Resonance Fluorescence in a Frequency Dependent Photonic Reservoir: 
An Exact Multi-Photon Scattering Theory

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We develop an exact theory of resonance fluorescence of a two-level atom embedded in a frequency dependent photonic reservoir exhibiting sharp features or gaps in the local density of states. In our approach, the fluorescence is treated as scattering of incident photons on the atomic system. Within the framework of the dipole, rotating-wave approximation, we derive a solution of the multi-photon scattering problem without recourse to other approximations. In ordinary vacuum, both the optical Bloch equations and the multi-photon scattering approach result in the well-known Mollow spectrum of fluorescence. Near a sharp feature in the photonic density of states, the non-Markovian character of radiative decay invalidates the usual optical Bloch equations approach. The multi-photon scattering problem approach is still applicable in this case due to a hidden symmetry of the scattering problem. For a frequency dependent reservoir characterized by a step-like discontinuity in the photonic density of states, we find that, in the limit of weak driving field, the scattering cross section becomes strongly non-Lorentzian, reflecting the non-exponential character of the atomic dipole decay. With increasing intensity of the driving field, the scattered field acquires new features associated with the dynamical decoupling of the atomic system from the radiation reservoir. The scattered spectrum suffers a global narrowing as the position of the sideband components of the spectrum is shifted towards the central component, and a local narrowing as the width of the individual components considerably decreases. The multi-photon scattering approach developed here can be directly applied to systems characterized by a frequency-dependent isotropic dispersion relation, as well as to realistic photonic crystal heterostructures characterized by an effective one-dimensional dispersion. We also derive the quantum version of the optical Bloch equations in the ordinary vacuum, without making use of the Born-Markov approximation. They provide a relationship between the quantum statistical properties of incident light and the scattered light.

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I. INTRODUCTION

The phenomenon of resonance fluorescence of a two-level atom in ordinary vacuum has been widely studied [1], [2]. In the case of an arbitrary strong incident field, a general solution of the problem has been found by Mollow [3]. In Mollow’s approach (as well in all other approaches based on the optical Bloch equations (OBE) [1,2]) resonance fluorescence is treated as emission of photons by an atom driven by a classical laser field. Moreover, the quantum nature of the vacuum field is eliminated from consideration, and all physical characteristics of the emitted radiation are expressed only in terms of operators for the atomic variables.

Another approach to the problem of resonance fluorescence has been developed in Refs.[4], [6]. In this approach, the atomic system (in the framework of the dipole, rotating-wave approximation) coupled to an isotropic radiation reservoir is mathematically mapped to an effective one-dimensional model. This type of one-dimensionalization is characteristic to several problems (e.g., in condensed matter, the theory of magnetic alloys [7], [8], and in quantum optics, the theory of Dicke superradiance [4]) in which a bosonic field interacts with a point-like impurity. In quantum optical problems, the role of the impurity is played by the atom. In contrast to the condensed matter problems, where of interest are the thermodynamic equilibrium properties of the whole particle-plus-impurity system, the physical formulation of the resonance fluorescence problem corresponds to a many-body scattering problem, which in the initial state contains only laser photons, i.e., contains only field excitations. Consequently, it can be solved exactly even though the physical model is not completely integrable.

In this paper, we develop an exact multi-photon scattering theory of the resonance fluorescence in a frequency-dependent photonic reservoir (colored vacuum). In this quantum approach, the resonance fluorescence consists in scattering of laser photons on the atomic system. We focus on the analysis of the observables associated with the quantum degrees of freedom of the electromagnetic field. In particular, we describe the spectral density of the field of the scattered photons. We assume that while the electromagnetic dispersion is strongly dependent on the frequency, it is essentially one-dimensional. A one-dimensional photonic reservoir can be physically realized in a waveguide channel embedded
in a three-dimensional photonic band-gap (PBG) material. By tuning the characteristics of the microstructure (geometry and index of refraction contrast), the linear defect in the three-dimensional PBG may support only two waveguide modes, one of which experiences a sharp cutoff in the middle of the three-dimensional PBG. In this case, the sub-gap generated by the waveguide channel has a true one-dimensional character, since there is only one direction available for wave propagation. For an infinite structure, there is a physical square-root singularity in the photonic density of states near the cutoff of one of the waveguide modes. For a finite structure, the divergence is cutoff by the finite size effects. However, the strong variation with frequency of the photonic DOS remains. For this system, standard perturbation theory is highly questionable and the formalism we develop in this paper may constitute the only reliable way of calculating the scattered spectrum from active elements embedded in the dielectric microstructure.

To maintain generality and enable comparison with previous theories of resonance fluorescence, we consider an isotropic radiation reservoir field with an optical density of modes frequency dependent (colored vacuum). In this model system, which is formally equivalent to a system characterized by a one-dimensional dispersion relation, the radiation is effectively described by a scalar wave vector \( k \). For an isotropic system, the problem presents spherical symmetry and it is convenient to use the spherical harmonic representation for the dynamical variables of the photonic field. The photon operators, \( \alpha_k \), corresponding to a photon of momentum \( k \) are expanded in terms of spherical functions \( Y_{jm}(\hat{k}) \),

\[
a_k = \sum_{jm} Y_{jm}(\hat{k})c_{jm}(k), \tag{1.1}
\]

where \( \hat{k} = k/k \) and \( k = |k| \). Since, in the dipole approximation, only the electric dipole harmonic with angular momentum \( j = 1 \) is assumed to be coupled to the atom, we can omit all higher harmonics. Then, the effective Hamiltonian contains only dipole operators \( c_{j=1,m}(k) \equiv c(k) \), where the magnetic quantum number \( m = 0, \pm 1 \) is fixed by the given polarization of the incident field. Because the dipole operators depend only on a single quantum number (the momentum modulus \( k \)), the effective model is one-dimensional in terms of auxiliary operators

\[
c(x) = \sum_k e^{ikx}c(k), \tag{1.2}
\]

and thus formally equivalent to an one-dimensional photonic reservoir system.

For a sufficiently low intensity of the incident laser field, such that the mean temporal inter-photon separation in the incident photon flux exceeds the characteristic time of the atom-photon interaction, fluorescence can be treated as scattering of independent photons on the atom. The elastic cross-section for the corresponding one-photon scattering problem is given by the Weisskopf formula [10]

\[
\sigma(\omega_L) \sim \frac{\gamma^2}{(\omega_L - \omega_A)^2 + (\gamma/2)^2}, \tag{1.3}
\]

where \( \omega_L \) is the frequency of the incident and the scattered photons and \( \gamma \) is the spontaneous decay constant of the atomic transition in vacuum. The inverse of the decay rate constant, \( \gamma^{-1} \), represents the characteristic time scale of atom-photon interaction. If the intensity of incident light increases, such that the mean inter-photon separation becomes comparable to or smaller than the time of atom-photon interaction, multi-photon processes become important and the atomic system is coherently driven between its bare states many times before spontaneously emitting a photon.

In ordinary vacuum, the solution of the one-particle scattering problem for the effective one-dimensional model leads to the Weisskopf result (Eq. (1.3)). Similarly to the quantum impurity problems in the theory of magnetic alloys [7, 8], at a sufficiently high intensity of the incident field, we need to solve a multi-photon scattering problem (MSP). In our model, the photon-atom scattering generates an effective photon-photon interaction [11], [6], and we deal with a strongly correlated system of dipole photons. Nevertheless, the effective one-dimensional model is diagonalized exactly [6] in an arbitrary \( N \)-particle sector of the total Hilbert space of the model Hamiltonian, by means of the Bethe ansatz technique [7, 8]. Since a given multi-particle wave function of the incident photons is symmetric with respect to permutations of photon coordinates in the \( x \)-space (the auxiliary \( x \)-space is generated by the Fourier transform of the \( k \)-space), we can limit our analysis to an ordered sector of the \( N \)-particle wave function, say \( x_N > \ldots > x_1 \). The two-photon scattering processes are strictly “forbidden”, because, due to the conservation of the total excitation number, the appearance of two photons, say, in the region \( (x_i, x_i + 1) \) would require no photons in the final interval \( (x_{N-1}, x_N) \). However, the causal nature of the propagation of the photons and the fact that a two-level atom cannot be excited twice implies that there is a non-vanishing probability of finding a photon arbitrarily far from its point of creation. The absence of two-photon scattering processes has an important consequence: the order of photons in an initial state of the scattering problem is preserved in the scattering process [4]. As a result, the only manifestation of the multi-particle character of scattering is a confinement of the wave function of the \( j \)-th scattered photon in the interval between the coordinates of the \( (j-1) \)-th and \( (j+1) \)-th scattered photons.

In ordinary vacuum, the solution of the multi-photon scattering problem results in the Mollow spectrum of the scattered field [12]. The solution obtained in the MSP formalism reproduces completely all the results of the theory of resonance fluorescence derived in the framework of the OBE, without making recourse to any additional approximations, such as the Born-Markov approximation.
used to derive the OBE. The MSP approach shows that the OBE in vacuum are exact in the framework of the dipole, rotating-wave approximation, used to derive the standard Hamiltonian of the problem.

The situation is drastically changed in the case of frequency dependent photonic reservoirs, such as confined photonic systems [13]. If the transition frequency of the atom, $\omega_A$, lies far enough from the photonic DOS discontinuity, the scattering problem can be linearized with a good accuracy in the vicinity of the transition frequency (on a frequency scale given by the Rabi frequency). This approximation leads to the Mollow spectrum of fluorescence. If the transition frequency lies close to a band-gap edge (or any other photonic DOS discontinuity) the linearization is in general impossible. Therefore, at first glance, both OBE and MSP approaches would not work in the case of a frequency dependent reservoir. In the case of OBE, the difficulty is related to the non-Markovian character of radiative decay of the excited atomic state [14], [15], [16], whereas in the case of MSP, photons of different frequencies propagate with different velocities, and, as a result, the effective photon-photon correlations generated by the photon-atom scattering must contribute essentially to the final state of the scattering problem.

Remarkably, the MSP approach is still applicable to a colored vacuum, due to a hidden symmetry of the problem. As in ordinary vacuum, the standard model reduces to an effective one-dimensional model, but now in an auxiliary operator $\tau$-space, related to photon frequency $\omega$ through Fourier transform. We demonstrate in this paper that the solution of the one-particle scattering problem leads to a non-Lorentzian cross-section of scattering of the incident photon. For the multi-photon scattering problem, we show that, even if we deal with a strongly correlated system of dipole photons, since all photons in the auxiliary $\tau$-space propagate with the same velocity, the order of photons in an initial state of the scattering problem in the auxiliary space is preserved in the scattering process. This property allows us to find an exact expression for the final state of the multi-photon scattering problem and to compute the spectral density of the scattered field.

This paper is organized as follows. In Sec. II we introduce a photonic crystal heterostructure [34] relevant to the formalism we develop. In Sec. III, we formulate the problem of resonance fluorescence as a multi-photon scattering problem of an incident flux of photons on a two-level atom. In Sec. IV, we analyze the one-photon scattering problem and generalize the Weiskopf solution to the case of a frequency dependent reservoir. Section V is dedicated to the analysis of the frequency dependence of the atomic Lamb shift and the scattering cross section for a model photonic density of states (DOS), presenting a step-like discontinuity in the photonic density of states. In Sec. VI, we find a solution of the multi-photon scattering problem and compute the spectral density of the scattered field for an arbitrary density of photon states. In Sec. VII, we study the ordinary-vacuum limit of the resonance fluorescence problem and show that for a laser field resonant with the atomic frequency, our general results reduce to the standard Mollow spectrum of resonance fluorescence. We then apply the theory formulated to simple models of colored vacua. We also present extended results for the ordinary vacuum case, by considering arbitrary detunings of the laser field frequency with respect to the atomic frequency. Section VIII presents a semi-classical interpretation of the resonance fluorescence in a frequency dependent photonic reservoir. We argue qualitatively that some features present in the spectrum of the scattered radiation can be associated with the positive atomic inversion of the atomic system described in [16] and [17]. In Sec. IX, we derive the quantum optical Bloch equations in ordinary vacuum without making use of the Born-Markov approximation. Since in these equations the incident laser field is represented by corresponding field operators rather than a classical field amplitude, they are valid for an arbitrary quantum state of incident field and allow the study of the dependence of physical properties of scattered light on quantum-statistical properties of incident light. Finally, in Sec. X we present some concluding remarks.

II. ONE-DIMENSIONAL PHOTON DISPERSION RELATION IN A THREE-DIMENSIONAL PHOTONIC CRYSTAL HETEROSTRUCTURE

In this section, we introduce a three-dimensional photonic crystal heterostructure [34] that has an one-dimensional photon dispersion relation. This kind of structure is suitable for the multi-photon scattering theory developed in the following sections. The structure consists of a two-dimensional PC slab sandwiched between two three-dimensional PCs with large absolute band-gaps. The cladding three-dimensional PCs can be square spiral [35], woodpile [36], slanted pores [37], or inverse Si opal [38]. The two-dimensional PC slab consists of dielectric rods in a square lattice, which is lattice matched with the three-dimensional claddings. One line of rods in the two-dimensional slab is removed to create an on-chip waveguide. It is the waveguide that provides the one-dimensional photon dispersion relation. Here, we use direct diamond:1-square spirals as the claddings because this kind of square spirals has already been synthesized using glancing angle deposition techniques [39]. In addition, this class of heterostructures exhibits exceptional strong tolerance against the structure disorders [34] . The parameters of the direct diamond:1-square spiral are $[L, c, r] = [0.7a, 1.3a, 0.2a]$ [35] (a is the square lattice constant). The square spiral of those parameters possesses a three-dimensional PBG of $\sim 14\%$ when the refractive index of the arms is 11.9.

