Market Design and Walrasian Equilibrium†

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Abstract

We establish the existence of Walrasian equilibrium for economies with many discrete goods and possibly one divisible good. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or course selection. We consider all economies with quasilinear gross substitutes preferences but allow agents to have limited quantities of the divisible good (limited transfers economies). We also consider economies without a divisible good (nontransferable utility economies). We show the existence and efficiency of Walrasian equilibrium in limited transfers economies and the existence and efficiency of strong (Walrasian) equilibrium in nontransferable utility economies. Finally, we show that various constraints on minimum and maximum levels of consumption and aggregate constraints of the kind that are relevant for school choice/course selection problems can be accommodated by either incorporating these constraints into individual preferences or by incorporating a suitable production technology into nontransferable utility economies.

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1. Introduction

In this paper, we establish the existence of Walrasian equilibrium in economies with many discrete goods and either with a limited quantity of one divisible good or without any divisible goods. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or course selection. To this end, we develop techniques for analyzing allocation problems in economies with or without transfers and for incorporating additional constraints into allocation rules. In particular, we show that distributional requirements (for example, a rule stating that every BA student must take at least 2 science courses, 2 humanities courses and 1 social science course for credit) can be incorporated into preferences in a manner consistent with the resulting economy having a Walrasian equilibrium. Such requirements are common in US universities. We also show that aggregate constraints that restrict the total number of seats in a set of classes can be rendered consistent with the existence of Walrasian equilibrium by incorporating a suitable production technology into the economy.

In Kelso and Crawford (1982)'s formulation of the competitive economy, there is a finite number of goods, and a finite number of consumers with quasilinear utility functions that satisfy the *substitutes property.*[1] Kelso and Crawford also assume that each consumer is endowed with enough of the divisible good to ensure that she can purchase any bundle of discrete goods at the equilibrium prices. This last condition would be satisfied, for example, if each consumer had more of the divisible good than the value she assigns to the aggregate endowment of indivisible goods. We call the Kelso-Crawford setting the *transferable utility economy.* Kelso and Crawford's ingenious formulation of the substitutes property facilitates their existence theorem as well as a tatonnement process/dynamic auction for computing Walrasian equilibrium. Subsequent research has identified various important properties of Walrasian equilibrium in transferable utility economies.

Our first goal is to do away with the assumption that each consumer has enough of the divisible good to purchase whatever she may wish at the equilibrium prices. In

[1] The substitutes property requires that if x is an optimal consumption bundle at prices p and prices increase (weakly) to some $\hat{p}$ then the agent must have an optimal bundle at $\hat{p}$ which has her consuming at least as much of every good that did not incur a price increase.

particular, we allow for arbitrary positive endowments of the divisible good. We call this the *limited transfers economy.* We also consider the *nontransferable utility* economy; that is, we consider economies in which there is no divisible good. This setting is particularly well-suited for the analyzing many allocation problems such as school choice or course selection.

Substitutes preferences have been used to analyze a variety of market design problems. Then, results suggesting that the Walras equilibrium correspondence is nearly incentive compatible when there are sufficiently many agents (see, for example, Roberts and Postlewaite, (1976)) have been invoked to argue that Walrasian methods can play a role in market design. In most of these applications it is unreasonable to assume that each agent has enough of the divisible good to acquire whatever she wishes. In many applications, transfers (i.e., the divisible good or equivalently, money) are ruled out altogether and the problem is one of assigning efficiently and fairly a fixed number of objects to individuals. Hence, both the limited transfers economy and the nontransferable utility economy are of interest.

Theorem 1 establishes the existence of a Walrasian equilibrium (henceforth, equilibrium) in random allocations for limited transfers economies. In the transferable utility case, randomization is not necessary since in such economies, a random equilibrium allocation at prices p is simply a probability distribution over deterministic competitive equilibria at prices p. However, in both limited transfers and nontransferable utility economies, randomization is necessary for the existence of equilibrium. The following simple example establishes this fact.

Example 1: There are two agents and a single good (initially owned by the seller). Both agent's utility for the good is 2 and both have one unit of the divisible good. Without randomization, if the price is less than or equal to 1, both agents will demand the good; if the price is greater than 1 neither will demand the good. Since there is exactly one unit of the good, there can be no deterministic equilibrium for this economy. If randomization is allowed, the equilibrium price of the indivisible good is 2 and each agent will get the indivisible good with probability $1/2$. Note that the above example is non-generic; that is, if the money endowments are not equal (but less than 2), there is an equilibrium without randomization in which the agent with the larger money endowment receives the good.

Example 1, above, features utilities that satisfy the substitutes property and, as a result, a competitive equilibrium exists. Example 2, below, illustrates the failure of existence when utilities do not satisfy the substitutes property.

Example 2: The economy has three agents and three indivisible goods. Initially, agents 1 and 2 each have 1 unit of the divisible good and no indivisible goods. Agent 3's initial endowment consists of the three indivisible goods and zero units of the divisible good. For agents 1 and 2,

$$u_i(A) = \begin{cases} 0 & \text{if } |A| < 2 \\ 2 & |A| \geq 2 \end{cases}$$

while $u_3(A) = 0$ for all A. Since the three goods are perfect substitutes, in equilibrium, all three must have the same price. Let r be this common price. Clearly, $r = 0$ is impossible in any equilibrium since both agents 1 and 2 would demand at least 2 goods with probability 1 and market clearing would fail. Then, in any equilibrium, $r > 0$ and agents 1 and 2 must consume the three goods with probability 1. This implies $r \cdot 3 \leq 2$ and hence $r \leq 2/3$. If $r \leq 2/3$, both agent 1 and 2 will want to consume any 2 of the 3 goods with probability $1/(2r)$ and 0 goods with probability $1 - 1/(2r)$. Market clearing requires, at a minimum, that the expected total consumption of the these two agents is 3. Hence, $2 \cdot 2 \cdot 1/(2r) = 3$ and therefore $r = 2/3$. This means that the unique optimal random consumption bundle for agent $i = 1, 2$ at these prices is the distribution that assigns her 2 goods with probability 3/4 and zero goods with probability 1/4. This pair of random consumptions is feasible in expectation but is not *implementable.* That is, there is no random allocation that yields this random consumption to both consumers. To see why, note that in any state of the world in which player 1 is allocated 2 goods, player 2 must be allocated either 1 good, which is never optimal for him, or 0 goods. However, consuming 0 goods with probability 3/4 is not optimal for player 2.

The utility function of agents 1 and 2 in Example 2 does not satisfy the *substitutes property.*[2] Our main results show that examples such as the one above cannot be constructed with utility functions that do.

[2] Consider price vector p such that $p^1 = p^2 = 0.5, p^3 = 3$. At this price, the unique optimal bundle is $\{1, 2\}$. Next, increase the price of good 2 to $q^2 = 3$ and keep the other prices unchanged. At the new price $q = (0.5, 3, 3)$, the unique optimal bundle of divisible goods is $\emptyset$. Hence, the demand for good 1 decreased despite the fact that its price remained the same while another good's price increased establishing that u_1 and u_2 do not satisfy the substitutes property.

The assignment of courses to students typically requires a mechanism without transfers, i.e., without a divisible good. To address this and related applications, in Theorem 2, we prove the existence of a competitive equilibrium for the nontransferable utility economy. Hylland and Zeckhauser (1979) first proposed Walrasian equilibria as an allocation mechanism for the unit demand nontransferable utility economy. They showed that some equilibria may be Pareto inefficient because local non-satiation need not hold in this setting. Nonetheless, Hylland and Zeckhauser (1979) noted that efficient equilibria always exist. Mas-Colell (1992) coins the term *strong equilibrium* for a competitive equilibrium in which every consumer chooses the cheapest utility maximizing consumption and shows that strong equilibria are efficient. Our Theorem 2 establishes the existence of a strong and, therefore, of a Pareto efficient equilibrium.

Allocation problems often feature constraints on individual or group consumption. In course assignment problems, university rules may constrain students' course selection either by imposing distributional requirements of the kind described above or by limiting the number of courses that a student can take for credit from a specified list of courses (as in Princeton's rule of 12 discussed below). In a school choice problem, administrators may restrict parents' choices based on the location of their residence; and, finally, in office allocation problems, choices may be constrained by employee seniority. We analyze such constrained allocation problems in Theorem 4. There, we consider a broad range of constraints on individual consumption and show that our model can incorporate them. In some applications, groups of individuals may face constraints on their joint consumption or there may be aggregate constraints. For example, a university may reserve a certain number of seats in a class for those students who must take this class as a requirement of their majors or there may be aggregate constraints on lab space that limit the total number of seats available in a collection of related courses. Theorem 4 permits such aggregate constraints as well.

1.1 Randomization and Efficiency

Many of the commonly used allocation mechanisms do not entail explicit randomization; randomization, if it takes place at all, does so to order the agents; that is, to assign them priorities. These mechanisms depend only on the ordinal preferences of the agents and not on their attitude towards random bundles. The randomly assigned priorities provide ex ante symmetry or fairness but do not facilitate gains from trades based on attitudes to uncertainty. We will call such allocation procedures ordinal mechanisms with random priorities.

In one-to-one matching, commonly used ordinal mechanisms such as the Gale-Shapley algorithm or the top-trading cycles procedure are efficient among ordinal mechanisms with random priorities. However, as Hylland and Zeckhauser (1979) note, these mechanisms are typically dominated by random Walrasian equilibrium allocations. In the more general multiple good setting, there is little consensus on the choice of mechanism and virtually no results regarding efficiency. The simple example below highlights the inefficiency of all ordinal mechanisms with random priorities. We conclude our discussion of randomization and efficiency by noting that deterministic Walrasian equilibrium with random endowments is an ordinal mechanism with random priorities, and hence, is also inefficient.

There are two indivisible goods (a and b), three consumers ($1, 2$ and 3) and no divisible good in the economy. The following table summarizes the utility functions of the consumers:

Consumer	$\{a, b\}$	$\{a\}$	$\{b\}$
1	11	10	8
2	11	10	2
3	11	10	2

Note that all three consumers have the same ordinal ranking over consumption bundles: they all strictly prefer $\{a, b\}$ to $\{a\}$ and $\{a\}$ to $\{b\}$. Under the Walrasian mechanism, the planner endows each student with a budget of 1 (unit of fiat money) and the goods initially belong to a fictitious agent (the designer/seller).[3] This agent values fiat money but does not value the divisible goods. Our mechanism allocates courses according to the resulting Walrasian equilibrium lotteries. In this example, the unique equilibrium prices are $p_a = 1$ and $p_b = 2$. At these prices, agent 1 purchases a deterministic lottery consisting

[3] Alternatively, we could eliminate the seller and endow all three consumers with equal random allocations of the indivisible goods; that is, each consumer would own each good with probability 1/3.

of good b. Agents 2 and 3 purchase the lottery that yields the good a with probability $1/2$. The equilibrium utility of agent 1 is 8 and the equilibrium utility of agents 2 and 3 is 5.

All three consumers are better off under the Walrasian mechanism than under any ordinal mechanism that treats all agents symmetrically since any mechanism that allocates the two goods based solely on the agents' ordinal preferences would have to give the same allocation to all three agents. The best symmetric allocation would give each agent each good with probability $1/3$ and nothing with probability $1/3$. This allocation yields utility 6 for agent 1 and 4 for agents 2 and 3; strictly lower utilities for all three agents than their utilities in the Walrasian mechanism above.

Example 1 above establishes the necessity of randomization for ensuring the existence of Walrasian equilibrium. Budish (2011) notes that the non-existence of deterministic Walrasian equilibrium is non-generic: when agents have unequal money endowments, a deterministic competitive equilibrium typically exists. Thus, an alternative approach to randomized allocations would be to assign agents random endowments of money so as to take advantage of the generic existence of equilibrium. While solving the existence problem, this approach would typically lead to an ex ante inefficient allocation. To see why, consider the example above: suppose that the designer draws a random allocation of fiat money for the three agents from the interval $[1, 2]$ according to some continuous symmetric distribution. Then, with probability 1, no two agents will have identical money endowments and, therefore, a deterministic Walrasian equilibrium will exist: the agent with the greatest endowment of money will receive good a and the agent with the second greatest money endowment will receive good b. Hence, deterministic Walrasian equilibrium (with randomized endowments of money) is an ordinal mechanism with random priorities. Thus, while it solves the existence problem, the random allocation of money does not achieve ex ante Pareto efficiency. Allowing endowments to be drawn from an interval of the form $[0, r]$ would only increase the inefficiency since it would make it possible for one agent to have more than twice as much money as any of the other agents. In that situation, there are deterministic Walrasian equilibria in which the agent with the largest endowment gets both goods. Symmetric allocations that assign both goods to the same agent with positive probability lead to greater inefficiency than the symmetric allocation in which each agent gets at most one good.

