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AN IDENTITY CONCERNING THE RELATION BETWEEN THE PAASCHE AND LASPEYRE INDICES

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Consider two occasions in which the prices and compositions of goods, n in number, are given by the pairs of vectors (p_o, x_o), (p_1, x_1), of order n. The expenditures are

$$e_o = p_o'x_o \ , \ e_1 = p_1'x_1 \ .$$

Let

$$u_o = \frac{p_o}{e_o} \ , \quad u_1 = \frac{p_1}{e_1} \ ,$$

so that

$$u_o'x_o = 1 \ , \quad u_1'x_1 = 1 \ .$$

With 0 and 1 as base and object occasions, the Laspeyre and Paasche indices are

$$L = \frac{p_o'x_1}{p_o'x_o} = u_o'x_1 \ , \quad P = \frac{p_1'x_1}{p_1'x_o} = \frac{1}{u_1'x_o} \ .$$

The method of limits[1] in index-number theory relies on the relation[2]

$$P < L \ ,$$

which is equivalent to

$$\Delta = 1 - (u_o'x_1)(u_1'x_o) < 0 \ .$$

An equivalent algebraical expression for Δ will be found which provides a geometrical interpretation for this condition. It appears that, though the relation $P < L$ follows, trivially, from the condition $u_o'x_1 > 1$, $u_1'x_o > 1$, when the condition does not hold, the relation has no general necessity. This is contrary to the widely established doctrine that the Paasche index is less than the Laspeyre index.

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[1] Wassily Leontief. Composite commodities and the problem of index numbers. Econometrica 4, 1(1936), 39-59.

[2] J. R. Hicks. A Revision of Demand Theory (Oxford 1956).

Any points u, x in the positive orthants B, C of real Euclidean spaces of dimension n define a balance and a composition. They have scalar product $u'x$; and the composition x is said to be within, on, or over the balance u according as $u'x \leq , = ,$ or > 1 . Now, with $u \in B$,

$$O_u = \{x \mid u'x = 1 , x \in C\}$$

denotes the set of compositions on a balance u . Then the given balances $u_o, u_1 \in B$ and compositions $x_o, x_1 \in C$ are such that $x_o \in O_{u_o}$, $x_1 \in O_{u_1}$.

Assume $x_o \neq x_1$; and let

$$D_{o1} = u_o'x_1 - 1 , D_{1o} = u_1'x_o - 1 .$$

Then Samuelson's Weak Axiom of Revealed Preference excludes the possibility

$$D_{o1} \leq 0 , \quad D_{1o} \leq 0 .$$

Hence either

$$\text{(I)} \quad D_{o1} > 0 , \quad D_{1o} > 0 ,$$

in which case $D_{o1}D_{1o} > 0$, or one or the other of two further possibilities holds, such as

$$\text{(II)} \quad D_{o1} > 0 , \quad D_{1o} \leq 0 ,$$

in which case $D_{o1}D_{1o} \leq 0$.

If (I) holds, that is,

$$u_o'x_1 > 1 , \quad u_1'x_o > 1 ,$$

then, by multiplication,

$$(u_o'x_1)(u_1'x_o) > 1 ,$$

so that $\Delta < 0$. So it remains to consider a case such as (II), in which $D_{o1}D_{1o} \leq 0$.

Now, in the Euclidean spaces, without restriction to the positive orthants, let U, X denote the spaces spanned by u_o, u_1 and x_o, x_1 . Their points are of the form

$$u = u_o\alpha_o + u_1\alpha_1 \ , \quad x = x_o\beta_o + x_1\beta_1$$

where the α's and β's are any scalars, and they do not necessarily belong to B, C . Given any vector u , without restriction to the positive orthant B , by <u>the hyperplane</u> u will be meant the locus $u'z = 1$, without restriction of z to the positive orthant C . Thus, in the case $u \in B$, the set O_u , of compositions on the balance u , is the intersection of the hyperplane u with the positive orthant C .

Let it now be asked of the hyperplane u that it pass through x_o, x_1 ; that is,

$$u'x_o = 1 \ , \ u'x_1 = 1 \ ;$$

then

$$\alpha_o + u_1'x_o\alpha_1 = 1 \ , \ u_o'x_1\alpha_o + \alpha_1 = 1 \ ,$$

so that, eliminating α_1 ,

$$\alpha_o + u_1'x_o(1 - u_o'x_1\alpha_o) = 1 \ ,$$

whence

$$\alpha_o = \frac{1 - u_1'x_o}{1 - u_o'x_1u_1'x_o} \ ,$$

that is,

$$\alpha_o = \frac{D_{1o}}{-\Delta} \ , \text{ and similarly, } \ \alpha_1 = \frac{D_{o1}}{-\Delta} \ .$$

