Review for Midterm 2

Problem 1.

Gasoline is restocked in a large tank once at the beginning of each week and then sold to individual customers. Let X denote the proportion of the capacity of the tank that is available after it is stocked at the beginning of the week. X varies from week to week. Let *Y* denote the proportion of the capacity of the tank that is sold during the week. Suppose the joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases} 3x, & 0 \le y \le x \le 1 \\ 0, & elsewhere \end{cases}$$

a) Find the probability that the tank will be less than half-filled at the beginning of the week, but more than ¼ of the tank will be sold (i.e. find the probability that

   $0 \le X \le .5$ and $Y \ge .25$)

b) The random variable $X - Y$ denotes the proportional amount of gasoline remaining at the end of the week. Find $\mathbf{E}(X - Y)$.

c) Find the variance of $X - Y$.

Solution.

a)

$$\mathbf{P}\left(0 \le X \le .5,\ \ Y \ge .25\right) = \iint_A f(x,y)\ \ dx\ \ dy$$

$$= \int_{1/4}^{1/2}\int_{1/4}^{x} 3x\ \ dx\ \ dy$$

$$= \int_{1/4}^{1/2} 3x\ \left(y\right)\Big|_{1/4}^{x}\ \ dx$$

$$= \int_{1/4}^{1/2} 3x\left(x - 1/4\right)\ \ dx$$

$$= \left(x^3 - \frac{3}{8}x^2\right)\Bigg|_{1/4}^{1/2}$$

$$= \left(\frac{1}{8} - \frac{3}{8}\cdot\frac{1}{4}\right) - \left(\frac{1}{64} - \frac{3}{8}\cdot\frac{1}{16}\right)$$

$$= \frac{5}{128}$$

b) $\mathbf{E}\left(X - Y\right) = \mathbf{E}\left(X\right) + \mathbf{E}\left(-Y\right) = \mathbf{E}\left(X\right) - \mathbf{E}\left(Y\right)$

$$\mathbf{E}\left(X\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\cdot f(x,y)\ \ dx\ \ dy = \int_0^1\int_0^x x\ \ \left(3x\ \right)\ \ dx\ \ dy = \int_0^1 x\cdot\left(3x^2\right)\ \ dx = \frac{3}{4}$$

$$\mathbf{E}\left(Y\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\cdot f(x,y)\ \ dx\ \ dy = \int_0^1\int_0^x y\ \ \left(3x\right)\ \ dx\ \ dy = \int_0^1 3x\left(\frac{y^2}{2}\Bigg|_0^x\right)dx = \int_0^1 \frac{3}{2}x^3 dx = \frac{3}{8}$$

$$\mathbf{E}\left(X - Y\right) = \mathbf{E}\left(X\right) - \mathbf{E}\left(Y\right) = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$$

c) $\mathbf{V}\left(X - Y\right) = \mathbf{V}\left(X\right) + \mathbf{V}(Y) - 2\,\mathrm{Cov}(X,Y)$

Let us first find the marginal distributions of X and Y

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\ \ dy = \int_0^x 3x dy = 3x^2$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\ \ dx = \int_y^1 3x dx = \frac{3x^2}{2}\Bigg|_y^1 = \frac{3}{2}\left(1 - y^2\right)$$

Now we can compute the variances of X and Y

$$\mathbf{V}\left(X\right) = \mathbf{E}\left(X^2\right) - \left[\mathbf{E}\left(X\right)\right]^2 \qquad\qquad \mathbf{V}\left(Y\right) = \mathbf{E}\left(Y^2\right) - \left[\mathbf{E}\left(Y\right)\right]^2$$

$$\mathbf{E}\left(X^2\right)=\int_{-\infty}^{+\infty}x^2 f_X(x)\ \ dx=\int_0^1 x^2\cdot 3x^2dx=3\int_0^1 x^4dx=\left.\frac{3x^5}{5}\right|_0^1=\frac{3}{5}$$

$$\mathbf{E}\left(Y^2\right)=\int_{-\infty}^{+\infty}y^2 f_Y(y)\ \ dy=\int_0^1 y^2\cdot\frac{3}{2}\left(1-y^2\right)dy=\frac{3}{2}\int_0^1\left(y^2-y^4\right)dy=\frac{3}{2}\left.\left(\frac{y^3}{3}-\frac{y^5}{5}\right)\right|_0^1=\frac{1}{5}$$

$$\mathbf{V}(X)=\frac{3}{5}-\left(\frac{3}{4}\right)^2=.04\qquad \mathbf{V}(Y)=\frac{1}{5}-\left(\frac{3}{8}\right)^2=.06$$