1.2 Related Literature

Kelso and Crawford (1982) establish the existence of a Walrasian equilibrium using an ascending tatonnement process. They show that this process converges to a Walrasian equilibrium price vector. Gul and Stacchetti (1999) show that given any utility function that does not satisfy the substitutes property, it is possible to construct an N-person economy consisting of an agent with this utility function and $N-1$ agents with substitutes utility functions that has no Walrasian equilibrium.[4] Hence, their result shows that it is impossible to extend Kelso and Crawford's existence result to a larger class of utility functions than those that satisfy the substitutes property.

Sun and Yang (2006) provide an alternative to the Kelso-Crawford existence result that allows for some complementarities in a transferable utility economy. They circumvent Gul and Stacchetti's impossibility result by excluding some substitutes preferences. In particular, they assume that the set of indivisible goods can be partitioned into two sets such that all agents consider goods within each element of the partition substitutes and goods in different partition elements complements. Shioura and Yang (2015) generalize the Sun-Yang result. Baldwin and Klemperer (2019) introduce the notion of a "demand type" to facilitate an existence result based on a unimodularity theorem. A demand type is a collection of vectors describing the possible ways in which the agent's demand can change in response to "a small generic price change". One of their main results establishes that a Walrasian equilibrium exists for any aggregate endowment if and only if agents' utility functions satisfy a discrete concavity property and are of a demand type corresponding to a unimodular set of vectors. They identify the demand type associated with substitutes preferences and the preferences of Sun and Yang (2006). Hence, their theorem provides a generalization of Kelso and Crawford (1982), Sun and Yang (2006) and the necessity result of Gul and Stacchetti (1999).

As noted above, a special class of substitutes preferences are the unit-demand preferences. These preferences are suitable for situations in which agents can consume at most one unit of the divisible good. Leonard (1983) studies transferable utility unit-demand economies and identifies an allocation rule that generalizes the second-price auction and

[4] Yang (2017) finds an error in Gul and Stacchetti's proof and supplies an alternative proof.

has strong incentive and efficiency properties. His allocation rule is the Walrasian rule together with the lowest equilibrium prices. Hylland and Zeckhauser (1979) are the first to study what we have called a nontransferable utility unit-demand economy. They establish the existence of an efficient Walrasian equilibrium in such economies. Hylland and Zeckhauser's work has led to a literature on competitive equilibrium solutions to market design problems: Ashlagi and Shi (2016) study competitive equilibrium with equal incomes in a market with continuum of agents. Le (2017), He, Miralles, Pycia and Yan (2015) and Echenique, Miralles and Zhang (2018) maintain the assumption of unit-demand preferences, but allow for general endowments, non-EU preferences[5] or priority-based allocations. Miralles and Pycia (2017) consider nontransferable utility economies (with indivisibilities). For the unit demand case, they show that every efficient random allocation is the Walrasian allocation of an economy with suitably chosen endowments. For the general, multi-unit demand case, they impose an assumption stronger than efficiency to ensure that the allocation is a Walrasian allocation. Mas-Colell (1992) and McLennan (2018) study more general convex economies with production.

There are two major differences between these last two papers and ours. First, these papers introduce some notion of "slackness" into the definition of Walrasian equilibrium to guarantee its existence, while the equilibrium notion in our paper is standard. Second, they focus on convex (or convexified) economies and thus there is no implementability problem. In our set-up implementability is the key issue. We provide a discussion of the second point after the statement of Theorem 1.

Budish, Che, Kojima and Milgrom (2013) study a variety of probabilistic assignment mechanisms. Our work relates to section 4 of their paper, where they define and show the existence of what they call pseudo-Walrasian equilibrium.[6] In that section, they consider fully separable preferences and establish the existence of efficient pseudo-Walrasian equilibrium. They also describe how individual constraints can be incorporated into pseudo-Walrasian equilibrium.

In Appendix B of their paper, they consider a richer class of preferences adopted from Milgrom (2009). These preferences amount to the closure of unit-demand preferences under satiation and convolution.[7] Ostrovsky and Paes Leme (2015) prove that the closure of

[5] That is, in the convexified economy, they allow utility functions that are nonlinear in probabilities.

[6] Presumably, the qualifier pseudo is to indicate the interjection of fiat money and also to acknowledge various additional constraints that are typically not a part on the definition of a competitive economy. By incorporating these constraints into preferences and technology and by assuming that the mechanism designer/seller values the fiat money, we are able to interpret our equilibria as proper Walrasian equilibria.

[7] See section 2.1 for a discussion of closures under substitutes preserving operations.

unit-demand preferences under endowment and convolution yields a strict subset of substitutes preferences. They identify a rich class of preferences that belong to the latter but not the former. It is easy to check that this class of preferences is also excluded from the class described in Appendix B of Budish, Che, Kojima and Milgrom (2013). Thus, compared to Budish, Che, Kojima and Milgrom (2013), we consider a richer class of preferences and a richer class of constraints. In particular, their analysis of pseudo-Walrasian equilibrium does not include individual lower bound constraints, group constraints or aggregate constraints other than bounds on the aggregate supply of each good.

Kojima, Sun and Yu (2018) study constraints in a transferable utility economy. They show that imposing upper and lower bounds on quantities consumed (i.e., interval constraints) on gross substitutes preferences preserves the gross substitutes property. They also show that a slightly more general class of constraints than interval constraints are the only ones that preserve the gross substitutes property for every gross substitute utility function. Lemma 1 below and the discussion prior to it is related to their first result regarding interval constraints. Our main results focus on limited transfers and nontransferable utility economies and allow for joint restrictions on the utility function and the constraints (and hence permit a larger set of constraints than Kojima, Sun and Yu (2018)).

2. The Substitutes Property and the Limited Transfers Economy

Let $H = \{1, \ldots, L\}$ be the set of goods. Subsets of H are consumption bundles.[8] We identify each $A \subset H$ with $x \in X := \{0, 1\}^L$ such that $x^j = 1$ if and only if $j \in A$. Hence, $o = (0, \ldots, 0) \in X$ is identified with the empty set.

For any $x \in X$, let $\text{supp}\,(x) = \{k \in H \,|\, x^k = 1\}$ and $\sigma(x) = \sum_j x^j$. A utility on X is a function $u : X \to I\!R \cup \{-\infty\}$. The *effective domain* of u is the set $\text{dom}\, u = \{x \in X \,|\, -\infty < u(x)\}$. Without loss of generality, we normalize u so that $u(x) \geq 0$ for all $x \in \text{dom}\, u$. Throughout, we adopt the following convention: $-\infty + (-\infty) = -\infty \geq -\infty$. We assume that every agent's overall utility function is quasilinear in the divisible good.

[8] We are assuming that there is a single unit of each good. This assumption makes the analysis of the implementability problem easier and is without loss of generality, since we can label each of the multiple units of a good as a distinct good. Equilibrium will ensure that each of these units has the same price.

Given any price vector $p \in I\!R^L_+$, we let $U_i(x,p) = u(x) - p \cdot x$ denote the agent's overall utility function.[9]

For $x, y \in I\!R^L$, we write $x \leq y$ to mean that each coordinate of x is no greater than the corresponding coordinate of y and let $x \wedge y$ denote $z \in X$ such that $z^j = \min\{x^j, y^j\}$ for all j. Similarly, let $x \vee y$ denote $z \in X$ such that $z^j = \max\{x^j, y^j\}$ for all j. Without risk of confusion, we sometimes refer to u as the utility function (instead of saying the utility index associated with the (overall) utility function U). We let $\chi^j \in X$ denote the good j; that is, $\chi^j(k) = 1$ if $k = j$; otherwise, $\chi^j(k) = 0$. Similarly, for any set of indivisible goods $A \subset H$, define $\chi^A \in X$ as follows:

$$\chi^A(k) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{otherwise} \end{cases}$$

Throughout, we will assume that $\operatorname{dom} u \neq \emptyset$ and u is monotone; that is, $x \leq y$ implies $u(x) \leq u(y)$.

2.1 The Substitutes Property and the Transferable Utility Economy

Define the transferable utility demand correspondence for u as follows:

$$D_u(p) := \{x \in X \mid u(x) - p \cdot z \geq u(y) - p \cdot y \text{ for all } y \in X\}$$

Since $\operatorname{dom} u \neq \emptyset$ and $p \in I\!R^L_+$, $D_u(p)$ will always lie in the effective domain. The substitutes property states the following: let x be an optimal consumption bundle at prices p and assume that prices increase (weakly) to some $\hat{p}$. Then, the agent must have an optimal bundle at $\hat{p}$ which has her consuming at least as much of every good that did not incur a price increase. The formal definition is as follows:

Definition[10]**:** *The function u has the substitutes property if $x \in D_u(p)$, $p \leq \hat{p}$, $\hat{p}^j = p^j$ for all $j \in A$ implies there exists $y \in D_u(\hat{p})$ such that $y^j \geq x^j$ for all $j \in A$.*

Kelso and Crawford (1982) introduced the substitutes property. Since then, numerous alternative characterizations have been identified. For example, Fujishige and Yang (2003) prove that the substitutes property is equivalent to $M^\natural$-concavity: the function u is $M^\sharp$-*concave* if for all $x, y \in \operatorname{dom} u$, $x^j > y^j$ implies $[u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$ or there is k such that $y^k > x^k$, $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)]$.[11] Gul and

[9] If the agent has an endowment of indivisible goods, the overall utility function is unchanged since the value of the endowment enters the utility function as a constant.

[10] Generally, the substitutes property should be defined for arbitrary price vector $p \in I\!R^L$. However, since we assume monotone utility functions, it suffices to define the property for nonnegative price vectors.

[11] See also Shioura and Tamura (2015), Theorem 4.1.

Stacchetti (1999) show that that if u satisfies the substitutes property, then it must be submodular:[12]

$$u(x) + u(y) \geq u(x \vee y) + u(x \wedge y)$$

If the inequality above always holds with equality, then u is additive.

Perhaps the best-known subclass of substitutes utility functions are unit demand utilities. These utility functions are appropriate for situations in which each agent can consume at most one indivisible good: u is a *unit demand utility* if

$$u(x) = \max\{u(\chi^j) \mid \chi^j \leq x\}$$

Below, we describe operations on gross substitutes utility functions that enable us to derive new gross substitute utility functions. In section 3, we use these operations to incorporate additional (curricular) restrictions in course selection and school choice problems. Let $k > 0$ be an integer, $z \in X$ and u, v be two substitutes utility functions. Define,

$$u^z(x) = u(x \wedge z)$$

$$u_z(x) = u(x \vee z) - u(z)$$

$$(u \odot v)(x) = \max_{y \leq x}\{u(y) + v(x - y)\}$$

$$[u]^k(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq k}} u(y)$$

$$[u]_k(x) = \begin{cases} \max_{\substack{y \leq x \\ \sigma(y) \geq k}} u(y) & \text{if } \sigma(x) \geq k \\ -\infty & \text{otherwise.} \end{cases}$$

Call u^z the z-constrained u, u_z the z-endowed u, $u \odot v$ the convolution (or aggregation) of u, v, $[u]^k$ the k-satiation of u and $[u]_k$ the k-lower bound u. It is easy to verify that a z-endowed u satisfies the substitutes property whenever u does and that the convolution of u and v satisfies the substitutes property whenever u and v both satisfy the substitutes property. Similarly, verifying that a z-constrained utility satisfies the gross substitutes property whenever $\operatorname{dom} u \cap \{x \mid x \leq z\} \neq \emptyset$ is straightforward.[13]

[12] Gul and Stacchetti (1999) show this result for u such that $\operatorname{dom} u = H$. Their result extends immediately to utilities functions u such that $\operatorname{dom} u \neq X$.

[13] In some cases, it is easier to verify that the new utility function satisfies $M^\natural$-concavity which, as we noted above, is equivalent to the substitutes property.

Bing, Lehmann and Milgrom (2004) prove that the k-satiation of the substitutes utility u is a substitutes utility (provided there is at least one k-element set in $\operatorname{dom} u$). The following lemma establishes the substitutes property for k-lower bound u. All proofs are in the Appendix.

Lemma 1: *If u satisfies the substitutes property and $\operatorname{dom} u \cap \{x \in X | \sigma(x) \geq k\} \neq \emptyset$, then $[u]_k$ satisfies the substitutes property.*

For any given class of utility functions, $\mathcal{U}$, and set of substitutes preserving operations, τ, let $\tau(\mathcal{U})$ denote the set of all utility functions that can be derived from the elements of $\mathcal{U}$ by repeatedly applying various operations in τ. We will call $\tau(\mathcal{U})$ the τ-closure of $\mathcal{U}$. In other words, $\tau(\mathcal{U})$ is the smallest family of utility functions that includes $\mathcal{U}$ and is closed under operations in τ. Clearly, if each element of $\mathcal{U}$ satisfies the substitutes property, then so does the τ-closure of $\mathcal{U}$.

