A NOTE ON USING PROFIT FUNCTIONS TO AGGREGATE PRODUCTION FUNCTIONS

Richard R. Cornwall

Econometric Research Program
Research Paper No. 26
March - 1971
Revised - September 1971

The research described in this paper was supported by NSF Grant GS 2799.

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PRINCETON UNIVERSITY
207 Dickinson Hall
Princeton, New Jersey

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Several recent articles ([4], [7], [8]) have followed Houthakker's procedure [3] for aggregating the production functions of the firms in an industry to get the production function of the industry. By production function of an industry, we mean a function which specifies, for each vector of inputs to the industry, that output which is gotten when these inputs are allocated to the firms in the industry so as to maximize the industry's output. This note points out that this aggregation can be carried out using the duality between profit functions and production functions and illustrates this procedure by applying it to the model developed by Houthakker [3].

Suppose firm a in industry A (A is a set of indices) has a production function $f(x;a)$. for each n-vector x of inputs, $f(x;a)$ is the maximum amount of the output commodity obtainable by firm a. For any n-vector p of input prices, and an output price p_o, the profit for firm a is $\pi(p_o,p,a) = \sup_x\{p_o f(x;a) - p\cdot x\}$.[1] $\pi(\cdot,\cdot;a)$ is the <u>profit function</u> of firm a. It is convenient to also define the <u>normalized profit function</u> of firm a:

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*Some of the ideas in this note came from conversations with Gregory Chow.

[1] $p\cdot x$ means $\sum_{i=1}^{n} p_i x_i$.

$$\pi^N(p;a) = \sup_x\{f(x;a) - p \cdot x\} .$$

Since $\pi(\cdot,\cdot;a)$ is clearly positively homogeneous of degree one,

$$(1) \qquad \pi(p_o,p;a) = p_o\, \pi^N\left(\frac{1}{p_o}\, p;a\right) .$$

From the theory of conjugate functions [6], we know that if $f(\cdot;a)$ is an uppersemicontinuous, concave function on $\mathbb{R}^n$,[2] then $\pi^N(\cdot;a)$ is a lowersemicontinuous convex function on $\mathbb{R}^n$ and $f(x;a)$ can be calculated from the function $\pi^N(\cdot;a)$ by the formula:[3]

$$f(x;a) = \inf_p\{\pi^N(p;a) + p \cdot x\} .$$

Suppose there are no external economies or diseconomies among the firms in industry A. Then it is plausible (and is demonstrated in the Appendix) that the maximum profit $\pi^N(p)$ obtainable by the industry as a whole at input prices p and output price one is the sum of the profits $\pi^N(p;a)$ over a in A. Since it is often analytically convenient to let A be infinite,

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[2] If $f(\cdot;a)$ is initially defined only on a subset Ω of $\mathbb{R}^n$, then it can be extended to all of $\mathbb{R}^n$ by letting $f(x;a) = -\infty$ for x not in Ω.

[3] These remarks can be explained as follows (where to ease notation we omit the index a): $\pi^N(\cdot)$ is the negative of the concave conjugate of $f(\cdot)$: $\pi^N(p) = \sup_x\{f(x) - p\cdot x\} = -\inf_x\{p\cdot x - f(x)\} = -f^*(p)$. But $f(x) = f^{**}(x)$ ([6], page 308) so

$$f(x) = \inf_p\{p \cdot x - f^*(p)\} = \inf_p\{\pi^N(p) + p \cdot x\} = -\sup_p\{(-x)\cdot p - \pi^N(p)\}$$
$$= -(\pi^N)^*(-x) .$$

we represent this sum as an integral:

$$\pi^N(p) = \int_A \pi^N(p;a)\, d\mu(a)$$

where μ is a nonnegative measure.[4] The industry production function f is then calculated