We also have to compute the covariance of X and Y

$$\mathrm{Cov}(X,Y)=\mathbf{E}(XY)-\mathbf{E}(X)\cdot\mathbf{E}(Y)$$

$$E(XY)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf(x,y)\ \ dx\,dy=\int_0^1\int_0^x xy\cdot 3x\ \ dy\ \ dx=\int_0^1 3x^2\left(\left.\frac{y^2}{2}\right|_0^x\right)dx=\int_0^1\frac{3}{2}x^4dx=\frac{3}{2}\frac{x^5}{5}=\frac{3}{10}$$

$$\mathrm{Cov}(X,Y)=\mathbf{E}(XY)-\mathbf{E}(X)\cdot\mathbf{E}(Y)=\frac{3}{10}-\frac{3}{4}\cdot\frac{3}{8}=.30-.28=.02$$

We now substitute everything in to get a final formula

$$\mathbf{V}(X-Y)=\mathbf{V}(X)+\mathbf{V}(Y)-2\,\mathrm{Cov}(X,Y)=.04+.06-2\cdot(.02)=.06\,.$$

Problem 2.

The reaction of an individual to a stimulus in a psychological experiment may take one of two forms: A or B. In a selected sample size of 230 individuals, 138 reacted in manner A.

a) Compute the 90% confidence interval for the true proportion of the individuals reacting in manner A to the stimulus in the experiment.

b) If the desired length of the 90% interval is .05, what sample size is necessary to ensure this?

Solution.

a) A 100(1-α)% confidence interval for a large sample of the population with population proportion p is $\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}$. In our case, $\hat{p} = \frac{138}{230} = 0.600$, so the 90% CI for p is

$$.600 \pm 1.645\left[\frac{(.6)(.4)}{230}\right]^{1/2} = 0.600 \pm 0.053 = (.547,\ \ .653).$$

b) Using $\hat{p} = \hat{q} = .5$, $L = 2z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n} = 2 \cdot 1.645\sqrt{\frac{(.5)(.5)}{n}} = 0.05$, which yields

$$n = \frac{4 \cdot (1.645)^2 \cdot 0.25}{(0.05)^2} = 1082\,.$$

Problem 3.

The flatwise bonding strength (lb./in.) was determined for dowel joints of two different types used in wood furniture frame construction, with the second type having greater rail thickness (1.5 in.) than the first type (1.25in.).

Rail Thickness	Sample size	Sample Mean	Sample St. Dev.
1.25 in.	m = 10	$\bar{x} = 1215.6$	$s_1 = 137.4$
1.5 in.	n = 10	$\bar{y} = 1376.4$	$s_2 = 155.0$

a) Does the larger rail thickness of the second type of dowel result in the greater true average bonding strength than for dowels of the first type? Test the relevant hypotheses at the level .05

b) Give upper and lower bounds on the P-value of the data.

Solution.

a) 1. Parameter of interest: $\mu_1 - \mu_2$.

2. H_0: $\mu_1 - \mu_2 = 0$

3. H_a: $\mu_1 - \mu_2 < 0$

4. Test statistics: $t = \dfrac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{m} + \frac{1}{n}}}$, where $s_p^2 = \dfrac{m-1}{m+n-2}s_1^2 + \dfrac{n-1}{m+n-2}s_2^2$

5. Reject H_0 if $t \le -t_{0.5,18}$, or $t \le -1.734$

6. $s_p^2 = \dfrac{10-1}{10+10-2}s_1^2 + \dfrac{10-1}{10+10-2}s_2^2 = \dfrac{s_1^2 + s_2^2}{2} = \dfrac{(137.4)^2 + (155.0)^2}{2} = 21{,}451.88$

$s_p = \sqrt{21{,}451.88} = 146.46$

$t = \dfrac{1215.6 - 1376.4}{146.46\sqrt{\frac{1}{10} + \frac{1}{10}}} = \dfrac{-160.8}{65.50} = -2.45 \le -1.734$

7. H_0 is rejected; the second type does appear to have greater true average bonding strength.

b) Because $t_{.025,\,18} < 2.45 < t_{.01,\,18}$, $.01 < P < .025$.