Hence, there is the unique determination

$$u = u_o\frac{D_{1o}}{-\Delta} + u_1\frac{D_{o1}}{-\Delta} \ .$$

Similarly, if it is asked of x that it be on the hyperplanes u_o, u_1 , that is

$$u_o'x = 1 \ , \quad u_1'x = 1 \ ,$$

then, uniquely,

$$x = x_o\frac{D_{o1}}{-\Delta} + x_1\frac{D_{1o}}{-\Delta} \ .$$

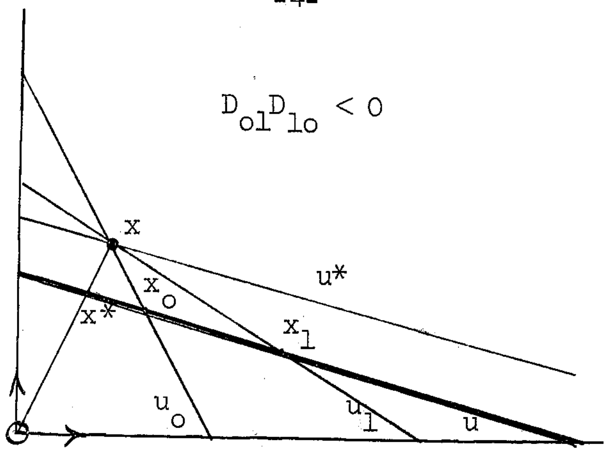

Fig. 1

It now appears that

$$\Delta^2 u'x = (u_o D_{1o} + u_1 D_{o1})(x_o D_{o1} + x_1 D_{1o})$$
$$= u_1'x_1 D_{1o}^2 + 2D_{1o}D_{o1} + u_1'x_o D_{o1}^2$$
$$= D_{1o}(u_o'x_1 D_{1o} + D_{o1}) + D_{o1}(u_1'x_o D_{o1} + D_{1o})$$
$$= -(D_{1o} + D_{o1})\Delta ,$$

so that

$$u'x = \frac{D_{o1} + D_{1o}}{-\Delta} .$$

Therefore,

$$u'x - 1 = \frac{u_o'x_1 - 1 + u_1'x_o - 1 + 1 - (u_o'x_1)(u_1'x_o)}{-\Delta}$$
$$= \frac{(u_o'x_1 - 1)(u_1 x_o - 1)}{\Delta} ,$$

and accordingly,

$$1 - (u_o'x_1)(u_1'x_o) = \frac{(u_o'x_1 - 1)(u_1'x_o - 1)}{u'x - 1} .$$

Hence, with

$$D = u'x - 1 ,$$

it has been established that

$$\Delta = \frac{D_{o1}D_{1o}}{D} .$$

It follows that, if it is given that $D_{o1}D_{1o} < 0$, then $D \neq 0$, and

$$\Delta < 0 \iff D > 0 \ .$$

While an interpretation of the condition $\Delta < 0$ is not immediately obvious, the condition $D > 0$ has a direct geometrical interpretation, as follows: The linear space X spanned by x_o, x_1 cuts the intersection of the hyperplanes u_o, u_1 in a unique point x. The linear space U spanned by u_o, u_1 contains a unique point u such that the hyperplane u passes through x_o, x_1. In two dimensions u is simply the join of x_o, x_1 and x is simply the intersection of u_o, u_1. The interpretation of the condition $\Delta < 0$ is that the origin 0 and the point x lie on opposite sides of the hyperplane u.

Now further observations can be made, as follows. The point

$$x^* = x_o \frac{D_{o1}}{D_{o1} + D_{1o}} + x_1 \frac{D_{1o}}{D_{o1} + D_{1o}}$$

lies on the line joining x_o, x_1. Hence also it lies on the hyperplane u, since this passes through x_o, x_1; that is, $u'x^* = 1$. Further

$$x = x^* \frac{D_{o1} + D_{1o}}{-\Delta}$$

$$= x^*(u'x) \ ,$$

from which directly it appears again that $u'x^* = 1$; and also that x^* lies on the join of $0, x$. Similarly, there is a dual system of relations in respect to an hyperplane u^*. Finally, since

$$u = u^*(u'x) \ ,$$

it appears that

$$u'x = u^{*\prime}x^*(u'x)^2 \ ,$$

so that

$$(u'x)(u^{*\prime}x^*) = 1 \ .$$

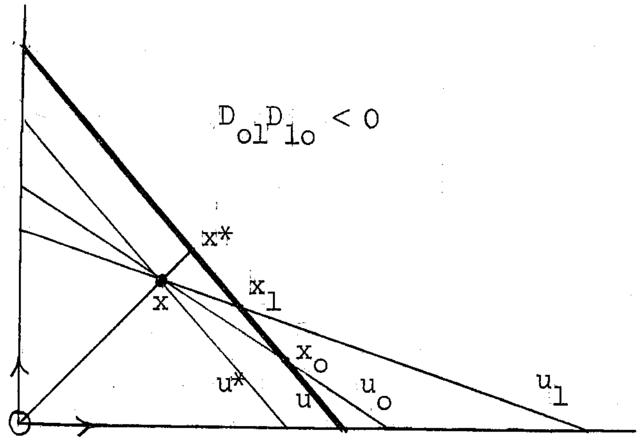

Fig. 2

Hence, if

$$D^* = u^{*\prime} x^* - 1 \; ,$$

then

$$D^* < 0 \iff D > 0 \; .$$

Now $D^* < 0$, which means that the origin 0 and the point x^* lie on the same side of the hyperplane u^* , and provides another geometrical interpretation of the condition $\Delta < 0$, in the case $D_{ol}D_{lo} < 0$.