Ostrovsky and Paes Leme (2015) show that the endowment and convolution closure of the set of unit demand preferences is a strict subset of the set of all substitutes preferences.[14] They also provide a rich class of examples that satisfy the substitutes property but are not in the endowment and convolution closure of the set of unit demand preferences. Let τ be the five substitutes preserving operations discussed above and let $\mathcal{U}$ be the set of all unit demand preferences. Then, using Ostrovsky and Paes Leme's arguments, it is easy to verify that $\tau(\mathcal{U})$ is a strict subset of substitutes preferences and excludes the same rich class of preferences that these authors have identified.

Next, we define a new class of utility functions that are useful for describing student preferences over class schedules. We call these utility functions academic preferences since they incorporate curricular requirements into the agent's utility function. Below, we provide an example of an academic preference.

Example 3: Students are required to take at least three courses for credit and satisfy a distributional requirement by taking courses from at least four out of the five categories

[14] They conjecture that the endowment and convolution closure of the set of all weighted matroids is the set of substitutes preferences. Ostrovsky and Paes Leme (2015) note that results from Murota (1996), Murota and Shioura (1999), and Fujishige and Yang (2003) ensure that every *weighted matroid* and hence every *rank function* satisfies the substitutes property. For the definitions of weighted matroid, rank function and other relevant terms and results from matroid theory, see Appendix A where we provide a short proof that weighted matroids satisfy the substitutes property based on Fujishige and Yang (2003)'s result that the substitutes property is equivalent to $M^\natural$-concavity.

a, b, c, d, e. Each class that a student takes for credit can potentially meet two of these four requirements. There are ten classes, $H = \{ab, ac, ad, ae, bc, bd, be, cd, ce, de\}$, each identified by the requirements that it might satisfy. Students can use at most one class to meet two different requirements simultaneously. The remaining two requirements must be met with distinct classes.[15] A student may get credit for an additional course if the four courses together meet all five requirements. Hence, a student can meet the distributional requirement by taking either three or four courses for credit, but students who take four courses for credit must cover all of the five categories.

For example, the schedules $\{ab, ac, ad\}$ and $\{ab, bc, cd, de\}$ are both feasible; the former yields 3 course credits, the latter yields 4. The schedule $\{ab, bc, ac\}$ is not feasible since it only meets 3 distributional requirements; the schedule $C = \{ab, bc, cd, da\}$ is feasible but only yields 3 course credits since it only meets four distributional requirements. Let Y be the set of all feasible schedules. That is, each $z \in Y$ either has 3 courses and meets four distributional requirements or has four courses and meets five distributional requirements. Let v be an additive utility function on X. A student's objective is to choose a feasible course schedule that maximizes the total utility of the courses she takes for credit.[16] That is, her objective is to maximize the utility function u such that

$$u(x) = \begin{cases} \max_{\substack{y \in Y \\ y \leq x}} v(y) & \text{if there is } y \in Y \text{ such that } y \leq x \\ -\infty & \text{otherwise} \end{cases}$$

In Appendix A, we offer a general definition of academic preferences, show that they include the example above and that they are gross substitutes preferences.

We conclude this section by discussing the existence of equilibrium in a transferable utility economy. Let N be the number of agents in the economy, $\xi \in X^N$ be an allocation and let ξ_i denote agent i's consumption in the allocation ξ. Then (ξ, p) is a deterministic Walrasian equilibrium in the transferable utility economy if $\sum_{i=1}^N \xi_i \leq \chi^H$, $u_i(\xi_i) - p\xi_i \geq u_i(x) - px$ for all $x \in X$, i and for all $j \in H$, $\sum_{i=1}^N \xi_i^j = 1$ whenever $p^j > 0$.

[15] Undergraduates need to satisfy distributional requirements in most universities. For example, Northwestern's version allows multiple requirements to be met with a single course in some but not all situations.

[16] The example is meant to show that a rich set of curricular restrictions can be accommodated with academic preferences. Simpler versions can easily be constructed. For example, replacing the condition "a student may get credit for an additional course if the four courses together meet all five distributional requirements" with "a student who fulfills the requirements for 3 course credits can get an additional credit by taking a fourth course" would also yield an academic preference.

Kelso and Crawford (1982) showed that if utility functions satisfy monotonicity, the substitutes property and have domain X, then there exists an deterministic equilibrium in the transferable utility economy. To see how the result can be extended to general effective domains, note that since $\operatorname{dom} u_i \neq \emptyset$, the demand satisfies $D_{u_i}(p) \subset \operatorname{dom} u_i$ for all i and p. Hence, for any candidate equilibrium allocation ξ, we have $\xi_i \in \operatorname{dom} u_i$ for each i. The existence of such an allocation is a necessary condition for existence of an equilibrium and we will incorporate it into the definition of a transferable utility economy.

Definition: *$\mathcal{E} = \{(u_i)_{i=1}^N\}$ is a transferable utility economy if u_i satisfies the substitutes property for all i and there is an allocation ξ such that $\sum \xi_i \leq \chi^H$, $\xi_i \in \operatorname{dom} u_i$ for all i.*

The following lemma states that the additional condition above is also sufficient for the existence of an equilibrium when preferences satisfy the substitutes property.

Lemma 2: *The transferable utility economy has a deterministic equilibrium.*

Notice that efficient allocations of indivisible goods, optimal demands and Walrasian equilibria are independent of the initial endowments in the transferable utility economy. Therefore, the definition of the transferable utility economy omits them. However, endowments will matter in the limited transfers economy, defined in the next section.

2.2 The Limited Transfers Economy

Example 1 shows that when agents have limited budgets, a deterministic equilibrium may not exist even in the simplest limited transfers economies. Thus, we need to extend our definition of an allocation to allow randomness: a random consumption (of indivisible goods) $\theta : X \to [0, 1]$ is a probability distribution on X; that is, $\sum_x \theta(x) = 1$. Let Θ denote the set of all random consumptions. For $\theta \in \Theta$, let $\bar{\theta} \in I\!R_+^L$ denote the coordinate-by-coordinate mean of θ; that is $\bar{\theta}^j = \sum_x \theta(x) \cdot x^j$. We assume that u is also the agent's von Neumann-Morgenstern utility function. Hence,

$$u(\theta) = \sum_z u(z)\theta(z)$$

The effective domain of u on Θ consists of all the random consumptions such that $\theta(x) > 0$ implies $x \in \operatorname{dom} u$.[17]

[17] The function $u : \Theta \to [0, 1]$ is continuous on the effective domain but not on the whole domain. For example, suppose that $\operatorname{dom} u = X \backslash \{o\}$, $x \in \operatorname{dom} u$ and take a sequence of random consumptions θ^n such that $\theta^n(o) = 1/n$ and $\theta^n(x) = 1 - 1/n$. Clearly, $u(\theta^n) = -\infty$ for all n, but $\lim \theta^n = \theta$ such that $\theta(x) = 1$ and therefore, $u(\theta) \neq -\infty$.

Quasilinearity allows us to ignore randomness in the consumption of the divisible good and identify each random consumption of the divisible good with its expectation. Hence, we define the von Neumann utility function U as follows:

$$U(\theta, p) = u(\theta) - p \cdot \bar{\theta}$$

Let $w_i \in X$ denote agent i's endowment of indivisible goods and let b_i denote her endowment of the divisible good. For some applications, it is useful to have an additional agent, the seller or market designer, who holds some or all of the aggregate endowment of the indivisible goods. We assume that the seller derives no utility from the indivisible goods; she only values the divisible good. The aggregate endowment of indivisible goods in the economy is $\chi^H := (1, \ldots, 1) \in X$ and, therefore, the seller's endowment of the indivisible goods is $w_0 = \chi^H - \sum_{i=1}^N w_i$. We will assume that $w_i \in \operatorname{dom} u_i$ for each i to guarantee that agent i can afford at least one bundle in the effective domain.

Definition: *$\mathcal{E}^o = \{(u_i, w_i, b_i)_{i=1}^N\}$ is a limited transfers economy if, for all i, u_i satisfies the substitutes property, $b_i > 0$ and $w_i \in \operatorname{dom} u_i$.*

A random allocation (of indivisible goods) for this economy is a probability distribution $\alpha : X^N \to [0, 1]$. For any such α, let α_i denote the i'th marginal of α; that is, $\alpha_i \in \Theta$ is the random consumption of agent i, where

$$\alpha_i(x) = \sum_{\{\xi : \xi_i = x\}} \alpha(\xi)$$

A random allocation α is feasible for the economy $\mathcal{E}^o$ if, for all ξ such that $\alpha(\xi) > 0$, $\xi_i \in \operatorname{dom} u_i$ for all i and $\sum_{i=1}^N \xi_i \leq \chi^H$.

The budget (set) of an agent with endowment w, b at prices p is

$$B(p, w, b) = \left\{ \theta \in \Theta \,\middle|\, p \cdot \bar{\theta} \leq p \cdot w + b \right\}$$

Then, $\theta \in B(p, w, b)$ is optimal for agent i given budget $B(p, w, b)$ if

$$U_i(\theta, p) \geq U_i(\theta_o, p)$$

for all $\theta_o \in B(p, w, b)$.

Definition: *A price $p \in I\!R^L_+$ and a random allocation α is an* equilibrium *for the limited transfers economy $\mathcal{E}^o$ if*

(1) α is feasible for the economy $\mathcal{E}^o$;

(2) for all i, α_i is optimal for agent i given budget $B(p, w_i, b_i)$;

(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i=1}^N \xi_i^j = 1$.

Theorem 1: *Every limited transfers economy has an equilibrium.*

Our proof relies on the existence of equilibrium in the transferable utility economy. We seek a $\lambda_i \in (0, 1]$ for each agent i and a Walrasian equilibrium (p, α) for the modified *transferable utility* economy (with random consumption) in which each u_i is replaced by

$$\tilde{u}_i = \lambda_i u_i$$

such that each agent i spends, in expectation, (1) no more that b_i on indivisible goods and (2) exactly b_i on indivisible goods if $\lambda_i < 1$. It is possible to decrease an agent's equilibrium spending as much as needed by decreasing that agent's λ_i. Hence, we can satisfy condition (1). A fixed-point argument ensures that we can also satisfy condition (2). We then show that Walrasian equilibria (of the modified economy) that satisfy conditions (1) and (2) are Walrasian equilibria of the original economy. We choose this proof strategy because it allows us to solve the implementability problem by using properties of the unconstrained economy: in the unconstrained economy every (randomized) Walrasian equilibrium must simply be a randomization over deterministic equilibria and, therefore, the allocation must be implementable.

3. Nontransferable Utility Economies

In this section, we will consider allocation problems in settings without a divisible good. We call this type of an economy a *nontransferable utility economy.* We will describe the nontransferable utility exchange economy and nontransferable utility economy with production, define a strong (Walrasian) equilibrium and establish its existence and efficiency. In many applications, nontransferable utility economies impose constraints on individual consumption, or on the consumption of groups or on aggregate feasibility. In the next section, we describe how and to what extent such constraints can be incorporated into our model.

3.1 Nontransferable Utility Exchange Economies

In a nontransferable utility economy, each agent i has a substitutes utility function u_i and a quantity b_i of fiat money. Initially, the entire aggregate endowment belongs to the market designer. Each agent's utility depends only on her consumption of indivisible goods. That is, agents solve the following utility maximization problem:

$$U_i(p, b_i) = \max u_i(\theta) \text{ subject to } p \cdot \bar{\theta} \leq b_i$$

Hence, U_i is the indirect utility function of agent i.

Definition: *$\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$ is a nontransferable utility economy if, for all i, u_i satisfies the substitutes property, $o \in \text{dom}\, u_i$ and $b_i > 0$.*

In the nontransferable utility setting, Walrasian mechanisms provide a rich menu of allocation rules with desirable properties. The designer may accommodate fairness concerns by choosing the agents' endowments of fiat money (the b_i's) appropriately. In particular, choosing the same b_i for every agent ensures that the resulting allocations are envy-free. This is the setting for many allocations problems such as school choice, course selection or office selection (for example, when a business or a department moves into a new building). In such markets, the Walras correspondence can serve both as real allocation mechanism and as a benchmark for evaluating other mechanisms.

Hylland and Zeckhauser (1979) note that in a nontransferable utility economy with unit-demand preferences, some Walrasian equilibria are inefficient. Specifically, nontransferable utility economies may have equilibria in which some agents do not purchase the least expensive optimal option in their budget sets and equilibria with this property may be inefficient. To address this problem, Mas-Colell (1992) introduces the concept of a *strong* equilibrium; that is, a Walrasian equilibrium in which every consumer chooses the least expensive optimal bundle and proves that strong equilibria are Pareto efficient.

Definition: *A price $p \in I\!R_+^L$ and a random allocation α is a strong equilibrium for the nontransferable utility economy $\mathcal{E}^*$ if*

(1) α is feasible for the economy $\mathcal{E}^*$;

(2) for all i, α_i is optimal given budget $B(p, b_i)$ and costs no more than any other optimal random consumption;

(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i \geq 1} \xi_i^j = 1$.

The theorem below establishes the existence of a strong and, therefore, Pareto efficient equilibrium for the nontransferable utility economy.