$$f(x) = \inf_p \{\pi^N(p) + p \cdot x\} .$$

This simple idea is illustrated by considering an n-factor version of a model originally considered by Houthakker [3]. We suppose there is a collection A of firms each of which has a Leontief-type technology with fixed input-output coefficients $a_1,\ldots,a_n$ (a_i is the quantity of input i needed to produce one unit of output). There is only one firm with any particular vector $(a_1,\ldots,a_n)$ of input-output coefficients, so we use the vectors $a = (a_1,\ldots,a_n)$ to label the firms. (Thus $A = \mathbb{R}^n_+$) Each firm a also has a capacity contraint limiting firm a to a maximum output of

$$\varphi(a) = \kappa_o \prod_{i=1}^{n} a_i^{\alpha_i - 1}$$

where κ_o and α_i are fixed nonnegative scalars. It will turn out that this nonnegativity constraint on the parameters α_i serves to keep the industry production function concave.

[4] This use of nonnegative measures to "count" economic agents has been explored by Debreu [1] and Hildenbrand [2].

The function φ is called the distribution density function. An alternative interpretation of φ is that there are $\varphi(a)$ firms each having input-output coefficients vector a and each having a capacity of one unit of output. Note that if all the α_i are greater than one, then in this industry the distribution of output capacity is directly proportional to the input-output coefficients and thus inversely proportional to "efficiency." If each of the α_i is less than one, output capacity is directly proportional to efficiency.

In this model, it is clear that firm a has the normalized profit function

$$\pi^N(p;a) = \begin{cases} \varphi(a) \quad [1-p \cdot a] & \text{if} \quad 1 > p \cdot a \\ 0 & \text{otherwise} \end{cases} .$$

From earlier remarks, we conclude that the profit function corresponding to A's technology is

$$\pi^N(p) = \int_0^{1/p_n} d\,a_n \int^{\frac{1}{p_{n-1}}(1-p_n a_n)} d\,a_{n-1} \cdots$$

$$\cdots \int^{\frac{1}{p_1}(1 - \sum_2^n p_i a_i)} \kappa_0 \prod_1^n a_i^{\alpha_i - 1} [1 - \sum_1^n p_i a_i] da_1 \quad .$$

Substituting $y_i = p_i a_i$, $i=1,\ldots,n$, this becomes

$$\pi^N(p) = \frac{1}{p_1 \cdots p_n} \int_0^1 dy_n \int_0^{1-y_n} dy_{n-1} \cdots \int^{1-\sum_2^n y_i} \kappa_o \prod_i^n \left(\frac{y_i}{p_i}\right)^{\alpha_i - 1} [1 - \sum_1^n y_i] dy_1$$

$$(2) \qquad = \kappa_1 \prod_1^n p_i^{-\alpha_i}$$

where κ_1 is the positive, finite constant (i.e. independent of p) defined by

$$\kappa_1 = \int_0^1 dy_n \int_0^{1-y_n} dy_{n-1} \cdots \int^{1-\sum_2^n y_i} \kappa_o \prod_1^n y_i^{\alpha_i - 1} [1 - \sum_1^n y_i] \, dy_1 .$$

The industry production function is wholly characterized by (2). It is a standard result that the production function corresponding to (2) is Cobb-Douglas. To actually compute this production function, we use (1) to derive the profit function corresponding to this normalized profit function:

$$\pi(p_o, p) = p_o \, \pi^N\left(\frac{1}{p_o} p\right)$$

$$= \kappa_1 \, p_o^{1-\lambda} \prod_1^n p_i^{-\alpha_1}$$

where $\lambda = -\sum_1^n \alpha_i$. We then use Shephard's Lemma [5] to derive the industry supply function $X_o(\pi_o, p)$ and derived demand functions $x_i(p_o, p)$, $i=1,\ldots,n$:

$$X_o(p_o,p) = \frac{\partial\pi}{\partial p_o}(p_o,p) = \frac{(1-\lambda)}{p_o}\pi(p_o,p)$$

$$x_i(p_o,p) = -\frac{\partial\pi}{\partial p_i}(p_o,p) = \frac{\alpha_i}{p_i}\pi(p_o,p) \quad . \qquad (3)$$

