty model.

The overwhelming majority of capital-budgeting literature argues that the goal of a firm's capital program is to maximize the value of the owner's equity or the value of the owner's income resulting from the business. This goal is then translated by capital-budgeting theorists into maximization of the present value of future net cash flows deriving from the projects. This is the objective function one finds in most of the programming work on capital budgeting. Maximization of discounted present value of the investment program means the goal should be

$$\text{Maximize } \sum_{t=1}^T D_t \left(\sum_{i=1}^n a_{ti} y_i \right) \quad (1)$$

where D_t is the discount factor to be applied to cash flows in the t^{th} period.

Typically, the literature tells us to discount cash flows at the "company's cost of capital". In the case of pure capital rationing, this "cost of capital" seems to have become a "weasel word," with nearly everyone using it but no one really able to define it precisely. The problem is that for a firm having no recourse to a capital market - or, more generally, having no recourse to a perfect capital market - the relevant discount factors, the D_t 's, are not independent entities. Rather they are themselves obtained as a product of the analysis that determines the firm's optimal investment position.⁷ Moreover, as Baumol and Quandt have

⁷ This was most clearly shown by J. Hirshleifer in [10].

shown, the relevant discount rates cannot be determined solely on the basis of the production-possibility information contained in the a_{ti} 's, the c_{ti} 's and the C_t 's.⁸

I conclude that the objective function to be optimized should be a utility function, thus treating the capital-budgeting problem as part of the general theory of choice where utility is to be maximized subject to opportunities and constraints. This is also the path followed by Hirshleifer and Baumol and Quandt. The utility function in question, I then argue, is management's perception of the utility to the owners of consumption alternatives available in different periods. Biases or imperfections in management's vision are not excluded as a possibility. In attempting to ascertain the owner's time preferences, management may err, consciously or unconsciously. Denoting by W_t (for $t = 1, 2, \dots, T$) the sum available for withdrawal from the firm in period t , the utility function to be maximized is

$$U = U(W_1, W_2, \dots, W_T). \quad (2)$$

It is also shown that this view of the capital program's goal can be reconciled with the behavioral theory of the firm.⁹

The utility function in (2) can be reduced to more basic elements of the investment decision, namely, the individual projects. Unfortunately, the literature has for the most part - the Baumol and Quandt article and Manne's paper being exceptions - not drawn together the consumption-alternative approach of

⁸[3].

⁹See [7], [22] for discussions of this theory of the firm.

neoclassical capital theory and the project approach of more recent capital-budgeting theory. Hirshleifer, on the one hand never leaves his world of indifference curves and production-possibility loci between consumption alternatives. The model-builders in capital budgeting, for their part, never forsake their individual-project formulations. Since we have assumed that net returns generated by projects in any period are available for distribution to the firm's owners, the consumption alternative available in period t is the sum of the projects' net returns in that period. Hence, we have

$$W_t = \sum_{i=1}^n a_{ti} y_i. \quad (3)$$

The utility function to be maximized can consequently be written as

$$U = U\left(\sum_i a_{1i} y_i, \sum_i a_{2i} y_i, \dots, \sum_i a_{Ti} y_i\right). \quad (4)$$

The consideration of the objective function under certainty concludes with an argument that the utility function in (2) is nonlinear in the W_t variables. Such nonlinearity is necessary: (1) to provide for project interactions in the utility function, (2) to avoid the untenable implication that the marginal rate of substitution between dollars in any two periods is a constant, and (3) to avoid the incorrect stance that discount rates can be determined ex ante in the presence of imperfect capital markets.

Returning to decision-making in the risk environment described earlier, the expected-utility maxim is taken as the fundamental basis for behavior. The decision-maker pursues that policy which maximizes the expected value of his

numerical-valued utility function.¹⁰ Using this rule, the objective function in the capital-budgeting-under-risk problem is

$$\text{Maximize } E(U) = E \left(U \left(\sum_i a_{1i} y_i, \sum_i a_{2i} y_i, \dots, \sum_i a_{Ti} y_i \right) \right), \quad (5)$$

where E is the expectations operator.

The presence of risk imposes further and more specific restrictions, besides nonlinearity in the W_t 's, on the utility function to be used in the objective function, (5). Proceeding in a quasi-axiomatic fashion, a set of properties is specified that it is thought such a utility function should possess. The implications of these properties for the capital-budgeting-under-risk objective function are then presented. After that a solution procedure for the problem of capital budgeting under risk can be examined.

