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THE METHOD OF LIMITS IN THE THEORY OF INDEX NUMBERS

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1. Balance and Composition

Consider two occasions, in which the prices and quantities of some n goods consumed are given by pairs of vectors (p_o, x_o), (p_1, x_1) of order n. The expenditures are

$$e_o = p_o'x_o \ , \quad e_1 = p_1'x_1 \ .$$

Let

$$u_o = \frac{p_o}{e_o} \ , \quad u_1 = \frac{p_1}{e_1} \ ,$$

so that

$$u_o'x_o = 1 \ , \quad u_1'x_1 = 1 \ .$$

With 0 and 1 as base and object occasions, the Laspeyre index is

$$L_{1o} = \frac{p_1'x_o}{p_1'x_1} = u_1'x_o \ ,$$

and the Paasche index is

$$P_{1o} = \frac{p_o'x_o}{p_o'x_1} = \frac{1}{u_o'x_1} \ .$$

Any points u, x in the positive orthants B, C of real Euclidean spaces of dimension n are to define a <u>balance</u> and a <u>composition</u>. They have scalar product $u'x$; and the composition x is said to be <u>on</u> the balance u if $u'x = 1$.

A balance u together with a composition x which is on it defines an <u>expenditure figure</u>, which is to be denoted by $E = (u \mid x)$, where, in this notation, it is to be automatically understood that $u'x = 1$. Any collection of expenditure figures defines an <u>expenditure configuration</u>.

Thus, in occasions $0, 1$ there are given balances $u_o \ , u_1 \in B$, together with compositions $x_o \ , x_1 \in C$ on them, providing a pair of expenditure figures $E_o = (u_o \mid x_o)$, $E_1 = (u_1 \mid x_1)$.

2. Boundaries

For any balance $u \in B$, the compositions $z \in C$ in the sets defined by

$$W_u = \{z \mid u'z \leq 1\} ,$$

$$M_u = \{z \mid u'z \geq 1\} ,$$

are said to be within and upon the balance u, respectively. Then the set of compositions

$$O_u = \{z \mid u'z = 1\} ,$$

on the balance u, is given by

$$O_u = W_u \cap M_u .$$

Given any balances $u, v, \ldots \in B$, they determine regions

$$W_{u,v,\ldots} = W_u \cup W_v \cup \ldots$$

and

$$M_{u,v,\ldots} = M_u \cap M_v \cap \ldots$$

in C, with a surface

$$O_{u,v,\ldots} = W_{u,v,\ldots} \cap M_{u,v,\ldots}$$

as their common boundary.

All these definitions may now be formulated in a dual fashion, with the roles of balance and composition interchanged.

Any surface of the form $I = O_{u,v,\ldots}$ defines a boundary in C. It bounds, from below, a region M_I, any point of which defines a supported composition; and any balance w such that $M_I \subset M_w$ defines a supporting balance. For example, $u, v, \ldots$ are supporting balances since $M_I \subset M_{u,v,\ldots}$; and also they are a generating set of supports for I, since $M_I = M_{u,v,\ldots}$. Let J be the surface in B which is the boundary of the region M_J of supporting balances. Then, for $u \in M_J$, $x \in M_I$,

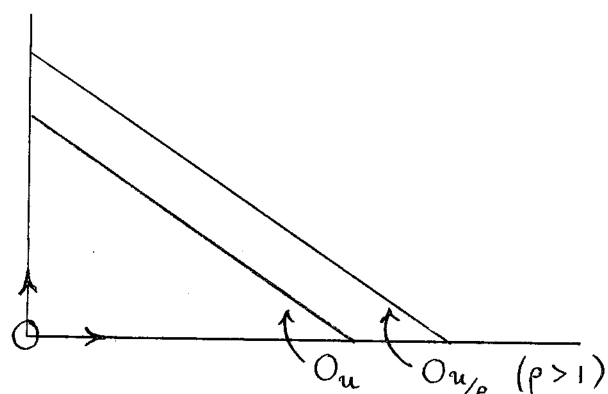
$$u'x \geq 1 , \text{ and } u'x = 1 \Rightarrow u \in J \text{ and } x \in I .$$

In an equivalent, dual formulation, the supporting and supported balances and compositions interchange their roles. The boundaries J, I in B, C are dual constructions; each is reconstructable from the other; and formulations can be made with reference primarily to the one or the other.

If J, I are dual boundaries in B, C and $u \in J$, $x \in I$, then I is said to be through x and to touch u; or, dually, J is through u and touches x.

Given any balance u, and a number $\rho > 0$, the balance $\frac{u}{\rho}$ defines the balance u expanded by the factor ρ. Expansion is positive or negative according as $\delta = \rho - 1$ is positive or negative.

Balances are called parallel if one is an expansion of the other.



