THE DYNAMIC STABILITY OF A CONTINUOUSLY ADJUSTING

COURNOT OLIGOPOLY WITH A PRODUCTION LAG

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This note is intended to complement some discussions which have appeared in the literature [1,2,3] concerning the stability of the Cournot Oligopoly Model. Specifically it deals with the influence of a finite time lag occurring between continuously adjusting decisions to alter output and the actual changes in output that ultimately ensue.

The model considered here is rather similar to the differential equation model discussed by McManus and Quandt [3] and defines the prices, total costs and profits of the ith firm as:

$$p(i,t) = a - b [x(i,t) + \sum_{j \neq i} x(j,t)]$$

$$C(i,t) = f(i) + c(i)x(i,t)$$

$$P(i,t) = -f(i) + [a - c(i)] x(i,t) - bx(i,t)^2 - bx(i,t \sum_{j \neq i} x(j,t) .$$

Hence the profit-maximizing output $x^*(i,t)$ estimated on the Cournot assumption is given by:

$$\frac{\partial P^*(i,t)}{\partial x^*(i,t)} = a - c(i) - 2bx^*(i,t) - b \sum_{j \neq i} x(j,t) = 0$$

or

$$x^*(i,t) = \frac{a - c(i)}{2b} - \frac{1}{2} \sum_{j \neq i} x(j,t) .$$

Further, we make the dynamic assumption that the production plans are continuously changed in proportion to any discrepancy between the actual output and the estimated profit-maximizing output. Hence we have:

$$\dot{x}(i,t+L) = s(i) [x^*(i,t) - x(i,t)]$$

$$= s(i) \frac{a - c(i)}{2b} - s(i) [x(i,t) + \frac{1}{2} \sum_{j \neq i} x(j,t)]$$

where L is the time lag between the change in production plans and the actual change in output, and $s(i)$ is a measure of the speed of reaction of the ith firm.

For all firms taken together this yields the system of equations:

$$2I\dot{X}(t+L) = SA - S(I + J)X(t)$$

where I is the identity matrix, J is a matrix all of whose elements are unity, S is a diagonal matrix of the $s(i)$ and A is a diagonal matrix of the $[a - c(i)]/b$.

Putting $\dot{X}(t+L)$ equal to zero yields the unique stationary solution

$$\bar{X} = (I + J)^{-1}A$$

which is the same as the equilibrium solution obtained by McManus and Quandt [3]. But in the present model the transient part of the system is not a set of differential equations, but the set of mixed difference-differential equations:

$$(1) \qquad 2S^{-1}\dot{X}(t+L) + (I + J)X(t) = 0 \qquad .$$

This can be solved by Euler's method, i.e. we substitute $X(t) = C.e^{zt}$, where C is a vector, into equation (1), whence it can be shown (see e.g. Bellman and Cooke [4]) that to each root z of the determinantal equation:

$$(2) \qquad \det|S^{-1}2z.e^{Lz} + I + J| = 0$$

there corresponds a non-zero characteristic vector $C(z)$ such that

$$(3) \qquad X(t) = C(z).e^{zt}$$

satisfies equation (1).

In order to simplify our study of the effect of L and the $s(i)$ on the permissible values of z we write:

(4) $$y(i) = \frac{2R}{s(i)} + 1$$

where R equals any one of the values of $z.e^{Lz}$ which satisfy equation (2). Then we can rewrite equation (2) as:

$$\det|Iy + J| = \prod_{i=1}^{n} y(i) + \sum_{i=1}^{n} \prod_{j\neq i} y(j) = 0 .$$

Hence, if we denote by $R(k)$ the kth distinct value of R in (4) and order the $s(i)$ in ascending magnitude, then by an argument similar to that employed by McManus and Quandt [3] we can establish:

RESULT (A)

$$0 > -\tfrac{1}{2}s(1) > R(1) > -\tfrac{1}{2}s(2) > R(2) > .... > -\tfrac{1}{2}s(k) > R(k) > ..... > -\tfrac{1}{2}s(n) > R(n)$$

We may also note that the symmetry of the determinant in equation (2) ensures that all $R(k)$ are real.

Thus we have shown the effect of the values of the $s(k)$ on the permissible values of R (i.e. $z.e^{Lz}$). But this does not complete the analysis, for we seek the values of z . We note (c.f. Bellman and Cooke [4]) that in order that the solutions of equation (1) given by (3) should be stable it is necessary and sufficient that the roots z of:

(5) $$H(z) = z.e^{Lz} - R(k) = 0$$

should have their real parts negative for all k . We can find the restrictions on $R(k)$ for this to be true using a special case of a very powerful

theorem[2] due to Pontryagin [5]:

> THEOREM: "In order that the zeros of $H(z)$ should be to the left of the imaginary axis, it is necessary and sufficient that the zeros of $G(y)$ are real and simple and that for each of these zeros $G'(y).F(y) > 0$, where $F(y)$ and $G(y)$ are defined by $H(iy) = F(y) + iG(y)$."

In our case we have:

$$F(y) + iG(y) = iy.e^{iLy} - R(k)$$

$$= -R(k) - y.\sin(Ly) + iy.\cos(Ly) .$$

Hence the roots of $G(y) = 0$ are:

$$y = 0, \quad (2n + 1)\pi/2L , \quad n = 0, \pm 1, \pm 2, \quad \ldots\ldots$$

which are clearly all real and simple. Hence our system is stable if and only if for all these roots we have:

$$G'(y).F(y) > 0$$

i.e.

$$-[-yL.\sin(Ly) + \cos(Ly)] [R(k) + y.\sin(Ly)] > 0 .$$

For $y = 0$, this gives $R(k) < 0$,

and for $y = (2n + 1)\pi/2L$, we have $R(k) > - \pi/2L$; i.e. overall we have:

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[2] The properties of equations like (5) were originally discussed in a more elementary fashion by Frisch and Holme [6], however the Pontryagin approach is introduced here because of its great potential for analyzing the stability conditions of more general difference-differential equation models.

RESULT (B)

The solutions of equation (1) given by (3) are stable if and only if:

$$0 > R(k) > - \pi/2L \qquad .$$

Combining Results (A) and (B) we can see that the system is stable if and only if $0 > -s(n)/2 > R(n) > - \pi/2L$.

We see that when L becomes zero, then, as in the differential equation model of McManus and Quandt [3], the system is stable for all finite values of $s(n)$. But for a non-zero lag L the system is stable only if $s(n) L < \pi$, (which is analogous to the behaviour of the difference equation model described by Fisher [2]).

Thus it would seem that apart from setting up yet another Cournot model, this note may be of interest in showing how the difference-differential equation approach can be of service in illustrating the discrepancies between differential and difference equation models of dynamic systems.

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