First, the marginal utility of each period's consumption income is assumed to be positive. All first partial derivatives of the utility function must, consequently, be positive.

Second, the utility function must be bounded. This condition, which is often overlooked, is required if the expected-utility maxim is to be used. Otherwise one falls into the trap of a version of the St. Petersburg paradox.¹¹

Our third axiom states that the firm's decision-making unit is a risk-averter. That is, suppose the firm expects (with certainty) the stream of future

¹⁰ See [19, Chapter 1, Section 3, especially pp. 26-29] for the derivation of this rule from the Von Neumann-Morgenstern axioms.

¹¹ For a description of the St. Petersburg paradox see [12, pp. 19-20]. Arrow [1, p. 26] and Markowitz [14, p. 154] emphasize the need for the boundedness requirement, first pointed out by Karl Menger.

internally generated cash throw-offs W_1, W_2, \dots, W_T . If management is now confronted with a vector \tilde{Z} of T risks - one in each period - each with expected value zero, it will prefer its certain status quo position (W_1, W_2, \dots, W_T) to the fair¹² risky result $(W_1, W_2, \dots, W_T) + (\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_T)$. The firm is a risk-averter since it will prefer its certain status quo to an "actuarially neutral" gamble. This risk-aversion axiom is equivalent to requiring that the firm's utility function in (4) be concave in the sums $\sum_i a_{ti} y_i, t = 1, 2, \dots, T$.¹³

The fourth condition we set for a utility function for capital budgeting under risk is that of decreasing risk aversion. K. J. Arrow and J. W. Pratt, working separately, have both discussed the question of decreasing risk aversion with regard to the decision-maker's single-period utility function.¹⁴ A decision-maker is decreasingly risk-averse if and only if the risk premium and the amount of money he would be willing to pay for insurance against a given absolute dollar risk decreases as his consumption income increases. The risk premium referred to is an amount of money such that the decision-maker is indifferent between subjecting himself to the risk and receiving the expected value of the stochastic return minus his risk premium. Both Arrow and Pratt have derived neat conditions under which a single-period utility function shows decreasing risk aversion. Extending the concept to the multi-period case - as to the utility

¹² A risk is called "fair" or "actuarially neutral" if the expected value of the change it causes in the decision-maker's position is zero; here, $E(\tilde{Z}) = 0$.

¹³ This can be shown by extending Jensen's Inequality to functions of vectors of variables. See [8, pp. 151-152] for a discussion of Jensen's Inequality in the case of functions of a single variable.

¹⁴ [1], [17].

function in (2) - engenders a number of conceptual and technical difficulties. Nevertheless, some progress has been made with the problem and the implications of some of my results on risk aversion over time will be useful shortly.

Perhaps a brief justification of this less familiar fourth axiom is in order. Why be concerned with decreasing risk aversion? Because "it seems likely that many decision-makers would feel they ought to pay less for insurance against a given risk the greater their assets"¹⁵ and this is possible only if their utility functions show decreasing risk aversion. Moreover, as Arrow notes, if decision-makers' utility functions show increasing risk aversion, risky investment becomes an inferior good. "This result is empirically implausible" and "we must reject the hypothesis of increasing absolute risk aversion. If, on the other hand, we assume decreasing absolute risk aversion, then risky investment becomes a normal good."¹⁶

The final requirement we will set upon the function $U(W_1, W_2, \dots, W_T)$ is of a somewhat different nature from the four preceding ones. This last property borders on the realm of analytical convenience as opposed to behavior description or rationality for its justification. Nevertheless, insofar as the problem would be extremely difficult, if not analytically impossible, to solve without this restriction, it seems reasonable to believe that actual decision-makers could not make their decisions within a framework where this property does not obtain. Perhaps this will be clearer after the restriction has been

¹⁵[17, p. 123].

¹⁶[1, p. 26].

stated. It is that the utility function not impose upon us the task of having to specify fully the frequency function $f(W_1, W_2, \dots, W_T)$ of the random variables W_t .