Given any balance $u \in B$, and a boundary I in C, with dual J in B, there is a unique balance $\frac{u}{\rho}$ which is parallel to u and which belongs to J, or equivalently, which touches I, where $\rho = \rho(u, I)$ defines the index of the balance u in respect to the boundary I, and is given by

$$\rho(u, I) = \min \{u'z \mid z \in M_I\}$$
$$= \min \{u'z \mid z \in I\} .$$

If $I = O_{v,w,\ldots}$, then

$$\rho(u, I) = \min \{u'z \mid v'z \geq 1 , w'z \geq 1, \ldots \} .$$

With $v = \frac{u}{\rho}$,

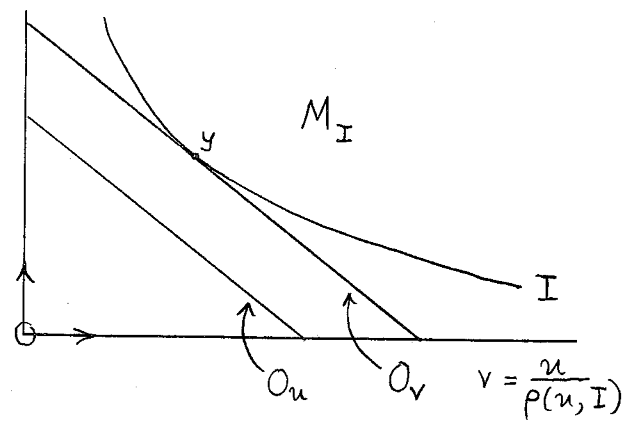
$$v'z \geq 1 \text{ for all } z \in M_I \text{ , whence } v \in M_J .$$

Also

$$v'y = 1 \quad \text{for some} \quad y \in M_J ,$$

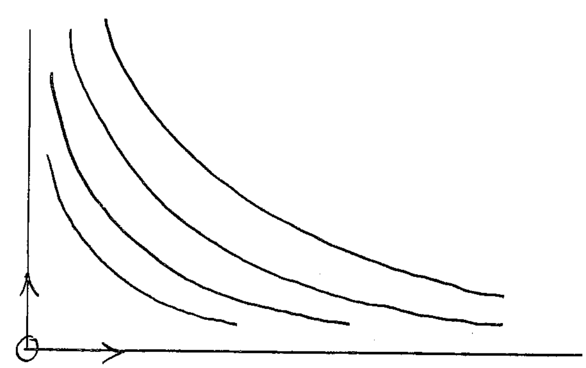
in which case $v \in J$ and $y \in I$. The condition for u to be a support of I is $\rho(u, I) \geq 1$. Thus

$$u \in M_J \Longleftrightarrow \rho(u, I) \geq 1 , \quad u \in J \Longleftrightarrow \rho(u, I) = 1 .$$



3. Maps .

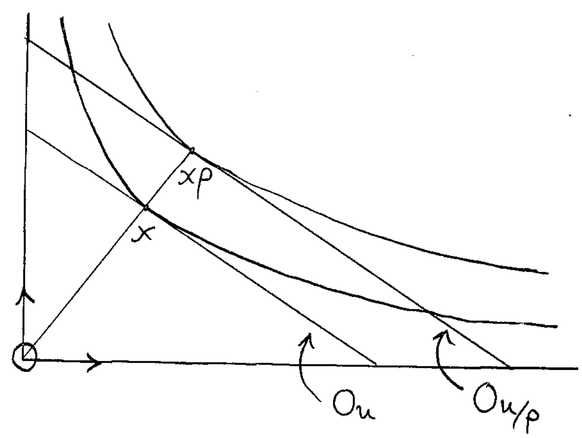
A map in C is defined by a system of boundaries, one and only one of which passes through any composition, and, equivalently, one and only one of which touches any balance. Since the boundaries do intersect, they are completely ordered, each being above or below another. Then the points of C are partially ordered relative to the map, any two compositions having the order of the unique boundaries through them. Also the point of B are partially ordered relative to the map, two balances having the order of the unique boundaries which touch them.



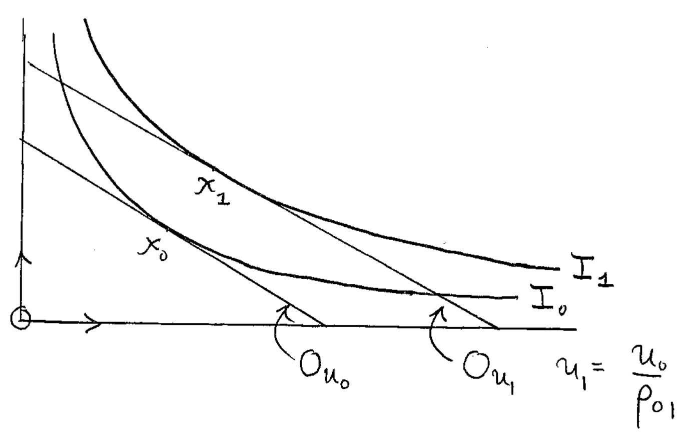
A balance together with a composition which is on it are said to be <u>conjugates</u>, relative to a map, or the expenditure figure they form is said to <u>belong</u> to the map, if the unique boundaries touching the one and through the other coincide.

Since the set of conjugates of a given balance or composition is an intersection of convex sets, of the form O_u and M_I, it is a convex set. It is possible that every balance or composition has a unique conjugate, so a one-to-one correspondence is determined between balances and compositions. This is the case when the boundaries are all smooth and strictly convex.

Any boundary I can be a boundary in a map. Thus, for any $\rho > 0$, another boundary I_ρ is defined by $I_\rho = \{x\rho \mid x \in I\}$. If J is the dual of I, then the dual of I_ρ is $J_\rho = \{\frac{u}{\rho} \mid u \in J\}$. Since, for any x, there is just one point of the form $x\rho$ on any of these boundaries, there is just one of these boundaries through any point, so they constitute a map.



Also, any two non-intersecting boundaries I_o, I_1 can be boundaries in the same map.



