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Betweenness Satisfying Preferences and Dynamic Choice*

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We consider the dynamic choice problem where uncertainty is resolved gradually (i.e., decision trees). We impose *consistency* on the decision maker by requiring that his behavior in trees conform to preference maximization over lotteries. We formulate and defend three different requirements on dynamic behavior and show that for any decision maker whose behavior is consistent with maximizing a continuous, monotone preference relation over lotteries, each one of these conditions is equivalent to the betweenness property of the underlying preferences. *Journal of Economic Literature* Classification Number: 026. © 1990 Academic Press, Inc.

1. INTRODUCTION

In this paper we discuss decision making in situations where uncertainty resolves in multiple stages. We call the objects of our study *decision trees.* As usual, the term refers to problems in which a decision is followed either by a final (deterministic) outcome, a probability distribution over them, or a partial resolution of uncertainty, which in turn is followed by another decision, etc. We limit ourselves to problems where uncertainty is removed in two stages, hence the name *two-stage decision trees*, but our approach extends immediately to trees with more than two stages. Different choices in the *decision nodes* of the tree induce different probability distributions, or lotteries, on the final outcomes. This allows us to view the decision problem as one of choosing actions at each decision node so as to obtain the most preferred lottery available in the tree.

It has been suggested[1] who decision makers that are not expected utility maximizers, in particular, those who violate the independence axiom "will

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[1] See Boder [1], Green [6], LaValle and Fishburn [10], Markowitz [15], Raiffa [16] who refers to a similar argument by Schlaifer, Seidenfeld [17], Shafer [18], and Yaari [19].

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be dynamically inconsistent in a manner which leaves them vulnerable to 'making book against themselves.'"[2] However, Machina argues that "such arguments implicitly invoke the additional assumption of consequentialism in the sense of Hammond, which is essentially a dynamic version of the very separability that non-expected utility maximizers reject."[3] Machina's argument is quite compelling; given the obvious desirability of dynamic consistency and the substantial body of empirical and theoretical work challenging the expected utility hypothesis, it would appear that consequentialism is not a particularly attractive restriction on behavior involving dynamic choice. The debate, however, suggests the possibility that an analysis of the dynamic choice problem might lead to restrictions on the set of acceptable (static) preferences over lotteries. The purpose of this paper is to explore that possibility. Specifically, Section 2 provides the definitions of the dynamic choice problem and dynamic consistency and consequentialism. Theorem 1 is a statement of the observation due to Hammond [7] that dynamic consistency and consequentialism imply expected utility maximization. Then we discuss three different restrictions on dynamic choice behavior. The first is what we call *weak consequentialism.* It is motivated by Machina's discussion of why non-expected utility maximizers would not satisfy consequentialism. Machina argues "that an agent with non-expected utility/non-separable preferences feels risk which is borne but not realized is gone in the sense of having been *consumed* (or "borne"), rather that gone in the sense of *irrelevant.*"[4] But this suggests that behavior after a partial resolution of uncertainty should differ from initial behavior only to the extent that the risk which is not realized represents a genuine loss or gain to the decision maker in terms of the change in the choice set. To put it differently, if the opportunities available to the decision maker are not altered by the risk which is not realized (hence the risk is really no risk at all) then we might expect a rational decision maker to behave as he would have done initially. This requirement is made formal in our definition of weak consequentialism.

The second requirement is related to what is called *folding back* or *rolling back* decision trees. This term refers to a particular technique of solving decision trees. When this technique is used, first the optimal action choices at final decision nodes are determined; then these nodes are replaced by the values of the optimal actions. Hence a reduced decision tree is obtained. The process is repeated until the optimal strategy and value for the decision tree is computed. For this technique to be usable it is essential that optimal action choices at, say, two final decision nodes be computable

[2] Machina [14], abstract.
[3] Machina [14], abstract.
[4] Machina [14], p. 43.

independently. That is, the optimal choice at final decision node d_1 should not depend on the planned course of action at some other final decision node d'_1. This requirement we call *dynamic programming solvability.*[5]

The final requirement that we consider is motivated by much of the empirical work on the stability of the utility function. We will examine the extent to which the "von Neumann–Morgenstern utility function" can depend on the choice set of distributions over final wealth levels. *Status quo dependent expected utility maximization* formalizes this dependence.

Our main result (Theorem 2) establishes that given dynamic consistency and the continuity and monotonicity of the underlying preferences (in the sense of preferring stochastically dominating prospects), all of the above requirements of dynamic choice behavior are equivalent to the *betweenness* property of the underlying static preferences. This property is the requirement that an agent who is indifferent between two lotteries, say, p and q, will also be indifferent between p and any mixture of p and q.

Betweenness satisfying preferences have been studied extensively by Dekel [5]. Such preferences can resolve many of the observed violations of the expected utility hypothesis. Consider lotteries over three monetary prizes $x<y<z$. These lotteries can be plotted on a graph such as the ones depicted in Fig. 1. The coordinates p_x and p_z determine the probabilities of obtaining prizes x and z, respectively. Hence the probability of obtaining y is $1-p_x-p_z$. The indifference curves in Fig. 1a are parallel straight lines; hence these are expected utility preferences. The indifference curves in Fig. 1b are also straight lines but they are not parallel. These preferences satisfy betweenness but not the expected utility hypothesis. A person who strictly prefers lottery q to both p and r (Fig. 1c) could not have a straight line indifference curve through lottery q and hence would not satisfy betweenness. We take Theorem 2 to be a statement that if we wish to develop a theory of choice under uncertainty in a way which permits a reasonable (i.e., normative) extension to dynamic choice theory then it is necessary to restrict attention to betweenness satisfying preferences. In Section 3 we discuss possible procedures that a decision maker whose preferences satisfy betweenness might use to arrive at his choice. We argue that such procedures need not be substantially more complicated than those that would be used by an expected utility maximizer and offer some opportunity for explaining some behavior which is not consistent with preference maximization. Section 4 is a brief summary of our conclusions.