If we then cleverly choose to evaluate the product $\prod_{i=1}^{n}[x_i(p_o,p)]^{\alpha_i/1-\lambda}$, it turns out that:

$$\prod_{i=1}^{n}[x_i(p_o,p)]^{\alpha_i/1-\lambda} = \prod_{i=1}^{n}\left[\frac{\alpha_i}{p_i}\pi(p_o,p)\right]^{\alpha_i/1-\lambda}$$

$$= \kappa_1^{\frac{-1}{1-\lambda}}\left[\prod_{1}^{n}\alpha_i^{\frac{\alpha_i}{1-\lambda}}\right]\frac{1}{p_o}\pi(p_o,p)$$

$$= \frac{1}{1-\lambda}\left[\kappa_1^{-1}\prod_{1}^{n}\alpha_i^{\alpha_i}\right]^{\frac{1}{1-\lambda}} X_o(p_o,p) \quad .$$

Thus the industry production function is

$$X_o = \kappa_2 \prod_{i=1}^{n} x_i^{\frac{\alpha_i}{1-\lambda}}$$

where

$$\kappa_2 = (1-\lambda)\left[\kappa_1 \prod_{1}^{n}\alpha_i^{-\alpha_i}\right]^{\frac{1}{1-\lambda}} \quad .$$

Thus the industry has a Cobb-Douglas production function with returns to scale $\frac{\lambda}{\lambda-1}$. It is easily checked that the usual conditions that the coefficients $\frac{-\alpha_i}{\lambda-1}$ be nonnegative and sum to no more than unity are equivalent to the condition that $\alpha_i \geqq 0$, $i=1,\ldots,n$.

This derivation of the industry production function differs from that of Houthakker [3] in that he derived each of the $n+1$ equations (3) (for the case $n=2$) by a separate integration similar to the integration used above to find the industry profit function. The conceptual advantage of using profit functions is that it demonstrates how to carry out this aggregation for less simple models and thus answers a question raised by Houthakker; namely, once one has derived the $n+1$ equations (3), how does one "eliminate" the variables p_o and p to find the implicit relation among $X_o, X_1, \ldots X_n$? The answer is to compute the negative of the convex conjugate of $\pi^N(p)$ (see footnote 3 above). In the Appendix, we indicate a procedure for making this computation, for this particular profit function, without the need to guess the answer in advance as we did above. In any case, even if an easily analysable form cannot be gotten for the convex conjugate of $\pi^N(p)$, it is possible to approximate this conjugate by numerical means.

APPENDIX

We show first that if each firm a in industry A has a profit function $\pi(q;a)$ and if there are no external economies or diseconomies among the firms, then the industry profit function is $\pi(q) = \int \pi(q\,;a)\, d\mu(a)$. We shall represent firm a's technology by a technology set $T(a)$ which is a subset of the commodity space S (where S is any separable Banach space or even Polish space). In the case considered in the text where firm a has a production function $f(\cdot;a)$ on $\mathbb{R}^n$, we have $S = \mathbb{R}^{n+1}$ and $T(a) = \{(x_o,-x) \in \mathbb{R}^{n+1}: x_o \leq f(x)\}$. For any price vector q in the dual space S', the supremum of the profits attainable by firm a is $\pi(q\,;a) = \sup\{q\cdot z: z \in T(a)\}$.

The assumption that there are no external economies or diseconomies among the firms means just that the industry's technology set is the sum of the firm technology sets: $\int T(a)\, d\mu(a)$. If A is finite, we have $\int T(a)\, d\mu(a) = \sum_{a\in A} T(a)$. Otherwise, to represent the industry technology set we must suppose A supports a nonnegative "counting" measure μ [1] and [2]. But then the industry profit function is

$$\begin{aligned}\pi(q) &= \sup\{q\cdot z: z \in \int T(a)\, d\mu(a)\} \\ &= \int \pi(q\,;a)\, d\mu(a)\end{aligned}$$

as was to be shown. The second equality holds whenever $\int T(a)\,d\mu(a)$ is not empty and the set $\{(a,z): z \in T(a)\}$ is measureable in

$A \times S$ (Theorem C, page 621 in [2]). In particular, it is true whenever A is finite.