Thus, let u_o be any tangent balance of I_o. Then $u_1 = \frac{u_o}{\rho_{o1}}$ is a parallel tangent balance of I_1, for some ρ_{o1}. If x_1 is any contact of u_1 with I_1, then

$$1 = u_1'x_1 = \frac{u_o'x_1}{\rho_{o1}} ,$$

so that

$$\rho_{o1} = u_o'x_1 .$$

So long as I_o and I_1 do not intersect, $\rho_{o1} \neq 1$. Then it is possible to consider a function defined by

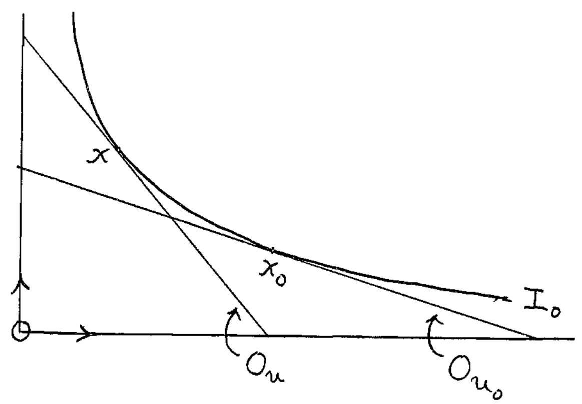
$$\varphi = \min_{u_o} \frac{u_o'x - 1}{\rho_{o1} - 1} ,$$

for all supporting balances u_o of I_o, and the corresponding ρ_{o1}. It is a concave increasing function, it has a maximum on every balance, and its levels describe a preference map P.

Since

$$u_o'x - 1 \geq 0$$

for every x on I_o, with the value 0 attained when u_o is a balance touching I_o at x_o, it follows that $\varphi(x) = 0$ on I_o.



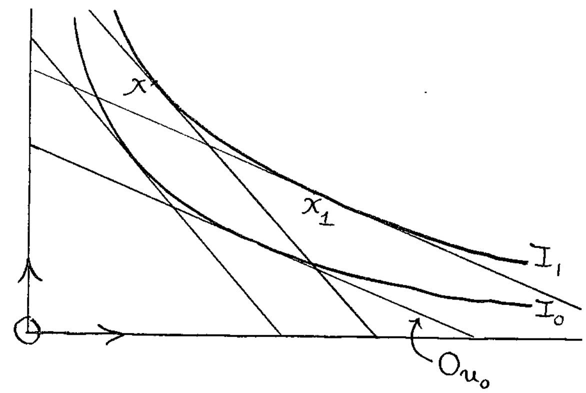
Now let x be on I_1. Then

$$u_o'x \geq u_o'x_1 = \rho_{o1} ,$$

where x_1 is the contact of the tangent balance of I_1 parallel to u_o, so that

$$\frac{u_o'x - 1}{\rho_{o1} - 1} \geq 1 ,$$

and the value 1 is attained when u_o is parallel to a balance touching I_1 at x . Hence $\varphi(x) = 1$ in I_1 .



Accordingly, I_o, I_1 are among the boundaries of the preference map P described by φ . Any other boundary of the map is a level $\varphi(x) = t$ of the function φ , and the boundary I_t of the region defined by the system of inequalities

$$u_o'x \geq 1 + t(\rho_{o1} - 1) ,$$

where u_o is any supporting balance of I_o , and ρ_{o1} is determined correspondingly.

Finally, any finite set of non-intersecting boundaries can be boundaries in the same map, for they fall in an order, according to on which side of each other they lie. Any consecutive pair of boundaries can be interpolated in the manner just described; and the first and last can then be used to complete the map below and above, in the manner first described for a single given boundary.

4. Admissibility

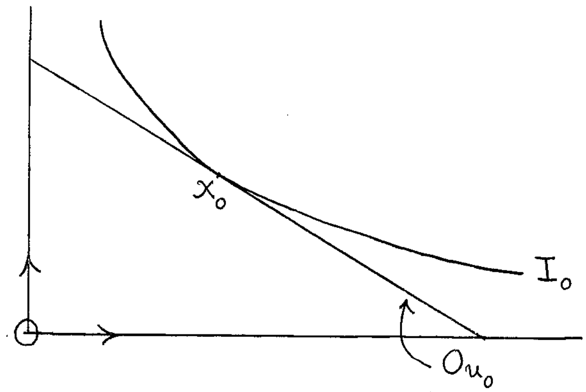
If any given expenditure figures belong to a given map, then that map defines an <u>admissible preference map</u> for the configuration they form.

The question arises as to the existence of an admissible preference map for a given expenditure configuration, in particular for the configuration formed by the two given figures E_o, E_1 .

