IDENTIFICATION OF NONLINEAR SYSTEMS: AN ALTERNATIVE TO FISHER

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1. Introduction

In recent years economists have increasingly relied upon nonlinear econometric systems to describe the economy.[1] Despite this increased use, however, theoretical results relating to such systems lag far behind the corresponding results for linear systems. A notable exception to this are the theoretical results given by Fisher (1959, 61, 65, 66) concerning the identification problem in a nonlinear system. Unfortunately, the material given by Fisher lacks an intuitive appeal, and further, the complexities involved in the theoretical framework essentially restrict Fisher's results from many economists. Therefore, the purpose of this paper is to rederive some of Fisher's results from an alternative point of view which, hopefully, is not subject to the same shortcomings.

2. The Model[2]

Consider an M equation econometric system which is linear in the parameters and which relates M (basic) endogenous and Λ (basic) predetermined variables to each other. Let this system be nonlinear in the sense that $M^o > M$ linearly independent functions of the endogenous variables explicitly appear as either regressors or dependent variables in the M equations. These functions, termed endogenous functions,

[1] See, for instance, Duesenberry, et al (1965), Evans and Klein (1967), Holt (1967), and Black and Kelejian (1968).

[2] Fisher's (1966, pp. 131-32) notation is used in order to facilitate comparisons.

may also involve predetermined variables. Finally, assume that $\Lambda^o \geq \Lambda$ linearly independent functions of the predetermined variables appear as regressors in the system - e.g., these are called predetermined functions.

Such a system may be formalized as

$$(1) \qquad A\, q_t = u_t ,$$

where A is an $M \times N^o$ matrix of parameters where $N^o = M^o + \Lambda^o$, and the rank of A is M : $\rho(A) = M$; u_t is an $M \times 1$ vector of disturbance terms at time t ; q_t is an $N^o \times 1$ vector of observations at time t on the N^o linearly independent endogenous and predetermined functions appearing in the system.

We assume that the predetermined variables and the disturbance term have been generated by a process with finite moments and that $E[u_t | x_t] = 0$, where x_t is the $\Lambda \times 1$ vector of observations at time t on the basic predetermined variables. Because nonlinear equations generally admit multiple solutions, we make the assumption of a single generating solution. That is, the generating process, implicit in (1), relating the M endogenous variables to the Λ predetermined variables and the disturbance term is, essentially, single valued.[3] Finally, we assume that the N^o functions of q_t have finite range, are single valued, and are differentiable with respect to each argument.

[3] The alternative to this, is to assume further information which would enable us to identify the generating solution.

3. Identification

The system described in (1) is essentially the theoretical system described by Fisher (1966, p. 131). We now assume that equation (1) can be solved for M endogenous functions in terms of the remaining $M^o - M$ endogenous functions, the Λ^o predetermined functions, and the disturbance vector u_t. Usually, these M endogenous functions will correspond to the variables for which equation (1) was written - i.e., these M variables will correspond to the normalization rules employed in (1). We now rearrange, if necessary, the elements of q_t so that it can be partitioned as $q_t' = (y_t' \; F_t' \; z_t')$, where y_t is the $M \times 1$ vector of endogenous functions at time t for which equation (1) can be solved; F_t' is a $(M^o - M) \times 1$ vector, at time t, containing the remaining endogenous functions; z_t is the $\Lambda^o \times 1$ vector at time t of functions of the elements of x_t alone - the predetermined functions. Using these definitions we write (1) as

$$A_1 y_t + A_2 F_t + A_3 z_t = u_t \;, \tag{2}$$

where A_1, A_2, and A_3 are the corresponding submatrices of A conformably defined. For future reference, we define y_t as the vector of <u>basic endogenous functions</u>, and F_t as the vector of <u>additional endogenous functions</u>.