[5] Note that this is *not* the same concept as folding back in Machina [14]. Also, our rolling back criterion is weaker than that of LaValle and Wapman [11]. By requiring that the rule for rolling back the trees be independent of the particular tree that is presented LaValle and Wapman are implicitly imposing the same consequentialism that Machina argues against. We make no such requirement.

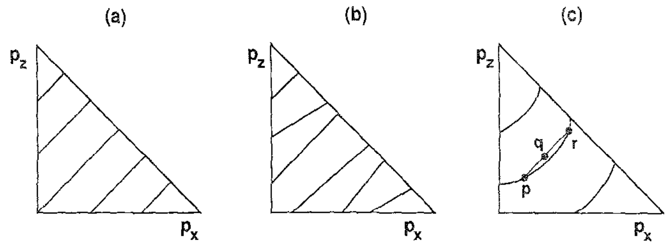

FIG. 1. Indifference curves corresponding to (a) expected utility preferences, (b) betweenness satisfying preferences, and (c) general preferences.

2. Two-Stage Decision Trees

Let $X=[w, b]$ for some $w, b \in \Re$, $w<b$. $\mathscr{L}_1$ is the set of all simple lotteries over X. That is, $p_1 \in \mathscr{L}_1$ if p_1 is a function from X to $[0, 1]$ such that $\operatorname{supp}(p_1) \equiv \{x \in X \mid p_1(x)>0\}$ is a finite set and $\sum_x p_1(x)=1$. A final decision node d_1 is a finite, non-empty subset of $\mathscr{L}_1$ and D_1 denotes the set of all final decision nodes. $\mathscr{L}_0$ is the set of all simple lotteries over D_1. An initial decision node d_0 is a finite, non-empty subset of $\mathscr{L}_0$, and D_0 denotes the set of all initial decision nodes. We sometimes refer to d_0 as a (two-stage) decision tree and to lotteries as chance nodes. We identify any degenerate lottery with the unique prize of that lottery. Hence we write $d_1 \in \mathscr{L}_0$ to denote $p_0 \in \mathscr{L}_0$ such that $p_0(d_1)=1$. Similarly, $x \in \mathscr{L}_1$ denotes $p_1 \in \mathscr{L}_1$ such that $p_1(x)=1$.

An example of a decision tree, d_0, is provided in Fig. 2a. Note that our notation requires that every decision node be followed by a chance node and every chance node be followed by either a decision node or a prize in X. This involves no loss of generality since a decision tree of the type depicted in Fig. 2b is naturally identified with d_0 of Fig. 2a. We sometimes use the abbreviated representation provided in Fig. 2c to depict d_0.

For any decision tree d_0, and any $p_0 \in d_0$, let $\Sigma_{p_0}(d_0)$ denote the set of all functions $\sigma\colon \operatorname{supp}(p_0) \to \mathscr{L}_1$ such that $\sigma(d_1) \in d_1$ for all $d_1 \in \operatorname{supp}(p_0)$. A strategy $\bar{\sigma}$ for the tree d_0 is a pair (p_0, σ) such that $p_0 \in d_0$ and $\sigma \in \Sigma_{p_0}(d_0)$. For $\bar{\sigma}=(p_0, \sigma)$ we refer to p_0 as the initial action and σ as the second stage action. Let $\Sigma(d_0)$ denote the set of all strategies for the decision tree d_0 and $\Sigma=\bigcup_{d_0 \in D_0} \Sigma(d_0)$. For any strategy $\bar{\sigma} \in \Sigma(d_0)$, let $s_{d_0}(\bar{\sigma})$ denote the lottery that is associated with employing the strategy $\bar{\sigma}$ in d_0. That is, $s_{d_0}(\bar{\sigma})=p_1 \in \mathscr{L}_1$ such that $p_1(x)=\sum_{d_1 \in \operatorname{supp}(p_0)} p_0(d_1)\sigma(d_1)(x)$, where $\bar{\sigma}=(p_0, \sigma)$. Let $S(d_0)=\{p_1 \in \mathscr{L}_1 \mid p_1=s_{d_0}(\bar{\sigma})$ for some $\bar{\sigma} \in \Sigma(d_0)\}$. Hence $S(d_0)$ is the set of all lotteries that can be obtained by employing different strategies available for d_0.

A decision maker, I, is a correspondence from D_0 to Σ such that

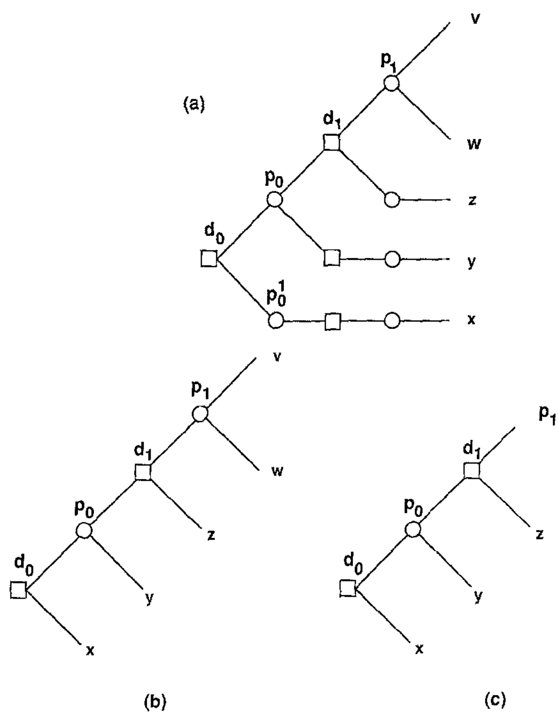

FIG. 2. An example of different representations of a decision tree d_0.