If such a map P exists, let I_o, I_1 be the boundaries through x_o, x_1 . Then any point within u_o is either on or in the inferior side of I_o . Therefore, $u_o'x_1 \leq 1$ means that x_1 is either on I_o , in the case $u_o'x_1 = 1$, or inferior to x_o , in the case $u_o'x_1 < 1$. In the one case $I_o = I_1$ and hence $u_1'x_o \geq 1$, and in the other, I_o is over u_1 , so that $u_1'x_o > 1$. It follows that

$$u_o'x_1 \leq 1 \text{ and } u_1'x_o \leq 1 \text{ only if } u_o'x_1 = 1 \text{ and } u_1'x_o = 1 .$$

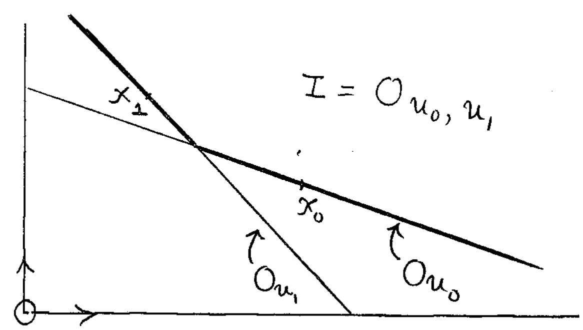
This is a weakened version of Samuelson's Axiom of Revealed Preference.[1]



Conversely, if this condition holds, an admissible preference map exists. There only has to be examined the cases

$$u_o'x_1 \geq 1 , \quad u_1'x_o \geq 1$$

$$u_o'x_1 < 1 , \quad u_1'x_o > 1 .$$



[1] P. A. Samuelson, "Consumption theory in terms of revealed preference," Economica 28 (1948), pp. 243-53.

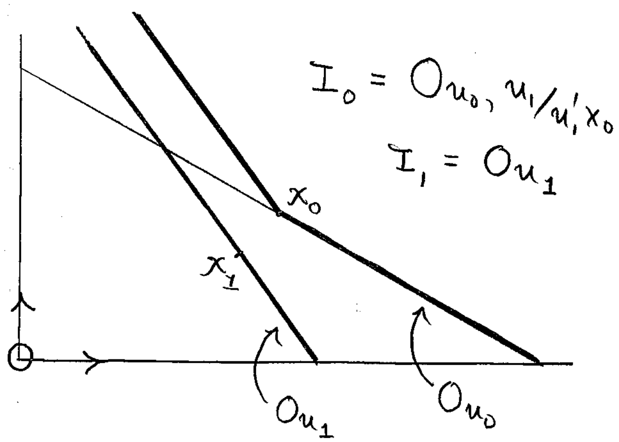
In the first case, it is possible to take

$$I_o = I_1 = O_{u_o,\, u_1}$$

and a map containing this boundary. In the second case, it is possible to take

$$O_{u_o,\, u_1/u_1'x_o} \quad \text{and} \quad I_1 = O_{u_1} ,$$

and a map containing these two boundaries.

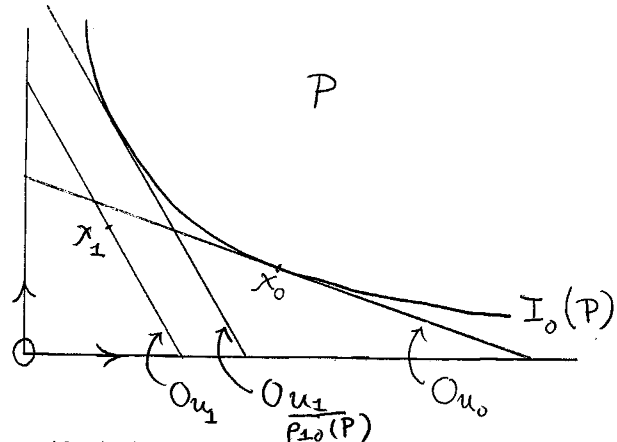


Thus it appears that the revealed preference axiom, for a configuration of two figures, is necessary and sufficient for the existence of an admissible preference map. A more general result for a configuration with any number of figures, has been obtained elsewhere.[2]

5. Limits

Now assume this axiom is satisfied, and that P is any of the therefore existing non-empty class $\mathcal{P}$ of admissible preference maps. Let I_o be the boundary of P through x_o. This must have u_o as a tangent, since it is admitted by E_o. It has a unique balance $\frac{u_1}{\rho_{1o}}$ parallel to u_o as a tangent. The number $\rho_{1o} = \rho_{1o}(P)$ defines the cost of living index, with E_o, E_1 as base and object figures, determined relative to the preference map P admitted by these figures. The problem now is to determine all the values which ρ_{1o} takes as P ranges throughout $\mathcal{P}$.

[2] S. N. Afriat, "The algebra of revealed preference." Research Paper No. 2 (May 1962), Econometric Research Program, Princeton University.



It will appear that these values form an interval, whose upper limit is the Laspeyre index, and whose lower limit is going to be determined.

The result is in accordance with the familiar proposition that the Laspeyre index provides an upper bound for the cost of living index.[3] It establishes it moreover as a least upper bound; that is, there always exists an admissible preference map in respect to which the cost of living index is determined within any assigned distance of the Laspeyre index, however small.

The other familiar proposition, that the Paasche index provides a lower bound, is rejected. This is in the first place because the Paasche index can be greater than the Laspeyre upper limit,[4] in which case the proposition becomes absurd; and in the second place, because, even when it is less than the Laspeyre limit, all that can generally be said of it is, contrary to its being a lower bound, that it is a point in the range of values, though not one of any special significance.

What is usually understood as the "method of limits" in index-number theory is the giving of limits to "the ideal index" in terms of the Paasche and Laspeyre indices. Such an understanding of the method has proved unworkable. The notion of an "ideal index," besides being unnecessary, is quite unsatisfactory because it has never been given a clear meaning. What

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[3] Wassily Leontief, "Composite commodities and the problem of index numbers," Econometrica 4, 1 (1936), pp. 39-59.