Let the j^{th} element of F_t be the vector function

$$f_{jt} = f_j\,(y_t, x_t) \;. \tag{3}$$

Then, the assumptions underlying (1) imply that f_{jt} is a random variable with finite mean and variance. Thus, because the conditional expectation of one variable upon a set of others is, in general, a function of the conditioning variables we have

(4) $$E[f_{jt} \mid x_t] = h_j(x_t) = h_{jt}, \qquad j=1,\dots M^o - M,$$

where h_{jt} is a function of the elements of x_t. From (4) we see that

(5) $$f_{jt} = h_{jt} + \Phi_{jt}, \qquad j=1,\dots M^o - M,$$

where Φ_{jt} is a random variable such that $E[\Phi_{jt}|x_t] = 0$. Substituting (5) into (2) we have

(6) $$A_1 y_t + A_2 H_t + A_3 z_t = v_t$$

where H_t is the $(M^o - M) \times 1$ vector whose j^{th} element is h_{jt}, and $v_t = u_t - A_2 \Phi_t$, where Φ_t is the $(M^o - M) \times 1$ vector whose j^{th} element is Φ_{jt}; it is clear that $E[v_t|x_t] = 0$. We also note that the elements of H_t are predetermined functions since they are functions of the elements of x_t alone. Although these functions will generally be unknown, they can be approximated in terms of polynomials estimated via the reduced-form equations in (5).[4]

The model described in (6) is a linear model relating the elements of the basic endogenous vector y_t to those of the predetermined vectors

[4] Because of the lack of invariance of the expectations operator with respect to nonlinear transformations, the functions of H_t will not generally be solutions of equation (1) for the elements of F_t in terms of the elements of x_t when $u_t = 0$ - see Kelejian (1968).

H_t and z_t. We assume, at first, that the $M^o - M + \Lambda^o = N^o - M$ elements of H_t and z_t are linearly independent. A few points are now noted concerning this assumption.

First, the original system (1) or (2) must contain at least one non-constant predetermined variable if the elements of H_t and z_t are to be linearly independent. The reason for this is that if $x_{0t} \equiv 1$ is the only predetermined variable, the conditional expectations in (4) imply that the elements of H_t are simply the unconditional means of the additional endogenous functions f_{jt}, $t=1,\ldots M^o - M$. Therefore, the maximum number of linearly independent predetermined variables would be one.

Another point to note is that the $N^o - M$ elements of H_t and z_t are the only predetermined variables that need be considered in relation to (2). This follows from (6) in that

$$(7) \qquad E[y_t | x_t] = \Pi_1 H_t + \Pi_2 z_t ,$$

where $\Pi_1 = -A_1^{-1} A_2$, and $\Pi_2 = -A_1^{-1} A_3$.

Finally, in comparing (2) with (6) we see that if the elements of H_t and z_t are <u>linearly independent</u>, each additional endogenous function in (2) may be considered, for identification purposes, as just another linearly independent predetermined variable - e.g., (6) is a linear model in the parameter matrices A_i, $i = 1, 2, 3$. Therefore, the conditions necessary for the identification of these parameter matrices are given by the standard results concerning linear systems - see Christ (1966, pp. 314 - 331) and Goldberger (1964, pp. 306-318). As an example, assuming zero restrictions, the order condition for the identification of the first equation is that $\gamma_2 \geq \gamma_1 - 1$ where γ_2 is the number of predetermined variables and additional endogenous functions excluded from the first equation, and γ_1 is the number of basic endogenous functions appearing in that equation.

We now consider the case in which the elements of H_t and z_t are linearly dependent. Clearly the above analysis will not go through since, for example, the reduced-form parameters in (7), Π_1 and Π_2 will not be identified because the corresponding regressor matrix will be singular.

Assume that the elements of H_t and z_t satisfy $j \leq M^o - M$ linear restrictions, namely

$$B\, H_t + C\, z_t = 0 \tag{8}$$

where B and C are constant matrices of orders $j \times (M^o - M)$ and $j \times \Lambda^o$, and the rank $(B) = j$. Substituting the matrix representation of (5) into (8) we see that

$$B\, F_t + C\, z_t = \psi_t \tag{9}$$

where $\psi_t = B\, \Phi_t$, and so $E\, \psi_2 \mid x_t = 0$. That is, the j linear restrictions in (8) imply j additional "structural" equations for the elements of F_t. Furthermore, these equations are linearly independent of the original M structural equations in (2). To see this, note that in (2) rank $(A_1) = M$. Therefore, pre-multiplication of (2) by a $j \times M$ matrix of rank j will result in a structure which contains the elements of y_t; however, these elements do not appear in (9). Fisher (1966, pp. 134-45) has shown that the implied equations of a nonlinear model are the results of nonlinear transformations of the original structural equations.