Our second objective in this Appendix is to give an alternative calculation of the negative of the convex conjugate of the normalized profit function: $\pi^N(p) = \kappa_1 \prod_{i=1}^{n} p_i^{-\alpha_i}$. This function is clearly positively homogeneous of degree $\lambda = -\sum_1^n \alpha_i$. In this case the following Proposition simplifies things.

<u>Proposition</u>: If g is a closed convex function[5] positively homogeneous of degree μ and if $z^* \in \partial g(z)$ (or, equivalently, $z \in \partial g^*(z^*)$)[6] then

a) $z^* \cdot z = \mu\, g(z)$

b) $g^*(z^*) = (\mu - 1)\, g(z)$

c) $t^{\mu-1} z^* \in \partial g(t z)$

d) $g^*(tz^*) = t^{\mu/\mu-1} g^*(z^*)$ for all $t > 0$.

Note that a) is just an extension of Euler's Theorem to convex functions; b) is the statement we shall use below in the calculation of the conjugate of the profit function; d) is close to the statement that g^* is positively homogeneous of degree $\frac{\mu}{\mu-1}$. More precisely, g^* is positively homogeneous of degree $\frac{\mu}{\mu-1}$ at each point z^* in the domain of g^* where g^* is subdifferentiable. This collection of z^*'s includes the relative

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[5] For the definitions of these terms and of the subdifferential $\partial f(x)$ of a function, see [6].

[6] g^* is the convex conjugate of g .

interior of the domain of g^* ([6], Theorem 23.4, page 217). In particular, d) implies that the relative interior of the domain of g^* is a cone (as is the domain of g).[7]

PROOF: a) Given z and z^* in $\partial g(z)$, define a convex function h of a scalar variable t by $h(t) = g(tz)$. Since $z^* \in \partial g(z)$, it is easily seen that $z^* \cdot z \in \partial h(1)$. On the other hand, for $t > 0$, $h(t) = t^\mu g(z)$, so h is differentiable at $t=1$ with $\frac{dh}{dt}(1) = \mu g(z)$. Thus $\partial h(1)$ contains just the number $\mu g(z)$ ([6], Theorem 25.1, page 242). Thus $z^* \cdot z = \mu g(z)$.

b) If $z^* \in \partial g(z)$, then $g^*(z^*) = z^* \cdot z - g(z)$ ([6], Theorem 23.5, page 218). Making use of part a), this gives b).

c) Since $z^* \in \partial g(z)$, then for any other vector x in S and any positive scalar t,

$$g\left(\frac{x}{t}\right) \geqq g(z) + z^* \cdot \left(\frac{x}{t} - z\right) .$$

But then

$$\begin{aligned} g(x) &= t^\mu g\left(\frac{x}{t}\right) \\ &\geqq t^\mu \left[g(z) + z^* \cdot \left(\frac{x}{t} - z\right)\right] \\ &= g(tz) + t^{\mu-1} z^* \cdot (x - tz) . \end{aligned}$$

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[7] Note that if $\lambda = \frac{\mu}{\mu-1}$, then $\frac{1}{\mu} + \frac{1}{\lambda} = 1$. Rockafellar has shown, using gauge functions, that when μ is greater than 1 and g is closed proper convex and positively homogeneous of degree μ, then g^* is closed, convex proper and positively homogeneous of degree $\mu/\mu-1$ ([6], Corollary 15.3.1, page 135).

Since this is true for all x , $t^{\mu-1} z^* \in \partial g(t z)$.

d) If $z^* \in \partial g(z)$, then by c), $t z^* \in g(t^{\frac{1}{\mu-1}} z)$ so that ([6], Theorem 23.5, page 218)

$$g^*(t z^*) = (t z^*) \cdot (t^{\frac{1}{\mu-1}} z) - g(t^{\frac{1}{\mu-1}} z)$$

$$= t^{\frac{\mu}{\mu-1}}(z^* \cdot z - g(z))$$

$$= t^{\frac{\mu}{\mu-1}} g^*(z^*)$$

by ([6], Theorem 23.5, page 218) again.