[4] S. N. Afriat, "An identity concerning the relation between the Paasche and Laspeyre indices," Research Paper No. 3 (May 1962), Econometric Research Program, Princeton University.

is plain is that every admissible preference map gives a determination of the index, and all such determinations are, without any further criterion to distinguish them, equally admissible. Therefore, instead of the usual adoption of the "theorem" that the, as yet undefined, ideal index always lies in the, sometimes non-existent, Paasche-Laspeyre interval,[5] the method will now be understood as the attempt to find the range of possible values of the index, determined relative to all admissible preference maps. With this, a definite question can be asked, and the answer found.

The popular assumption is that, if $\rho_{1o}{}^*$ is the ideal index, whatever it could be, then

$$\frac{1}{u_o{}'x_1} \leq \rho_{1o}{}^* \leq u_1{}'x_o \ .$$

From this it is concluded that the Fisher index, which is the geometric mean

$$F_{1o} = \left(\frac{u_1{}'x_o}{u_o{}'x_1} \right)^{\frac{1}{2}}$$

of these limits given by the Paasche and Laspeyre indices, is closer to the ideal index than either of these limits. While the rejection of this argument destroys the usual justification for the Fisher index, still another, more serious objection has been made elsewhere,[6] in which an observation of Buscheguennce[7], often considered to constitute a justification, becomes the basis for a rejection.

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[5]J. R. Hicks. A Revision of Demand Theory (Oxford, 1956).

[6]S. N. Afriat. Preference Analysis: A General Method with Application to the Cost of Living Index. Research Memorandum No. 29 (August 1961), Econometric Research Program, Princeton University.

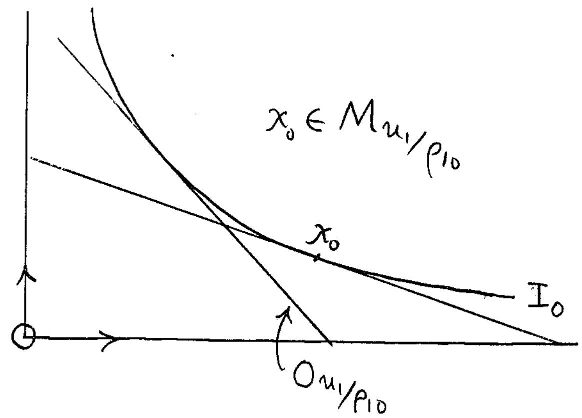
[7]S. S. Buscheguennce, "Sur une class des hypersurfaces. A propos de'l'index idéal de M. Irv. Fisher," Recueil Mathématique, XXXII, 4 (1925), Moscow.

6. Evaluation of Limits

Consider the case in which x_o is revealed preferred to x_1 $(u_o'x_1 < 1 , u_1'x_o > 1)$, and a remaining case, in which x_o is not revealed preferred to x_1 $(u_o'x_1 \geq 1)$.

If I_o is any preference boundary through x_o , and ρ_{1o} is determined correspondingly, then I_o , in particular x_o , lies on or above $\frac{u_1}{\rho_{1o}}$, that is $\frac{u_1'x_o}{\rho_{1o}} \geq 1$, or

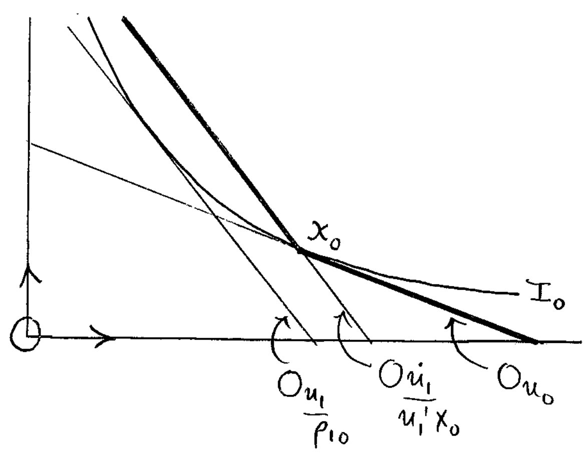
$$\rho_{1o} \leq u_1'x_o .$$



Let $I_o = O_{u_o, u_1/u_1'x_o}$. Then I_o is a preference boundary through x_o . If $u_o'x_1 < 1$, $u_1'x_o > 1$, then x_1 is below I_o , and if $u_o'x_1 > 1$, $u_1'x_o < 1$, then x_1 is above I_o , as required for admissibility. But

$$\rho_{1o} = \min_x \{u_1'x \mid u_o'x \geq 1 , u_1'x \geq u_1'x_o\} = u_1'x_o$$

so that $u_1'x_o$ is an attainable value of ρ_{1o} .