$I(d_0) \subset \Sigma(d_0)$ and $I(d_0) \neq \varnothing$ for all $d_0 \in D_0$. Let $S(d_0, I) = \{p_1 \in \mathscr{L}_1 \mid p_1 = s_{d_0}(\bar{\sigma})$ for some $\bar{\sigma} \in I(d_0)\}$. Obviously, $S(d_0, I) \subset S(d_0)$.

A decision maker I is said to be *consistent* iff there exists a preference relation (i.e., complete and transitive) $\succcurlyeq_I$ on $\mathscr{L}_1$ such that for all d_0, $I(d_0) = \{\bar{\sigma} \in \Sigma(d_0) \mid s_{d_0}(\bar{\sigma}) \succcurlyeq_I s_{d_0}(\bar{\sigma}')$ for all $\bar{\sigma}' \in \Sigma(d_0)\}$.

Hence a consistent decision maker is one whose choice of strategies for every d_0 is consistent with preference maximization. Since for every $\{p_1, q_1\} \subset \mathscr{L}_1$ we can construct a d_0 such that $S(d_0) = \{p_1, q_1\}$, $\succcurlyeq_I$ is uniquely determined for any consistent I.

Some authors who have studied the dynamic choice problem described above have concluded that if the behavior of the decision maker is to be consistent with preference maximization (that is, if I chooses those strategies in $\Sigma(d_0)$ which maximize some preference relation $\succcurlyeq$ (over $\mathscr{L}_1$)) then the implied preference relation must satisfy the independence axiom. But, they argue, given that the independence axiom together with a mild continuity requirement on $\succcurlyeq$ yields expected utility preferences, the only preferences which can be extended to dynamic choice problems are expected utility preferences.

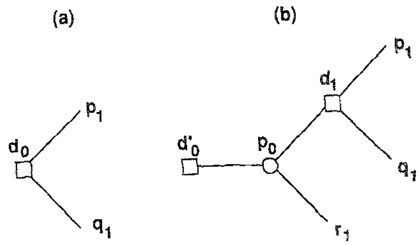

FIG. 3. An example illustrating a situation in which the independence axiom might be violated.

The argument is as follows: Consider the decision trees d_0 and d'_0 depicted in Fig. 3. If the decision maker is behaving as if he is maximizing some preference relation $\succcurlyeq$ over $\mathscr{L}_1$ and if he chooses p_1 in d_0 then he must choose p_1 in the tree d'_0. But this establishes that p_1 is preferred to q_1 implies $\alpha p_1 + (1-\alpha) r_1$ is preferred to $\alpha q_1 + (1-\alpha) r_1$ which is the independence axiom.

Machina [14] argues that this line of argument is flawed since it implicitly assumes *consequentialism*, that is, that if the final decision node d_1 $(= d_0)$ appears in two different trees d_0 and d'_0 then the optimal action at d_1 should be the same in both trees. The notion of consequentialism was first introduced by Hammond [7, 8]. Machina [14] argues that consequentialism is not a particularly compelling assumption within the context of the dynamic choice problem defined above and provides examples to illustrate why it might be violated. More precisely he argues that a person who is not an expected utility maximizer would not be consequentialist.

Alternatively, we can formulate the discussion above as follows:

(a) How is the dynamic choice problem defined by decision trees related to the (static) theory of preferences over $\mathscr{L}_1$?

(b) What are reasonable restrictions on the nature of the solutions to the dynamic choice problem?

(c) Given the answers to (a) and (b) above what restrictions are imposed on preferences over $\mathscr{L}_1$?

Machina's argument shows that even if one answers (a) by requiring that the decision maker behave *consistently* (i.e., as if he is maximizing a preference relation) then no restriction is imposed on the answer to (c) unless some further restriction is imposed on the answer to (b).

Below we present formal definitions of (monotone continuous) consistency (MCC) and consequentialism. Theorem 1 states that MCC and consequentialism imply expected utility maximization. Then we consider alternatives to consequentialism and investigate their implications for the underlying preference relation over $\mathscr{L}_1$ given MCC (Theorem 2).

I is said to be *monotone continuously consistent* (MCC) iff I is consistent and

(i) p_1 stochastically dominates q_1 implies $p_1 \succ_I q_1$

(ii) $\succcurlyeq_I$ is continuous, that is, $p_1 \succ_I q_1 \succ_I r_1$ implies there exist $\alpha, \beta \in (0, 1)$ such that $\alpha p_1 + (1-\alpha) r_1 \succ_I q_1 \succ_I \beta p_1 + (1-\beta) r_1$.

A decision maker I is said to be *consequentialist* iff there exists $I_*: D_1 \to D_1$ such that $I_*(d_1) \subset d_1$ for all $d_1 \in D_1$ and for all $d_0 \in D_0$, $(p_0, \sigma) \in I(d_0)$ implies $[(p_0, \sigma') \in I(d_0)$ iff $\sigma'(d_1) \in I_*(d_1)$ for all $d_1 \in \operatorname{supp}(p_0)]$. Thus for a conequentialist decision maker the optimal choice, $I(d_1)$, at any final decision node d_1 does not depend on the particular decision tree in which d_1 appears.

It is obvious from Fig. 3 that consistency and consequentialism implies that $\succcurlyeq_I$ satisfies the independence axiom. Then continuity and standard proofs of the von Neumann–Morgenstern theorem yield an expected utility representation of $\succcurlyeq_I$. This is stated in Theorem 1.