The band structure of the waveguide is calculated by plane wave expansion method [40] and super cell method [41]. The result is depicted in Fig. 1. Plane waves with
waves are used [43]. This difference will disappear if enough number of plane waves are used as the claddings (1a). The height of the two-dimensional slab is 0.3a. The rod radius is 0.2a.

The details of the properties of the heterostructure can be found in Ref. 34. The relevant features are summarized below. When there is no on-chip waveguide inside the intercalated two-dimensional microchip, some two-dimensional planar waveguide bands (upper shaded regions in Fig. 1) will occupy the upper fraction of the original three-dimensional PBG. However, a complete on-chip three-dimensional PBG is retained (unshaded region between two-dimensional bands and three-dimensional bands in Fig. 1) below these planar modes. The size of this on-chip band-gap decreases with the thickness of the two-dimensional microchip layer, disappearing completely when the two-dimensional slab is about 1.1a. Meanwhile, removing one line of rods in the two-dimensional microchip will create a linear air-waveguide mode inside the on-chip band-gap. Light localization by the PBG materials surrounding this air-waveguide enables flow of light only along the waveguide. Consequently, the photon dispersion relation is one-dimensional as long as the light frequency is within the on-chip band-gap.

When the wave vector is $0.5(2\pi/a)$, the waveguide mode exhibits a cut-off frequency around $0.362(2\pi c/a)$, where $c$ is the light velocity in the vacuum. The cut-off frequency obtained here is slight different from that obtained by the finite-difference time-domain (FDTD) method [42] because the number of plane waves used in our calculation is limited by our computational abilities. This difference will disappear if enough number of plane waves are used [43].

Near the cut-off frequency $\omega_c$, the waveguide mode dispersion relation can be approximated by a parabolic function, i.e., $\omega = \omega_c - A(k_y - k_c)^2$, where $k_c$ is the wave vector corresponding to $\omega_c$ and $A$ is a fitting constant. Consequently, the photon DOS will be $\rho_1(\omega) = (1/2)A(\omega_c - \omega)^{-1/2}$, which is divergent at $\omega_c$. However, in practice, the size of the hetero-structure (and length of the corresponding waveguide) is always finite. As a result, the LDOS divergence is replaced by a large but finite sharp peak near $\omega_c$.

In order to calculate the LDOS within the microchip, a dipole is placed near the center of the rod adjacent to the waveguide half-way along the length of the waveguide. The dipole is perpendicular to the plane of the two-dimensional slab because the electric field of the waveguide mode is almost in this direction [34]. The finite structure has three units of direct diamond-1-square spirals as the claddings above and below the two-dimensional slab. The height of the two-dimensional slab is 0.3a. There are 5 lines of rods on each side of the air waveguide. The finite-size structure is placed in vacuum, which we simulate with the second-order Mur absorbing boundary condition [44]. Since the energy emitted from the dipole is proportional to the LDOS at the dipole position [45], we integrate the Poynting vector over a closed surface outside the structure to obtain the LDOS. The results are depicted in Fig. 2. In our numerical calculations, a cubic FDTD grid [47] is used with 10 points per lattice constant $a$, i.e., $\Delta x = \Delta y = \Delta z = \Delta = 0.1a$.

As the length of the waveguide increases, the peak value of the LDOS increases rapidly. The LDOS in the waveguide of 15a in length can be 1000 times than that of the vacuum. At the same time, the peak width decreases. Besides the peak near the cutoff frequency, there are some smaller fringes in the LDOS due to finite-size effects [45]. This is a Fabry-Perot effect arising from the heterostructure to free space boundary at the edges of our sample. Near the cutoff frequency, the LDOS varies rapidly, forming a sharp peak. In this case, the non-Markovian character of the radiative decay is crucial and invalidates the usual optical Bloch equations approach. To overcome this obstacle, we develop an exact multi-photon scattering theory.

![Graph showing band structure](image1.png)

**FIG. 1:** The band structure of a waveguide in a threedimensional heterostructure. The wave vector $k_y$ is along the waveguide direction. The height of the two-dimensional slab is 0.3a. The rod radius is 0.2a.

![Graph showing LDOS](image2.png)

**FIG. 2:** The local density of states of waveguides in a three-dimensional heterostructure. Here, $\rho_0$ is the local density of states in the vacuum. The length of the waveguide is indicated in the legend.
III. GENERAL FORMALISM

We consider a physical system consisting of a two-level atom interacting with a quantized radiation field and embedded in a frequency dependent photonic reservoir, described by the total Hamiltonian

$$H = H_A + H_F + H_{AF},$$  (3.1)

where

$$H_A = \frac{1}{2} \omega_A (\sigma_3 + 1),$$  (3.2)

$$H_F = \sum_k \omega(k) a_k^\dagger a_k,$$  (3.3)

are the Hamiltonians of the free atom and electromagnetic field in a medium characterized by an isotropic dispersion $\omega(k)$, respectively, while $H_{AF}$ describes the atom-field interaction Hamiltonian. For simplicity, throughout this paper we use a system of units in which $\hbar = c = 1$.

Here, $\omega(k)$ represents the spectrum of elementary electromagnetic excitations of the medium, which we refer to loosely as “photons” (the spectrum of elementary excitations of the electromagnetic field in the presence of a photonic crystal is strongly modified from its free space counterpart.). Their creation and annihilation operators obey the commutation relations:

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}.$$  (3.4)

The photon polarization is considered fixed by the external laser field.

The two-level atom with the transition frequency $\omega_A$ between the ground ($1$) and excited ($2$) levels is described by the Pauli matrices $\sigma_i = (\sigma_x, \sigma_y, \sigma_z)$, with the commutator

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k,$$  (3.5)

where $\epsilon_{ijk}$ is the unit anti-symmetric tensor.

In the continuum limit, when the quantization volume becomes very large, $V \to \infty$, the free field Hamiltonian takes the form

$$H_F = \int \frac{d^3k}{(2\pi)^3} \omega(k) a_k^\dagger a_k, \quad \text{(3.6a)}$$

with

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta(k - k').$$  (3.6b)

For the purpose of this paper, it is convenient to expand the field operators in terms of spherical harmonics,

$$a(k) = \sum_{j,m} \int \frac{d\omega}{2\pi} \Psi_{\omega jm}(k) c_{jm}(\omega),$$  (3.7)

where $\Psi_{\omega jm}(k)$ are a complete set of eigenfunctions of the free field Hamiltonian, $H_F$,

$$[\omega(k) - \omega] \Psi_{\omega jm}(k) = 0.$$  (3.8)

The operator $c_{jm}(\omega)$ corresponds to a photon of frequency $\omega$, angular momentum $j = 1, \ldots, \infty$, and projection thereof $m = -j, \ldots, +j$. The integration contour $C$ contains all allowed photon states in the respective medium. The functions $\Psi_{\omega jm}(k)$ are chosen to be

$$\Psi_{\omega jm}(k) = B(\omega) \delta(\omega - \omega(k)) Y_{jm}(n),$$  (3.8)

with the normalization coefficient

$$B(\omega) = \frac{(2\pi)^2}{k(\omega)} \frac{1}{\sqrt{\rho(\omega)}}$$  (3.9)

specified by the condition

$$\int [c_{jm}(\omega), c_{jm'}^\dagger(\omega')] = 2\pi \delta_{jj'} \delta_{mm'} \delta(\omega - \omega').$$  (3.10)

Here, $\rho(\omega) = d\rho(\omega)/d\omega$ is the density of states for the photonic reservoir.

The radiation field Hamiltonian becomes

$$H_F = \sum_{jm} \int_C \frac{d\omega}{2\pi} \omega c_{jm}^\dagger(\omega)c_{jm}(\omega),$$  (3.11)

In what follows, we will assume that the photon spectrum, $\omega(k)$, of the medium has a frequency pseudo-gap (or a fill-gap), where the density of states is much less than outside a pseudo-gap (or vanishes inside a full-gap).

In deriving the coupling operator in the spherical harmonic representation, we take into account that, in the optical spectral range, the length scale of the atomic system, $a$, is much less than the characteristic wavelength $c/\omega_A$, $a \omega_A \ll 1$. As a result, for a dipole allowed atomic transitions, we account only for the coupling of the transition to electric dipole photons $(j = 1)$, while all higher electric harmonics and all magnetic ones can be omitted [10]. Thus

$$H_{AF} = \sum_{m = \pm, \pm} \int_C \frac{d\omega}{2\pi} \sqrt{z(\omega)} [c_m(\omega) + c_m^\dagger(\omega)] [\sigma_+ + \sigma_-]$$  (3.12)

where the operators $c_m(\omega) \equiv c_{j=1,m}(\omega)$ correspond to the electric dipole photons, and $\sigma_\pm = (\sigma_x \pm i \sigma_y)/2$. In the resonance approximation, we assume that the main contribution to the coupling parameter $\gamma(\omega)$ comes from the neighborhood of the resonance transition $\omega_A$ and we replace $\gamma(\omega)$ by the constant value $2\omega_A^2/4d^2$, where $d$ is the dipole matrix element of the transition. The atomic factor $z(\omega) = 1/(d\omega/dk)$ [18] accounts for corrections in a frequency dependent reservoir and is proportional to the photon density of states $\rho(\omega)$:

$$\rho(\omega) = 1/(d\omega/dk) = 2\pi^2 \frac{\rho(\omega)}{k(\omega)^2}$$  (3.13)

The operators of incident plane wave photons propagating along an arbitrary axis passing through the atomic system are also represented in terms of operators $c_{jm}(\omega)$,

$$a_{kL}^\dagger = \frac{1}{\sqrt{V}} a_{kL}^\dagger = A_{kL}^\dagger + \beta^{1/2} c_{L}(\omega),$$  (3.14)
where the operator $A_{kL}$ is the sum of higher spherical harmonics with $j > 1$ (not coupled to the atomic transition), and $\beta \equiv \sigma_1/S L \rho_L$. Here, $\rho_L = \rho(\omega_L)$ is the density of photon states at the frequency of incident light. $S$ and $L$ are the cross-section and the length of the incident flux ($V = SL$), and $\sigma_1 = 3\pi/k^2(\omega_L)$ is the impact area of the dipole harmonic [19].

If the polarization of the incident laser pulse is fixed, we can fix the magnetic quantum number $m = 0, \pm 1$ of dipole photons. For example, if the incident plane wave pulse is linearly polarized, we can choose the axis of quantization of atomic states directed along the vector of the polarization of the incident field. Then, only electric dipole photons with $m = 0$ are coupled to the atomic transition. In this case, the operator $c(\omega_L)$ in Eq. (3.15) corresponds to the electric-dipole photon of frequency $\omega_L$, angular momentum $j = 1$ and projection $m = 0$.

The Hamiltonian of the atom and dipole photons system can be cast in the form

$$H_D = \frac{1}{2} \omega_A (\sigma_3 + 1) + \int_{-\infty}^{\infty} d\omega \frac{d}{2\pi} \omega \sigma^\dagger(\omega) c(\omega)$$

$$- \sqrt{\gamma} \int_{-\infty}^{\infty} d\omega \frac{d}{2\pi} \sqrt{\gamma} \left[ c(\omega) \sigma_+ + \sigma_- c(\omega) \right],$$

(3.16)

where $c(\omega) \equiv c_{j=1,m=0}(\omega)$. In deriving Eq. (3.16), we made use of the rotating-wave and resonance approximations. That is to say, we have (i) omitted the terms, which contain $c(\omega \sigma_+ c(\omega)$ and $c \sigma (i)$ neglected the frequency dependence of the coupling parameter $\gamma(\omega)$ and set $\gamma = \gamma(\omega_A) = \frac{4}{3} \omega_A^2 d^2$, and (ii) extended the lower limit of integration to $-\infty$.

We emphasize the Hamiltonian (3.16) can also be used to describe a one-dimensional system, in which a two-level atom interacts with the radiation reservoir of a photonic crystal heterostructure presenting an effective one-dimensional dispersion relation, such as the one introduced in Sec. II. The only difference from the case of an isotropic reservoir is that the “non-dipolar” contribution (the $A_{kL}$ term) and the $\sqrt{\beta}$ coefficient in front of the “dipolar” contribution in Eq. (1.1) would have different values. However, these changes will have only a minor influence on our formalism and will result in an overall normalization constant for the scattering cross-sections evaluated in Sec. IV and Sec. VI.

The main purpose of our study is to solve the scattering problem for a general initial state and then to compute expectation values of the scattered field in the limit $N \to \infty$, $L \to \infty$, but $N/L = $ constant.

IV. ONE-PARTICLE SCATTERING PROBLEM

The operator of the number of excitations of the total system (atom plus dipole photons),

$$\hat{N} = \frac{1}{2} (\sigma_3 + 1) + \int_{C} d\omega \frac{d}{2\pi} \omega c(\omega),$$

(4.1)

commutes with the model Hamiltonian (Eq. (3.16)), and the number of excitations is a good quantum number and we can analyze separately the sectors of Hilbert space corresponding to different numbers of excitations. We start by considering the one-particle scattering problem with an initial state containing a single plane-wave photon

$$|\text{In}, 1\rangle = a_k^\dagger |0\rangle = A_k^\dagger |0\rangle + \beta^{1/2} |\psi_{in}\rangle,$$

(4.2a)

where

$$|\psi_{in}\rangle = c^\dagger(\omega_L)|0\rangle = \int_{-\infty}^{\infty} d\tau e^{i\omega_L \tau} c(\tau)|0\rangle.$$  

(4.2b)

Here, and throughout the paper, $|\psi \ldots\rangle$ is used to label the dipole component in the frequency representation of a given state $|\ldots\rangle$.