The problem is that specifying this joint density function involves specifying the joint frequency function of the T sums $\sum_{i=1}^n a_{ti} y_i$, since $W_t = \sum_{i=1}^n a_{ti} y_i$ by (3). But $\sum_{i=1}^n a_{ti} y_i$ is an unknown random variable. It is a sum of a subset of random variables, the a_{ti} 's. But the random variables to be included in the sum are only determined when the unknown y_i values are found. The frequency function required can only be determined simultaneously with the optimal solution. The optimization process would be, to say the least, extremely difficult if the full frequency function $f(W_1, W_2, \dots, W_T)$ had to be included in the objective function.

The fifth property stipulated, that of avoiding full specification of $f(W_1, W_2, \dots, W_T)$, is most easily met by restricting consideration to polynomial utility functions. Such functions have an additional desirable property. They permit formulation of an objective function reducible to individual projects and moments of the probability distributions of their net returns. This is advantageous in terms of our goal of bringing together the approaches of neoclassical capital theory and capital budgeting. Applying the rule of Occam's Razor, it is desirable to meet the requirements set on the multi-period utility function with the polynomial function of lowest possible degree.

The lowest-order polynomial function which satisfies the conditions set out above is a cubic defined over a restricted range of consumption incomes. Limiting the values of consumption income considered bounds the function just as, for

example, one would have to arbitrarily bound a logarithmic utility function from above to meet our second axiom. The range of consumption-income (or net-returns) values to which attention is restricted is dictated by the decision-maker's views as to the range he believes relevant and over which his attitudes are described by our axioms. While the first requirement - that of positive marginal utility - can be met by a linear function, the risk-aversion (concavity) axiom requires at least a second-degree function. A quadratic utility function is not appropriate, though, since it fails to satisfy the decreasing risk-aversion requirement. A cubic with suitably restricted parameter signs and values can, however, meet all the requirements.

The objective function for the capital-budgeting-problem-under-risk thus takes the following form:

$$\begin{aligned} \text{Maximize } E(U) = & \sum_{i=1}^n (p_i y_i + p_{ii} y_i^2 + p_{iii} y_i^3) \\ & + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (q_{ij} y_i y_j + q_{iij} y_i^2 y_j) + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \sum_{\substack{k=1 \\ i \neq j \neq k \neq i}}^n R_{ijk} y_i y_j y_k. \end{aligned} \quad (6)$$

The p, q, and R coefficients are products of the parameters of the utility function and the moments (about the origin) of the joint distribution of the a_{ti} 's. For example,

$$p_i = \sum_{t=1}^T b_t \mu_{ti} \text{ where } b_t \text{ is the coefficient of } W_t \text{ in the expanded form of}$$

the utility function in (2) and $\mu_{ti} = E(a_{ti})$. As another example,

$$q_{iij} = \sum_{t=1}^T b_{ttt} \mu_{ti,ti,tj} + \sum_{\substack{s=1 \\ s \neq t}}^T \sum_{t=1}^T b_{sst} \mu_{si,si,tj} + \sum_{\substack{v=1 \\ v \neq s \neq t \neq v}}^T \sum_{s=1}^T \sum_{t=1}^T b_{hst} \mu_{vi,si,tj} \text{ where}$$

b_{ttt} is the coefficient of the W_t^3 term, b_{sst} is the coefficient of the $W_s^2 W_t$ term, and b_{vst} is the coefficient of the $W_v W_s W_t$ term while $\mu_{ti,ti,tj} = E(a_{ti}^2 a_{tj})$, $\mu_{si,si,tj} = E(a_{si}^2 a_{tj})$, and $\mu_{vi,si,tj} = E(a_{vi} a_{si} a_{tj})$.

The expression in (6) may be simplified by combining terms since with each y_i variable required to be either zero or one we have $y_i = y_i^2 = y_i^3$. Using this fact, (6) may be written as:

$$\text{Maximize } E(U) = \sum_{i=1}^n P_i y_i + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n Q_{ij} y_i y_j + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \sum_{\substack{k=1 \\ k \neq i, j}}^n R_{ijk} y_i y_j y_k \quad (7)$$

where $P_i = p_i + p_{ii} + p_{iii}$ and $Q_{ij} = q_{ij} + q_{iij}$. The programming model of capital budgeting under risk framed here thus consists of maximizing the objective function in (7) subject to the T budget constraints and the zero-one restrictions on the n project variables. That is, it is desired to maximize (7) subject to

$$\sum_{i=1}^n c_{ti} y_i \leq C_t \quad t = 1, \dots, T$$

$$y_i = 0 \quad \text{or} \quad 1 \quad i = 1, \dots, n. \quad (8)$$

This is a cubic 0-1 programming problem, with a cubic objective function and linear constraints. The method proposed for solving this problem is based upon the partitioning procedure for mixed-variables problems developed by J. F. Benders.¹⁷ In order to simplify the exposition we shall assume that the third-order interactions incorporated in the $\sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \sum_{\substack{k=1 \\ k \neq i, j}}^n R_{ijk} y_i y_j y_k$ can be