Theorem 2: *The nontransferable utility economy has a strong equilibrium.*

Our proof of Theorem 2 relies on Theorem 1: we consider the sequence of limited transfers economies $\mathcal{E}_n^o = \{(nu_i, w_i, b_i)_{i=1}^k\}$ for $n = 1, 2 \ldots$ where $w_i^j = 0$ for all j and i. Hence, $\mathcal{E}_n^o$ is a limited transfers economy in which agent i's endowment of goods is equal to her endowment of goods in $\mathcal{E}^*$ (i.e., zero), her endowment of the divisible good is the same as her endowment of fiat money in $\mathcal{E}^*$ and her utility function is n-times her utility function in $\mathcal{E}^*$. Then, we appeal to Theorem 1 to conclude that each $\mathcal{E}_n^o$ has an equilibrium (p^n, α^n). Since this sequence lies in a compact set, it has a limit point which we show to be an equilibrium of $\mathcal{E}^*$. This equilibrium must be a strong since it is a limit point of equilibria in which money has intrinsic value.

3.2 Nontransferable Utility Economies with Production

In this subsection, we describe how a production set can be incorporated into a nontransferable utility economy in a manner that ensures the existence of a Walrasian equilibrium. In the next section, we describe how aggregate feasibility constraints induce a production technology in course selection problems.

We assume, without loss of generality, that there is a single firm. The function $f : X \to \{0, 1 \ldots\}$ describes the technology of this firm: the set of feasible production plans is $\mathcal{I} = \{x \in X \,|\, \sigma(x) \leq f(x)\}$. We say that the technology has a submodular bound if f satisfies the following three properties: (1) $f(o) = 0$; (2) $f(x) \geq f(y)$ if $x \geq y$ and (3) $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$.[18]

[18] The set I is a matroid if and only if f satisfies(1)-(3). In Appendix A, we offer two alternative definitions and a few other relevant notions and results from matroid theory.

Definition: $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$ *is a production economy with nontransferable utility if* $\mathcal{I}$ *has submodular bound and for all* i, u_i *satisfies the substitutes property,* $b_i > 0$ *and* $o \in dom\, u_i$.

In the economy with production, a random allocation α is a probability distribution over $X^N \times \mathcal{I}$. For any such α, the marginal α_i is the random consumption for agent $i = 1, \ldots, N$ and the marginal α_{N+1} is the production plan for the producer. A random allocation α is *feasible* for the economy $\tilde{\mathcal{E}} = \{(u_i, 1)_{i=1}^N, \mathcal{I}\}$ if, for all (ξ, z) such that $\alpha(\xi, z) > 0$, $\sum_{i=1}^N \xi_i \leq z$, $z \in \mathcal{I}$ and $\xi_i \in \text{dom}\, u_i$ for all i. The definitions of budget sets and consumer optimality remain unchanged. The random allocation α is *producer optimal* if $\alpha(\xi, z) > 0$ implies $pz \geq pz'$ for all $z' \in \mathcal{I}$.

Definition: *A price* $p \in I\!R_+^L$ *and a random allocation* α *is a strong equilibrium for the production economy with nontransferable utility* $\tilde{\mathcal{E}}$ *if*

(1) α *is feasible for* $\tilde{\mathcal{E}}$*;*

(2) for all i, α_i *is optimal given budget* $B(p, b_i)$ *and costs no more than any other optimal random consumption;*

(3) α *is producer optimal;*

(4) $p^j > 0$ *and* $\alpha(\xi, z) > 0$ *imply* $\sum_{i=1}^N \xi_i^j = z^j$.

Hence, with production, a Walrasian equilibrium specifies prices, a random allocation and a random production plan. The implied random consumption and production plans must be feasible and optimal for both the consumers and the producer. The definition of a strong equilibrium is as in the previous section: the Walrasian equilibrium (p, α) is a strong equilibrium if for each agent i, α_i is the cheapest optimal random consumption for i given the budget constraint.

Theorem 3: *The production economy with nontransferable utility has a strong equilibrium.*

In the next section, incorporate individual constraints into the market design problem by building them into the utility function of agents and allow for aggregate constraints by incorporating a suitable production technology into the market design problem.

4. Individual, Group and Aggregate Constraints

An individual constraint restricts the number of goods that a single agent can consume from a specified set of goods. A group constraint restricts the total number of goods that a particular group of agents can consume from a specified set of perfect substitutes. Finally, aggregate constraints restrict the various combinations of goods available for the entire population.

An example of an individual constraint is Princeton University's *rule of 12.* According to this rule, no more than 12 courses in a student's major may be counted towards the 31 courses needed to obtain the A.B. degree. Distribution requirements are a second type of individual constraint. For example, Art and Archaeology students at Princeton University must take at least one course in each of the following three areas: group 1 (ancient), group 2 (medieval/early modern), and group 3 (modern/contemporary). An example of a group constraint is the requirement that at least 50 percent of the slots in each school must go to students who live in the school's district. Similarly, the "controlled choice" constraints in school assignment that require schools to balance the gender, ethnicity, income, and test score distributions among their students, are group constraints.[19] Aggregate constraints define the feasible allocations for the entire economy. For example, suppose two versions of introductory physics are being offered: Phy 101, the version that does not require calculus and Phy 103, the version that does require calculus. Suppose each of these classes can accommodate 120 students, but because both courses have lab requirements and lab facilities are limited, the total enrollment in the two courses can be no greater than 200 students.

4.1 Group Constraints

In many applications, one group is given priority over another. For example, suppose that the maximal enrollment in a particular physics class is n and there are $m < n$ physics majors who are required to take that class. Thus, at most $n - m$ non-majors can enroll in the class. More generally, a group constraint (A, n) for the group $I \subset \{1, \ldots, N\}$ states that the agents in I can collectively consume at most n units from the set A, where A is a collection of *perfect substitutes* (for all agents).

[19] See Abdulkadiroğlu, Pathak and Roth (2005) for examples of such constraints in practice.

To accommodate this constraint, pick any $|A| - n$ element subset B of A. Then, replace each u_i for $i \in I$ with u_i' such that

$$u_i'(x) = u_i\left(x \wedge \chi^{B^c}\right)$$

Thus, the new utility for members of group I is the their original utility restricted to the complement of B. As we noted above, restrictions of utilities to a subset of choices satisfy the substitutes property if the original utility satisfies the substitutes property. Moreover, since elements of A are perfect substitutes, restricting members of group I to B^c is equivalent to restricting their aggregate consumption from the set A. Thus, a group constraint can be accommodated by modifying utility functions of the group's members.

4.2 Individual Constraints

The simplest individual constraints are bounds on the number of goods an agent may consume from a given set of goods. For example, a student may be required to take 4 classes each semester, but may be barred from enrolling in more than 6. We can incorporate this constraint by modifying the student's unconstrained utility function u as follows:

$$[\hat{u}]^6(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq 6}} \hat{u}(y)$$

where

$$\hat{u}(x) = [u]_4(x) = \begin{cases} u(x) & \text{if } \sigma(x) \geq 4 \\ -\infty & \text{if } \sigma(x) < 4 \end{cases}$$

The modified utility $[\hat{u}]^6 = [[u]_4]^6$ incorporates the lower bound constraint by restricting the effective domain of u to those bundles that satisfy the constraint. It incorporates the upper bound by imposing satiation above the constraint. Next, we generalize these constraints and impose bounds on overlapping subsets of goods. To preserve the gross substitutes property, we require the utility function to be separable across subsets of goods that must satisfy a constraint. A collection of goods, $A \subset H$, is a *module* for the utility u if

$$u(x) = u(x \wedge \chi^A) + u(x \wedge \chi^{A^c})$$

Note that this condition is symmetric: if A is a module, then so is A^c. For example, suppose that A is the set of all humanities courses and A^c is the set of all other courses.

If a student's utility for various combination of humanities courses is independent of her utility over various combinations of the other courses, then A is a module. A collection of sets, $\mathcal{H}$, is a *hierarchy* if $A, B \in \mathcal{H}$ and $A \cap B \neq \emptyset$ implies $A \subset B$ or $B \subset A$. Given any u, we say that the hierarchy $\mathcal{H}$ is *modular* if each element of $\mathcal{H}$ is a module of u.

A modular constraint places bounds on the agent's consumption for subsets of items that form a modular hierarchy. The collection $c = \{(A(k), (l(k), h(k))_{k=1}^{K}\}$ is a *constraint* if $l(k), h(k)$ are integers, $A(k) \subset H$, and $X(c) \neq \emptyset$, where the set

$$X(c) := \{x \in X | \forall k, \sigma(x \wedge \chi^{A(k)}) \geq l(k)\} \cap \{x \in X | \forall k, \sigma(x \wedge \chi^{A(k)}) \leq h(k)\}$$

is the set of consumptions that satisfy both the lower bounds and upper bounds of c. The constraint $c = \{(A(k), (l(k), h(k))_{k=1}^{K}\}$ is a *modular constraint* for u if $\mathcal{H} = \{A(1), \ldots, A(K)\}$ is a modular hierarchy for u and $X(c) \cap \operatorname{dom} u \neq \emptyset$.

As an example of a modular constraint, suppose that students must take at least 3 humanities classes and at least 4 social science classes; moreover, each student is required to take at least 8 but no more than 12 classes overall. In this case, the constraint is modular if the student's utility over combinations of science courses is independent of her utility over combinations of humanities courses.

Given utility u, define $u(c, \cdot)$, the c-constrained u, as follows:

$$u(c, x) = \begin{cases} \max_{\substack{y \in X(c) \\ y \leq x}} u(y) & \text{if } X(c) \cap \{y \,|\, y \leq x\} \neq \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

Then, the effective domain of $u(c, \cdot)$ is $\operatorname{dom} u(c, \cdot) = X(c) \cap \operatorname{dom} u \neq \emptyset$.

Lemma 3: *If u satisfies the substitutes property and c is a modular constraint for u, then $u(c, \cdot)$ satisfies the substitutes property.*

To see how the substitutes property may fail if the constraints are not modular, consider the utility function described in equation (1) below. Let $H = \{0, 1, 2, 3\}$. Then,

$$u(x) = \begin{cases} 2 & \text{if } x^j \cdot x^{j \oplus 1} > 0 \text{ for some } j \in H \\ 0 & \text{if } x = o \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

where $\oplus$ denotes addition modulo 4. Note that u satisfies the substitutes property.[20] Let $A = \{0, 1\}$ and suppose that the agent is restricted to consuming at most one unit from A. To see that the resulting utility function does not satisfy the substitutes property, set $p^0 = p^1 = p^2 = p^3 = 1/2$. Then, $\{0, 3\}$ is an optimal consumption set at prices p. The substitutes property fails since at prices q such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$, there is no optimal bundle that contains item 0.

To see how the substitutes property may fail if the modular constraints do not form a hierarchy, consider the following utility: $u(x) = \sigma(x)$. Let $H = \{0, 1, 2, 3\}$, then any subset of H is a module of u. Suppose the constraints are $(\{1, 2\}, 0, 1)$, $(\{0, 1\}, 0, 1)$ and $(\{0, 1, 2, 3\}, 0, 2)$. Then, at $p^j = 1/2$ for all $j \in H$, $\{1, 3\}$ is an optimal consumption set at p. Again, the substitutes property fails since at prices q such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$, there is no optimal consumption set that contains 1.

4.3 Aggregate Constraints

Suppose an economics department schedules classes in labor economics, intermediate microeconomics and corporate finance. There are two types of TAs, those that can cover labor economics and microeconomics, and those that can cover microeconomics and corporate finance. There are 6 TAs of each type. The university would like a student-TA ratio of ten-to-one in every class. TA time is fungible across different classes. Hence, at most 60 students can enroll in labor economics, at most 60 students can enroll in corporate finance and at most 120 students can enroll in any of the three types of classes. Alternatively, consider a company that must design a new office building. The building has at most 200 offices each either small, medium or large; no more than 10 can be large and no more than 25 can be large or medium.

In both of the examples above, we can describe the aggregate constraint as a hierarchy $\mathcal{H}$ that limits the supply of available items. That is, the aggregate constraint has the form $c = \{(A(k), n(k))_{k=1}^K\}$ where $A(k) \subset H$ for all k, $\{A(k)\}_{k=1}^K$ is a hierarchy and each $n(k)$ is a natural number describing the maximal quantity of indivisible goods that can be supplied from set $A(k)$.

[20] It is a convolution of the two unit demand preferences v and $\hat{v}$ where v takes the value 1 at any x such that $x^0 > 0$ or $x^2 > 0$ and is equal to zero otherwise and $\hat{v}$ takes the value 1 at any x such that $x^1 > 0$ or $x^3 > 0$ and is equal to zero otherwise.