We introduce the one-particle eigenstates in the form

$$|\lambda\rangle = \left[ \xi(\lambda) \sigma_+ + \int_{C} d\omega \frac{d}{2\pi} \sqrt{\gamma(\omega)} \phi(\omega) |\lambda\rangle c^\dagger(\omega) \right] |0\rangle,$$

(4.3)

where $\lambda$ is the eigen-energy of the $|\lambda\rangle$ state. Then, the Schrödinger equation, $(H_D - \lambda)|\lambda\rangle = 0$, becomes

$$\begin{align}
(\omega - \lambda) \phi(\omega) |\lambda\rangle - \sqrt{\gamma} \xi(\lambda) = 0 & \quad (4.4a) \\
(\omega - \lambda) \xi(\lambda) - \sqrt{\gamma} \int_{C} d\omega \frac{d}{2\pi} \phi(\omega) |\lambda\rangle = 0. & \quad (4.4b)
\end{align}$$

Inserting the general solution of Eq. (4.4a),

$$\phi(\omega) |\lambda\rangle = 2\pi \delta(\omega - \lambda) + \frac{\sqrt{\gamma}}{\omega - \lambda - i0} \xi(\lambda),$$

(4.5)

into Eq. (4.4b), we find a closed system of equations for atomic wave functions,

$$|\omega_A - \lambda - \Sigma(\lambda)| \xi(\lambda) = \sqrt{\gamma} \xi(\lambda),$$

(4.6)

where the atomic self-energy is given by

$$\Sigma(\lambda) = \gamma \int_{C} d\omega \frac{d}{2\pi} \frac{\gamma(\omega)}{\omega - \lambda - i0}.$$ (4.7)

The imaginary part of the self-energy, $\Sigma''(\lambda)$, can be computed for an arbitrary function $\gamma(\omega)$,

$$\Sigma''(\lambda) = \frac{i\sqrt{\gamma}}{2} \xi(\lambda),$$

(4.8)

whereas its real part is formally represented as

$$\Sigma'(\lambda) = \gamma P \int_{C} d\omega \frac{\gamma(\omega)}{2\pi} (\omega - \lambda),$$

(4.9)
where \( P \) stands for the principal part of the integral.

Thus, the photon eigenfunction is found to be

\[
\phi(\omega|\lambda) = 2\pi \delta(\omega - \lambda) - i \frac{\varphi(\lambda)}{\omega - \lambda - i\delta}, \tag{4.10a}
\]

where

\[
\varphi(\lambda) = \frac{-i\gamma}{\hbar(\lambda) + i\gamma/2}, \tag{4.10b}
\]
\[
h(\lambda) = \frac{\lambda - |\omega_A - \Sigma'(\lambda)|}{\sqrt{\lambda(\lambda - 2\pi)}}, \tag{4.10c}
\]

The solution of the one particle scattering problem can be cast into a more familiar form by introducing the auxiliary \( \tau \)-space. In the auxiliary \( \tau \)-space, the Hamiltonian

\[
H_D = \frac{1}{2} \omega_A(\sigma_3 + 1) - i \int_{-\infty}^{\infty} d\tau c^\dagger(\tau) \frac{\partial}{\partial \tau} c(\tau)
\]
\[
+ \sqrt{\pi} \int_{-\infty}^{\infty} d\tau w(\tau)[c(\tau)\sigma_+ + \sigma_- c^\dagger(\tau)]
\]

(4.11)

describes particles which propagate in the positive direction of the \( \tau \) axis and are scattered by the atom located at the point \( \tau = 0 \). The atomic "potential",

\[
w(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sqrt{\pi(\omega - \lambda)} e^{i\lambda \tau}, \tag{4.12}
\]

tends to zero as \( \tau \to \pm\infty \), because \( \sqrt{\pi(\omega)} \to 1 \) as \( \omega \to \pm\infty \). Negative (positive) values of the coordinate \( \tau \) correspond to incoming (outgoing) dipole photon.

In order to find the solution of the one-particle scattering problem in the \( \tau \)-space, we construct first a eigenstate of the Hamiltonian \( H_D \), which asymptotically coincides with the initial state given by Eq. (4.2a) as \( \tau \to -\infty \), and then study the asymptotics of this state as \( \tau \to +\infty \), to obtain the scattered field [19].

The photon wave function in the auxiliary \( \tau \)-space is the Fourier transform of the direct space wave function (Eq. (4.5)), and is given by [20]

\[
\Phi(\tau|\lambda) = \frac{h(\lambda) - i\gamma/2}{h(\lambda) + i\gamma/2} \exp(i\lambda \tau). \tag{4.13}
\]

In solving the scattering problem, we are interested only in the asymptotic behavior of the photon wave functions as \( \tau \to \pm\infty \). In these limits, the corresponding Fourier images take the form

\[
\Phi(\omega|\lambda) = 2\pi \delta(\omega - \lambda) \cdot \begin{cases} 
1 & \text{if } \lambda(\lambda - \gamma/2) > 0, \quad \tau \to -\infty, \\
\frac{h(\lambda) - i\gamma/2}{h(\lambda) + i\gamma/2} & \text{if } \lambda(\lambda - \gamma/2) < 0, \quad \tau \to +\infty.
\end{cases} \tag{4.14}
\]

The state \( c^\dagger(\tau)|0\rangle \) (of central importance in evaluating the dipole component of the incident photon states Eq. (4.2a)) can be represented as a linear superposition of one-particle eigenstates, \( |\lambda\rangle \), (see Eq. (4.3)) of the Hamiltonian \( H_D \):

\[
c^\dagger(\tau)|0\rangle = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} A(\tau|\lambda)|\lambda\rangle. \tag{4.15}
\]

where the expansion coefficients are given by

\[
A(\tau|\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{\pi(\omega)}} \Phi^*(\omega|\lambda) e^{-i\omega \tau}. \tag{4.16}
\]

As \( \tau \to \infty \), we find the following expansion for the state \( c^\dagger(\tau) \)

\[
c^\dagger(\tau)|0\rangle \to \int_{-\infty}^{\infty} d\lambda \frac{h(\lambda) - i\gamma/2}{2\pi \hbar(\lambda) + i\gamma/2} \int_{-\infty}^{\infty} d\tau' e^{i\lambda(\tau' - \tau)} c^\dagger(\tau')|0\rangle. \tag{4.17}
\]

Therefore, in the auxiliary \( \tau \)-space the out-state of the one-particle scattering problem is

\[
|\psi_{\text{out}}\rangle = \int_{-\infty}^{\infty} d\tau \psi_{\text{out}}(\tau) c^\dagger(\tau)|0\rangle, \tag{4.18a}
\]

where

\[
\psi_{\text{out}}(\tau) = \int_{-\infty}^{\infty} d\tau' s(\tau|\tau') e^{i\lambda \tau}, \tag{4.18b}
\]

and the scattering matrix

\[
s(\tau|\tau') = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{h(\lambda) - i\gamma/2}{\hbar(\lambda) + i\gamma/2} e^{i\lambda(\tau - \tau')} \tag{4.19}
\]

relates the photon wave functions in the in- and out-states.

As a result of the scattering on the "atomic potential" \( w(\tau) \), the photon experiences a delay in propagation and the photon wavefunction in the auxiliary \( \tau \)-space, acquires a scattering phase,

\[
\frac{h(\lambda) - i\gamma/2}{h(\lambda) + i\gamma/2} = 1 + \varphi(\lambda). \tag{4.20}
\]

This relates the scattering matrix (\( s \)-matrix)

\[
s(\tau|\tau') = \delta(\tau - \tau') + t(\tau|\tau'), \tag{4.21a}
\]

to the \( t \)-matrix, given by

\[
t(\tau|\tau') = \int_{-\infty}^{\infty} d\lambda \varphi(\lambda) \exp[i\lambda(\tau - \tau')]. \tag{4.21b}
\]

In Eq. (4.21a), the first term corresponds to the transmitted dipole photon, while the second one represents the scattered dipole photon. Since the scattering amplitude, \( \varphi(\lambda) \), is analytic (this property of the function \( \varphi(\lambda) \) is also evident from the analytical properties of the self-energy \( \Sigma(\lambda) \) (Eq. (4.7))) in the upper half-plane of
the complex variable $\lambda$, the $t$–matrix must vanish for $\tau - \tau' > 0$.

The out-state of the one-particle scattering problem is expressed as a sum of states corresponding to the transmitted and scattered dipole photon components,

$$|\psi_{\text{out}}\rangle = |\psi_{\text{in}}\rangle + |\psi_{\text{scatt}}\rangle,$$

where

$$|\psi_{\text{scatt}}\rangle = \int_{-\infty}^{\infty} d\tau \psi_{\text{scatt}}(\tau) e^{i \lambda \tau} |0\rangle,$$

and

$$\psi_{\text{scatt}}(\tau) = \int_{-\infty}^{\infty} d\tau' t(\tau|\tau') e^{i \omega L \tau'}.$$(4.22c)

For example, in ordinary vacuum ($z(\epsilon) = 1$) $h(\lambda) = \lambda - \omega_A$, and

$$t^{(\text{vac})}(\tau|\tau') = -\gamma \theta(\tau' - \tau) \exp[i(\omega_A - i\gamma/2)(\tau - \tau')],$$

and the integral in Eq. (4.22c) is computed in an explicit form,

$$\psi_{\text{scatt}}^{(\text{vac})}(\tau) = \frac{-i\gamma}{\omega_L - \omega_A + i\gamma/2} \exp(i\omega_L \tau).$$

In a frequency dependent reservoir, the $t$-matrix cannot, in general, be expressed in a simple analytical form. Nevertheless, we can evaluate the wave function of scattered photons for an arbitrary isotropic or one-dimensional density of states $z(\lambda)$. Inserting the integral representation for the $t$-matrix given in Eq. (4.21b) into Eq. (4.22c), and computing first the integral over $\tau'$, we obtain

$$\psi_{\text{scatt}}(\tau) = e^{i \omega_L \tau} \int_{-\infty}^{\infty} d\lambda \frac{\varphi(\lambda)}{2\pi i \lambda - \omega_L + i\delta}.$$

Since the scattering amplitude is analytical in the upper half-plane, the $\lambda$–integration yields

$$\psi_{\text{scatt}}(\tau) = \varphi(\omega_L) \exp(i\omega_L \tau).$$

Therefore, the out-state of the one-particle scattering problem takes the form

$$|\text{Out}, 1\rangle = a_k^\dagger |0\rangle + \beta |\psi_{\text{scatt}}\rangle = \left[ a_k^\dagger + \beta \frac{1}{2} \varphi(\omega_L) c^\dagger(\omega_L) \right] |0\rangle,$$

where the first term represents the transmitted plane wave component, and $\varphi(\omega_L)$ is the scattering amplitude of incident dipole photon of frequency $\omega_L$.

The spectral density of the scattered field,

$$G(\omega) = \langle \text{Out}, 1 | c^\dagger(\omega) c(\omega) | \text{Out}, 1 \rangle,$$

can be expressed as

$$G(\omega) = 2\pi \delta(\omega - \omega_L) \sigma(\omega),$$

where the scattering cross section is given by:

$$\sigma(\omega) = \frac{\sigma_1}{S(\omega)} \frac{\gamma^2}{h^2(\omega) + (\gamma/2)^2}.$$

In general, this function may have a non-Lorentzian character and may exhibit several peaks depending on the behavior of the atomic form-factor $z(\omega)$ and the density of photon states $\rho(\omega)$. We note that by setting $z(\omega) = 1$, Eq. (4.30) reduces to the Weisskopf result (1.3).

V. LAMB SHIFT AND SCATTERING CROSS-SECTION FOR A MODEL DOS

In this section, we consider a specific model of frequency dependent photonic reservoir and evaluate the Lamb shift and the scattering cross section. Our toy model is based on a step-like discontinuity in the photonic density of states, which is present in a number of realistic band structure calculations [21], [22].

The model density of states considered is assumed to gives rise to an atomic form factor

$$z(\omega) = 1 - \frac{\hbar}{2} \left\{ \tanh \left[ \frac{\omega - \omega_l}{\gamma_c} \right] - \tanh \left[ \frac{\omega - \omega_r}{\gamma_c} \right] \right\},$$

(5.1)

where the frequencies $\omega_l$ and $\omega_r$ identify the (left and right) positions of the step discontinuities (surrounding a gap or a pseudo band gap) and the constants $\hbar$ and $\gamma_c$ determine the magnitude of jump in the density of states and the steepness of the discontinuity, respectively.