¹⁷[4]; [2] also contains an excellent exposition of Bender's method.

neglected. In effect, we posit that the projects are such that $R_{ijk} = 0$ for all $i \neq j \neq k \neq i$, and the problem becomes a 0-1 quadratic programming problem. Such interactions can be treated in a manner exactly analogous to our treatment of pairwise interactions in what follows.

The programming problem as given in (7) and (8) must be modified in order to make it amenable to the use of Benders. In making the appropriate changes, we not only transform the problem into a completely linear 0-1 programming problem but we also (and this is where Benders' method comes into the picture) end with two distinct sets of variables. The members of one set are restricted to be either 0 or 1 but the elements of the second set can be treated as the ordinary continuous variables one would find in a linear programming problem.

The change in the programming formulation centers on the pairwise interaction terms of the form $Q_{ij} y_i y_j$.¹⁸ The interaction term Q_{ij} affects the value of the objective function in (7) if and only if both y_i and y_j are included in the investment program considered. Define a new variable x_{ij} for each ordered i, j pair, $i \neq j$, as follows. The new variable has a value of 1 if and only if both y_i and y_j are each 1. As a result, the pairwise interaction portion of the objective function may be rewritten as $\sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n Q_{ij} x_{ij}$. The value taken on by x_{ij} will equal that of x_{ji} . Both are zero unless both proposal i and proposal j are accepted in which case $x_{ij} = x_{ji} = 1$. But, x_{ij} and x_{ji} do not necessarily have the same objective function coefficient. While $\mu_{si,tj} = \mu_{tj,si}$ for all s, t , it is not necessarily

¹⁸ Recall that the third-order interactions $R_{ijk} y_i y_j y_k$ are being ignored.

true that $\mu_{v_i, s_i, t_j} = \mu_{v_i, s_j, t_j}$ for any v, s, t . Hence x_{ij} 's objective function coefficient, Q_{ij} , need not equal x_{ji} 's, Q_{ji} .

Three structural constraints and one non-negativity constraint are needed to define each x_{ij} variable. For each ordered i, j pair, $i \neq j$, we must have

$$x_{ij} \leq y_i, \quad x_{ij} \leq y_j, \quad x_{ij} \geq y_i + y_j - 1, \quad x_{ij} \geq 0. \quad (9)$$

No integer restriction need be placed on the x_{ij} variables since the 0-1 requirement for each y_i necessitates that each x_{ij} will be either 0 or 1.¹⁹ For example,

if project i is undertaken but j is not, the series of constraints in (9) becomes

$x_{ij} \leq 1, x_{ij} \leq 0, x_{ij} \geq 0, x_{ij} \geq 0$ and hence $x_{ij} = 0$. If, on the other hand, both are accepted, we have $x_{ij} \leq 1, x_{ij} \leq 1, x_{ij} \geq 1, x_{ij} \geq 0$ which yield $x_{ij} = 1$.

We have, in effect, created $n(n - 1)$ new projects, x_{ij} , all i, j for $i \neq j$.

Project ij has an expected-utility return of Q_{ij} and involves no cash outlays in any period. Each such project is analogous to a contingent project, that is, one whose acceptance depends on the acceptance of one or more other projects. But the artificial projects represented by the x_{ij} variables differ from the ordinary contingent projects. While a contingent project may be accepted or rejected if the project(s) upon which it is contingent is (are) accepted, a project represented by an x_{ij} variable must be accepted if the proposals upon which it depends are included in the investment program.

¹⁹ If one is willing to impose the requirement that x_{ij} be an integer (which we do not want to do), the two constraints $x_{ij} \leq y_i$ and $x_{ij} \leq y_j$ can be replaced by the single constraint (their sum) $2x_{ij} \leq y_i + y_j$. For an application of this alternative approach, developed independently by W.I. Zangwill, see [23, p. 33].