Problem 4.

A firm purchases two types of industrial chemicals. Type I chemical costs a = 3 dollars per gallon, whereas type II chemical costs b = 5 dollars per gallon. The mean and the variance of the amount (in gallons) of type I chemical purchased X, are 40 and 4 respectively. The amount of type II chemical purchased, Y, has **E**(Y)=65 and **V**(Y)=8.

a) Assume that X and Y are independent, and find the mean and the variance of the total amount of money T spent per week on the two chemicals.

b) Suppose now that the costs of type I and type II chemicals are random variables. Denoting them by A and B, respectively, assume that **E**(A)=2, **V**(A)=0.1, **E**(B)=6,

$\mathbf{V}$(B)=0.2, and that A, B, X and Y are independent of one another. What are the new mean and variance of T?

<u>Solution</u>

a) $T = aX + bY$

$\mathbf{E}(T) = \mathbf{E}(aX + bY) = \mathbf{E}(aX) + \mathbf{E}(bY) = a\mathbf{E}(X) + b\mathbf{E}(Y) = 3 \cdot 40 + 5 \cdot 65 = 445$, note that we don't have to assume independence of X and Y in order to compute the expected value of the linear combination. We will need the independence of X and Y to calculate the variance of T.

Under the assumption of independence, we get

$\mathbf{V}(T) = \mathbf{V}(aX + bY) = \mathbf{V}(aX) + \mathbf{V}(bY) = a^2\mathbf{V}(X) + b^2\mathbf{V}(Y) = 9 \cdot 4 + 25 \cdot 8 = 236$

b) Assuming the independence of A, B, X, and Y, we get

$\mathbf{E}(T) = \mathbf{E}(AX + BY) = \mathbf{E}(AX) + \mathbf{E}(BY) = \mathbf{E}(A)\mathbf{E}(X) + \mathbf{E}(B)\mathbf{E}(Y) = 2 \cdot 40 + 6 \cdot 65 = 470$

Computing the variance of T:

$\mathbf{V}(T) = \mathbf{V}(AX + BY) = \mathbf{V}(AX) + \mathbf{V}(BY)$

$$
\begin{aligned}
\mathbf{V}(AX) &= \mathbf{E}(A^2X^2) - [\mathbf{E}(AX)]^2 \\
&= \mathbf{E}(A^2)\mathbf{E}(X^2) - [\mathbf{E}(A)]^2 \cdot [\mathbf{E}(X)]^2 \\
&= \left[\mathbf{V}(A) + (\mathbf{E}(A))^2\right] \cdot \left[\mathbf{V}(X) + (\mathbf{E}(X))^2\right] - [\mathbf{E}(A)]^2 \cdot [\mathbf{E}(X)]^2 \\
&= (.1 + 2^2)(4 + 40^2) - 2^2 \cdot 40^2 \\
&= 176.4
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\mathbf{V}(BY) &= \mathbf{E}(B^2Y^2) - [\mathbf{E}(BY)]^2 \\
&= \mathbf{E}(B^2)\mathbf{E}(Y^2) - [\mathbf{E}(B)]^2 \cdot [\mathbf{E}(Y)]^2 \\
&= \left[\mathbf{V}(B) + (\mathbf{E}(B))^2\right] \cdot \left[\mathbf{V}(Y) + (\mathbf{E}(Y))^2\right] - [\mathbf{E}(B)]^2 \cdot [\mathbf{E}(Y)]^2 \\
&= (.2 + 6^2)(8 + 65^2) - 6^2 \cdot 65^2 \\
&= 1134.6
\end{aligned}
$$

So, we get

$\mathbf{V}(T) = \mathbf{V}(AX + BY) = \mathbf{V}(AX) + \mathbf{V}(BY) = 175.4 + 1134.6 = 1310$.