![FIG. 3: The “atomic form factor” for various choices of the $h$ and $\gamma_c$ parameters. The pseudo-gap “band-edges” are given by $\omega_l = \omega_A - 0.05 \omega_A$, $\omega_r = \omega_A + 0.05 \omega_A$. The continuous curve corresponds to $\gamma_c = 0.01 \omega$, and $h = 0.99$, the dashed curve corresponds to $\gamma_c = 0.1 \gamma$ and $h = 0.99$, the dot-dashed curve corresponds to $\gamma_c = 0.1 \gamma$ and $h = 0.5$, respectively. The inset shows a close-up (on a $\gamma$ scale) of the right “band-edge” region.](image)
In Fig. 3, we plot the “atomic form factor”, for some specific values of the parameters entering its definition. For frequencies located deep into the pseudo-gap, \( \omega \in (\omega_e, \omega_c) \), \( z(\omega) \approx 1 - h \), whereas for frequencies far away from the pseudo-gap, \( z(\omega) \approx 1 \). The free space case is regained for \( h = 0 \), while the case of a two dimensional photonic band gap (a true step discontinuity) corresponds to \( h = 1 \), and \( \gamma_c \rightarrow 0 \). This model of the “atomic form factor” allows us to include in our study a broader class of photonic reservoirs, including those associated with dielectric structures that do not possess a full band gap in the density of states, but their density of states exhibits rapid variation with frequency on specific spectral ranges (photonic crystals).

The Lamb shift is defined as the real part of the self energy (Eq. (4.9)). In the resonance approximation, we assume that the main contribution to the \( \omega \)-space integrals comes from frequencies in the neighborhood of the atomic frequency and the integration contour can be then extended to \( C = (-\infty, \infty) \). It follows that in free space, the principal part in Eq. (4.9) vanishes. If we had not extended the integration contour to \(-\infty\), the real part of the self energy would become divergent. This divergence can be removed by introducing a cutoff in the frequency integration at the electron Compton’s frequency, \( \omega_e = m_e c^2 / h \), with \( m_e \) the electron rest mass (higher energies of the photonic reservoir excitations would probe the relativistic structure of the electronic wave-packet and can therefore be neglected in our analysis). This simplification together with the pole approximation, \( \Sigma'(\omega) \approx \Sigma'(\omega_A) \), gives the usual Wigner-Weisskopf result for the free space Lamb shift, \( \Sigma'(\omega_A) \approx \gamma \ln(\omega_e / \omega_A) / 2\pi \). In the visible spectral range, \( \gamma \approx 10^{8} \text{Hz}, \omega_A \approx 10^{15} \text{Hz} \), such that the Lamb shift is approximatively given by \( \Sigma'_S(\omega) \approx \Sigma'_S(\omega_A) = 2.15 \times 10^{8} \text{Hz} \). Since the Lamb shift is approximately constant, it can be easily incorporated in the definition of the atomic state energy.

In Fig. 4 and 5 we plot the relative difference between the Lamb shifts evaluated for the pseudo-gap model and the free space Lamb shift, \( \delta_{Lamb}(\omega) = - (\Sigma'(\omega) - \Sigma'_S(\omega_A)) / \Sigma'_S(\omega_A) \). Clearly, the frequency dependence of Lamb shift is inherited from the DOS, i.e., important corrections from the free space values are obtained only if the photonic DOS varies with frequency on the scale of \( \gamma \). As shown in Fig. 4, the magnitude of the Lamb shift is strongly dependent on both the depth and the sharpness of the pseudo-gap. For a sufficiently strong pseudo-gap (\( h = 0.99 \) and \( \gamma_c = 0.1 \gamma \)), the maximum departure from the free space Lamb shift can be as high as \( \pm 125\% \). This value corresponds to the “band edge” frequencies for which the photonic DOS exhibits the greatest asymmetry. However, if the “band edges” of the pseudo-gap are “smooth” and defined on spectral regions comparable to the atomic frequency \( \omega_A \) (rather than \( \gamma \)), the pseudo-gap emission phenomena can be treated by means of a modified Wigner-Weisskopf approximation involving the density of states at the atomic frequency position, \( \rho(\omega_A) \).

Further insight is gained by analyzing the scattering cross-section (4.30). Clearly, the zeros of the function \( b(\omega) \) determine the position of the peaks in the scattering cross section, while the widths of these individual peaks are determined by the value of the “atomic form factor” \( z(\omega) \) at the peak position. The relative magnitude of the individual peaks is determined by the “envelope” fac-
FIG. 6: The scaled Lamb shift function $\Sigma'(x + \omega_r)/\gamma$ for $\omega_l - \omega_A = -0.05 \omega_A$, $\omega_r - \omega_A = 0.05 \omega_A$. The figure on the left corresponds to a large depth pseudo-gap, $h = 0.99$. Different curves correspond to different values of the parameter $\gamma_c$, which varies from $\gamma_c = \gamma/20$ for the continuous curve to $\gamma_c = \gamma$ for the double-dotted-dashed curve. The figure on the right corresponds to a relatively moderate depth pseudo-gap, $h = 0.5$. The straight line represents the function $y(x) = -x$ and the intersection points with the bell-shaped curves represent solutions of Eq. (5.4).

It can be shown [18] that $\rho(\omega) \propto z(\omega)$. Finding the solutions of the equation $h(\omega) = 0$ is equivalent to identifying the dressed (by the real and virtual photons) atomic frequencies. The zeros of the function $h(\omega)$ are found from the equation

$$\omega_A - \omega = \Sigma'(\omega) \tag{5.2}$$

Using the fact that the real part of the self energy is maximal for the “band edge” frequencies, $\omega_l$, $\omega_r$, we introduce

$$\bar{\omega}_A = \omega_A - \Sigma'(\omega_r) \tag{5.3a}$$

$$\Sigma'(\omega) = \Sigma'(\omega) - \Sigma'(\omega_r) \tag{5.3b}$$

and hereafter treat $\bar{\omega}_A$ as the transition frequency of the atom in the medium. The analysis is further simplified by assuming “resonant” conditions, i.e., $\bar{\omega}_A = \omega_r$. Then Eq. (5.2) becomes

$$\Sigma'(x + \omega_r) = -x \tag{5.4}$$

where we introduced the variable $x \equiv \omega - \omega_r$. Equation (5.4) may have one, two or three distinct solutions, depending on the characteristics of the photonic DOS. Clearly, in the Wigner-Weisskopf approximation (in which the frequency dependence of the self-energy is neglected), Eq. (5.4) has a unique solution $\bar{\omega}_A = \omega_A - \Sigma'(\omega_A)$, which reproduces the usual free space Lamb shift of the atomic frequency. As shown in Fig. 6, there is a direct correlation between the frequency dependent character of the photonic DOS and features of the scattering cross-section. The most important influence is related to the scale over which the photonic DOS varies. Even if the depth of the pseudo-gap is not very large (Fig. 6, the plot on the right), for well defined “band edges” ($\gamma_c = \gamma/20$, for the continuous bell shaped curve), Eq. (5.4) possesses three distinct solutions, leading to the scattering cross-section exhibits three peaks. Since these distinct peaks occur in the scattering of just a single photon, this effect can be interpreted as vacuum Rabi splitting.

FIG. 7: The scattering cross section, $\sigma(\omega)$, for a strong pseudo-gap, $h = 0.99$ and for various values of the parameter $\gamma_c$. For all curves $\omega_l - \omega_A = -0.05 \omega_A$, $\omega_r - \omega_A = 0.05 \omega_A$.

FIG. 8: The scattering cross section, $\sigma(\omega)$, for a relatively modest pseudo-gap, $h = 0.5$ and for various choices of the parameter $\gamma_c$. For all curves $\omega_l - \omega_A = -0.05 \omega_A$, $\omega_r - \omega_A = 0.05 \omega_A$.

The scattering cross-section is plotted in Fig. 7 and Fig. 8. As expected, for important deviations of the photonic DOS from the free space value, the spec-
trum becomes non-Lorentzian, leading to a strong non-exponential decay of the atomic system [21]. For the specific choices of parameters used here, some of the spectral features predicted by Eq. (5.4) overlap, and even for strong pseudo-gaps (large values of $h$ and small values of $\gamma$), the spectrum presents at most two distinct peaks. We also note the important reduction of the radiation emitted in the spectral range of the pseudo-gap.

VI. MULTI-PARTICLE SCATTERING PROBLEM

We now consider the scattering on the atomic system of a multi-particle state describing an incident pulse containing $N$ plane wave photons. The choice of an initial state containing $N$ plane wave photons has a two-fold motivation. From a formal point of view, it is very practical, since, as seen in the single-particle excitation problem, it allows for a direct separation of the dipole component of the field from the contribution of higher spherical harmonics, which are not influenced by the interaction with the atomic system (3.15). On the other hand, it is a natural extension of the single-particle problem presented in the preceding section, and the calculations are greatly simplified by making use of the results obtained in Sec. IV. More specifically, we start with an initial ($t \rightarrow -\infty$) state containing $N$ plane-wave photons (see Appendix A)

$$|\text{In}, N \rangle = \frac{1}{\sqrt{N!}}|a_k|^N|0\rangle,$$

(6.1)

(with $a_k$ defined by Eq. (3.15)), which is scattered on the atomic system, and we then seek its asymptotic behavior,

$$|\text{Out}, N \rangle = \lim_{\tau \rightarrow -\infty} \left[ \exp (-iH\tau) |\text{In}, N \rangle \right].$$

(6.2)

Once the out-state of the scattering problem is found, analogous to the single photon case, we can evaluate the spectral properties of the scattered radiation from the atomic system.

Since only the dipole photons interact with the atomic system, the problem reduces to the scattering of $n$ dipole photons in an initial symmetrized state,

$$|\text{In}, n \rangle = \sqrt{n!} \int d\vec{r}'\theta(\tau_1' > \ldots > \tau_n')|\Psi_{\text{in}}(\vec{r}')\rangle \prod_{j=1}^{n} c^{\dagger}(\tau_j')|0\rangle.$$

(6.3)

Here, the integration is carried out over the ordered particle coordinates $\tau_1' > \ldots > \tau_n'$. As shown in Appendix A, the out-state of the $n$ scattered dipole photons, $|\Psi_{\text{out}}, n \rangle$, is obtained from Eq. (6.3) by replacing $\Psi_{\text{in}}(\vec{r}')$ by

$$\Psi_{\text{out}}(\vec{r}) = \int d\vec{r}' S(\vec{r} | \vec{r}') \Psi_{\text{in}}(\vec{r}'),$$

(6.4a)

where the integration range is specified by $\tau_{j-1} < \tau_j' < \tau_j$, and the multi-photon $S$-matrix is given by

$$S(\vec{r} | \vec{r}') = \prod_{j=1}^{n} s(\tau_j | \tau_j').$$

(6.4b)

Here, the single particle $s$-matrix $s(\tau_j | \tau_j')$ is expressed through a $t$-matrix, as in Eq. (4.21). As discussed in the introduction, the characteristic feature of the interaction between the photons and the atomic system is that the two-photon scattering processes are “forbidden”, and, as a consequence, the order of incident particles on the $\tau$ axis is preserved in the scattering process. Furthermore, making use of the solution of the scattering problem for $n$ dipole photons presented in Appendix A, we can isolate in the out-state of the scattering problem with $N$ incident plane wave photons

$$|\text{Out}, N \rangle = \sum_{n=0}^{N} \sqrt{C_N^n \beta^n/2} |\Psi_{\text{scatt}, n}\rangle \otimes \frac{1}{\sqrt{(N-n)!}}|a_k|^{N-n}|0\rangle,$$

(6.5)

the state

$$|\Psi_{\text{scatt}}, n\rangle = \sqrt{n!} \int d\vec{r} |\Psi_{\text{scatt}}(\vec{r})\rangle \prod_{j=1}^{n} c(\tau_j)|0\rangle,$$

(6.6a)

which contains only scattered photons. Here,

$$\Psi_{\text{scatt}}(\vec{r}) = \prod_{j=1}^{n} \psi(\tau_{j-1}, \tau_j) \exp (i\omega_L\tau_j),$$

(6.6b)

and the wave function

$$\psi(\tau_{j-1} - \tau_j) = \int_{-\infty}^{\infty} d\lambda \frac{\varphi(\lambda)}{2\pi i (\lambda - \omega_L + i0) \times \{ 1 - \exp [-i(\lambda - \omega_L) (\tau_{j-1} - \tau_j)] \}}$$

(6.6c)

vanishes for negative argument, $\tau_{j-1} - \tau_j < 0$.

Therefore, despite the factorization of the multi-particle scattering into a product of one-particle scatterings, the multi-particle wave function of scattered dipole photons is not factorized into a product of one-particle wave functions. This strongly correlated state of the scattered field explains multi-photon effects in resonance fluorescence found by Mollow in the case of ordinary vacuum [3].

All information about the medium is contained in

$$\varphi(\lambda - \omega_L) \times \{ 1 - \exp [-i(\lambda - \omega_L) (\tau_{j-1} - \tau_j)] \}$$

through $\varphi(\lambda)$, and its knowledge fully solves the multi-particle scattering problem. In ordinary vacuum and on-resonance case ($\omega_L = \omega_A$), the scattering amplitude reduces to the simple form

$$\varphi_{\text{vac}}(\lambda) = \frac{-i\gamma}{\lambda + i\gamma/2}.$$

(6.7)
Then, the integral over \( \lambda \) in Eq. (6.6c) yields

\[
\psi_{\text{vac}}(\tau) = \left[ 1 - \exp \left( -\frac{\gamma}{2} \tau \right) \right] \theta(\tau). \tag{6.8}
\]

In a frequency dependent reservoir, such a simple expression does not in general occur. Nevertheless, this formal difficulty can be overcome by noticing that the expectation values of the scattered field are expressed in terms of the Fourier transform of the function \( \psi(\tau) \),

\[
\psi(\omega) = \int_{0}^{\infty} d\tau \psi(\tau) e^{i\omega \tau} = \frac{i}{\omega + i0} \varphi(\omega + \omega_L). \tag{6.9}
\]

This functional is analytical in the upper half-plane (since \( \psi(\tau < 0) = 0 \)) and can be evaluated for an arbitrary frequency dependent photonic reservoir.