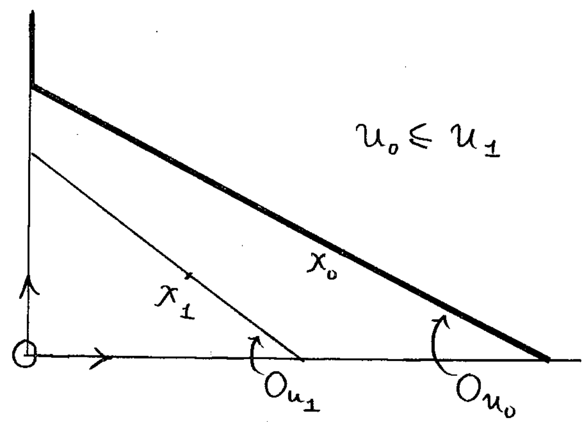


Consider the case in which x_o is not revealed preferred to x_1. Thus, O_{u_o} is an admissible preference boundary through x_o, and gives

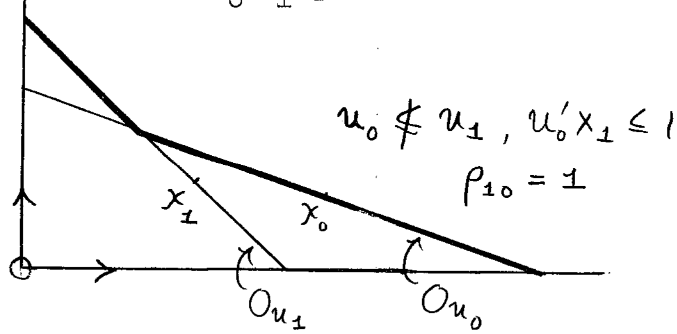
$$\rho_{1o} = \min \{u_1'x \mid u_o'x \geq 1\}$$

as an admissible value. Also, any admissible preference boundary I_o through x_o lies in M_{u_o} and therefore cannot give a determination less than this. Hence

$$\rho_{1o} \geq \min \{u_1'x \mid u_o'x \geq 1\} .$$



Next, consider the case $u_o'x_1 \leq 1$.



Any boundary through x_o must lie in M_{u_o,u_1}. A limiting position is O_{u_o,u_1} which provides the value

$$\rho_{1o} = \min \{u_1'x \mid u_o'x \geq 1 , u_1'x \geq 1\}$$

as the lower limit.

Now

$$u_o \leq u_1 \iff u_o'x \leq u_1'x , \text{ all } x ,$$

so that

$$u_o' \leq u_1 \iff u_o'x \geq 1 \implies u_1'x \geq 1 .$$

Hence, in the case $u_o \leq u_1$

$$\min \{u, x \mid u_o'x \geq 1 , u_1'x \geq 1\}$$

$$= \min \{u_1'x \mid u_o'x \geq 1\} .$$

Otherwise,

$$\min \{u, x \mid u_o'x \geq 1 , u_1'x \geq 1\}$$

$$= \min \{u_1'x \mid u_1'x \geq 1\} = 1 .$$

Accordingly, assuming

$$u_o'x_1 \leq 1 \text{ and } u_1'x \leq 1 \text{ only if } u_o'x_1 = u_1'x_o = 1 ,$$

in the case $u_o'x_1 \geq 1$ or $u_o \leq u_1$, the limits of ρ_{1o} are

$$\min \{u_1'x \mid u_o'x \geq 1\} \text{ and } u_1'x_o ,$$

and otherwise (in the case $u_o'x_1 < 1$ without $u_o \leq u_1$), the limits are 1 and $u_1'x_o$.

It remains to evaluate $\min \{u_1'x \mid u_o'x \geq 1\}$, or equivalently $\min \{u_1'x \mid u_o'x = 1\}$. Any point $x = \{x_r\}$ of C on $u_o = \{u_{or}\}$ can be written $x = \{\frac{\alpha_r}{u_{or}}\}$ where $\alpha_r \geq 0$, $\Sigma \alpha_r = 1$. Then

$$u_1'x = \sum_r \alpha_r \frac{u_{1r}}{u_{or}} ,$$

for which the minimum value is $\min_r \frac{u_{1r}}{u_{or}}$. Hence

$$\min_{x \in C} \{u_1'x \mid u_o'x \geq 1\} = \min_r \frac{u_{1r}}{u_{or}} .$$

Now the following is established. Given (p_o, x_o), (p_1, x_1) as the prices and quantities of goods consumed on two occasions, a necessary and sufficient condition for the existence of an admissible preference map is that $p_o'x_1 \leq p_o'x_o$ and $p_1'x_o \leq p_1'x_1$ only if $p_o'x_1 = P_o'x_o$ and $p_1'x_o = p_1'x_1$.

Given this condition, the cost of living index ρ_{1o} with o and 1 as base and object occasions, determined relative to any of the existing admissible preference maps, ranges between lower and upper limits $\check{\rho}_{1o}$, $\hat{\rho}_{1o}$, where always

$$\hat{\rho}_{1o} = \frac{p_1'x_o}{p_1'x_1} ,$$

where, if $p_o'x_1 \geq p_o'x_o$ or $\frac{p_{or}}{p_o'x_o} \leq \frac{p_{1r}}{p_1'x_1}$ $(r = 1, \ldots, n)$,

$$\check{\rho}_{1o} = \frac{p_o'x_o}{p_1'x_1} \min_r \frac{p_{1r}}{p_{or}} ,$$

and otherwise

$$\check{\rho}_{1o} = 1 .$$

This solves the problem of limits for a cost-of-living index between two occasions, based on the price and quantity data for these occasions.