The capital-budgeting-under-risk programming model considered here has now been transformed into the following mixed-variables programming problem:

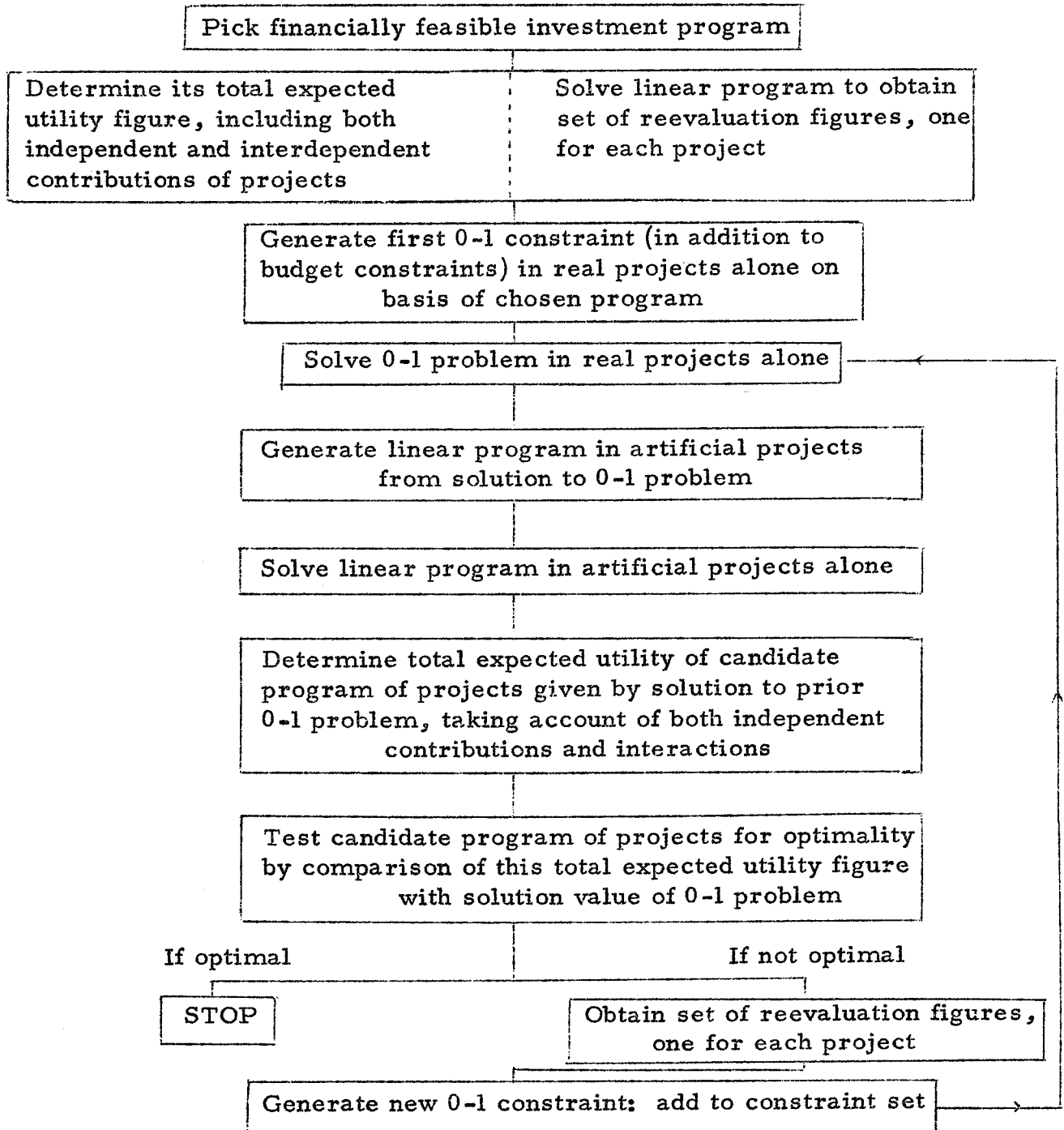
$$\begin{aligned}
 \text{Maximize } E(U) &= \sum_{i=1}^n P_i y_i + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n Q_{ij} x_{ij} \\
 \text{Subject to: } \sum_{i=1}^n c_{ti} y_i &\leq C_t && t = 1, \dots, T \\
 -y_i + x_{ij} &\leq 0 && \text{all } i \neq j \\
 -y_j + x_{ij} &\leq 0 && \text{all } i \neq j \\
 y_i + y_j - x_{ij} &\leq 1 && \text{all } i \neq j \\
 y_i &= 0 \text{ or } 1 && \text{all } i \\
 x_{ij} &\geq 0 && \text{all } i \neq j
 \end{aligned} \tag{10}$$

The y_i variables are required to be 0 or 1 while the x_{ij} variables are continuous.