We will re-interpret aggregate constraints as a production technology; that is, define a production possibility set for the economy that includes only the output combinations consistent with the given constraints. To see how we can embed a collection of aggregate restrictions into a production set, let (A, n) denote a single aggregate restriction. Hence, the set of feasible production plans given any X and the restriction (A, n) is:

$$X(A, n) = \{x \in X \,|\, \sigma(x \wedge \chi^A) \leq n\}$$

We can nest aggregate constraints the same way that we nested individual and group constraints; that is, we can construct a hierarchy of aggregate constraints. Given any hierarchy of aggregate restrictions $d = \{(A(k), n(k))_{k=1}^K\}$, let $\mathcal{I}_d$ denote the set of all production plans consistent with d; that is,

$$\mathcal{I}_d = \bigcap_{a \in d} X(a)$$

The following lemma reveals that hierarchical constraints are a special case of constraints that can be described by a submodular bound.

Lemma 4: *If d is a hierarchical collection of aggregate constraints, then $\mathcal{I}_d$ has a submodular bound.*

As the following example illustrates, production sets with submodular bounds are more general than production sets corresponding to hierarchical constraints:

Example: Let $N = \{1, \ldots, 6\}$, let $A = \{1, 2\}, B = \{3, 4\}, C = \{5, 6\}$ and let $\mathcal{I}$ be the set defined by the constraints

$$\sum_{A \cup B} x^j \leq 2; \sum_{A \cup C} x^j \leq 2; \sum_{A} x^j \leq 1; \sum_{N} x^j \leq 3$$

Then, $\mathcal{I}$ has a submodular bound but the constraint is not a hierarchy.

To establish existence of an equilibrium that meets all constraints, we must not only assume the existence of production plan that enables consumptions in the effective domain of every each agent's modified utility but we must also guarantee the these consumption are in the "interior" of each agent's budget set. To address this issue, we add two assumptions to our earlier model. First, we assume that aggregate resources can be divided into $N+1$ consumption bundles that meet every consumer's lower bound constraint. Second, we assume that all agents have equal endowments of fiat money, which we normalize to 1.

Definition: $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$ *is a production economy with nontransferable utility and modular constraints if*

(1) u_i *satisfies the substitutes property for all* i*;*

(2) c_i *is a modular constraint for* u_i *for all* i*;*

(3) for some $z \in \mathcal{I}$*, there are* $x_1, \cdots, x_{N+1}$ *such that* $\sum_{k=1}^{N+1} x_k \leq z$ *and* $x_k \in \operatorname{dom} u_i(c_i, \cdot)$ *for all* k, i*.*

Theorem 4 below establishes the existence of a strong equilibrium in a market design problem with modular constraints and aggregate constraints corresponding to a production technology with submodular bound.

Theorem 4: *A strong equilibrium for the production economy with nontransferable utility and modular constraints* $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$ *exists if* $\mathcal{I}$ *has a submodular bound. Strong equilibrium allocations are Pareto efficient.*

In Theorem 4, the assumption of equal money endowments ensures that every consumer can afford some element in the effective domain of her utility function. If money endowments were arbitrary, we would need to add an assumption that preserves this feature.

5. Conclusion

Our results suggest that Walrasian methods can be employed in a variety of market design problems whenever preferences satisfy the substitutes property. Gul and Stacchetti (1999) show that given any utility function that does not satisfy the substitutes property, it is possible to construct a transferable utility economy with N agents, one with the preference in question and $N-1$ with a substitutes preference such that no equilibrium exists. Hence, it seems unlikely that a general existence result for the nontransferable utility economy that permits a larger set of preferences than the substitutes class can be proved.

However, Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in consumption. In particular, they show that if the goods can be partitioned into two classes such that all agents consider goods within each element of the partition substitutes and consider goods in different elements complements, then an equilibrium exists in the corresponding transferable utility economy. Shioura and Yang (2015) generalize the Sun-Yang result and Baldwin and Klemperer (2019) provide an even more general result. They identify classes of utility functions for which existence of equilibrium in transferable utility economies is guaranteed given any aggregate endowment. One possible extension of the current work would be the see if the results of this paper can be extended to the classes of preferences for which Baldwin-Klemperer prove their existence result.[21]

6. Appendix A

Unless indicated otherwise, the definitions and results below can be found in Oxley (2011):

A *matroid* $\mathcal{I} \subset X$ is a collection of sets such that (I1) $\emptyset \in \mathcal{I}$, (I2) $y \in \mathcal{I}$, $x \leq y$ implies $x \in \mathcal{I}$ and (I3) $x, y \in \mathcal{I}$, $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x + \chi^j \in \mathcal{I}$.

There are various alternative ways to describe a matroid. One is the definition in the text with a non-decreasing, submodular function $f : X \to \{0, 1, \ldots\}$ that satisfies $f(0) = 0$. The matroid corresponds to the set $\mathcal{I} = \{x \in X \,|\, \sigma(y) \leq f(y), \forall y \leq x\}$.

Alternatively, we can describe the matroid $\mathcal{I}$ via its maximal elements. Let $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \geq x \text{ and } y \in \mathcal{I} \text{ implies } y = x\}$ be the set of all maximal elements of $\mathcal{I}$. Then, $\mathcal{B}(\mathcal{I})$ is a *basis system*; that is, (B1) $\mathcal{B}(\mathcal{I})$ is nonempty and (B2) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k \in \mathcal{B}(\mathcal{I})$.

If $\mathcal{B} \subset X$ satisfies (B1) and (B2), then $\mathcal{I} = \{x \in X \,|\, x \leq y \text{ for some } y \in \mathcal{B}\}$ is a matroid and $\mathcal{B} = \mathcal{B}(\mathcal{I})$. Every basis system $\mathcal{B}$ satisfies the following stronger version of (B2): (B2*) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in$

[21] Since multiplying a utility function by a positive constant preserves its discrete concavity and demand type and since Lemma B3 in the appendix holds whenever the set of equilibria is nonempty, we would expect that Theorems 1-3 generalize to the Baldwin-Klemperer setting. It is hard to known if imposing modular constraints preserves the demand type of a utility function. Hence, we do not know what kind of individual constraints can permitted in this more general setting.

$\mathcal{B}(\mathcal{I})$. Also, all elements of a basis system have the same cardinality; that is, if $x, y \in \mathcal{B}$ and $\mathcal{B}$ is a basis system, then $\sigma(x) = \sigma(y)$. Hence, for any matroid $\mathcal{I}$, $\mathcal{B}(I)$ is the set of elements of $\mathcal{I}$ with the maximal cardinality; $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \in \mathcal{I} \text{ implies } \sigma(x) \geq \sigma(y)\}$.

Gul and Stacchetti (2000) show that if u satisfies the substitutes property, then the set of elements of $D_u(p)$ with the smallest cardinality is a basis system for every p.

For any $\mathcal{B}$, let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. If $\mathcal{B}$ is a basis system, then $\mathcal{B}^\perp$ is also a basis system and is called the *dual* of $\mathcal{B}$.

A function $r : X \to I\!N$ is a *rank function* if (R1) $0 \leq r(x) \leq \sigma(x)$, (R2) $x \leq y$ implies $r(x) \leq r(y)$ and (R3) $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$. For any rank function, r, the set of all minimal (in the natural order on $I\!R^L$) maximizers of r is a basis system. Also, given any matroid $\mathcal{I}$, the function r defined by $r(x) = \max\{\sigma(y) \,|\, y \leq x, y \in \mathcal{I}\}$ is a rank function.

A *weighted matroid* is a function ρ, defined as follows: given an additive and monotone utility function v and matroid $\mathcal{I}$, let $\rho(x) = \max_{\substack{y \leq x \\ y \in \mathcal{I}}} v(y)$. A rank function is a special case of a weighted matroid, one in which $v(x) = \sigma(x)$.

To define academic preferences, we adopt the following concept from Yokote (2017): $Y \subset X$ is an $M^\sharp$-*convex set* if $x, y \in Y$ and $x^j > y^j$ implies either $x - \chi^j, y + \chi^j \in Y$ or there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. It is easy to see that a set Y is $M^\sharp$-convex if and only if the function I_Y^* define below is $M^\sharp$-concave:

$$I_Y^*(x) = \begin{cases} t & \text{if } x \in Y \\ -\infty & \text{otherwise} \end{cases}$$

for some $t \in I\!R_+$.

The utility function u is an academic preference if there exists an additive and monotone utility function v and an $M^\sharp$-convex set Y such that

$$u(x) = \begin{cases} \max_{\substack{y \in Y \\ y \leq x}} v(y) & \text{if there is } y \leq x,\, y \in Y \\ -\infty & \text{otherwise} \end{cases}$$

Fact: Every academic preference satisfies the substitutes property.

Proof: Murota (2009) shows that a weighted matroid is $M^\sharp$-concave. The same argument establishes that an academic preference is $M^\sharp$ concave. Since $M^\sharp$-concavity is equivalent to the substitute property, the fact follows. □

We will conclude Appendix A by showing that the utility function in example 3 is an academic preference. Identify H with the edges of an undirected graph with vertices $\{a, b, c, d, e\}$. Then, the set of feasible schedules, Y, is the collection of all sets of edges with 3 or 4 elements that contain no cycles. To prove that the utility function in example 3 is an academic preference, we we need to show that Y is $M^\natural$-convex. Let Z be the set of all subsets of H that contain no cycles. It is well-known that Z is a matroid. Then, let r be the rank function of the matroid Z. Murota (2009) shows that a weighted matroid (and in particular, a rank function) is $M^\natural$-concave. Hence, by Lemma 1 (its proof is in Appendix B), $[r]_3$, the 3-lower bound of r satisfies the substitutes property. Then, by Bing, Lehman and Milgrom (2004), $I_Y^* = [[r]_3]^3$, the 3-satiation of $[r]_3$, satisfies the substitutes property; that is $M^\natural$-concavity. Then, by the observation above, Y is $M^\natural$-convex.

7. Appendix B

7.1 Proof of Lemma 1

First, we will extend the definition of the single improvement property (SI) (Gul and Stacchetti, 1999) to include u such that $o \notin \operatorname{dom} u$ as follows:

Definition: *The function u has the single improvement property (SI) if for all p such that $D_u(p) \subset \operatorname{dom} u$ and all $x \in \operatorname{dom} u - D_u(p)$, there is y such that $U(x, p) < U(y, p)$, $|supp\,(x) - supp\,(y)| \leq 1$ and $|supp\,(y) - supp\,(x)| \leq 1$.*

Theorem 4.1 and Theorem 5.1 in Shioura and Tamura (2015) establish that the substitutes property, (SI) and $M^\natural$-concavity are equivalent. Also, a utility function u is submodular if it satisfies the substitutes property. Gul and Stacchetti (1999) show that (SI) is equivalent to the substitutes property for the effective domain X. Their proof reveals that the above modified definition of (SI) is equivalent to the substitutes property for a general effective domain.

The following proof is similar to the proof that k-satiation preserves substitutes property in Bing, Lehmann and Milgrom (2004). We first prove two auxiliary lemmas. Lemma B1 provides an alternative characterization of $M^\natural$-concavity.

Lemma B1: *Let u be a utility that satisfies the substitutes property. If $x, y \in \operatorname{dom} u$ with $x \not\geq y$ and $y \not\geq x$, then there is j, k such that $x^j > y^j$, $y^k > x^k$ and $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$.*

Proof: Since u satisfies the substitutes property, u is $M^\natural$-concave. Since $y \not\geq x$, there exists j with $x^j > y^j$. Since $x, y \in \operatorname{dom} u$, $M^\natural$-concavity implies that either there is k such that $y^k > x^k$ and $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$, hence we are done, or that $u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$. That is,

$$u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$$

Similarly, since $x \not\geq y$, there exists l with $y^l > x^l$. It follows from $M^\natural$-concavity that either there is k with $x^k > y^k$ such that $u(x - \chi^k + \chi^l) + u(y + \chi^k - \chi^l) \geq u(x) + u(y)$ and we are done, or

$$u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$

The above two inequalities together with the submodularity of u imply

$$u(x - \chi^j + \chi^l) - u(x - \chi^j) \geq u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$
$$u(y - \chi^l + \chi^j) - u(y - \chi^l) \geq u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$$

Hence,

$$u(x - \chi^j + \chi^l) + u(y - \chi^l + \chi^j) \geq u(x) + u(y)$$

as desired. □

The following lemma states that if a bundle with n elements does not maximize utility among all bundles with at least n elements, then we can increase its utility either by adding an element to it or replacing one of its elements with a different one.

Lemma B2: *Let u be a utility that satisfies the substitutes property. Let A, B be such that $|B| \geq n = |A|$, $\chi^A \in \operatorname{dom} u$ and $U(\chi^B, p) > U(\chi^A, p)$. Then, either there exists $l \notin A$ such that $U(\chi^A + \chi^l, p) > U(\chi^A, p)$ or there exists $k \in A$, $l \notin A$ such that $U(\chi^A + \chi^l - \chi^k, p) > U(\chi^A, p)$.*

Proof: Let $\mathcal{D}$ denote the utility maximizing bundles, at price p, among all bundles with at least n elements. Let B^* minimize the Hausdorff distance ($d(\hat{A}, \hat{B}) = |\hat{A} - \hat{B}| + |\hat{B} - \hat{A}|$) from A among the elements of $\mathcal{D}$. By assumption, $U(\chi^{B^*}, p) > U(\chi^A, p)$. Clearly $B^* - A \neq \emptyset$, otherwise, since $|A| = n, |B^*| \geq n$ and $B^* \subset A$, we have $A = B^*$, a contradiction. Since $\chi^A \in \operatorname{dom} u$, we have $\chi^{B^*} \in \operatorname{dom} u$.