In order to make closer contact with experimental relevant situations (usually, the resonance fluorescence experiments involve strong incident fields) and to compare our results to previous calculations for the resonance fluorescence spectrum (done, in general, in the approximation that the incident field is strong enough to be treated as a classical variable), we rewrite the multi-particle scattering problem in terms of coherent states of the incident field (as shown in Appendix A, this involves just a change of basis in the photonic Hilbert space).

An initial coherent state pulse with a mean photon number \( \bar{N} \) of frequency \( \omega_L \) and wave vector \( \mathbf{k}_L = \mathbf{k}(\omega_L) \) is described by

\[
|\text{In}\rangle = e^{-\bar{N}/2} \exp \left( \sqrt{\bar{N}} \hat{a}_{L}^{\dagger} \right) |0\rangle. \tag{6.10}
\]

After scattering from the atomic system, the final \( (\tau \to \infty) \) state of the field is given by

\[
|\text{Out}\rangle = \sum_{n=0}^{\infty} \frac{(\rho_0)^{n/2}}{\sqrt{n!}} |\Psi_{\text{scatt}}, n\rangle \otimes |\text{In}\rangle, \tag{6.11}
\]

where

\[
\rho_0 \equiv \bar{N} \left( 1 \right) = \frac{\sigma_1}{S} \frac{\bar{N}}{\rho(\omega_L)} \tag{6.12}
\]

is the linear density of dipole photons in the incident pulse (see Appendix A for a full derivation of Eq. (6.11)).

As usual in the scattering theory [19], we assume that the incident beam has a finite cross section and that the measurements of the scattered field are carried out outside this cross section. For this reason, the out-state of the multi-particle scattering problem has to be renormalized and, as shown in Appendix B, the renormalized effective out-state of the scattered field is given by

\[
|\text{Out}\rangle = \sum_{n=1}^{\infty} (\rho_0)^{n/2} \int_{-\infty}^{\infty} d\tau' v(\tau_n + \tau_0) \times \prod_{j=1}^{n} u(\tau_{j-1} - \tau_j) c^{\dagger}(\tau_j) |0\rangle, \tag{6.13}
\]

where the Fourier transforms of the functions \( u(\tau) \) and \( v(\tau) \) (evaluated in Appendix B) are found to be

\[
v(\omega) = \frac{i}[\hbar(\omega + \omega_L) + i\gamma/2] - \gamma \rho_0, \tag{6.14a}
\]

\[
u(\omega) = \frac{\hbar(\omega + \omega_L) + i\gamma/2 - \gamma \rho_0}{\hbar(\omega + \omega_L) + i\gamma/2 - \gamma \rho_0}. \tag{6.14b}
\]

The spectral density of the scattered field per unit of time, \( G(\omega) \), is given by the Fourier image over the difference of coordinates of the correlation function

\[
G(\tau, \tau') = \frac{1}{2\pi} \langle \text{Out}|c^{\dagger}(\tau)c(\tau')|\text{Out}\rangle, \tag{6.15}
\]

where \( N \) is the normalization constant (C2).

In Appendix C, we show that for an arbitrary frequency dependent radiation reservoir, the spectral density of the scattered field can be cast in the form

\[
G(\omega) = \frac{\rho_0}{F_0} \left[ \rho_0^{-1} \Gamma(2)(\omega) + \frac{\Gamma(1)(\omega)\Gamma(1)^{*}(-\omega)}{1 - \rho_0 U(\omega)} + \frac{\Gamma(1)^{*}(-\omega)\Gamma(1)(\omega)}{1 - \rho_0 U(-\omega)} \right]. \tag{6.16}
\]

with \( F_0, U(\omega), \Gamma(1)(\omega), \Gamma(2)(\omega) \) defined by Eqs. (C3), (C14b) and (C16b). We emphasize that, in contrast to the analysis of the one-particle scattering problem [see Eq. (4.28)], we define here the function \( G(\omega) \) in such a manner that \( dN(\omega) = G(\omega) d\omega/2\pi \) represents the number of photons scattered by the atom per unit of time in the frequency interval \( d\omega \).

VII. SPECTRAL DENSITY OF THE SCATTERED FIELD

A. Ordinary Vacuum: Analytical Results

In this section, we demonstrate the ability of our approach to recapture the ordinary vacuum results. To avoid more complicated expressions, we restrict our consideration to the case of exact resonance, \( \omega_L = \omega_A \).

The information about the dispersion law of photons in the frequency dependent reservoir is contained only in the function \( h(\omega) \) and the linear density of dipole photons in the incident pulse, \( \rho_0 \), given by Eq. (6.12). In ordinary vacuum, \( z(\omega) = 1 \), and in exact resonance, the effective wave function of scattered photon, \( u(\omega) \), takes a simple form:

\[
u_{\text{vac}}(\omega) = \frac{\gamma}{(\omega - \omega_+)(\omega - \omega_-)} \tag{7.1}
\]

where

\[
\omega_{\pm} = -i\frac{\gamma}{4} \pm \sqrt{\gamma \rho_0 - (\gamma/4)^2}. \tag{7.2}
\]

Using Eqs. (7.1) and (C3), the function \( U(\omega) \) (the function \( U(\omega) \) is defined as a Fourier transform of \( U(\tau) =
\( |u(\tau)|^2 \), which represents the probability of having a photon scattered in the time interval \( \tau \) is then found to be

\[
U(\omega) = \frac{-2i\gamma^2}{(\omega + i\gamma/2)(\omega - 2\omega_+)(\omega - 2\omega_-)},
\]

while the function \( H(\omega) \) (defined in Appendix B) is given by

\[
H(\omega) = -\frac{2i\gamma^2 \rho_0}{(\omega + i0)(\omega - \Omega_+)(\omega - \Omega_-)},
\]

with the complex frequencies defined by

\[
\Omega_{\pm} = -i\frac{3}{4} \gamma \pm \sqrt{\Omega_R^2 - (\gamma/4)^2},
\]

and where \( \Omega_R = (4\gamma \rho_0)^{1/2} \) is the Rabi frequency. In free space, we also obtain a simple expression for the coefficient \( F_0^{-1} \)

\[
1 - \frac{2\gamma^2 \rho_0}{\gamma \Omega_R^2} = \frac{1}{2} \frac{\gamma \Omega_R^2}{\Omega_R^2 + \gamma^2/2}. \tag{7.6}
\]

Since the correlation function \( G(\tau) \equiv G(\tau - \tau', 0) \) is an even function, we restrict our further computations to the case \( \tau > 0 \). Then, we obtain [48]

\[
G(\tau) = G^{(1)}(\tau) + G^{(2)}(\tau), \tag{7.7a}
\]

\[
G^{(1)}(\tau) = \rho_0^2 \int_0^\infty d\omega H(\omega)D(\omega)e^{-i\omega\tau}, \tag{7.7b}
\]

\[
G^{(2)}(\tau) = \rho_0^2 \int_\infty^0 d\omega \Gamma^{(2)}(\omega)e^{-i\omega\tau}, \tag{7.7c}
\]

where we introduced

\[
D(\omega) = \frac{\Gamma^{(1)}(\omega)\Gamma^{(1)*}(-\omega)}{U(\omega)}. \tag{7.7d}
\]

At the first glance, the behavior of the correlator is much richer than in the Mollow theory, because, apart from the contributions of three poles of the function \( H(\omega) \) corresponding to the elastic component, \( \omega = 0 \), and two sideband components, \( \omega = \Omega_{\pm} \), Eqs. (7.7) contain also contributions from singular points of the functions \( D(\omega) \) and \( \Gamma^{(2)}(\omega) \). By direct computation we obtain

\[
\Gamma^{(1)}(\omega) = \frac{-\omega + i\gamma}{2\gamma \rho_0} U(\omega), \tag{7.8a}
\]

\[
\Gamma^{(2)}(\omega) = \frac{1}{2\rho_0} \frac{i}{\omega + i\gamma/2} - \frac{(\omega + i\gamma)^2}{4\gamma^2 \rho_0} U(\omega) + \text{c.c.,} \tag{7.8b}
\]

and hence

\[
\Gamma^{(2)}(\omega) = \frac{1}{2\rho_0} \frac{i}{\omega + i\gamma/2} + D(\omega) + \text{c.c.}. \tag{7.8c}
\]

By substituting these expressions in Eqs. (7.7), we finally obtain in the ordinary vacuum limit

\[
G(\tau) = \frac{1}{2} \frac{\Omega_R^2}{\gamma^2/2} \text{Re} \left[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ -\frac{2(\omega + i\gamma)^2}{\Omega_R^2} H(\omega) \right\} e^{-i\omega\tau} \right]. \tag{7.9}
\]

This expression for the spectral density of the scattered field reproduces the Mollow results for the spectrum of the scattered radiation. We note that the central broadened line of the Mollow spectrum is represented by the second term in Eq. (7.9).

As we argue below, the value of the Rabi frequency used in this paper differs by a \( \sqrt{2} \) factor from the value used in Mollow’s work. This difference is caused by the different normalization of the atom-field interaction Hamiltonian. We start by writing the Rabi frequency \( \Omega_R = \sqrt{\gamma \rho_0} \) in terms of the amplitude of the electric field of the incident laser pulse. Inserting the explicit expressions for \( \gamma \) and \( \rho_0 \), we obtain

\[
\Omega_R^2 = 4 \left( \frac{4}{3} \omega_A^3 d^2 \right) \left( \frac{3\pi}{\omega_A^2 SL} \right) = 16\pi d^2 \omega_A^2 \tilde{N} \frac{S L}{\pi}, \tag{7.10}
\]

where we used the expression for the impact area of dipole photons, \( \sigma_1 = 3\pi/\omega_A^2 \), and set \( \omega_L = \omega_A \). The last factor in this expression represents the density of energy in the incident pulse. If the electric field of the incident pulse is given by

\[
E(x) = \frac{1}{2} E_0 \left( e^{ikLx} + e^{-ikLx} \right), \tag{7.11}
\]

the density of energy is found to be

\[
\omega_A \tilde{N} = \frac{E_0^2}{8\pi}, \tag{7.12}
\]

and hence

\[
\Omega_R = \sqrt{2} d E_0. \tag{7.13}
\]

Therefore, in our approach the Rabi frequency is \( \sqrt{2} \) times bigger than in the Mollow theory, because, apart from the contributions of three poles of the function \( H(\omega) \) corresponding to the elastic component, \( \omega = 0 \), and two sideband components, \( \omega = \Omega_{\pm} \), Eqs. (7.7) contain also contributions from singular points of the functions \( D(\omega) \) and \( \Gamma^{(2)}(\omega) \). By direct computation we obtain

\[
H_{AF} = -\sqrt{\gamma} \int \frac{d\omega}{2\pi} [\sigma(\omega)A(s_+ + s_-)c(\omega)], \tag{7.14a}
\]

over the state of the incident field. We find

\[
|\langle \text{in} | H_{AF} |\text{in} \rangle| = -\sqrt{\gamma} \rho_0 (s_+ + s_-) = -\frac{1}{\sqrt{2}} d E_0 (s_+ + s_-), \tag{7.14b}
\]

in which the coupling magnitude \( dE_0 \) is again \( \sqrt{2} \) times bigger than \( 1/2 d E_0 \) used in OBE. Clearly, the difference in the values of the Rabi frequency is caused by the different normalization of the atom-field interaction Hamiltonian.
B. Numerical Results

The resonance fluorescence theory developed in this paper is generally valid, regardless the magnitude of the quantum driving field and the specifics of the photonic reservoir considered. In this section, we consider a specific model for the photonic reservoir DOS and evaluate the resonance fluorescence spectrum associated with the scattered radiation from a two-level atomic system driven by an external laser field. Generally, the spectral density of the scattered field cannot be obtained in an analytical form and the integrals involved in Eq. (6.16) have to be evaluated numerically. The elastic and inelastic [2] parts of the scattered spectrum are separated in the usual way:

\[ G(\omega) = G_{el}(\omega) + G_{in}(\omega) \]  

(7.15a)

\[ G_{el}(\omega) = 2 \text{Re} \left\{ \frac{1}{z} \lim_{z \to 0} z \cdot G(z) \right\} \]  

(7.15b)

\[ G_{in}(\omega) = 2 \text{Re} \left\{ G(z) - \frac{1}{z} \lim_{z \to 0} z \cdot G(z) \right\} \]  

(7.15c)

where

\[ G(z) = \frac{\rho_0^2}{2F_0} G^{(2)}(z) + \frac{\rho_0^2}{F_0} \frac{\Gamma^{(1)}(z)\Gamma^{(1)*}(z)}{1 - \rho_0 U(z)} \]  

(7.15d)

Further, using the expansion of the function \( U(\omega) \) in the neighborhood of the origin, Eq. (C7b), the frequency is measured from \( \omega_L \), the elastic part of the scattered spectrum gives the usual delta component:

\[ G_{el}(\omega) = 2\pi F_0 \left| (\Gamma^{(1)}(0))^{-1} \delta(\omega) \right| \]  

(7.16)

1. Ordinary Vacuum

We now extend the results obtained in the previous section to arbitrary intensities and detunings of the driving laser field. The ordinary vacuum results are well known [2], [3] and are used here as a consistency check of our approach.