We return now to the computation of the conjugate of π^N .[8] Letting $z = p$, $g(p) = \pi^N(p) = \kappa_1 \prod_{i=1}^{n} p_i^{-\alpha_i}$, $-z^* = x$ and $g^*(z^*) = -f(x)$,[9] the condition $z^* \in \partial g(z)$ becomes (for positive p):

$$(4) \quad x_i = -z_i^* = -\frac{\partial \pi^N}{\partial p_i}(p) = \frac{\alpha_i}{p_i} \pi^N(p) \qquad i=1,\ldots,n \quad .$$

Since g is closed, convex and positively homogeneous of degree $\lambda = \sum_1^n -\alpha_i$, part b) of the Proposition asserts (4) implies $(\lambda-1)\, \pi^N(p) = g^*(z^*) = -f(x)$ so that (4) gives

[8] This derivation follows the solution given to this problem in an unpublished paper by Lawrence J. Lau.

[9] The motivation for these definitions is given in footnote 3 in the text.

(5) $$p_i = \frac{\alpha_i}{1-\lambda} \frac{f(x)}{x_i} \quad .$$

Finally, the condition $z^* \in \partial g(z)$ is equivalent to $z \in \partial g^*(z^*)$ which is translated as $p_i = \frac{\partial f}{\partial x_i}(x)$, $i=1,\ldots,n$, so (5) becomes

$$\frac{\partial f}{\partial x_i} = \frac{\alpha_i}{1-\lambda} \frac{f(x)}{x_i} \qquad i=1,\ldots,n \quad .$$

This set of differential equations clearly has the solution

$$f(x) = \kappa_3 \prod_1^n x_i^{-\alpha_1/\lambda-1}$$

for some constant κ_3. Thus f is a Cobb-Douglas function with returns to scale $\frac{\lambda}{\lambda-1}$.

To evaluate κ_3, we use again the relation

$$\pi^N(p) = g(z) = z^* \cdot z - g^*(z^*)$$

(6) $$= f(x) - p \cdot x$$

which holds whenever (4) holds. But

$$p \cdot x = -z^* \cdot z$$

$$= -\lambda\, g(z) \qquad \text{by part a) of the Proposition}$$

(7) $$= -\lambda\, \pi^N(p)$$

Furthermore,

$$f(x) = \kappa_3 \prod_1^n x_i^{\frac{\alpha_i}{1-\lambda}}$$

(8) $$= \kappa_3 \prod_1^n [\frac{\alpha_i}{p_i} \pi^N(p)]^{\frac{\alpha_i}{1-\lambda}} \quad \text{by (4)}$$

$$= \kappa_3 [\prod_1^n \alpha_i^{\frac{\alpha_i}{1-\lambda}}] [\prod_1^n p_j^{-\alpha_j}]^{\frac{1}{1-\lambda}} [\pi^N(p)]^{\frac{-\lambda}{1-\lambda}}$$

(8) $$= \kappa_3 [\prod_1^n \alpha_i^{\frac{\alpha_i}{1-\lambda}}] \kappa_1^{\frac{-1}{1-\lambda}} \pi^N(p) .$$

Substituting (7) and (8) into (6) gives

$$\pi^N(p) = \lambda \pi^N(p) + \kappa_3 \kappa_1^{\frac{-1}{1-\lambda}} [\prod_1^n \alpha_i^{\frac{\alpha_i}{1-\lambda}}] \pi^N(p) .$$

Since this is true for all positive p and $\pi^N(p)$ is positive when p is positive, then we can divide this equation by $\pi^N(p)$ to give:

$$\kappa_3 = (1-\lambda) [\kappa_1 \prod_1^n \alpha_i^{-\alpha_i}]^{\frac{1}{1-\lambda}}$$

so that κ_3 equals the constant κ_2 found in the text. This completes this alternative derivation of the production function f .

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