It is to the capital-budgeting problem in this form that I have applied Benders' decomposition. Having indicated which set of variables is restricted to a subset of the reals (the y_i variables) and which set is unrestricted aside from the linear constraints in (10) (the x_{ij} variables), the application of Benders' method is almost completely straightforward. Hence, rather than present a more mathematical discussion of how the procedure is carried out, I shall discuss verbally the way the algorithm proceeds. The chart that follows should aid in understanding the decision mechanism.

We begin by selecting any financially feasible investment program, that is, we start with any list of proposals meeting all the budget constraints. In this initial step, the expected utility deriving from both the individual project's contributions (the P_i 's) as well as from their interactions (the Q_{ij} 's) is considered. At the same time, set $y_i = 1$ for all projects in this initial investment program

THE ITERATIVE DECISION PROCEDURE



and $y_i = 0$ for all rejected proposals. The program in (10) then becomes a linear program in the x_{ij} variables. By solving the dual of this linear program, a set of re-evaluation figures (one for each real project) is derived. These re-evaluation values are used to modify the independent contribution each project makes to expected utility, P_i . They modify the P_i 's by the change in total expected utility that would take place if the project's status were changed from what it is in the candidate program.

Using these modified valuations, a new constraint involving only real-project variables (in addition to the budget constraints) is derived. It states that the optimal value of total expected utility can be no greater than the expected utility of the candidate solution plus an upper bound on the possible change in expected utility. This possible change refers to the effect on expected utility if projects are added to or removed from the candidate budget.

At each step after this initial one, a 0-1 problem involving only the real projects is first solved. It locates an optimal investment program taking into account the budget constraints and some information about the effects of project interactions. This information about inter-relationships is based on the solutions to the duals of previously generated linear programs in the artificial projects. Each of these linear programs has been generated in the same way as the initial one described above. Specifically, the y_i variables corresponding to projects included in the most recently found candidate capital budget are set equal to 1. All other y_i variables are set equal to zero. Then (10) becomes a linear program in the x_{ij} variables and it is the dual to each such linear program that increases our knowledge of the effects of inter-relationships.

This information about the effects of project interactions is embodied in constraints of the type discussed above, that is, constraints providing upper bounds on the maximum attainable value of expected utility. When the optimal investment program, taking into account the budget constraints and these constraints concerning interactions, is found in the 0-1 problem, a new candidate capital budget is at hand. Solving the dual to the artificial-projects' linear program derived from this candidate generates the relevant interdependence information about this allocation and a new set of re-evaluation figure. This information enables us to decide whether or not this most recently proposed set of projects is optimal for the entire capital-budgeting problem. If it is, we have solved the problem. If it is not, a new set of re-evaluation figures is at hand from the linear program. Using these new figures, a new constraint in real projects is obtained. It is added to the constraint set of the 0-1 problem and the whole procedure is repeated again.

A word about the re-evaluation figures is in order. They constitute a peculiar mixture of optimism and pessimism with regard to the merit of individual projects. One example of optimism is that except in the case where both projects are in the candidate capital budget, no project's return is ever debited for a negative interaction. No blame is ever assessed for harmful joint acceptance, $Q_{ij} < 0$ and/or $Q_{ji} < 0$, unless both projects involved are presently accepted ones. As an example of the pessimistic aspect of the re-evaluation process, suppose both project i and j are accepted in the candidate capital budget. Assume, moreover, that their interaction is such as to diminish expected utility, for example, $Q_{ij} < 0$ and $Q_{ji} < 0$. Then each one's return would be debited by the full

amount $Q_{ij} + Q_{ji}$ of the decrement their interaction causes in expected utility. This is in accord with marginal analysis for removing one of them would increase expected utility by $-(Q_{ij} + Q_{ji})$. But each of the two projects is undervalued in that each is debited for the full decrement they jointly cause.