THEOREM 1 (Hammond). *If I is monotone continuously consistent and satisfies consequentialism then there exists a function $u: X \to \Re$ such that $p_1 \succcurlyeq_I q_1$ iff $\sum_{x \in \operatorname{supp}(p_1)} p_1(x) u(x) \geqslant \sum_{x \in \operatorname{supp}(q_1)} q_1(x) u(x)$.*

Proof. Omitted.

The first alternative to consequentialism we will consider is called *weak consequentialism* (WCON). Roughly, it requires that if, in some decision tree d_0, by employing various strategies the decision maker can choose among a set of lotteries $(S(d_0))$ and has some optimal strategy $\bar{\sigma}$ which leads to the lottery p_1 (i.e., $\bar{\sigma} \in I(d_0)$ and $p_1 = s_{d_0}(\bar{\sigma})$) and at some final decision node d_1 he confronts a subset, including p_1, of the choices that he had initially (i.e., $p_1 \in d_1 \subset S(d_0)$) then choosing p_1 at d_1 must be optimal. Consider the decision tree d_0 depicted in Fig. 4. Obviously there are three available strategies in d_0. Denote these by (a, h), (a, l), and (b). These strategies lead to the probability distributions $\alpha p_1 + (1-\alpha) q_1$, q_1, and p_1 respectively. Now, if (a, l) and (b) are optimal then (a, h) must also be optimal by WCON since at decision node d_1 the decision maker faces a choice between p_1 and q_1 (a subset of $S(d_0) = \{p_1, q_1, \alpha p_1 + (1-\alpha) q_1\}$) and therefore choosing p_1 again must be optimal. Observe that Fig. 4 shows that WCON implies that the preferences $\succcurlyeq_I$ are quasiconcave. Weak consequentialism addresses a major component of the criticism of consequentialism that might be developed along the lines of Machina [14],

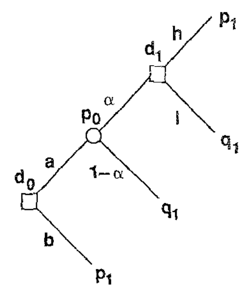

FIG. 4. By weak consequentialism, if p_1 is a preferred lottery at decision node d_0, it must be a preferred lottery at d_1 as well. In other words, if (a, l) and (b) are optimal strategies, (a, h) is also optimai.

namely those effects due to "elation" or "disappointment."[6] In explaining why the individual might behave differently in d_0 of Fig. 3a and d_1 of Fig. 3b Machina suggests that in arriving at d_1 in Fig. 3b the decision maker might be influenced by the experience (possibly disappointment or elation) of not receiving r_1, which has no counterpart in Fig. 3a. But it can be argued that whatever the decision maker experienced in reaching d_1 in Fig. 4 is of no consequence; the optimal choice p_1 is still available at d_1, hence there can be no disappointment at reaching d_1. Furthermore, the set of choices available at d_1 is a subset of the initial set of choices; hence there can be no elation.

Formally, I is said to be *weakly consequentialist* (WCON) iff for all d_0: $(p_0, \sigma) \in I(d_0)$, $d_1^* \in \text{supp}(p_0)$, $p_1^* \in d_1^* \subset S(d_0)$, $p_1^* \in S(d_0, I)$, $\sigma^*: \text{supp}(p_0) \to D_1$, $\sigma^*(d_1) = \sigma(d_1)$ for all $d_1 \neq d_1^*$, and $\sigma^*(d_1^*) = p_1^*$ implies $(p_0, \sigma^*) \in I(d_0)$.

Another possibility is to impose restrictions on I by imposing simplicity on the nature of the optimization problem that is to be solved by I such as being able to "fold back" or "roll back" the decision tree. In particular, if we wish to employ the techniques of dynamic programming it must be that the optimal decision at any final decision node should not depend on what decision is made at the other final decision nodes. Consider the following example:

Assume you have to decide how to get to work on Monday. Your options are to walk, to drive, to bike, or to take the bus. After some reflection you decide that the following two strategies are optimal: (1) drive if it rains, bike if it is sunny; (2) take the bus if it rains, walk if it is sunny.

[6] Arguments for violating even this weak version of consequentialism (such as those implicit in the "Mom" example of Machina [14, pp. 35–38]) appear to be intimately connected with equity and fairness considerations arising from the indivisibility of the prize(s) and are perhaps of less relevance for a theory of rational behavior under uncertainty.

Then, by dynamic programming solvability, the strategy "drive if it rains, walk if it is sunny" must also be optimal. Hence we say:

I is *dynamic programming solvable* (DPS) iff for all $d_0 \in D_0$, $(p_0, \sigma) \in I(d_0)$, $(p_0, \sigma') \in I(d_0)$, $\sigma^*\colon \operatorname{supp}(p_0) \to D_1$ such that $\sigma^*(d_1) \in \{\sigma(d_1), \sigma'(d_1)\}$ for all $d_1 \in \operatorname{supp}(p_0)$ implies $(p_0, \sigma^*) \in I(d_0)$.

Another possible simplicity requirement on the nature of the optimization problem that is confronted by the decision maker is a condition that is often invoked in evaluating evidence against expected utility theory. Machina [13] in discussing Markowitz [15] and other experimental work notes that the change in choice behavior as the initial or customary wealth level is varied is not consistent with a stable utility function over distributions of final wealth. In particular, he mentions the Markowitz hypothesis which states that the inflection point of the utility curve is always at the customary or status quo level of wealth. However, since we require that individual behavior be consistent with preference maximization (over distributions on final wealth levels) we can allow the decision maker to condition on the status quo only when it influences the choice set of distributions over final wealth levels.

