A NOTE ON UTILITY AND ATTITUDES TO RISK

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1. Most theories of economic decision-making under uncertainty recognize that the value of an uncertain prospect may differ from its "actuarial value", or more generally that the utility of the prospect differs from the utility of the mathematical expectation. For instance the certainty of receiving a gain A may have a utility different from the prospect of gaining either 0 or $2A$ with equal probability.

This idea can be formalized by assuming that the utility of a prospect is given by a function

$$f(E,V)$$

where

E = the mathematical expectation of the gain

V = the variance of the gain.

This model has been used extensively by Allais (2), Farrar (5), and Markowitz (7), (8) and appears to be extremely popular, particularly in studies of investment and portfolio selection.

2. None of the authors who have used this model appear to have investigated the conditions under which such a function can exist, and they usually avoid specific assumptions as to the shape of $f(E,V)$. In general only the following assumptions are made explicitly:

$$\frac{\partial f}{\partial E} > 0, \qquad \frac{\partial f}{\partial V} < 0 \ . \tag{1}$$

The latter inequality states that uncertainty, measured by the variance is undesirable, or that the decision-maker has an aversion to risk. Allais (2) explicitly recognizes that the inequality has to be reversed for a gambler, i.e., for a person who has a preference for risk.

A tacit assumption usually made by these authors is: For any positive E, there exists a number V_0 so that

(2) $$f(E,V) < f(0,0) \qquad \text{for } V > V_0 \ .$$

This means that some prospects are so risky that they are less desirable than the certainty of a gain equal to zero.

3. A more pedestrian approach to the problem is to consider only prospects of the simplest possible kind, i.e. "lottery tickets" with two prizes.

Let such a ticket give the prize x with probability p, and the "prize" 0 with probability $1-p$. One can then postulate that the utility of this ticket is given by a function

$$g(x,p).$$

Authors who use this model generally assume

(3) $$\frac{\partial g}{\partial x} > 0, \quad \frac{\partial g}{\partial p} > 0 \qquad \text{for } x > 0 \text{ and } 0 < p \leq 1.$$

These assumptions appear eminently sensible. They are in fact almost identical with any reasonable definition of "rationality" in assigning preferences. However, as we shall show in the following, assumption (3) contradicts (2).

4. If a function F(E,V) exists for all prospects, it must of course also exist for the special class of prospects which we referred to as lottery tickets, and it must assign the same value as g(x,p) to any prospect of this class.

For a lottery ticket we have

$$E = px \qquad \text{and} \qquad V = p(1-p)x^2 \;;$$

hence it follows that

$$g(x,p) = f(px,\; p(1-p)x^2),$$

and

$$g(0,0) = f(0,0) \;.$$

It is easy to see that

$$x = \frac{E^2 + V}{E} \qquad \text{and} \qquad p = \frac{E^2}{E^2 + V}$$

so that we also have

$$g\left(\frac{E^2 + V}{E},\; \frac{E^2}{E^2 + V} \right) = f(E,V) \;.$$

From assumption (3) it follows that it is not possible for an arbitrary positive E to find a V such that

$$f(E,V) < f(0,0) = g(0,0) \;.$$

This is a contradiction of assumption (2).

5. Having posed a paradox, we shall now try to resolve it.

It is clear that our problem in general is to establish a preference ordering over the set of all probability distributions $F(x)$, and to represent this ordering by a utility functional $U(F)$.

Von Neumann amd Morgenstern (9) have proved that any preference ordering which satisfies certain consistency conditions can be represented by the utility functional

$$U(F) = \int_{-\infty}^{+\infty} u(x)dF(x) .$$

In this expression the function $u(x)$ can be interpreted as the classical "utility of money", i.e., the utility attached to an amount of money equal to x.

A more formal definition is:

$$u(a) = U(F) \quad \text{if} \quad F(x) = \epsilon(x-a)$$

i.e., if $F(x)$ degenerates to that the whole mass of probability is concentrated in the point $x = a$.

6. The assumption in paragraph 1 implies that we must have

$$f(E,V) = \int_{-\infty}^{+\infty} u(x)dF(x) .$$

If $F(x)$ is an arbitrary probability distribution, the right-hand side can be a function of E and V alone only if $u(x)$ is a polynomial of second degree.

Since $u(x)$ is determined only up to a positive linear transformation, there is no loss of generality in assuming that $u(x) = x - ax^2$, so that we obtain

$$f(E,V) = \int_{-\infty}^{+\infty} (x - ax^2)dF(x) = E - aE^2 - aV$$

where $a > 0$ in order to satisfy the second inequality in condition (1).

This is the only form the function $f(E,V)$ can have if the consistency requirements of von Neumann and Morgenstern shall be fulfilled. These requirements are not sacrosant, but most authors who have assumed the existence of a function $f(E,V)$, with the eloquent expception of Allais (1), seem to assume tacitly or explicitly that the requirements must hold.

7. It is now quite clear why we reached an apparant paradox in paragraph 4. The function $u(x) = x - ax^2$ gives a reasonable representation of "utility of money" for $x < \frac{1}{2a}$. For $x > \frac{1}{2a}$ utility will decrease with increasing amounts of money. This is a contradiction of condition (3), and of our intuitive feelings of what utility of money should be.

If we only consider probability distributions which satisfy the condition $F(\frac{1}{2a}) = 1$, i.e., if the stochastic variables we deal with can take values only in the interval $-\infty \leq x \leq \frac{1}{2a}$, we will encounter no contradictions by assuming that $u(x) = x - ax^2$.

It has been shown in another paper (3) that it is fairly safe to assume that this condition holds for an insurance company. An insurance company can never make profits exceeding the gross premium income, and there will in general be no lower limit to the losses which the company may suffer. However, in any situation where large positive gains are possible, we will run into contradictions if we assume that the utility of a prospect can be represented by a function $f(E,V)$.

8. If we consider only probability distributions which are completely specified by the two first moments, it is clear that a representation

$$\int_{-\infty}^{+\infty} u(x)dF(x) = f(E,V)$$

will exist for any utility function for which the integral converges.

A case of this kind has been studied by Freund (6), who assumed that $F(x)$ is a normal distribution, and that $u(x) = 1 - e^{-x}$.

9. It appears that even if the "EV-criterion" in principle can lead to nonsense decisions, the method can be applied as a useful approximation to a wide range of problems. The most impressive example is probably Farrar's (5) analysis of the portfolios held by investment trusts.

The obvious limitation of the method is that only the two first moments are considered when the ordering is established over the set of probability distributions. There seems to be good reason for bringing in the third moment, and thus assuming that the skewness of the probability distributions is considered in decision-making under uncertainty. This will mean that the utility function $u(x)$ can be polynomial of third degree, and this class of functions agrees fairly well with our intuitive ideas of the "utility of money". A model in which $u(x) = (x-a)^3$ has been discussed in some detail in another paper (4), and seems to lead to reasonable results.

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