Consider again the function $u_1'x$ subject to the constraint $u_o'x = 1$. Its minimum value has been established, and, in a similar way, its maximum value is $\max_r \frac{u_{1r}}{u_{or}}$. Since $u_o'x_o = 1$, between its minimum and maximum value lies the value $u_1'x_o$. Hence

$$\min_r \frac{u_{1r}}{u_{or}} \leq u_1'x_o \leq \max_r \frac{u_{1r}}{u_{or}} .$$

Similarly, interchanging 0 and 1,

$$\min_r \frac{u_{or}}{u_{1r}} \leq u_o'x_1 \leq \max_r \frac{u_{or}}{u_{1r}} .$$

But

$$\min_r \frac{u_{or}}{u_{1r}} = \frac{1}{\max_r \frac{u_{1r}}{u_{or}}} ,$$

and similarly, with o and 1 interchanged. It follows that

$$\min_r \frac{u_{1r}}{u_{or}} \leq \frac{1}{u_o'x_1} \leq \max_r \frac{u_{1r}}{u_{or}} .$$

Accordingly,

$$\min_r \frac{u_{1r}}{u_{or}} \leq u_1{}'x_o \ , \quad \frac{1}{u_o{}'x_1} \leq \max_r \frac{u_{1r}}{u_{or}} \ ,$$

showing common limits for both the Laspeyre and Paasche indices $L_{1o} = u_1{}'x_o$ and $P_{1o} = \frac{1}{u_o{}'x_1}$.

Let

$$\check{M}_{1o} = \min_r \frac{u_{1r}}{u_{or}} = \frac{p_o{}'x_o}{p_1{}'x_1} \min_r \frac{p_{1r}}{p_{or}}$$

and

$$\hat{M}_{1o} = \max_r \frac{u_{1r}}{u_{or}} = \frac{p_o{}'x_o}{p_1{}'x_1} \max_r \frac{p_{1r}}{p_{or}}$$

so that

$$\check{M}_{1o} \leq \hat{M}_{1o} \ ,$$

and

$$\check{M}_{1o} = \hat{M}_{1o}$$

if and only if the prices p_o are proportional to the prices p_1 .

The condition $u_o \leq u_1$ is equivalent to the condition $1 \leq M_{1o}$. Hence neither $u_o \leq u_1$ nor $u_o \geq u_1$ means $\check{M}_{1o} \leq 1 \leq \hat{M}_{1o}$.

The case $u_o{}'x_1$ or $u_o \leq u_1$, in which $\rho_{1o} = \check{M}_{1o}$, thus requires

$$\check{M}_{1o} \leq 1 < \hat{M}_{1o} \ , \text{ or } \ 1 \leq \check{M}_{1o} \ ,$$

and the contrary case $u_o{}'x_1 < 1$ and $u_o \nleq u_1$, in which $\hat{\rho}_{1o} = u_1{}'x_o > 1$ and $\check{\rho}_{1o} = 1$, requires

$$M_{1o} \leq 1 \leq u_1{}'x_o \leq \hat{M}_{1o} \ .$$

If the <u>spread</u> of ρ_{1o} is defined as the length of the interval $\check{\rho}_{1o}, \hat{\rho}_{1o}$, thus,

$$\text{spread } \rho_{1o} = \hat{\rho}_{1o} - \check{\rho}_{1o} \ ,$$

then, in each case,

$$\text{spread } \rho_{1o} \leq \hat{M}_{1o} - \check{M}_{1o} \ .$$

The spread is thus zero, or ρ_{10} is completely determined, if the prices p_o and p_1 are proportional. In this case, moreover, the quantities $\check{M}_{1o}$, $u_1{}'x_o$, $\frac{1}{u_o{}'x_1}$, $\hat{M}_{1o}$ and the now determinate values of ρ_{1o} and $\frac{1}{\rho_{o1}}$, are all equal.

7. Generalization

Instead of asking for the range of admissible values of an index ρ_{1o} between two occasions, based just on the price and quantities data for those occasions, and the preference maps admitted by them, it is possible, more generally, to ask for this in regard to such data for any number of further occasions. As more data are included, the class of admissible preference maps becomes more confined, and there is a corresponding narrowing of the range of admissible values of an index.

To find this more generally determined range is an elaborate problem.[8] Just the general result will be stated here, and it will be shown how it contains as a special case the results just obtained.

Let (p_r, x_r) be the price and quantity data for occasion r $(r = 1, \ldots, k)$. Let $e_r = p_r{}'x_r$ and $u_r = \frac{p_r}{e_r}$. Let $D_{rs} = u_r{}'x_s$, and let (Λ, Φ) , where $\Lambda = \{\lambda_r\}$, $\Phi = \{\varphi_r\}$, denote any non-trivial solution of the system of inequalities

$$\lambda_r \geq 0 \; ; \quad \lambda_r D_{rs} \geq \varphi_s - \varphi_r \quad (r, s = 1, \ldots, k) .$$

Let

$$\check{\rho}_{rs}(\Lambda, \Phi) = \min_x \{u_r{}'x \mid u_t{}'x \geq 1 + \frac{\varphi_s - \varphi_t}{\lambda_t} , t = 1, \ldots, k\}$$

and

$$\check{\rho}_{rs} = \min_{\Lambda,\Phi} \check{\rho}_{rs}(\Lambda,\Phi) .$$

Now, with any $\alpha = \{\alpha_r\}$ such that $\alpha_r \geq 0$, $\sum_r \alpha_r = 1$, let

[8] Afriat, op. cit.