Thus we are lead to a model of *status quo dependent expected utility maximization*, where the individual associates a utility function with each decision tree and maximizes utility. Formally,

I is a *status quo dependent expected utility maximizer* (SQDEUM) iff there exists a function $u\colon X \times D_0 \to \mathfrak{R}$ such that for all $d_0 \in D_0$, $I(d_0) = \arg\max_{\bar\sigma \in \Sigma(d_0)} \sum_x u(x, d_0) s_{d_0}(\bar\sigma)(x)$.

Finally we will consider an explicit restriction on $\succcurlyeq_I$. A preference relation $\succcurlyeq_I$ satisfies *betweenness* iff $p_1 \sim_I q_1$ implies $\alpha p_1 + (1-\alpha) q_1 \sim_I p_1$ for all $p_1, q_1 \in \mathscr{L}_1$ and $\alpha \in [0, 1]$.

THEOREM 2. *For any MCC decision maker I the following conditions are equivalent*:

(i) *I is weakly consequentialist*;

(ii) *I is dynamic programming solvable*;

(iii) *I is a status quo dependent expected utility maximizer*;

(iv) *$\succcurlyeq_I$ satisfies betweenness.*

Proof. See appendix.

Theorem 2 establishes that a variety of different and apparently appealing conditions on I are all equivalent to the betweenness of $\succcurlyeq_I$ whenever $\succcurlyeq_I$ is well-defined and satisfies monotonicity and continuity. Preferences which satisfy betweenness, monotonicity, and continuity have been studied extensively by Dekel [5]. Theorem 2 suggests that if we wish to concentrate on static preferences which can be reasonably extended to dynamic choice problems then we must restrict attention to preferences which satisfy Dekel's conditions. In what follows we will provide additional arguments for restricting attention to betweenness preferences.

3. Approximate Solutions and Status Quo Dependent Solutions to the Preference Maximization Problem

Dekel [5] has shown that if a preference relation $\succcurlyeq$ on $\mathscr{L}_1$ satisfies monotonicity and continuity then there exists a unique function $u: X \times [0, 1] \to [0, 1]$ such that

(i) $u(\cdot, \cdot)$ is increasing in both arguments and continuous in the second argument.

(ii) There exists a unique function $v: \mathscr{L}_1 \to [0, 1]$ such that $v(p_1)b + (1 - v(p_1))w \sim p_1$ and $\sum_x u(x, v(p_1))\, p_1(x) = v(p_1)$ for all $p_1 \in \mathscr{L}_1$.

Thus v represents $\succcurlyeq$ and the local utility function u provides an implicit representation of $\succcurlyeq$. However, in general obtaining the explicit representation of v from u is difficult. In this section we will argue that this difficulty is not as severe as one might think and furthermore might be used in explaining many observed inconsistencies (i.e., violations of preference maximization, intransitivities, etc.).

Dekel [5] provides the following observation: If u is the local utility function associated with the (betweenness satisfying) preference $\succcurlyeq$ and $\sum_x u(x, v)\, p_1(x) = v$, $\sum_x u(x, v) q_1(x) = \bar{v}$ then $p_1 \succcurlyeq q_1$ iff $v \geqslant \bar{v}$.

Note that both expected utilities above are computed at the same v which is the utility of one of the lotteries. But this immediately shows that $\sum_x u(x, v)\, p_1(x) \geqslant v \geqslant \sum_x u(x, v) q_1(x)$ implies $p_1 \succcurlyeq q_1$ and that $p_1 \succ q_1$ if at least one of the inequalities is strict. Hence to determine if p_1 is preferred to q_1 we do not have to solve the implicit equation in (ii) for both p_1 and q_1 but only need to find a value v which is between $v(p_1)$ and $v(q_1)$ and choose the lottery which yields the higher expected utility at that v. Finding such a v will be difficult if $v(p_1)$ is close to $v(q_1)$. But this is precisely the situation in which the "cost" of making an incorrect choice from $\{p_1, q_1\}$ is not very high.[7]

Now consider the following rule for choice between pairs of lotteries. Given $\{p_1, q_1\}$ the decision maker chooses a utility function u, computes the expected utility of both lotteries, and then chooses the lottery with the higher expected utility. Such a rule has the status quo dependent expected

[7] Naturally this cost must be measured in dollar amounts via certainty equivalents and not in utils, and we will do so.

utility maximization feature described in the preceding section and will be consistent with preference maximization whenever underlying preferences satisfy betweenness and the utility function that is chosen is $u(\cdot, v)$ for some v such that $v(p_1) \leqslant v \leqslant v(q_1)$ or $v(q_1) \leqslant v \leqslant v(p_1)$. To the extent that the individual uses unfit values for v (e.g., $v < v(p_1)$, $v < v(q_1)$) it will be possible to encounter preference reversals or intransitivities often discussed in the literature.

Although one may argue that the business of finding a v that will work may be troublesome and in real life decisions are made without going through such an exercise, it is equally clear that most of us do not actually maximize an (arbitrarily complicated) utility function when asked to choose among lotteries. Realistically, all that can be done is to outline a reasonable (normative) theory that can at best be used to gauge behavior. Kyburg [9] makes this point as follows: "We do not expect our subjects to conform completely to the normative theory but when they fall short we expect them to fall short in understandable ways." The decision model above suggests a theory which maintains expected utility maximization, allows for non-expected utility preferences, and offers an endogeneous model of the status quo and/or intransitivities (i.e., the failure of the normative theory). One (and by no means only) weakness of this approach as a normative theory is finding an appropriate v. Now we will provide a partial solution to this problem.

Take the case in which $\succcurlyeq$ satisfies monotonicity (i.e., p_1 stochastically dominates q_1 implies $p_1 \succ q_1$) and the following stronger continuity requirement:

The preference relation $\succcurlyeq$ is said to be *strong continuous* iff $\{p_1^i\}_{i=0}^{\infty}$ converges to p_1 (in distribution) implies $p_1 \succcurlyeq q_1$ if $p_1^i \succcurlyeq q_1$ for all i and $q_1 \succcurlyeq p_i$ if $q_1 \succcurlyeq p_1^i$ for all i.