Consider now the converse of the above. That is, assume the existence of j linearly independent equations in addition to those given by (2), say

$$D_1 y_t + D_2 F_t + D_3 z_t = w_t \,, \tag{10}$$

where D_1, D_2, and D_3 are constant matrices of orders $j \times M$, $j \times M^o - M$, and $j \times \Lambda^o$, and $E\, w_t \,|\, x_t = 0$. Then, solving (2) for y_t and substituting into (10) we have

$$B F_t + C z_t = \psi_t \tag{11}$$

where we have taken $B = (D_2 - D_1 A_1^{-1} A_2)$, $C = (D_3 - D_1 A_1^{-1} A_3)$, and $\psi_t = w_t - D_1 A_1^{-1} u_t$. If we now take conditional expectations in (11) with respect to x_t we have (8). In brief, we have shown that the elements of H_t and z_t are collinear if and only if there exists an implied equation.

We now assume a result given by Fisher (1966, pp. 143-45). In particular, consider the nonstochastic counterpart of (1)

$$A q_t = 0 \,. \tag{12}$$

Let Q_t^1 be the $N^o \times N\text{-}1$ matrix whose i,j element is the partial derivative of the i^{th} element of q_t with respect to the j^{th} basic variable (M endogenous + $(\Lambda - 1)$ predetermined since differentiation with respect to the constant term is impossible). Let $h = (D_1\ D_2\ D_3)$. Then the rows of h form a basis for the row kernel of Q_t^1 if all values of the basic variables satisfying (12) are considered:

$$h\, Q_t^1 = 0 \,. \tag{13}$$

Therefore, the implied equations may be obtained in terms of the solutions of (13).[5]

Thus, we proceed by assuming knowledge of both (9) and (2). First, partition F_t into F_{1t} and F_{2t} where F_{1t} is any vector of j elements of F_t for which equation (9) can be solved, and F_{2t} is the $(M^o - M - j) \times 1$ vector of remaining elements of F_t. Letting $E[F_{2t}|x_t] = H_{2t}$ (see equation 4) we note that the elements of H_{2t} and z_t are linearly independent. We therefore proceed by combining (2) and (9) into one system of $M + j$ equations in $M + j$ basic endogenous functions (elements of y_t and F_{1t}), $M^o - M - j$ additional endogenous functions (elements of F_{2t}), and Λ^o predetermined variables (elements of z_t):

$$(14) \qquad \begin{bmatrix} A_1 & A_{21} \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} y_t \\ F_{1t} \end{bmatrix} + \begin{bmatrix} A_{22} \\ B_2 \end{bmatrix} F_{2t} + \begin{bmatrix} A_3 \\ C \end{bmatrix} z_t = \begin{bmatrix} u_t \\ \psi_t \end{bmatrix}$$

where A_{21} and A_{22} are the corresponding partitions of A_2 in (2), and B_1 and B_2 are the corresponding partitions of B in (9). From the discussion prior to equation (8), it is clear that (14) may be regarded, for identification purposes, as a <u>linear</u> system in $y_t^* = (y_t' \, F_{1t}')$ since F_{2t} may be taken as predetermined. For instance, the order condition under the assumption of zero restrictions, for the first equation of (14) is $\gamma_3 \geq \gamma_4 - 1$ where γ_3 is the number of elements of z_t <u>and</u> F_{2t} <u>excluded</u> from that equation, and γ_4 is the number of elements of y_t^* appearing in the first equation.

[5] A point to note is that, in general, $H \equiv 0$ if the elements of q_t are each functions of only <u>one</u> basic variable - see Fisher (1966, pp. 147-48). In this case there are <u>no</u> implied equations and so the argument prior to equation (8) still holds. Another case in which there will generally be no implied equations is the case in which the original disturbance terms in (1) are assumed to be generated by a <u>particular process</u> - e.g., the assumption of normality. The reason for this is that nonlinear transforms of, say, normal variables are not themselves normally distributed.

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