First, assume $A - B^* \neq \emptyset$. By Lemma B1, there exists k, l with $k \in A - B^*$, $l \in B^* - A$ such that

$$u(\chi^A) + u(\chi^{B^*}) \leq u(\chi^A - \chi^k + \chi^l) + u(\chi^{B^*} - \chi^l + \chi^k)$$

Since the total cost of bundles on either side of the above inequality is the same, we have,

$$U(\chi^A, p) + U(\chi^{B^*}, p) \leq U(\chi^A - \chi^k + \chi^l, p) + U(\chi^{B^*} - \chi^l + \chi^k, p)$$

By assumption, $U(\chi^{B^*}, p) \geq U(\chi^{B^*} - \chi^l + \chi^k, p)$. If $U(\chi^{B^*}, p) = U(\chi^{B^*} - \chi^l + \chi^k, p)$, then $\chi^{B^*} - \chi^l + \chi^k$ is also optimal at price p among all bundles with at least n elements and $d(A, B^* \cup \{k\} - \{l\}) < d(A, B^*)$, which contradicts the definition of B^*. Thus, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^l + \chi^k, p)$ and by the inequality above, $U(\chi^A, p) < U(\chi^A - \chi^k + \chi^l, p)$.

Second, assume $A - B^* = \emptyset$. Then, since $A \neq B^*$, A is a strict subset of B^* and $|B^*| \geq n + 1$. For any $j \in B^* - A$, $d(B^* - \{j\}, A) < d(B^*, A)$ and $|B^* - \{j\}| \geq n$. Then, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^j, p)$ and therefore,

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j)$$

Since u is submodular u has decreasing marginal returns. Recall that $\chi^A + \chi^j \in \operatorname{dom} u$, $\chi^A \leq \chi^{B^*}$ and $j \notin A$. Hence,

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j) \leq u(\chi^A + \chi^j) - u(\chi^A)$$

That is, $U(\chi^A, p) < U(\chi^A + \chi^j, p)$. □

Proof of Lemma 1: Since $M^\natural$-concavity is equivalent to (SI), it suffices to show that $[u]_k$ satisfies the latter. Since the effective domain of $[u]_k$ is nonempty, there is x^* such that $\sigma(x^*) \geq k$, $u(x^*) = [u]_k(x^*)$ and x^* is optimal at price p for the utility function $[u]_k$.

Take any $x \in \operatorname{dom}[u]_k - D_{[u]_k}(p)$. Hence, $\sigma(x) \geq k$. We need to show there exists y such that $[U]_k(x,p) < [U]_k(y,p)$, $|\operatorname{supp}(x) - \operatorname{supp}(y)| \leq 1, |\operatorname{supp}(y) - \operatorname{supp}(x)| \leq 1$.

If $\sigma(x) = k$, then since $\sigma(x^*) \geq k$ and $U(x^*, p) > U(x,p)$, Lemma B2 yields the desired conclusion.

If $\sigma(x) > k$, then $\sigma(z) \geq k$ for any z such that $|\operatorname{supp}(x) - \operatorname{supp}(z))| \leq 1$ and $|\operatorname{supp}(z) - \operatorname{supp}(x)| \leq 1$. Recall that u satisfies (SI) and $x \notin D_u(p)$, $x \in \operatorname{dom}[u]_k \subset \operatorname{dom} u$. Then, there exists y such that $U(x,p) < U(y,p)$, $|\operatorname{supp}(x) - \operatorname{supp}(y)| \leq 1, |supp(y) - \operatorname{supp}(x)| \leq 1$. Since $\sigma(y), \sigma(x) \geq k$, $[U]_k(x,p) = U(x,p)$, $[U]_k(y,p) = U(y,p)$ and therefore, $[U]_k(x,p) < [U]_k(y,p)$. □

7.2 Proof of Lemma 2

This is a corollary of Theorem 8.2 in Shioura and Tamura (2015). They assume $\{o, \chi^H\} \in \operatorname{dom} u_i$, we instead assume that there exists a feasible allocation ξ such that $\sum \xi_i \leq \chi^H$ and $\xi_i \in \operatorname{dom} u_i$ for all i. By monotonicity, our assumption implies that there exists ξ^* such that $\xi^*_i \in \operatorname{dom} u_i$, $\sum_{i=1}^N \xi^*_i = \chi^H$ and

$$\sum_{i=1}^N u_i(\xi^*_i) = \max \left\{ \sum_{i=1}^N u_i(x_i) \middle| x_i \in \operatorname{dom} u_i, \forall i, \sum_{i=1}^N x_i = \chi^H \right\}$$

Then, the remainder of the proof follows Theorem 8.2 in Shioura and Tamura (2015). □

7.3 Proof of Theorem 1

Our existence proof relies on Lemma 2, a modification of Kelso and Crawford's proof of existence of an equilibrium for the transferable utility economy with substitutes. In a transferable utility economy, the set of equilibrium prices and the set of equilibrium allocations of divisible goods are independent of the initial endowments and we can state the consumers' problem as maximizing (over x)

$$U_i(x,p) = u_i(x) - p \cdot x$$

By assumption, $w_i \in \operatorname{dom} u_i$ for each i. Then, Lemma 2 establishes the existence of an equilibrium in deterministic allocations for the transferable utility gross substitutes economy. It is easy to see that given an equilibrium allocation α, the set of prices that support α; that is, p such that (p, α) is a an equilibrium, is defined by a finite set of linear weak inequalities and therefore is a compact and convex set. Since we are in a transferable utility setting, any Pareto efficient allocation must maximize total surplus. It is also easy to verify that if (p, ξ) is a deterministic equilibrium, and $\hat{\xi}$ is a social surplus maximizing allocation, then $(p, \hat{\xi})$ is also a Walrasian equilibrium. The following *exchangeability property* is a consequence of the last two observations: if (p, α) and $(\hat{p}, \hat{\alpha})$ are both equilibria, then $(p, \hat{\alpha})$ is also an equilibrium. Then, it also follows that the set of random equilibrium allocations is simply the convex hull of the set of deterministic equilibrium allocations and hence, the set of equilibrium prices for random allocations is the same as the set of equilibrium prices for deterministic allocations. It follows that for any transferable utility economy, there is a set of prices P^* and a set of random an allocations $\mathcal{A}^*$ such that the set of equilibria is $P^* \times \mathcal{A}^*$.

Since every price in P^* supports the same allocation, P^* is a nonempty, convex and compact set as it is defined by a finite set of linear weak inequalities. Since $\mathcal{A}^*$ is the set of surplus maximizing allocations, it is also a a nonempty, convex and compact set. We summarize these observation in Lemma B3 below.

Lemma B3: *For any transferable utility gross substitutes economy $\mathcal{E}$, the set of equilibria is $P^* \times \mathcal{A}^*$ for some nonempty compact convex set of prices P^* and the nonempty compact convex set of total surplus-maximizing random allocations $\mathcal{A}^*$.*

For the transferable utility economy $\mathcal{E}$, let $\mathcal{A}_o$ be the set of all feasible random allocations such that $\alpha(\xi) > 0$ implies $\xi_i \in \operatorname{dom} u_i$ for all i. Restricting attention to $\mathcal{A}_o$ is without loss of generality; for any $\alpha \notin \mathcal{A}_o$, there must be some agent with utility $-\infty$ which cannot be efficient in a transferable utility economy. Since the set of deterministic feasible allocations is finite and $w_i \in \operatorname{dom} u_i$ for each i, $\mathcal{A}_o$ is a nonempty, compact, convex subset of a Euclidian space. For any $\lambda = (\lambda_1, \ldots, \lambda_N) \in [0,1]^N$, we define the maximization problem:

$$M(\lambda) = \max_{\alpha \in \mathcal{A}_o} \sum_i \lambda_i u_i(\alpha_i)$$

We set $-\infty \times 0 = -\infty$; that is, when $\lambda_i = 0$, $\lambda_i u_i(\cdot)$ has the same effective domain as u_i and is 0 on the effective domain. Note that $M(\lambda)$ is a linear programming problem. Let $\Delta(\lambda)$ denote the set of solutions to this problem.

For $\lambda \in [0,1]^N$, define the transferable utility economy $\mathcal{E}(\lambda) = \{\lambda_1 u_1, \ldots, \lambda_N u_N\}$. Note that the transferable utility demand of u^j at price p is the same as the transferable utility demand of $\lambda_i u_i$ at price $\lambda_i p_i$ and hence $\mathcal{E}(\lambda)$ is a transferable utility gross substitutes economy.

By Lemma B3, $\Delta(\lambda)$ is the set of equilibrium allocations for the economy $\mathcal{E}(\lambda)$. Let $a = \sum_i u_i(\chi^H)$ and $\mathcal{P} = [0,a]^N$. Hence, any equilibrium price p of the transferable utility economy $\mathcal{E}(\lambda)$ must be in $\mathcal{P}$. Let $P^*(\lambda)$ be the set of all equilibrium prices for $\mathcal{E}(\lambda)$. Let $\mathcal{E}(\lambda)$ denote the transferable utility economy in which each agent i has utility $\lambda_i u_i$. Since the set of Walrasian equilibria in a transferable utility economy does not depend on initial endowments, we suppress them.

Lemma B4: *For any limited transfers economy $\mathcal{E}^o$, there exists $\lambda \in [0,1]^N$ and an equilibrium (p, α) of the corresponding transferable utility economy $\mathcal{E}(\lambda)$ such that if $\lambda_i < 1$, then $p\bar{\alpha}_i = b_i$ and if $\lambda_i = 1$, then $p\bar{\alpha}_i \leq b_i$.*

Proof: Let U_i^λ be consumer i's utility function in the transferable utility economy $\mathcal{E}(\lambda)$; that is, $U_i^\lambda(\theta, p) = \lambda_i u_i(\theta) - p\bar{\theta}$.

By Lemma B3, the correspondences Δ and P^* are nonempty, compact and convex valued. Since $\Delta(\lambda)$ is also the solution of the maximization problem defined above, Berge's Theorem ensures that Δ is uhc. Next, we will show that P^* is uhc as well. Since P^* compact-valued, it is enough to show that $\lambda(t) \in [0,1]^N$, $p(t) \in P^*(\lambda(t))$ for all $t = 1, 2, \ldots$, $\lim \lambda(t) = \lambda$ and $\lim p(t) = p$ implies $p \in P^*(\lambda)$.

Choose $\alpha(t) \in \Delta(\lambda(t))$ for all t. Since $\mathcal{A}$ is compact, we can assume, by passing to a subsequence if necessary, that $\alpha(t)$ converges. Let $\alpha = \lim \alpha(t)$. Since Δ is uhc, $\alpha \in \Delta(\lambda)$. Let θ be any random consumption. Then, since $\alpha(t)$ and is an equilibrium random allocation for the transferable utility economy $\mathcal{E}(\lambda(t))$,

$$U_i^{\lambda(t)}(\alpha_i(t), p(t)) = \lambda_i(t) u_i(\alpha_i(t)) - p(t) \cdot \bar{\alpha}_i(t) \geq \lambda_i(t) u_i(\theta) - p(t) \cdot \bar{\theta} = U_i^{\lambda_i(t)}(\theta, p(t))$$

Then, the continuity of U_i ensures that $U_i^{\lambda(t)}(\alpha_i, p) \geq U_i^{\lambda(t)}(\theta, p)$ for all θ and for all i. This implies that p is an equilibrium price for the transferable utility economy $\mathcal{E}(\lambda(t))$ and establishes the upper hemi-continuity of P^*.

Next, define correspondence Γ_i as follows:

$$\Gamma_i(p,z) = \begin{cases} [0,1] & \text{if } p(z-w_i) = b_i \\ 0 & \text{if } p(z-w_i) > b_i \\ 1 & \text{if } p(z-w_i) < b_i \end{cases}$$

Clearly, Γ_i is nonempty, convex and compact valued, and uhc.

Let $S = \mathcal{P} \times \mathcal{A}_o \times [0,1]^N$ and let

$$f(p,\alpha,\lambda) = P^*(\lambda) \times \Delta(\lambda) \times \Gamma_1(p,\bar{\alpha}_1) \times \cdots \times \Gamma_N(p,\bar{\alpha}_N)$$

Since P^*, Δ and the Γ_i's are nonempty, convex and compact valued, and uhc and the mapping $\alpha \to \bar{\alpha}_i$ is continuous, f is also nonempty, convex and compact valued, and uhc. Then, by Kakutani's Fixed-Point Theorem, there is an $s^* = (p^*, \alpha^*, \lambda^*)$ such that $f(s^*) = s^*$. Thus, (p^*, α^*) is a Walrasian equilibrium of the transferable utility economy $\mathcal{E}(\lambda^*)$.