For any $\succcurlyeq$ which satisfies betweenness and monotone strong continuity the local utility function u can be taken to be a mapping from X^2 to $[0, 1]$, where $u(x, v)$ is identified with $u(x, y)$ such that $v(y) = v$. Hence, for a preference which satisfies monotone strong continuity, certainty equivalents are well-defined; therefore the above defined rule for choosing between lottery pairs can be replaced by a rule in which the decision maker deals with certainty equivalents rather than v's. By allowing repeated expected utility calculations the difficulty of finding an appropriate v can essentially be eliminated.

In particular, there exists an algorithm which guarantees for any choice set containing n lotteries, that in at most k steps (where a step consists of at most n expected utility calculations), all lotteries except those with certainty equivalents no more than $(b-w)/2^k$ below that of the most preferred lottery in the set have been eliminated.

The algorithm works as follows: Let $C \subset \mathscr{L}_1$ such that $|C| = n$. Let $i = 1$.

Step i. Compute the expected utilities of all lotteries in C with the utility function $u(\cdot, (b+w)/2)$. Let $I_h = \{p_1 \in C \mid \sum_x u(x, (b+w)/2)\, p(x) \geqslant u((b+w)/2, (b+w)/2)\}$ and $I_l = C \backslash I_h$. Then set $C_i = I_h$, $b_i = b$, $w_i = (b+w)/2$ if $I_h \neq \varnothing$ and $C_i = I_l$, $b_i = (b+w)/2$, $w_i = w$ if $I_h = \varnothing$.

Step i′. If $|C_i| > 1$ let $b = b_i$, $w = w_i$, $C = C_i$, $i = i+1$, and return to step i.

Note that at the end of step 1 the lottery with the highest certainty equivalent will be in C_1 and the difference between the certainty equivalents of any two lotteries in C_1 is at most $(b-w)/2$. This follows from the fact that $p_1 \in I_h$, $q_1 \in I_l$ implies $p_1 \succcurlyeq (b+w)/2 \succcurlyeq q_1$. But completing step k yields a set $C_k \subset C$ such that C_k contains the lottery with the highest certainty equivalent and $p_1, q_1 \in C_k$ implies that the difference between the certainty equivalents of p_1 and q_1 is at most $(b-w)/2^k$ which is the desired conclusion.

4. Conclusion

The experimental evidence on the betweenness hypothesis appears to be inconclusive (see Chew and Waller [2], Conlisk [3], Coombs and Huang [4]). Furthermore, it is unlikely that any theory of choice under uncertainty which is consistent with what an economist might consider rational behavior will be capable of accommodating all observed behavior in actual choice situations, nor is it clear that a purely descriptive theory (i.e., one based solely on the criterion of accommodating all observed behavior) will add substantially to our understanding of choice under uncertainty. One approach would be to formalize a general theory in which the weakest set of rationality axioms are imposed initially, then additional restrictions are derived from *observed* behavior and their implications on the general theory are investigated.[8]

An alternative approach would be to consider possible applications of the theory and to formulate additional *normative* restrictions based on such applications, incorporate those restrictions to the theory and then confront the theory with empirical and experimental evidence. This approach would, it is hoped, ultimately lead to a uniform and coherent theory of choice under uncertainty.

This paper is meant to be an example of the second approach. In particular, we have considered the application of the theory of choice under uncertainty to problems in which there is gradual resolution of uncertainty and formulated what we believe to be reasonable requirements on behavior

[8] Machina [12] epitomizes this approach.

in that context. Our conclusion is that this line of analysis offers strong support for the betweenness axiom.

APPENDIX: PROOF OF THEOREM 2

(i) $\Leftrightarrow$ (iv)

$\Rightarrow$ Consider a tree d_0 such that $d_0 = \{p_0, p_1\}$, $p_0(d_1) = \alpha$, $p_0(q_1) = 1-\alpha$, $\alpha \in (0, 1)$, and $d_1 = \{p_1, q_1\}$. This tree is depicted in Fig. 4. Let $p_1 \sim q_1$. Assume for the moment that $\{p_1, q_1\} \subset S(d_0, I)$. Since $p_1 \in \text{supp}(p_0)$ by (i) we get $\bar{\sigma}^* = (p_0, \sigma^*) \in I(d_0)$, where $\sigma^* = p_1$. Therefore, $s_{d_0}(\bar{\sigma}^*) = \alpha p_1 + (1-\alpha) q_1 \sim_I p_1 \Rightarrow$ (iv). Now assume $\{p_1, q_1\} \not\subset S(d_0, I)$. Then by consistency $\{p_1, q_1\} \subset S(d_0) \setminus S(d_0, I)$. This implies $\alpha p_1 + (1-\alpha) q_1 \succ_I p_1$ by completeness of $\succcurlyeq_I$. Let $\alpha^* = \inf H$, $H = \{\hat{\alpha} \mid \hat{\alpha} p_1 + (1-\hat{\alpha}) q_1 \succcurlyeq_I \alpha' p_1 + (1-\alpha') q_1 \ \forall \alpha' \in [0, 1]\}$. By continuity $\alpha^* p_1 + (1-\alpha^*) q_1 \succcurlyeq_I \alpha p_1 + (1-\alpha) q_1$. By transitivity $\alpha^* p_1 + (1-\alpha^*) q_1 \succ_I p_1 \sim_I q_1$, and therefore $\alpha^* \in (0, 1)$. Now, consider a tree d_0' such that $d_0' = \{p_0', p_1\}$, $p_0'(d_1') = \alpha^*$, $p_0'(q_1) = 1-\alpha^*$, $d_1' = \{p_1^1, p_1\}$, $p_1^1(p_1) = \alpha^*$, $p_1^1(q_1) = 1-\alpha^*$ (see Fig. 5). Note that $S(d_0') = \{p_1, \alpha^* p_1 + (1-\alpha^*) q_1, \alpha^{*2} p_1 + (1-\alpha^{*2}) q_1\}$. By the definition of α^*, $\alpha^* p_1 + (1-\alpha^*)\, q_1 \in S(d_0, I)$. But (i) implies that also $\alpha^{*2} p_1 + (1-\alpha^{*2}) q_1 \in S(d_0, I)$, hence $\alpha^* p_1 + (1-\alpha^*) q_1 \sim_I \alpha^{*2} p_1 + (1-\alpha^{*2}) q_1$. But $\alpha^* \in (0, 1)$ implies $\alpha^{*2} < \alpha^*$ which contradicts the fact that α^* is the infimum of H. Therefore, it must be that $\{p_1, q_1\} \subset S(d_0, I)$.