In Fig. 9 we plot the exact resonance inelastic part of the scattered spectrum, evaluated by numerical integration of Eq. (6.16), for some arbitrarily chosen values of the laser field intensity. In the case of strong driving field, \( \Omega_R \gg \gamma \), the delta-function component gives only a very small contribution to the central component, while in the opposite regime, \( \Omega_R \ll \gamma \), it can dominate the spectrum [2]. In the case of weak driving field, illustrated in Fig. 9(curve (a)), the inelastic spectrum consists of only one central component. As the driving field intensity becomes larger than the critical value \( \Omega^C_R = \gamma/2 \), the spectrum develops two additional sidebands at frequencies \( \Omega_{\pm} = \pm \sqrt{\Omega^C_R - (\gamma/2)^2} \), as shown by curves (b) and (c). In the limit of strong driving laser field, the classical Mollow spectrum is regained in Fig. 9 (curve (c)). The peak heights of the center and sidebands are in ratio 3:1,

the integrated intensities are in the ratio 2:1, the central component width is \( \gamma/2 \), and the sideband widths are \( 3\gamma/4 \), respectively.

![Fig. 9](image1)

**FIG. 9:** Inelastic scattered spectrum in the free space case. The atomic frequency is resonant with the laser field frequency. The three curves correspond to different values of the driving field intensity: (a) \( \Omega_R/\gamma = 0.2 \), (b) \( \Omega_R/\gamma = 3.0 \), (c) \( \Omega_R/\gamma = 8.0 \), respectively.

In Fig. 10, we plot the peak height of the central component as a function of the Rabi frequency. As expected, we obtain that the weight of the inelastic scattered radiation increases with the increase of the strength of the laser field. For higher intensities, the atomic system becomes saturated, and the maximum value of the central component becomes insensitive to further changes in the driving field intensity.

![Fig. 10](image2)

**FIG. 10:** The peak height of the central component as a function of the scaled Rabi frequency, \( \Omega_R/\gamma \). The atomic frequency is resonant with the laser field frequency.

This picture is drastically changed for a non-resonant driving laser field. As the laser field is further detuned from the atomic resonant frequency, the center-to-sideband peak ratio decreases, and can become much
smaller than 1 (Fig. 11). Also, the larger the detuning of the laser field from the resonant atomic frequency, the less effective the coupling between the atomic system and the driving field. Consequently, the amount of scattered radiation diminishes considerably, as exemplified in Fig. 12, where we plot the maximum value of the central component as a function of the driving laser field strength. Our results, (Fig. 9, Fig. 10, Fig. 11 and (Fig. 12) are clearly in agreement with those obtained by other approaches [2], [3] for ordinary vacuum.

The analytical properties of the functions \( u(\lambda) \) are of central importance in our study. As argued before, the function \( u(\lambda) \) is analytical in the upper half-pane (UHP), provided that a full expression for the self-energy \( \Sigma(\lambda) \) is used. As seen from its definition (Eq. (4.7)), the self-energy is analytical in the upper half-plane, since the integration contour lies below the pole \( \omega = \lambda + i0 \). This property is the mathematical expression of the Causality Principle. The formal divergence of the self-energy can be removed through a regularization procedure and then incorporated into the definition of the physical atomic transition frequency, \( \bar{\omega}_A \), as is done in the Wigner-Weisskopf theory. We note that if, for a given frequency dependent photonic reservoir, the scattering amplitude acquires a singular point in the UHP (as a result of the chosen regularization procedure), this inconvenience can be addressed by slightly modifying the definition of the function \( u \) in the time domain (Eq. (B5a)). In all such cases, the corresponding Fourier transform is defined assuming that the integration contour in the Fourier integral lies upper to all singular points of the scattering amplitude in the frequency domain. This assumptions must be treated as a part of the regularization procedure, which, as a result, preserves causality. This renormalization procedure adds more complexity to the numerical methods used to evaluate the spectral density of the scattered field. The parameter space range that can be explored becomes strongly limited, since, for particular choices of the atomic form factor \( z(\omega) \), its complex plane extension may cause loss of convergence of the numerical algorithms used to evaluate the spectral density of the scattered field.

We restrict our study for colored vacua to the simple case of the laser field resonant with both the atomic transition and the “band edge” frequency: \( \omega_L = \bar{\omega}_A = \omega_r \). We choose the atomic form factor corresponding to a model density of states given by Eq. (5.1) with \( \gamma_0 = \gamma/100, h = 0.99 \). In this case we find that the spectral features are considerably narrowed (at low intensities of the driving field, the central component width is about five times narrower than its free space correspondent, as shown in Fig. 13). Moreover, the shift of the sideband components relative to the central component is also less.

These effects indicate a dynamical decoupling of the atomic system from the radiation reservoir of the photonic crystal, confirming earlier calculations made in the context of driven atoms coupled to frequency-dependent photonic reservoirs [23], [24], [25], [26]. The suppression of vacuum fluctuations and atomic decay rate over a appreciable spectral range (corresponding to the region \( \omega \in (\omega_L, \omega_r) \)) explains the narrowing of the spectral density components of the scattered field. Meanwhile, the displacement of the sidebands components can be explained by the less effective coupling between the atomic system and the driving field that produces a smaller effective Rabi frequency and, implicitly, modifies the position...
of the sidebands components of the spectrum.

Also, we note that the maximum value of the central peak exhibits a resonance-like behavior (see Fig. 14), at a “threshold” value of the intensity of the driving field (for the specific choice of parameters here, $\Omega^R_0 \approx 0.3 \gamma$). Below the “threshold” intensity, the maximum value of the central peak increases, if the driving field intensity is increased, whereas above the “threshold” value, the central maximum decreases. This dependence of central peak maximum on the driving field intensity is reminiscent of the switching behavior of atomic systems coupled to frequency dependent radiation reservoirs described in Ref. [27], [16], [28]. This connection is further analyzed in Sec. VIII.

VIII. SEMI-CLASSICAL PICTURE

In this section, we review the semi-classical picture of the resonance fluorescence in colored vacua. The purpose of this section is to justify the assertion made at the end of the previous section that the resonance-like feature present in Fig. 14 is associated with mechanisms similar to those responsible for the atomic population inversion. The analysis is based on the dressed-atom picture [29] used in [16], [17] and assumes that the atomic system is driven by an classical external single-mode laser field and that this “dressed” atomic system is coupled to the frequency-dependent photonic reservoir. We adopt the formalism and the notations in order to make connection to the multi-photon scattering approach developed in [17].

In the dipole and RWA approximations and in a rotating frame of reference (rotating with the laser frequency, $\omega_L$), the effective Hamiltonian of the system is given by [16], [28]

$$H = H_0 + H_1,$$

where

$$H_0 = \frac{1}{2} \Delta_{AL} \sigma_3 + \int_0^\infty \frac{d\omega}{2\pi} \Delta_{\omega} c(\omega) c(\omega) + \epsilon [\sigma_+ + \sigma_-],$$

represents the Hamiltonian of the atomic system and the external driving field, and

$$H_1 = i \sqrt{\gamma} \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\gamma(\omega)} [c(\omega)\sigma_3 - c(\omega)\sigma_3^\dagger]$$

is the interaction Hamiltonian between the atomic system and the frequency-dependent photonic bath. Here, $\epsilon = \bar{E} / \gamma$. $\bar{E}$ is the Rabi frequency, $\Delta_{\omega}$ is the laser electrical field at the atom position, $\Delta_{\omega} = \omega - \omega_L$ and $\Delta_{AL} = \omega_A - \omega_L$, are the detuning of the $\omega$-mode with respect to the laser field frequency, and the detuning of the atomic frequency, respectively.

The Hamiltonian $H_0$ can be diagonalized by transforming to the dressed atom basis defined in [16]. This transformation leads to the non-interacting dressed state Hamiltonian:

$$H_0 = h \Omega R_3 + \int_0^\infty \frac{d\omega}{2\pi} \Delta_{\omega} c(\omega)^\dagger c(\omega),$$

with $\Omega = [\bar{E}^2 + \Delta_{AL}^2 / 4]^{1/2}$ the generalized Rabi frequency.

We define the time-dependent interaction-picture Hamiltonian, $\tilde{H}_1 = U(t)^\dagger H_1 U(t)$, with the unitary transformation operator $U(t) = \exp(-iH_0 t / \hbar)$. In this pic-
ture, the interaction Hamiltonian $\tilde{H}_1$ takes the form:

$$
\tilde{H}_1 = i\sqrt{\pi} \int_0^\infty \frac{dw}{2\pi} \sqrt{2}(\omega) \left[c(\omega)t \right] \left[csR_3e^{i\Delta t} + c^2R_12e^{i(\Delta-2\Omega)t} - s^2R_21e^{i(\Delta+2\Omega)t}\right] \text{ + h.c.} \tag{8.3}
$$

The dressed atomic operators in the interaction picture exhibit the time dependence given by: $\tilde{R}_{12}(t) = \frac{R_{12}(0)}{\sqrt{1-\gamma^2}} \exp(-2i\Omega t)$, $\tilde{R}_{21}(t) = \frac{R_{21}(0)}{\sqrt{1-\gamma^2}} \exp(2i\Omega t)$, and $R_3(t) = R_3(0)$. Hereafter, we drop the tilde on the interaction picture operators.

In the case of frequency-dependent photonic reservoir, we assume that the “atomic” form factor (and, implicitly, the photonic density of modes) exhibits a step-like discontinuity, so that the Mollow spectral components experience very different photonic mode densities. For a simplified model of a photonic crystal, in which the photonic DOS, while rapidly varying at a given frequency, is constant over the spectral regions surrounding the dressed-state frequencies $\omega_L$, $\omega_{L-2\Omega}$ and $\omega_{L+2\Omega}$. Then one can make the Markov approximation, and associate spontaneous emission rates $\gamma_0$ and $\gamma_\pm$ with each of the frequencies $\omega_L$ and $\omega_L \pm 2\Omega$, where

$$
\gamma_0 = \gamma \cdot z(\omega_L), \quad (8.4) \\
\gamma_\pm = \gamma \cdot z(\omega_L \pm \Omega). \quad (8.5)
$$

Moreover, we further simplify our analysis by neglecting the energy shifts caused by atom-reservoir interaction, and discard the fast oscillating terms with frequencies $\pm 2\Omega$, $\pm 4\Omega$.

The connection between the resonance-like behavior exhibited by peak height of the central line obtained in Sec. VII B and the atomic switching behavior analyzed in [16], [17], can be understood using the intuitive dressed atom picture. In the secular approximation, the dressed atom approach leads to a simple interpretation of the dynamics of the atomic system and the characteristics of the scattered radiation [29], [30]. The bare (dressed) picture diagram is presented on the left (right) side of Fig. 15. The equations of evolution for the dressed atomic populations are given by [29], [28]

$$
\dot{\pi}_1 = -\pi_1\Gamma_{1\to2} + \pi_2\Gamma_{2\to1}, \quad (8.6a) \\
\dot{\pi}_2 = -\pi_2\Gamma_{2\to1} + \pi_1\Gamma_{1\to2}, \quad (8.6b)
$$

where $\pi_1$, $\pi_2$ represent the populations on the ground and excited dressed atomic states, respectively, $\Gamma_{1\to2} = |\langle 1, n | H_1 | 2, n - 1 \rangle|^2 = \gamma_\pm s^4$ is the transition rate for $|1, n \rangle \rightarrow |2, n - 1 \rangle$ and $\Gamma_{2\to1} = |\langle 2, n | H_1 | 1, n - 1 \rangle|^2 = \gamma_+ c^4$ is the transition rate for $|2, n \rangle \rightarrow |1, n - 1 \rangle$. The steady state solution of these equations is

$$
\pi_{1s} = \frac{\Gamma_{2\to1} - \Gamma_{1\to2}}{\Gamma_{1\to2} + \Gamma_{2\to1}} = \frac{\gamma_+ c^4}{\gamma_- s^4 + \gamma_+ c^4}, \quad (8.7a) \\
\pi_{2s} = \frac{\Gamma_{1\to2}}{\Gamma_{1\to2} + \Gamma_{2\to1}} = \frac{\gamma_- s^4}{\gamma_- s^4 + \gamma_+ c^4}. \quad (8.7b)
$$

As shown in [17], in the case of negative detunings of

the atomic frequency with respect to the frequency of the laser field, the condition to preferentially populate the ground state $|1 \rangle$ is

$$
\Gamma_{1\to2} < \Gamma_{2\to1} \implies \gamma_+ c^4 > \gamma_- s^4. \quad (8.8)
$$

and the threshold Rabi frequency is given by

$$
\frac{\epsilon_{thr}}{|\Delta_{AL}|} = \frac{\sqrt{\gamma_+ \gamma_-}}{\sqrt{\gamma_+} - \sqrt{\gamma_-}}. \quad (8.9)
$$

In the secular approximation, the dressed state picture renders the analysis of the spectral characteristics of the scattered radiation highly intuitive and simple. Here, we follow the analysis presented in Ref. [29] and extend it to the case of a frequency dependent photonic reservoir. By using the quantum regression theorem, it can be shown that the spectrum of the scattered radiation is symmetric about the laser field frequency $\omega_L$ and exhibits three lines centered on the Mollow triplet frequencies $\omega_{L-2\Omega}$, $\omega_L$, $\omega_{L+2\Omega}$. The central line appears as a result of the transitions $|1, n \rangle \rightarrow |1, n - 1 \rangle$ and $|2, n \rangle \rightarrow |2, n - 1 \rangle$ (which occur at a rate $\Gamma_{1\to1} = \Gamma_{2\to2} = \gamma_0 c^2 s^3$) and its structure is given the sum of an elastic component,

$$
G_{el}(\omega) = \gamma B \delta(\omega - \omega_L), \quad (8.10)
$$

with a weight of $\gamma B$ and an inelastic component, centered at $\omega_L$, with a weight $\gamma A$ and a FWHM of $2\Gamma_{pop}$.