We claim that $\lambda_i^* > 0$ for all i. Suppose $\lambda_i^* = 0$ for some i. Then, agent i's utility is identically 0 on the effective domain of u_i and since s^* is a fixed point of f, $p(\bar{\alpha}_i^* - w_i) \geq b_i > 0$. But, since $w_i \in \operatorname{dom} u_i$ and every consumption in the effective domain gives the agent utility 0, spending more than pw_i is inconsistent with utility maximization. Hence, α_i cannot be optimal, a contradiction.

To complete the proof of the lemma, we will show that $p^*(\bar{\alpha}_i^* - w_i) \leq b_i$ for all i and that the inequality is an equality whenever $\lambda_i^* < 1$. Since s^* is a fixed point of f and $\lambda_i^* > 0$, we must have $p^*(\bar{\alpha}_i^* - w_i) \leq b_i$. Similarly, since s^* is a fixed point of f, if $\lambda_i^* < 1$, we must have $p^*(\bar{\alpha}_i^* - w_i) = b_i$. □

To conclude the proof of Theorem 1, we will show that (p^*, α^*) is an equilibrium of the limited transfers economy $\mathcal{E}^o$. Consider any θ that i can afford (in the limited transfers economy). The optimality of α_i^* for i in the transferable utility economy implies

$$\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^* \cdot \bar{\theta} - p^* \cdot \bar{\alpha}_i^* \tag{2}$$

If $\lambda_i^* = 1$, then equation (2) implies that α_i is solves the utility maximization problem of agent i in the limited transfers economy $\mathcal{E}^o$. If $\lambda_i^* < 1$, since s^* is a fixed-point of

f, the right-hand side of equation (2) must be less than or equal to zero. Then, we have $\lambda_i^*(u_i(\theta)-u_i(\alpha_i^*)) \leq 0$ and hence $u_i(\theta)-u_i(\alpha_i^*) \leq \lambda_i^*(u_i(\theta)-u_i(\alpha_i^*)) \leq p^*\bar{\theta}-p^*\bar{\alpha}_i^*$ proving, again, that α_i is solves the utility maximization problem of agent i in $\mathcal{E}^o$. □

7.4 Proof of Theorem 2

Fix the nontransferable utility economy $\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$. Let $\mathcal{E}_n^o = \{(nu_i, w_i, b_i)_{i=1}^N\}$ for $n = 1, 2 \ldots$ be a sequence of limited transfer economies such that $w_i^j = 0$ for all j, i. Since $o \in \operatorname{dom} u_i$ for all i, by Theorem 1, each $\mathcal{E}_n^o$ has an equilibrium (p_n, α_n).[22] By monotonicity, $o \in \operatorname{dom} u_i$ implies that $\operatorname{dom} u_i = X$. Let $P = [0, \sum_i b_i]^L$. Note that p^n must be an element of P. Hence, the sequence (p^n, α^n) lies in a compact set and therefore has a limit point, (p, α). By passing to a subsequence if necessary, we may assume that (p, α) is its limit.

To complete the proof of Theorem 2, we will show that (p, α) is a strong equilibrium of $\mathcal{E}^*$. Clearly, α is feasible for $\mathcal{E}^*$. Since $p^n\bar{\alpha}_i^n \leq b_i^n$ for all n, $p\bar{\alpha}_i \leq b_i$, α_i is affordable for i in $\mathcal{E}^*$. Take any other affordable random allocation θ for i in $\mathcal{E}^*$; that is, $p\bar{\theta} \leq b_i$. We need to show that $u_i(\theta) \leq u_i(\alpha_i)$. First, assume $p\bar{\theta} < b_i$. Then, there exists $\epsilon > 0$ such that $p'\bar{\theta} < b_i$ for any $p' \in B_\epsilon(p) \cap I\!R_+^L$. Since $\lim p^n = p$, we can find $M > 0$ such that for all $n \geq M$, $p^n\bar{\theta} < b_i$. Hence, θ is affordable for i in $\mathcal{E}_n^o$ for $n \geq M$. Since α_i^n is an optimal consumption, $nu_i(\theta) - p^n\bar{\theta} \leq nu_i(\alpha_i^n) - p\bar{\alpha}_i^n$; that is, $u_i(\theta) - u_i(\alpha_i^n) \leq (p^n\bar{\theta} - p^n\bar{\alpha}_i^n)/n \leq b_i/n$ for all $n \geq M$. Then, the continuity of u_i ensures $u_i(\theta) \leq u_i(\alpha_i)$ as desired.

Next, assume $p\bar{\theta} = b_i$. Choose $\epsilon \in (0, 1)$. Then, $\theta_\epsilon := (1-\epsilon)\theta + \epsilon\delta_o$ satisfies $p\bar{\theta}_\epsilon < b_i$ and, therefore, by the argument in the previous paragraph $u_i(\theta_\epsilon) \leq u_i(\alpha_i)$. Since $u_i(\theta_\epsilon)$ is continuous in ϵ and ϵ was arbitrary, it follows that $u_i(\theta) \leq u_i(\alpha_i)$. Thus, α_i is optimal for i at prices p in $\mathcal{E}^*$.

To prove that all goods with strictly positive prices are allocated to the agents, it is enough to show that $p^j(1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$ whenever $p^j > 0$. This follows since $p^{nj}(1 - \sum_{i=1}^N \bar{\alpha}_i^{nj}) = 0$ for all j, n and hence, the same equality holds in the limit as n goes to infinity. Thus, (p, α) is an equilibrium of $\mathcal{E}^*$.

[22] When there is no risk of confusion, we use superscripts to specify the particular indivisible good (with generic element j) and the particular element in a sequence of prices or allocations (with generic element n, m). Otherwise, we use double superscripts.

To conclude, we will show that (p, α) is a strong equilibrium; that is, for all i, $u_i(\theta) = u_i(\alpha_i)$ implies $p\bar{\theta} \geq p\bar{\alpha}_i$. If not, assume that $p\bar{\theta} < p\bar{\alpha}_i$ for some θ such that $u_i(\theta) = u_i(\alpha_i)$ and consider two cases: (1) agent i is satiated at θ; that is, $u_i(\theta) = u_i(\alpha_i) = u_i(\chi^H)$ or (2) she is not satiated at θ. If (1) holds, then for sufficiently large n, purchasing θ instead of α_i is affordable for i at all p^n and $b_i - p\bar{\theta} > b_i - p\bar{\alpha}_i^n \geq 0$, contradicting the optimality of α_i^n for i in $\mathcal{E}_n^o$. If (2) holds, then choose $0 < r < 1$ such that $p(r\chi^H + (1-r)\bar{\theta}) < p\bar{\alpha}_i$. Again, for n sufficiently large, the random consumption $r\delta_{\chi^H} + (1-r)\theta$, where δ_{χ^H} is the degenerate lottery that yields χ^H for sure, is affordable at p^n and yields a higher utility than α_i^n, contradicting its optimality in $\mathcal{E}_n^o$. $\square$

7.5 Proof of Lemma 3 and Lemma 4

Proof of Lemma 3: By assumption, the effective domain of $u(c, \cdot)$ is nonempty. Recall that the operations that take u to u^z (the z-constrained u), $[u]^k$ (the k-satiation of u) and $[u]_k$ (the k-lower bound u) all preserve the substitutes property. Similarly, the binary operation $u \odot v$ (the convolution of u, v) also preserves the substitutes property. Then, to complete the proof of the lemma, we note that given any modular constraint $c = \{(A(k), (l(k), h(k))_{k=1}^K\}$ for u, we can express $u(c, \cdot)$ as a finite composition of these operations applied to u. This is straightforward; for example, let $c = \{\{(A(k), (l(k), h(k))_{k=1}^4\}$ where $A_1, A_2, A_3 \subset H$ are disjoint sets and $A_4 = A_1 \cup A_2$. Then, define

$$\hat{u} = \big([[v \odot w]^{h_4}]_{l_4} \odot [[u^{z_3}]^{h_3}]_{l_3}\big) \odot u^{z_5} \qquad \text{where}$$
$$v = [[u^{z_1}]^{h_1}]_{l_1}$$
$$w = [[u^{z_2}]^{h_2}]_{l_2}$$

$z_i = \chi^{A_i}$ for all $i = 1, 2, 3, 4$ and $z_5 = \chi^{H-A}$ for $A = \bigcup_{i=1}^4 A_i$. Since each utility function on the right-hand side of the equation above satisfies the substitutes property and all of the operations applied to them preserve the substitutes property, $\hat{u}$ satisfies the substitutes property as well, and since c is a modular constraint for u, $\hat{u} = u(c, \cdot)$. $\square$

Proof of Lemma 4: Clearly, $o \in \mathcal{I}_d$ and $x, y \in \mathcal{I}_d$ and $x \leq y$ implies $x \in \mathcal{I}_d$. Hence, we need only prove that $x, y \in \mathcal{I}_d$ and $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x^j + \chi^j \in \mathcal{I}_d$.

We order d, the hierarchy of constraints, in the obvious way: $(A,k) \succ (B,n)$ if $B \subset A$. Call j a *free* element in d if j is not an element of any A such that $(A,n) \in d$ for some n. Otherwise, call j a constraint element. Let $F \subset H$ be the set of free elements in d and let $F^c = H \backslash F$ be the set of constraint elements. Suppose there is $j \in F$ such that $y^j > x^j$. Then, clearly $x + \chi^j \in \mathcal{I}_d$ and we are done. Otherwise, $x^j \geq y^j$ for all $j \in F$ and hence there must be some $\succ$-maximal constraint, (A,k), such that $\sigma(y \wedge \chi^A) > \sigma(x \wedge \chi^A)$. Let $A_1 = A$ and $n_1 = k$.

Then, there is either $j \in A_1$ such that $y^j > x^j$, $j \notin B$ for any B such that $(A_1, n_1) \succ (B, n')$ in which case we have $x + \chi^j \in \mathcal{I}_d$ and we are done, or there is no such j. In the latter case, there must be an immediate predecessor of (A_1, n_1) such that $\sigma(y \wedge \chi^B) > \sigma(x \wedge \chi^B)$. Let (A_2, n_2) be this immediate predecessor and continue in this fashion until we end up with (A_l, n_l) and $j \in A_l$ such that $\sigma(y \wedge \chi^{A_k}) > \sigma(x \wedge \chi^{A_k})$ for all $k = 1, \ldots, l$ and $y^j > x^j$. Hence, $x + \chi^j \in \mathcal{I}_d$. □

7.6 Proof of Theorem 3

Let $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{I}\}$ be a nontransferable utility economy with production. Then, let $\mathcal{B}$ be the production possibility frontier of the technology $\mathcal{I}$; that is, $\mathcal{B} = \{x \in \mathcal{I} \,|\, y \geq x, y \in \mathcal{I} \text{ implies } y = x\}$. Hence, $\mathcal{B}$ is a basis system (Appendix A). Define

$$H = \{1, \ldots, L\} = \{j \,|\, x^j > 0 \text{ for some } x \in \mathcal{B}\}$$

Let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. Then $\mathcal{B}^\perp$ is a basis system since $\mathcal{B}$ is a basis system (Appendix A). Let r be the rank function associated with $\mathcal{B}^\perp$; that is,

$$r(x) = \max\{\sigma(x \wedge y) \,|\, y \in \mathcal{B}^\perp\}$$

Since every weighted matroid satisfies the substitutes property, so does r (Appendix A). Then, let $r^* := r(\chi^H) = r(x)$ for all $x \in \mathcal{B}^\perp$. Also, let $B = \sum_{i=1}^N b_i$ and $u_{N+1} = 2Br$ and define the following sequence of limited transfer economies:

$$\mathcal{E}_n^o = \{(nu_i, o, b_i)_{i=1}^N, (u_{N+1}, \chi^H, 1)\}$$

Lemma B5: *The economy $\mathcal{E}_n^o$ has an equilibrium (p^n, α^n) such that $\alpha(\xi) > 0$ implies $\chi^H - \xi_{N+1} \in \mathcal{I}$ and there is a real number K such that $p^{nj} \leq K$ for all $j \in H$ and n.*

Proof: Theorem 1 implies that the economy $\mathcal{E}_n^o$ has an equilibrium (α^n, p^n). Since $r(\chi^H) = r^*$, the equilibrium utility of agent $N+1$ must be at least $r^* + 1$. If $\alpha(\xi, z) > 0$ for some $\xi = (\xi_1, \ldots, \xi_{N+1})$ such that $r(\xi_{N+1}) < r^*$, then since the total money endowment of the economy is $B + 1$, we must have

$$u_{N+1}(\theta_{N+1}) + B + 1 = 2Br(\xi_{N+1}) + B + 1 \geq u_{N+1}(\chi^H) + 1 = 2Br^* + 1$$

for some ξ_{N+1} such that $r(\xi_{N+1}) \leq r^* - 1$. Then, the display equation above yields $B \leq 0$ a contradiction. Hence, $\chi^H - \xi_{N+1} \in \mathcal{I}$.