$\Leftarrow$ Let $\bar{\sigma} = (p_0, \sigma) \in I(d_0)$, $d_1^* \in \text{supp}(p_0)$, $p_1^* \in d_1^* \subset S(d_0)$, $p_1^* \in S(d_0, I)$, $\sigma^*: \text{supp}(p_0) \to D_1$, $\sigma^*(d_1) = \sigma(d_1)$ for all $d_1 \neq d_1^*$, and $\sigma^*(d_1^*) = p_1^*$. Let $\bar{\sigma}^* = (p_0, \sigma^*)$. Then $s_{d_0}(\bar{\sigma}) \sim_I p_1^*$ and $p_1^* \succcurlyeq_I \sigma(d_1^*)$ since $d_1^* \subset S(d_0)$. Since $\succcurlyeq_I$ satisfies betweenness and monotonicity by Dekel [5, Theorem 1], $\succcurlyeq_I$ also satisfies strict betweenness (SB); i.e., $p_1 \succ_I q_1$ implies $p_1 \succ_I \alpha p_1 + (1-\alpha) q_1 \succ_I q_1$ for all $\alpha \in (0, 1)$. Let $\alpha = p_0(d_1^*)$ and define $q_1(x) =$

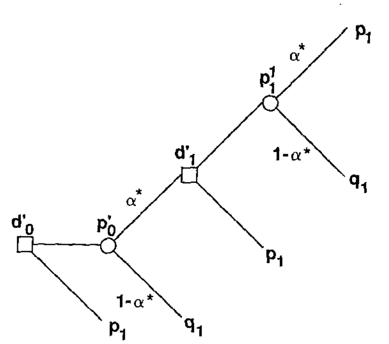

FIG. 5. A representation of a decision tree.

$1/(1-\alpha)\sum_{d_1 \in \operatorname{supp}(p_0): d_1 \neq d_1^*} p_0(d_1)\sigma(d_1)(x)$. Observe that $s_{d_0}(\bar{\sigma}) = \alpha\sigma(d_1^*) + (1-\alpha)q_1$ and $s_{d_0}(\bar{\sigma}^*) = \alpha p_1^* + (1-\alpha)q_1$. If $p_1^* \succ_I q_1$ then $p_1^* \succcurlyeq_I \sigma(d_1^*)$ and SB imply $p_1^* \succ_I \alpha\sigma(d_1^*) + (1-\alpha)q_1$. But this contradicts $s_{d_0}(\bar{\sigma}) \sim_I p_1^*$. So $q_1 \succcurlyeq_I p_1^*$. Then by SB and (iv) $\alpha p_1^* + (1-\alpha)q_1 \succcurlyeq_I p_1^*$, so $\bar{\sigma}^* \in I(d_0)$.

(ii) $\Leftrightarrow$ (iv)

$\Rightarrow$ Take $p_1 \sim_I q_1$. Consider a tree d_0 such that $d_0 = \{p_0\}$, $p_0(d_1^1) = \alpha$, $p_0(d_1^2) = 1-\alpha$, and $d_1^1 = \{p_1, q_1\}$, $d_1^2 = \{p_1, q_1, w\}$. See Fig. 6(a) for a depiction of this tree. Let $\sigma(d_1^1) = \sigma(d_1^2) = p_1$ and $\sigma'(d_1^1) = \sigma'(d_1^2) = q_1$. The strategies (p_0, σ) and (p_0, σ') yield lotteries p_1 and q_1, respectively. Assume for now that $\{p_1, q_1\} \subset S(d_0, I)$, hence $\{(p_0, \sigma), (p_0, \sigma')\} \subset I(d_0)$. By (ii) $(p_0, \sigma^*) \in I(d_0)$, where $\sigma^*(d_1^1) = p_1$ and $\sigma^*(d_1^2) = q_1$. Note that (p_0, σ^*) yields the lottery $\alpha p_1 + (1-\alpha)q_1$ and therefore we have $\alpha p_1 + (1-\alpha)q_1 \sim_I p_1$. $\Rightarrow$(iv). Now assume $\{p_1, q_1\} \subset S(d_0) \setminus S(d_0, I)$. By monotonicity it must be that either $\alpha p_1 + (1-\alpha)q_1 \succ_I p_1$ or $\alpha q_1 + (1-\alpha)p_1 \succ_I p_1$. Let $\alpha^* \in H$, $H = \{\hat{\alpha} \mid \hat{\alpha} p_1 + (1-\hat{\alpha})q_1 \succcurlyeq_I \alpha' p_1 + (1-\alpha')q_1 \ \forall \alpha' \in [0,1]\}$. By continuity, H is non-empty hence α^* is well-defined. Then, $\alpha^* p_1 + (1-\alpha^*)q_1 \succcurlyeq_I \alpha p_1 + (1-\alpha)q_1 \succ_I p_1$ and therefore $\alpha^* \in (0,1)$. Assume $\alpha^* \leqslant \frac{1}{2}$ (otherwise use $1-\alpha^*$ instead of α^*). Now consider a tree d_0' such that $d_0' = \{p_0'\}$, $p_0'(d_1^1) = \alpha^*$, $p_0'(d_1^2) = \alpha^*$, $p_0'(q_1) = 1-2\alpha^*$ (see Fig. 6(b)). Now, let $\sigma(d_1^1) = \sigma'(d_1^2) = p_1$ and $\sigma(d_1^2) = \sigma'(d_1^1) = q_1$. Note that by monotonicity and the definition of $\alpha^* \{(p_0', \sigma), (p_0', \sigma')\} \in I(d_0)$. Then by (ii) $(p_0', \sigma^*) \in I(d_0)$, where $\sigma^*(d_1^1) = \sigma^*(d_1^2) = q_1$. But this implies $q_1 \in S(d_0, I)$, hence a contradiction.