$$
G_{in}(\omega) = \gamma A \frac{\Gamma_{pop}}{\pi |\omega - \omega_L|^2 + \Gamma_{pop}^2}. \quad (8.11)
$$

FIG. 15: Bare Atom and Dressed Atom States Diagram.
Here, $1/\Gamma_{\text{pop}}$ is the time constant that governs the transient regime of the dressed atomic populations, and is given by

$$\Gamma_{\text{pop}} = \Gamma_{1\rightarrow2} + \Gamma_{2\rightarrow1} = \gamma_{-} s^4 + \gamma_{+} c^4. \quad (8.12)$$

The total weight of the central line is defined as number of photons emitted per second in this line [29] and is calculated using the dressed state diagram in Fig. 15:

$$\gamma (A + B) = \frac{\pi}{\Gamma_{\text{pop}}} \cdot \Gamma_{1\rightarrow1} + \frac{\pi}{\Gamma_{\text{pop}}} \cdot \Gamma_{2\rightarrow2} \quad (8.13)$$

Further calculations [29], [28] show that the weight of elastic line takes the form

$$\gamma B = \Gamma_{1\rightarrow1} \left( \pi_{1}^s - \pi_{2}^s \right)^2 = \Gamma_{2\rightarrow2} \left( \pi_{2}^s - \pi_{1}^s \right)^2, \quad (8.14)$$

while the maximum value of the inelastic component of the central line is given by

$$G_{\text{in}}(\omega_L) = \frac{\gamma A}{\pi} \frac{1}{\Gamma_{\text{pop}}} = \frac{2\Gamma_{1\rightarrow1}}{\pi} \frac{\pi_{1}^s \cdot \pi_{2}^s}{\Gamma_{1\rightarrow2} + \Gamma_{2\rightarrow1}} \quad (8.15)$$

It is thus clear that the dependence of the atomic population on the Rabi frequency will leave their mark on the dependence of the characteristics of the scattered radiation spectrum on the laser field intensity (the weights and widths of the spectral components are determined by atomic population and dressed-states transition rates).

In the case of positive atomic detunings, we obtain a monotonic dependence of the peak height of the inelastic component of the central line on the Rabi frequency, shown in Fig. 17, whereas in the case of the negative detunings, presented in Fig. 16, the peak height of the inelastic component of the central line has a resonance-like shape, associated with the ability of the atomic system to reach positively inverted states.

Due to the non-Markovian interaction between the atomic system and the frequency dependent reservoir, even in the case of exact resonance between the atomic frequency in the medium $\omega_A$ and the band edge frequency $\omega_r$, a small part of the radiation spontaneously emitted by the atomic system (about 2.5% for a frequency dependent reservoir characterized by $\gamma_c = \gamma/100$ and $h = 0.01$) falls in the spectral range of the pseudo-gap (see Fig. 7), which is negatively detuned from the band edge frequency $\omega_r$. We suggest that this contribution gives rise to the characteristic behavior of maximum value of the inelastic component of the central line $G_{\text{in}}(\omega_L)$ as a function of Rabi frequency shown in Fig. 14. We infer from Fig. 16 and Fig. 17, that the magnitude of this maximum value for negative detunings, is much larger than in the opposite regime of positively detuned atomic frequencies (in a ratio of about 1000). Thus, even if only a small percentage of the characteristic atomic radiation is negatively detuned, its contribution to the total emitted scattered radiation is still important.

\[FIG. 16: \text{The peak height} H^0_{\text{max}} \text{of the central component of the scattered spectrum (in arbitrary units) as a function of the Rabi frequency} \epsilon/|\Delta_{AL}| \text{for} |\Delta_{AL}|/\gamma_+ = 3, \text{sign}(\Delta_{AL}) = -1, \text{and the magnitude of the jump in the photonic DOS,} \gamma_-/\gamma_+, \text{in the absence of the dipolar dephasing,} \gamma_p/\gamma_+ = 0. \text{The atomic resonant frequency is detuned negatively from the laser field frequency} \Delta_{AL} = \omega_A - \omega_L < 0.\]

\[FIG. 17: \text{The peak height of the inelastic component of the central line} G_{\text{in}}(\omega_L) \text{as a function of} \epsilon/|\Delta_{AL}| \text{for positive detuning of the atomic frequency} \omega_A \text{with respect to the laser field frequency} \omega_L, \text{sign}(\Delta_{AL}) = 1 \text{and the ratio of the decay constants} \gamma_-/\gamma_+ \text{for} \gamma_0/\gamma_+.\]

\section{IX. QUANTUM OPTICAL BLOCH EQUATIONS}

As it has been shown in the previous sections, the Mollow spectrum for resonance fluorescence in vacuum is derived directly from the dipole-approximation Hamiltonian, $H_D$ (Eq. (3.16)), with no additional assumptions, such as the Born-Markov approximation. Therefore, the Optical Bloch equations should also be derived directly from the Hamiltonian $H_D$, without any additional approximations. In this section, we derive general form of the quantum optical Bloch equations. In order to simplify the calculations, we limit ourselves to the free space.
In the ordinary vacuum, the Hamiltonian $H_D$ can be cast in the form

$$H_D = \frac{1}{2} \omega_A (\sigma_3 + 1) - i \int_{-\infty}^{\infty} dx \, c^\dagger(x) \frac{\partial}{\partial x} c(x) + \sqrt{\gamma} \int_{-\infty}^{\infty} dx \, \delta(x) [c(x) \sigma_+ + \sigma_- c^\dagger(x)].$$  \hspace{1cm} (9.1a)

Since in ordinary vacuum $\omega = k$, we introduce in Eq. (9.1a) the radiation operators

$$c(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} c(k) \exp(i k x).$$  \hspace{1cm} (9.1b)

The Heisenberg equations of motion are then found to be

$$i \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) c(x, t) = -\sqrt{\gamma} \delta(x) \sigma_-(t),$$  \hspace{1cm} (9.2a)

$$i \frac{d}{dt} \sigma_-(t) = \omega_A \sigma_-(t) + \sqrt{\gamma} \sigma_3(t) c(0,t),$$  \hspace{1cm} (9.2b)

$$i \frac{d}{dt} \sigma_3(t) = 2 \sqrt{\gamma} \left[ c^\dagger(0,t) \sigma_-(t) - \sigma_+(t) c(0,t) \right].$$  \hspace{1cm} (9.2c)

These equations describe particles which propagate to the positive direction of the $x$ axis and are scattered by a two-level atom located at the point $x = 0$. The negative $x$ correspond to incoming dipole photon, while the positive $x$ to outgoing one.

From the Maxwell equation (9.2a), we can infer that the radiation field operator is a discontinuous function of the coordinate $x$, and hence this operator at the atom (assumed a point-like system) position,

$$c(0,t) = \int_{-\infty}^{\infty} dx \, \delta(x) c(x,t),$$  \hspace{1cm} (9.3)

is not well defined. To correct this unphysical behavior, we replace the $\delta$-function by a smooth function, $\delta_\epsilon(x)$, and set $\delta_\epsilon(x) \to \delta(x)$ only after Eqs. (9.2) have been solved. This physical redefinition results in

$$c(0,t) = \frac{1}{2} [c(0+,t) + c(0-,t)],$$  \hspace{1cm} (9.4a)

i.e., the discontinuous field at the atom position is found to be the semi-sum of the fields at the points $x \to \pm 0$. Thus, the atom interacts with both incoming and outgoing components of the field. The jump of the field at the point $x = 0$ is found by integrating Eq. (9.2a) over an infinitely small interval around the point $x = 0$,

$$i[c(0+,t) - c(0-,t)] = -\sqrt{\gamma} \sigma_-(t).$$  \hspace{1cm} (9.4b)

In the scattering problem of a given incident field, the operator of the incoming photon, $c(0-,t)$, is associated with the incident field at the point $x = 0$, i.e., $c(0-,t) = c_{in}(t)$. Its time evolution does not depend on the state of the atom, and is completely determined by the state of the incident field. Similarly, the operator of outgoing dipole photon, $c(0+,t)$, is associated with the out-field of the scattering problem, $c(0+,t) = c_{out}(t)$. Correspondingly, the difference of these operators defines the scattered field,

$$c_{scat}(t) = c_{out}(t) - c_{in}(t) = i \sqrt{\gamma} \sigma_-(t),$$  \hspace{1cm} (9.5a)

while the total field at the point $x = 0$ involved in the Bloch equations (9.2b) and (9.2c) is found to be

$$c(0,t) = \frac{1}{2} [c_{out}(t) + c_{in}(t)] = c(t) + \frac{i}{2} \sqrt{\gamma} \sigma_-(t).$$  \hspace{1cm} (9.5b)

We note that the operators of the out- and scattered fields at a given moment of time $t$ are determined by time evolution of atomic operators for all previous moments of time. Therefore, in the Bloch equations determining time evolution of the atomic operators, the atomic and field operators on the r. h. s. must be ordered as in Eqs. (9.2b) and (9.2c). Inserting then Eq. (9.5b) into Eqs. (9.2b) and (9.2c) we obtain

$$i \frac{d}{dt} \sigma_-(t) = \left( \omega_A - i \gamma/2 \right) \sigma_-(t) + \sqrt{\gamma} \sigma_3(t) c_{in}(t),$$  \hspace{1cm} (9.6a)

$$i \frac{d}{dt} \sigma_3(t) = -\gamma [1 + \sigma_3(t)] - 2i \sqrt{\gamma} \left[ c_{in}(t) \sigma_-(t) - \sigma_+(t) c_{in}(t) \right],$$  \hspace{1cm} (9.6b)

where we used $\sigma_3(t) \sigma_-(t) = -\sigma_-(t)$ and $\sigma_+(t) \sigma_-(t) = \frac{1}{2} [1 + \sigma_3(t)]$.

Equations (9.6) are quantum optical Bloch equations, because the external field is represented here by the operator $c_{in}(t)$ rather than its expectation value. These equations are valid for an arbitrary given quantum state of the incident field. Similar to the classical optical Bloch equations [2], they allow us to derive a closed set of equations for the correlation functions of atomic operators, and to obtain the Mollow spectrum of fluorescence. However, in this case, the Mollow expression for the spectrum is in an operator form and is valid for an arbitrary statistics of an incident laser field. To obtain the fluorescence spectrum for a given quantum state of the incident field, one has to average the operator Mollow spectrum over this quantum state. If and only if the incident light is in a coherent state, the averaging simply results in the replacement of the operator $c_{in}(t)$ by its expectation value, $E_{in}(t)$, and this way we derive the standard Mollow expression for fluorescence spectrum. In all other cases, the averaging can dramatically change the analytical structure of the Mollow expression, and may lead to a strong dependence of the fluorescence spectrum on quantum-statistical properties of incident light.

**X. CONCLUSIONS**

In this paper, we have extended the multi-photon theory of resonance fluorescence to colored vacua. Our
approach is valid for an arbitrary driving field intensity, and an arbitrary photonic reservoir bath. This approach is not restricted to weak coupling between atom and reservoir required by the perturbational methods [31], and can describe a general non-Markovian system. The only restriction of our theoretical framework is that, strictly speaking, it requires either an isotropic or an one-dimensional dispersion relation for the photonic reservoir.

For a strongly colored vacuum, the scattering cross section becomes strongly non-Lorentzian, reflecting the non-exponential character of the atomic dipole decay. As the intensity of the driving field increases, the scattered spectrum suffers a global narrowing (the position of the central component) and a local narrowing (the width of the sidebands components of spectrum is shifted towards the frequency of incident light). In all other cases, the averaging can dramatically change the analytical structure of the scattered spectrum. To obtain the fluorescence spectrum for a given quantum state of the incident field, one has to average the operator Mollow spectrum over this quantum state. If and only if the incident light is in a coherent state, the averaging simply results in the replacement of the incident field operators by their expectation values, and this results in the standard Mollow expression for the fluorescence spectrum. In all other cases, the averaging can dramatically change the analytical structure of the Mollow expression, and may lead to a strong dependence of the fluorescence spectrum on quantum-statistical properties of incident light.

The strength of our approach is given by the simplicity of its results (the scattered density of the scattered field is obtained by simple integration over analytical functions of its results (the scattered density of the scattered field is obtained by simple integration over analytical functions of its results), and a local narrowing (the width of the sidebands components of spectrum is shifted towards the frequency of incident light). In all other cases, the averaging can dramatically change the analytical structure of the Mollow expression, and may lead to a strong dependence of the fluorescence spectrum on quantum-statistical properties of incident light.