If, for some j, the sequence p^{nj} where unbounded, then the equilibrium utility of agent $N+1$ would also have to be unbounded, since she owns good j. However, a utility greater than $Br^* + 1 + B$ is not feasible for player $N+1$ given the aggregate endowment. Hence, there exists a real number K such that $p^{nj} \leq K$ for all j, n. $\square$

Lemma B5 ensures that the sequence (p^n, α^n) lies in a compact set. Then, it has a limit point (p, α). Set $b_{N+1} = p\chi^H + 1$ and define the following nontransferable utility economy:

$$\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N, (u_{N+1}, b_{N+1})\}$$

We can repeat the arguments in the proof of Theorem 2 to conclude that (p, α) is a strong equilibrium $\mathcal{E}^*$[23]. Then, the following lemma suffices to conclude the proof of Theorem 3:

Lemma B6: *If (p, α) is a strong equilibrium of $\mathcal{E}^*$, then $(p, \hat{\alpha})$ such that $\hat{\alpha}(\xi, z) = \alpha(\xi, \chi^H - z)$ is a strong equilibrium of $\tilde{\mathcal{E}}$.*

Proof: Let (p, α) be a strong equilibrium of $\mathcal{E}^*$. Note that agent $N+1$ is satiated at her initial endowment χ^H; that is, $2Br$ is maximal. Moreover, z' maximizes $r(\cdot)$ if and only if $z' \geq \chi^H - z$ for some $z \in \mathcal{B}$; that is, if and only if $z \in \mathcal{I}$. Thus, consumer optimality of agent $N+1$ implies $z \in \mathcal{I}$ whenever $\hat{\alpha}(\xi, z) > 0$. Since the equilibrium is strong, it follows that $p \cdot (\chi^H - z) \leq p \cdot (\chi^H - z')$ for all $z' \in \mathcal{I}$ and, therefore, $pz \geq pz'$ for all $z' \in \mathcal{I}$.

[23] These arguments carry over despite the fact that agent $N+1$'s utility of indivisible goods is not going to infinity along the sequence $\mathcal{E}_n^o$.

Parts (1), (2) and (4) of the definition of a strong equilibrium of $\tilde{\mathcal{E}}$, then follow from the definition of a strong equilibrium of $\mathcal{E}^*$. $\square$

7.7 Proof of Theorem 4

Let $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$ be a nontransferable utility production economy with modular constraints. As in the proof of Theorem 3, let $\mathcal{B}$ be the production possibility frontier of the technology $\mathcal{I}$, $H = \{1, \dots, L\} = \{j \,|\, x^j > 0 \text{ for some } x \in \mathcal{B}\}$ and $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. Then, let r be the rank function associated with $\mathcal{B}^\perp$ and $u_{N+1} = 2Nr$. As we noted in the proof of Theorem 3, u_{N+1} satisfies the substitutes property. By assumption, there exists $x_i \in \operatorname{dom} u_i$ for all $i = 1, ..., N$ such that $y = \sum_{k=1}^{N+1} x_k \in \mathcal{I}$. Define the random consumptions θ and θ' as follows:

$$\theta = \frac{1}{N+1} \sum_{k=1}^{N+1} \delta_{x_k}$$

$$\theta' = \frac{1}{N+1} \sum_{k=1}^{N+1} \delta_{\chi^H - y + x_k}$$

By assumption, every realization of θ lies in the effective domain of every u_i. It is straightforward to verify that there exists an allocation α with marginals θ for $i = 1, \dots, N$ and marginal θ' for $i = N+1$. Define a sequence of $N+1$ person limited transfer economies with random endowments,

$$\mathcal{E}_n^o = \{(nu_i(c_i, \cdot), \theta, 1)_{i=1}^N, (u_{N+1}, \theta', 1)\}$$

Lemma B7: *The economy $\mathcal{E}_n^o$ has an equilibrium (p^n, α^n) such that $\alpha(\xi) > 0$ implies $\chi^H - \xi_{N+1} \in \mathcal{I}$ and there is a real number K such that $p^{nj} \leq K$ for all $j \in H$ and n.*

Proof: Theorem 1 establishes the existence of equilibrium for limited transfers economy with deterministic endowments whenever agent i's endowment, w_i, is in the effective domain of u_i. Since every possible realization agent i's endowment, θ, is in the of the effective domain of u_i, the proof of Theorem 1 applies without change to the random endowment economy $\mathcal{E}_n^o$ to establish the existence of an equilibrium (α^n, p^n) for the economy $\mathcal{E}_n^o$. Repeating the arguments of Lemma B5 establishes that $\alpha(\xi) > 0$ implies $\chi^H - \xi_{N+1} \in \mathcal{I}$ and that the sequence p^n is bounded, now because agent $N+1$ owns at least $1/(N+1)$ portion of each divisible good. $\square$

Hence, the sequence (α^n, p^n) has a limit point (p, α). Let $b = p \cdot \bar{\theta} + 1$ and let $b' = p\bar{\theta}' + 1$. Then, define the following nontransferable utility economy:

$$\mathcal{E}^* = \{(u_i(c_i, \cdot), b)_{i=1}^N, (u_{N+1}, b')\}$$

Next, we will prove that (p, α) is a strong equilibrium of $\mathcal{E}^*$: clearly, α is feasible for $\mathcal{E}^*$. Since $p^n \bar{\alpha}_i^n \leq p^n \bar{\theta} + 1$ for all n, $p\bar{\alpha}_i \leq p\bar{\theta} + 1 = b$ and therefore, α_i is affordable for i in $\mathcal{E}^*$. Also, since $\alpha_i^n(x) > 0$ implies $x \in \operatorname{dom} u_i$ for all i, we have $\alpha_i(x) > 0$ implies $x \in \operatorname{dom} u_i$ for all i.

Take any affordable random allocation θ^1 for i in $\mathcal{E}^*$; that is, $p\bar{\theta}^1 \leq b = p\bar{\theta} + 1$. We need to show that $u_i(\theta^1) \leq u_i(\alpha_i)$. First, assume $p\bar{\theta}^1 < b$. Then, there exists $\epsilon > 0$ such that $p'\bar{\theta}^1 < b$ for any $p' \in B_\epsilon(p) \cap I\!R_+^L$. Since $\lim p^n = p$, we can find $M > 0$ such that for all $n \geq M$, $p^n\bar{\theta}^1 < p^n\bar{\theta}+1$. Hence, θ^1 is affordable for i in $\mathcal{E}_n^o$ for $n \geq M$. Since α_i^n is an optimal consumption, $nu_i(\theta^1) - p^n\bar{\theta} \leq nu_i(\alpha_i^n) - p\bar{\alpha}_i^n$; that is, $u_i(\theta^1) - u_i(\alpha_i^n) \leq (p^n\bar{\theta} - p^n\bar{\alpha}_i^n)/n \leq (p^n\bar{\theta} + 1)/n$ for all $n \geq M$. Then, the continuity of u_i ensures $u_i(\theta) \leq u_i(\alpha_i)$ as desired.

Next, assume $p\bar{\theta}^1 = b = p\bar{\theta}+1$. Choose $\epsilon \in (0, 1)$. Then, $p\bar{\theta}_\epsilon < b$ for $\theta_\epsilon := (1-\epsilon)\theta^1 + \epsilon\theta$ and, therefore, by the argument in the previous paragraph $u_i(\theta_\epsilon) \leq u_i(\alpha_i)$. Since $u_i(\theta_\epsilon)$ is continuous in ϵ and ϵ was arbitrary, it follows that $u_i(\theta) \leq u_i(\alpha_i)$. Thus, α_i is optimal for i at prices p in $\mathcal{E}^*$.

Repeating the corresponding arguments in the proof of Theorem 2 ensures that all goods with strictly positive prices are allocated and thus, (p, α) is an equilibrium of $\mathcal{E}^*$. Similarly, repeating the corresponding arguments in the proof of Theorem 2 establishes that (p, α) is a strong equilibrium of $\mathcal{E}^*$.

To conclude the proof of Theorem 4, let $\hat{\alpha}(\xi, z) = \alpha(\xi, \chi^H - z)$ for all z. Repeating the arguments of Lemma B6, establishes that $(p, \hat{\alpha})$ is a strong equilibrium of the nontransferable utility production economy with modular constraints $\{(u_i, b)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$. It follows that $(p/b, \hat{\alpha})$ is a strong equilibrium of $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$. $\square$

References

Abdulkadiroğlu, A., P. A. Pathak and A. E. Roth (2005) "The New York City High School Match," *American Economic Review,* 95, 364–67.

Ashlagi, I., and S. Peng (2016) "Optimal Allocation without Money: an Engineering Approach," *Management Science,* 62,1078–1097.

Baldwin, E. and P. Klemperer (2019) "Understanding Preferences: 'Demand Types,' and the Existence of Equilibrium with Indivisibilities," *Econometrica,* 87, 867–932.

Bing, M., D. J. Lehmann, Paul Milgrom (2004) "Presentation and Structure of Substitutes Valuations," In Proceedings of the 5th ACM Conference on Electronic Commerce, 238–239, ACM, New York.

Budish, E., (2011), "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes," *Journal of Political Economy*, 119, 1061-1103.

Budish, E., Y-K. Che, F. Kojima and P. Milgrom, (2013): "Designing Random Allocation Mechanisms: Theory and Applications," *American Economic Review,* 103, 585–623.

Echenique, F., A. Miralles, and J. Zhang (2018) "Fairness and Efficiency for Probabilistic Allocations with Endowments," mimeo, Caltech.

Fujishige, S. (2005) Submodular Functions and Optimization. Second Edition. Elsevier, Amsterdam.

Fujishige, S. and Z. Yang (2003) "A Note on Kelso and Crawford's Gross Substitutes Condition," *Mathematics of Operations Research,* 28, 463–469.

Gul, F., and E. Stacchetti (1999) "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory,* 87, 95–124.

Gul, F., and E. Stacchetti (2000) "The English Auction with Differentiated Commodities," *Journal of Economic Theory,* 92, 66–95.

He, Y., A. Miralles, M. Pycia, J. Yan, et al. (2017) "A Pseudo-Market Approach to Allocation with Priorities," mimeo, UCLA.

Hylland, A. and R. J. Zeckhauser (1979) "The Efficient Allocation of Individuals to Positions," *Journal of Political Economy,* 87, 293–314.

Kelso, A. S. and V. P. Crawford (1982) "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica,* 50, 1483–504.

Kojima, F., Sun, N. and N. N. Yu (2018) "Job Matching Under Constraints," working Paper.

Le, P. (2017) “Competitive Equilibrium in the Random Assignment Problem,” *International Journal of Economic Theory*, 13(4), 369–385.

Leonard, H. B. (1983). “Elicitation of Honest Preferences for the Assignment of Individuals to Positions,” *Journal of Political Economy*, 91(3), 461–479.

Mas-Colell, A., (1992) “Equilibrium Theory with Possibly Satiated Preferences,” Equilibrium and Dynamics: Essays in Honour of David Gale, edited by M. Majumdar, London, Macmillan, 201–213.

Milgrom, P. (2009) “Assignment Messages and Exchanges,” *American Economic Journal: Microeconomics*, 1, 95–113.

Miralles A., and M. Pycia, (2017) “Prices and Efficient Assignments Without Transfers,” working paper.

Murota, K. (1996) “Convexity and Steinitz's Exchange Property,” *Advances in Mathematics*, 124, 272–310.

Murota, K. (2009) “Recent developments in discrete convex analysis,” *Research Trends in Combinatorial Optimization*, Springer, 219–260.

Murota, K., and A. Shioura (1999) “M-convex Function on Generalized Polymatroid” *Mathematics of Operations Research*, 24, 95–105.

Ostrovsky, M and R. Paes Leme (2015) “Gross Substitutes and Endowed Assignment Valuations,” *Theoretical Economics*, 10, 853–865.

Oxley, James G. (2011) Matroid Theory. Second Edition. Oxford University Press, New York.

Roberts, D. J. and A. Postlewaite (1976), “The Incentives for Price-Taking Behavior in Large Exchange Economies,” *Econometrica*, 44, 115–127.

Shiorura, A and Z. Yang (2015) “Equilibrium, Auction and Generalized Gross Substitutes and Complements, *Journal of the Operations Research Society of Japan*, 58, 410–435.

Sun, N., and Z. Yang (2006), “Equilibria and Indivisibilities: Gross Substitutes and Complements,” *Econometrica*, 74, 1385–1402.

Yang, Y-Y. (2017) “On the Maximal Domain Theorem: A Corrigendum to ‘Walrasian Equilibrium with Gross Substitutes,” *Journal of Economic Theory*, 172, 505–511.

Yokote, K. (2017) “Application of the Discrete Separation Theorem to Auctions,” working paper, Waseda University.