$x_\alpha = \sum_r x_r \alpha_r$ and $\varphi_\alpha = \Sigma \varphi_r \alpha_r$. Let

$$\hat{\rho}_{rs}(\Lambda, \Phi) = \min_\alpha \{u_r{}'x_\alpha \mid \varphi_\alpha \geq \varphi_s\}$$

and let

$$\hat{\rho}_{rs} = \max_{\Lambda,\Phi} \hat{\rho}_{rs}(\Lambda, \Phi) .$$

Then $\check{\rho}_{rs}$ and $\hat{\rho}_{rs}$ are the lower and upper limits for the index ρ_{rs}, with s and r as base and object occasions, determined relative to all the preference maps which are admissible on the given data. The condition

$$D_{rs} \leq 0 , D_{st} \leq 0 , \ldots, D_{qr} \leq 0 \Longrightarrow D_{rs} = D_{st} = \ldots = D_{qr} = 0 ,$$

which generalizes the condition already stated for the case of two occasions, and is a weakened form of the revealed preference axiom of Houthakker, is necessary and sufficient for the existence of any such maps, and also for the existence of any solutions (Λ, Φ) of the systems of inequalities which enter into these formulae.

Now, in the case of two occasions $r = 0, 1$,

$$\check{\rho}_{10}(\Lambda, \Phi) = \min_x \{u_1{}'x \mid u_o{}'x \geq 1 , u_1{}'x \geq 1 + \frac{\varphi_o - \varphi_1}{\lambda_1} \}$$

where

$$\lambda_o D_{o1} \geq \varphi_1 - \varphi_o \geq - \lambda_1 D_{1o} ,$$

or equivalently,

$$u_1{}'x_o \geq 1 + \frac{\varphi_o - \varphi_1}{\lambda_1} \geq - \frac{\lambda_o}{\lambda_1} D_{o1} .$$

Therefore,

$$\check{\rho}_{1o} = \min_x \{u_1{}'x \mid u_o{}'x \geq 1 , u_1{}'x \geq \boldsymbol{\theta}\}$$

where

$$\begin{aligned} \boldsymbol{\theta} &= \inf \left(1 + \frac{\varphi_o - \varphi_1}{\lambda_1}\right. \\ &= \inf \left(1 - \frac{\lambda_o}{\lambda_1} D_{o1}\right) . \end{aligned}$$

If $D_{o1} \leq 0$, then $\theta = 1$; in which case

$$\check{\rho}_{1o} = \min_x \{u_1'x \mid u_o'x \geq 1\} .$$

Hence, if $D_{o1} \leq 0$, it follows that if $u_o \leq u_1$ then ρ_{1o} has the same expression as for the case $D_{o1} > 1$, and otherwise has the value 1 . Thus, the general formula for ρ_{1o} contains as a special case the results which have just been established.

Consider now the expression

$$\hat{\rho}_{1o}(\Lambda, \Phi) = \min_\alpha \{u_1'x_\alpha \mid \varphi_\alpha \geq \varphi_o\}$$

with

$$x_\alpha = x_o(1 - \alpha) + x_1\alpha , \quad \varphi_\alpha = \varphi_o(1 - \alpha) + \varphi_1\alpha ,$$

and hence

$$u_1'x_\alpha = u_1'x_o(1 - \alpha) + \alpha = (1 - u_1'x_o)\alpha + u_1'x_o ,$$

where $\alpha \geq 0$. It is required to evaluate the minimum of $u_1'x_\alpha$ subject to $\varphi_\alpha \geq \varphi_o$, that is, subject to

$$(\varphi_1 - \varphi_o)\alpha \geq 0 .$$

If $u_1'x_o \leq 1$, then, since $\lambda_1(u_1'x_o - 1) \geq \varphi_o - \varphi_1$, where $\lambda_1 \geq 0$, it follows that $\varphi_1 - \varphi_o \geq 0$, so that α is unrestricted, and $u_1'x_\alpha$, which decreases as α decreases, attains its minimum value $u_1'x_o$ when α has its minimum value 0 . In this case, always $\hat{\rho}_{1o}(\Lambda, \Phi) = u_1'x_o$, and therefore $\hat{\rho}_{1o} = u_1'x_o$. Now consider the case $u_1'x_o \geq 1$, and distinguish two classes of solutions, such that $\varphi_1 < \varphi_o$, and $\varphi_1 \geq \varphi_o$. For the first class, necessarily $\alpha = 0$, and thus $\hat{\rho}_{1o}(\Lambda, \Phi) = u_1'x_o$. For the second class, α is unrestricted. Since $u_1'x_\alpha$ is non-decreasing as α increases, its minimum value 1 is attained for $\alpha = 1$; so that $\hat{\rho}_{1o}(\Lambda, \Phi) = 1$.

Thus, in the case $u_1'x_o \geq 1$, the only values of $\hat{\rho}_{1o}(\Lambda, \Phi)$ are $u_1'x_o$ and 1 , and hence

$$\hat{\rho}_{1o} = \max_{\Lambda, \Phi} \rho_{1o}(\Lambda, \Phi)$$

$$= \max \{u_1{}'x_o, 1\}$$

$$= u_1{}'x_o .$$

Thus in all cases $\hat{\rho}_{1o} = u_1{}'x_o$, in accordance with the result already obtained.

The general formulae for $\check{\rho}_{rs}$ and $\hat{\rho}_{rs}$ which have been given can be shown to reduce to simpler formulae which generalize those obtained for data of two occasions. But they have a further importance, in that they are based on a general method of constructing and characterizing the entire class of preference maps which are admissible on any consistent data, or which are approximately admissible on any approximately consistent data. This opens the way to a further development of the index number problem, in the same framework as the rigid method of limits, but of a statistical character.