The above are just two examples of the mixture of pessimism and optimism in individual project re-evaluation. This mixture as it appears in the 0-1 constraints generated at each step, leads to an overoptimistic view of the overall advantage to be gained by altering the candidate budget allocation upon which the re-evaluation is based.

Nevertheless, the iterative procedure locates the globally optimal budget allocation. The best capital investment program is found for the firm operating in the risk environment we have described.

REFERENCES

1. Arrow, K. , "Comments on Duesenberry's 'The Portfolio Approach to the Demand for Money and Other Assets'," Review of Economics and Statistics XLV No. 1, Part 2, Supplement (February 1963) pp. 24-27.
2. Balinski, M. L. , "Integer Programming: Methods, Uses, Computation", Management Science XII, No. 3 (October 1965), pp. 253-313.
3. Baumol, W. J. and R. E. Quandt, "Investment and Discount Rates Under Capital Rationing - A Programming Approach", Economic Journal LXXV, No. 298 (June 1965), pp. 317-329.
4. Benders, J. F. , "Partitioning Procedures for Solving Mixed-Variables Programming Problems", Numerische Mathematik IV, No. 4, (December 1962), pp. 238-252.
5. Charnes, A. , W. W. Cooper and M. H. Miller, "Application of Linear Programming to Financial Budgeting and the Costing of Funds", Journal of Business XXXII, No. 1 (January 1959), pp. 20-46.
6. Cord, J. , "A Method for Allocating Funds to Investment Projects When Returns Are Subject to Uncertainty", Management Science X, No. 2 (January 1964), pp. 335-341.
7. Cyert, R. M. and J. G. March, A Behavioral Theory of the Firm (Prentice-Hall, Inc. , Englewood Cliffs, New Jersey, 1963).
8. Feller, W. , An Introduction to Probability Theory and Its Applications Volume II (John Wiley and Sons, Inc. , New York, 1966).
9. Hillier, F. S. , "The Evaluation of Risky Interrelated Investments", Technical Report No. 73, Department of Statistics, Stanford University, Stanford, July 24, 1964.
10. Hirshleifer, J. , "On the Theory of Optimal Investment Decisions", Journal of Political Economy LXVI, No. 4 (August 1958), pp. 329-352.
11. Lintner, J. , "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets", Review of Economics and Statistics XLVII, No. 1 (February 1965), pp. 13-37.
12. Luce, R. D. and H. Raiffa, Games and Decisions: Introduction and Critical Survey (John Wiley and Sons, Inc. , New York, 1957).

13. Manne, A. S. , "Optimal Dividend and Investment Policies for a Self-Financing Business Enterprise", Graduate School of Business Working Paper No. 93, Stanford Graduate School of Business, February 4, 1966.
14. Markowitz, H. M. , "The Utility of Wealth", Journal of Political Economy LX, No. 2 (April 1952), pp. 151-158.
15. Näslund, B. , Decisions Under Risk: Economic Applications of Chance-Constrained Programming, Carnegie Institute of Technology Graduate School of Industrial Administration (Ph. D. Thesis), November 1964.
16. Näslund, B. , "A Model of Capital Budgeting Under Risk", Journal of Business XXXIX, No. 2 (April 1966), pp. 257-271.
17. Pratt, J. W. , "Risk Aversion in the Small and in the Large", Econometrica XXXII, Nos. 1-2 (January-April 1964), pp. 122-136.
18. Reiter, S. , "Choosing An Investment Program Among Interdependent Projects", Review of Economic Studies XXX, No. 1 (January 1963), pp. 32-36.
19. Von Neumann, J. and O. Morgenstern, Theory of Games and Economic Behavior, (Princeton University Press, Princeton, 1944).
20. Weingartner, H. M. , "Capital Budgeting of Interrelated Projects: Survey and Synthesis", Management Science XII, No. 7 (March 1966), pp. 485-516.
21. Weingartner, H. M. , Mathematical Programming and the Analysis of Capital Budgeting Problems (Prentice-Hall, Inc. , Englewood Cliffs, New Jersey, 1963).
22. Williamson, O.E., The Economics of Discretionary Behavior: Managerial Objectives in a Theory of the Firm (Prentice-Hall. , Inc. , Englewood Cliffs, New Jersey, 1964).
23. Zangwill, W. I. , "Media Selection by Decision Programming", Journal of Advertising Research 5, No. 3 (September 1965), pp. 30-36.