$\Leftarrow$ Let $u : [w, b] \times [0,1]$ be the local utility function of Dekel [5], where $v : \mathscr{L}_1 \to [0,1]$ representes $\succcurlyeq_1$ and $v(p_1) = \sum_x u(x, v(p_1))\, p_1(x) \forall p_1 \in \mathscr{L}_1$. Then $q_1 \succcurlyeq_I p_1$ iff $\sum_x u(x, v(p_1)) q_1(x) \geqslant v(p_1)$ (see the discussion on u on p. 10). But this proves that I will behave as if he is maximizing expected

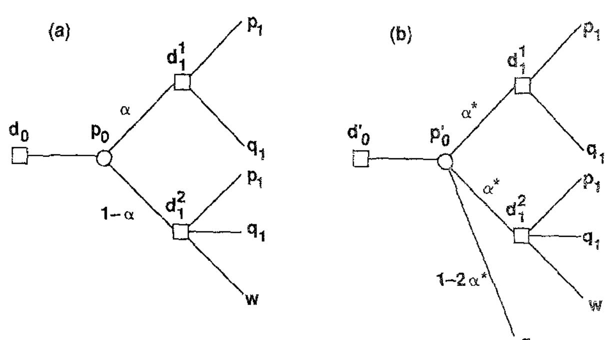

FIG. 6. Representations of two decision trees.

utility with respect to $u(\cdot, v(p_1))$ for some $p_1 \in S(d_0, I)$. But then (ii) follows from the fact that expected utility maximizing I's are DPS.

(iii) $\Leftrightarrow$ (iv)

$\Rightarrow$ Consider a tree d_0 such that $\Sigma(d_0) = \{\bar{\sigma}, \bar{\sigma}', \bar{\sigma}^*\}$, $s_{d_0}(\bar{\sigma}^*)(x) = \alpha s_{d_0}(\bar{\sigma})(x) + (1-\alpha) s_{d_0}(\bar{\sigma}')(x)$ for all $x \in X$ and for some $\alpha \in [0, 1]$ and $s_{d_0}(\bar{\sigma}) \sim_I s_{d_0}(\bar{\sigma}')$. Assume $\{\bar{\sigma}, \bar{\sigma}'\} \subset I(d_0)$ which implies $s_{d_0}(\bar{\sigma}) \sim_I s_{d_0}(\bar{\sigma}')$ by the definition of $I(d_0)$. (iii) yields $\sum_x u(x, d_0) s_{d_0}(\bar{\sigma})(x) = \sum_x u(x, d_0) s_{d_0}(\bar{\sigma}')(x)$. Now, $\sum_x u(x, d_0) s_{d_0}(\bar{\sigma}^*)(x) = \alpha \sum_x u(x, d_0) s_{d_0}(\bar{\sigma})(x) + (1-\alpha) \sum_x u(x, d_0)$ $s_{d_0}(\bar{\sigma}')(x) = \sum_x u(x, d_0) s_{d_0}(\bar{\sigma})$. Therefore, $s_{d_0}(\bar{\sigma}^*) \sim_I s_{d_0}(\bar{\sigma})$. Therefore, (iv) holds. Now assume $\{\bar{\sigma}, \bar{\sigma}'\} \not\subset I(d_0)$. Therefore $\bar{\sigma}^* \in I(d_0)$. But since $\sum_x u(x, d_0) s_{d_0}(\bar{\sigma}^*)$ is a convex combination of $\sum_x u(x, d_0) s_{d_0}(\bar{\sigma})$ and $\sum_x u(x, d_0) s_{d_0}(\bar{\sigma}')$ it cannot be strictly greater than both of them. Hence a contradiction.

$\Leftarrow$ Consider any tree d_0. Let $\bar{\sigma}^*$ be any strategy in $I(d_0)$. Let $v^* = \sum_x u(x, v^*) s_{d_0}(\bar{\sigma}^*)(x)$ where u is the local utility function. From Dekel [5] we know that for any $p_1 \not\succsim_I s_{d_0}(\bar{\sigma}^*)$, $\sum_x u(x, v^*) p_1(x) < v^*$ which implies $\sum_x u(x, v^*) s_{d_0}(\bar{\sigma})(x) < \sum_x u(x, v^*) s_{d_0}(\bar{\sigma}^*)(x)$ for all $\bar{\sigma} \in \Sigma(d_0) \backslash I(d_0)$. Therefore, choosing $u(x, d_0) = u(x, v^*)$ yields $I(d_0) = \arg\max_{\bar{\sigma} \in \Sigma(d_0)} \sum_x u(x, d_0)$ $s_{d_0}(\bar{\sigma})(x)$.

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