The “rapidities” are, in turn, defined by the Bethe ansatz [32] for the multi-particle wave function [4, 6]:

\[ | \vec{\lambda} \rangle = \int_{-\infty}^{+\infty} d\tau_1 \ldots d\tau_n \]
\[ \times \prod_{k<l} \left[ 1 + \frac{i\gamma}{h(\lambda_i) - h(\lambda_j)} \text{sign} (\tau_k - \tau_l) \right] \]
\[ \times \prod_{j=1}^n [\Phi(\tau_j|\lambda_j) e^{i\lambda_j \tau_j} \]
\[ \times \left[ e^{i(\tau_j)} - \frac{\sqrt{\gamma}}{h(\lambda_j)} \delta(\tau_j) \sigma_+ \right] ] |0\rangle. \]  

(A2)

Here, the Bethe factor,

\[ \prod_{k<l} \left[ 1 + \frac{i\gamma}{h(\lambda_i) - h(\lambda_j)} \text{sign} (\tau_k - \tau_l) \right], \]  

(A3)

is induced by the photon-photon correlations in the process of the scattering of photons on the atomic system [12].

Consider now an initial state of the system, \(|\Psi_{in}\rangle\), which in the \(\vec{\lambda}\) representation, has the expansion:

\[ |\Psi_{in}\rangle = \int d\vec{\lambda} \ |\vec{\lambda}\rangle \langle \vec{\lambda}|\Psi_{in}\rangle. \]  

(A4)

This initial state evolves according to \(|\Psi(\tau)\rangle = \exp (-iH\tau)|\Psi_{in}\rangle\), and its asymptotic behavior for \(\tau \to \infty\) is given by

\[ |\Psi_{out}\rangle = \lim_{\tau \to \infty} |\Psi(\tau)\rangle = \]
\[ \lim_{\tau \to \infty} \int d\vec{\lambda} \ |\vec{\lambda}\rangle \langle \vec{\lambda}|\Psi_{in}\rangle \exp (-i\tau \sum_{j=1}^n \lambda_j). \]  

(A5)

In the following, we consider the asymptotic behavior \((\tau \to \infty)\) of a specific multi-particle state that initially \((\tau \to -\infty)\) has the form

\[ |\in, N\rangle = \frac{1}{\sqrt{N!}} |n_k\rangle^N |0\rangle \]
\[ = \frac{1}{\sqrt{N!}} \sum_{n=0}^N C_N^n \beta^{n/2} \left[ A_k \right]^{N-n} [c^+(\omega)]^n |0\rangle. \]  

(A6)

(here, \(C_N^n = N! / n!(N - n)!\) are the binomial coefficients and \(A_k, \beta\) are defined in (3.15)) and describes an incident pulse containing \(N\) plane wave photons.

Due to the specific form of the interaction Hamiltonian (8.1c), only the dipole photons interact with the atomic

APPENDIX A: MULTI-PARTICLE SCATTERING PROBLEM SOLUTION

In the case of multi-photon scattering, the \(n\)-particle eigenstate of the Hamiltonian (3.16), \(|\vec{\lambda}_{\tilde{\nu}}\rangle \equiv |\lambda_1, \ldots, \lambda_n\rangle\), parameterized by the set of “rapidities” \(\{\lambda_j, j = 1, n; \lambda_i \neq \lambda_j \text{ for } i \neq j\}\) is defined by

\[ H_D |\vec{\lambda}\rangle = \left( \sum_{j=1}^n \lambda_j \right) |\vec{\lambda}\rangle, \]  

(A1a)

\[ N |\vec{\lambda}\rangle = n |\vec{\lambda}\rangle. \]  

(A1b)

The “rapidities” are, in turn, defined by the Bethe ansatz [32] for the multi-particle wave function [4, 6]:

\[ |\vec{\lambda}\rangle = \int_{-\infty}^{+\infty} d\tau_1 \ldots d\tau_n \]
\[ \times \prod_{k<l} \left[ 1 + \frac{i\gamma}{h(\lambda_i) - h(\lambda_j)} \text{sign} (\tau_k - \tau_l) \right] \]
\[ \times \prod_{j=1}^n [\Phi(\tau_j|\lambda_j) e^{i\lambda_j \tau_j} \]
\[ \times \left[ e^{i(\tau_j)} - \frac{\sqrt{\gamma}}{h(\lambda_j)} \delta(\tau_j) \sigma_+ \right] ] |0\rangle. \]  

(A2)

Here, the Bethe factor,

\[ \prod_{k<l} \left[ 1 + \frac{i\gamma}{h(\lambda_i) - h(\lambda_j)} \text{sign} (\tau_k - \tau_l) \right], \]  

(A3)

is induced by the photon-photon correlations in the process of the scattering of photons on the atomic system [12].

Consider now an initial state of the system, \(|\Psi_{in}\rangle\), which in the \(\vec{\lambda}\) representation, has the expansion:

\[ |\Psi_{in}\rangle = \int d\vec{\lambda} \ |\vec{\lambda}\rangle \langle \vec{\lambda}|\Psi_{in}\rangle. \]  

(A4)

This initial state evolves according to \(|\Psi(\tau)\rangle = \exp (-iH\tau)|\Psi_{in}\rangle\), and its asymptotic behavior for \(\tau \to \infty\) is given by

\[ |\Psi_{out}\rangle = \lim_{\tau \to \infty} |\Psi(\tau)\rangle = \]
\[ \lim_{\tau \to \infty} \int d\vec{\lambda} \ |\vec{\lambda}\rangle \langle \vec{\lambda}|\Psi_{in}\rangle \exp (-i\tau \sum_{j=1}^n \lambda_j). \]  

(A5)

In the following, we consider the asymptotic behavior \((\tau \to \infty)\) of a specific multi-particle state that initially \((\tau \to -\infty)\) has the form

\[ |\in, N\rangle = \frac{1}{\sqrt{N!}} |n_k\rangle^N |0\rangle \]
\[ = \frac{1}{\sqrt{N!}} \sum_{n=0}^N C_N^n \beta^{n/2} \left[ A_k \right]^{N-n} [c^+(\omega)]^n |0\rangle. \]  

(A6)

(here, \(C_N^n = N! / n!(N - n)!\) are the binomial coefficients and \(A_k, \beta\) are defined in (3.15)) and describes an incident pulse containing \(N\) plane wave photons.

Due to the specific form of the interaction Hamiltonian (8.1c), only the dipole photons interact with the atomic
system. Therefore, we analyze the scattering of \( n \) dipole photons in an initial state

\[
|\Psi_{in}, n\rangle = \frac{1}{\sqrt{n!}} \int d\vec{r}' \Psi_{in}(\vec{r}') \prod_{j=1}^{n} c^{\dagger}(\tau_j')|0\rangle,
\]

(A7)

where \( \vec{r}' = (\tau_1', \ldots, \tau_n') \). In the case of steady-state resonance fluorescence, the initial multi-photon wave function is given by

\[
\Psi_{in}(\vec{r}') = \prod_{j=1}^{n} \exp(i\omega_L \tau_j').
\]

(A8)

Since the integrand in Eq. (A7) is symmetric with respect to permutations of particle coordinates, the in-state can be represented as

\[
|\Psi_{in}, n\rangle = \sqrt{n!} \int d\vec{\theta}(\tau_1 > \ldots > \tau_n) \prod_{j=1}^{n} c^{\dagger}(\tau_j)|0\rangle
\]

\[
\times \Psi_{out}(\vec{\theta}) \prod_{j=1}^{n} c^{\dagger}(\tau_j)|0\rangle.
\]

(A9)

After scattering, the out-state of the \( n \) scattered dipole photons can be parameterized in an analogous way,

\[
|\Psi_{out}, n\rangle = \sqrt{n!} \int d\vec{\theta}(\tau_1 > \ldots > \tau_n)
\]

\[
\times \Psi_{out}(\vec{\theta}) \prod_{j=1}^{n} c^{\dagger}(\tau_j)|0\rangle.
\]

(A10)

Similar to the one-particle scattering case, we use the expansions

\[
\prod_{j=1}^{n} c^{\dagger}(\tau_j)|0\rangle \equiv \prod_{j=1}^{\infty} \int_{-\infty}^{\infty} d\lambda_j \int_{D_j} d\tau_j' e^{i\lambda_j (\tau_j' - \tau_j)}
\]

\[
\times [1 + \phi(\lambda_j)] c^{\dagger}(\tau_j')|0\rangle
\]

(A11)

where the scattered amplitude \( \phi(\lambda_j) \) is defined by Eq. (4.20). The function \( \Psi_{out}(\vec{\theta}) \) in Eq. (A10) becomes

\[
\Psi_{out}(\vec{\theta}) = \prod_{j=1}^{n} e^{i\omega_L \tau_j} \int_{-\infty}^{\infty} d\lambda_j \int_{D_j} d\tau_j' e^{i(\omega_L - \lambda_j)(\tau_j' - \tau_j)}
\]

\[
\times [1 + \phi(\lambda_j)].
\]

(A12)

The impossibility of double excitation of a single atom (which eliminates two-photon scattering processes) implies that the integration over the coordinate of an incident particle \( \tau_j' \) has to be carried out in the domain \( D_j \equiv (\tau_j-1, \tau_j) \) [4], and Eq. (A12) simplifies to

\[
\Psi_{out}(\vec{\theta}) = \prod_{j=1}^{n} e^{i\omega_L \tau_j} \psi_{out}(\tau_j-1, \tau_j),
\]

(A13)

where

\[
\psi_{out}(\tau_j-1, \tau_j) = \int_{-\infty}^{\infty} d\lambda_j \int_{\tau_{j-1}}^{\tau_j} d\tau_j' e^{i(\omega_L - \lambda_j)(\tau_j' - \tau_j)}
\]

\[
\times [1 + \phi(\lambda_j)]
\]

\[
= \int_{-\infty}^{\infty} d\lambda_j \frac{1 + \phi(\lambda_j)}{2\pi i \lambda_j - \omega_L + i0}
\]

\[
\times \left\{ 1 - e^{i(\omega_L - \lambda_j)(\tau_j-\tau_j')} \right\}.
\]

(A14)

Consider now an initial state containing \( N \) plane wave photons, \( |\text{In}, N\rangle \). From these \( N \) plane wave photons, only \( n \) (the dipole photons) interact with the atomic system. Using the definitions (6.1), (A6) we obtain

\[
|\text{Out}, N\rangle = \frac{1}{\sqrt{N!}} \sum_{n=0}^{\infty} \frac{N!}{(N-n)!n!} \beta^{n/2} \left[ a_k \right]^{N-n} \otimes |\Phi_{\text{scatt}}, n\rangle,
\]

(A15)

with \( |\Phi_{\text{scatt}}, n\rangle \) the unsymmetrized state vector of the scattered dipole photons. We isolate the contribution of the \( N-n \) plane wave photons and the contribution of the \( n \) dipole photons, and symmetrize them with respect to permutations of the respective photons. We obtain

\[
|\text{Out}, N\rangle = \sum_{n=0}^{\infty} \frac{\sqrt{C_N^k}}{\beta^{n/2}} |\Psi_{\text{scatt}}, n\rangle
\]

\[
\times \otimes \frac{1}{(N-n)!} \left[ a_k \right]^{N-n},
\]

(A16)

where \( |\Psi_{\text{scatt}}, n\rangle = 1/\sqrt{n!} |\Phi_{\text{scatt}}, n\rangle \) is now the symmetrized state vector of the scattered dipole photons.

If the incident electric field is in a coherent state containing an average number of photons \( \bar{N} \) and given by

\[
|\text{In} \rangle = e^{\bar{N}/2} \sum_{N=0}^{\infty} \frac{\bar{N}^N}{N!} \left[ a_k \right]^N |0\rangle = \sum_{N=0}^{\infty} A_N |\text{In}, N\rangle,
\]

(A17)

where

\[
A_N = \sqrt{\frac{e^{-\bar{N}}\bar{N}^N}{N!}},
\]

(A18)
the out-state of the field becomes

\[ |\text{Out} \rangle = \sum_{N=0}^{\infty} A_N |\text{Out}, N \rangle = e^{\hat{N}/2} \sum_{N=0}^{\infty} \frac{\hat{N}^{N/2}}{\sqrt{N!}} |\text{Out}, N \rangle \]

\[ = e^{\hat{N}/2} \sum_{N=0}^{\infty} \frac{\hat{N}^{N/2}}{\sqrt{N!}} \left[ \frac{1}{n!} \beta_n \right]^{N-n} \sum_{n=0}^{\infty} \sqrt{\frac{C_N}{N}} \langle \Psi_{\text{scatt}}, n \rangle \]

\[ \otimes \left[ \frac{1}{\sqrt{(N-n)!}} \right]^{N-n} \]

\[ = e^{\hat{N}/2} \sum_{N=0}^{\infty} \sum_{n=0}^{N} \frac{\hat{N}^{(N-n)/2}}{(N-n)!} \beta_n \left\{ \frac{1}{n!} \right\}^{N-n} \]

\[ \otimes \left[ \frac{1}{\sqrt{n!}} \beta_n \right]^{N-n} \]

\[ = e^{\hat{N}/2} \sum_{N=0}^{\infty} \sum_{n=0}^{N} \frac{\hat{N}^{(N-n)/2}}{(N-n)!} \beta_n \left\{ \frac{1}{n!} \right\}^{N-n} \]

\[ \otimes \left[ \frac{1}{\sqrt{n!}} \beta_n \right]^{N-n} , \]

(A19)

with \( \rho_0 = \beta \hat{N} \). Now, using the identity

\[ \sum_{N=0}^{\infty} \sum_{n=0}^{N} F(N-n) \cdot g(n) = \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} F(N) \cdot g(n), \]

(A20)

we have

\[ |\text{Out} \rangle = e^{\hat{N}/2} \sum_{N=0}^{\infty} \left[ \frac{\hat{N}^{(N)/2}}{N!} \right] \left[ \frac{1}{n!} \beta_n \right]^{N-n} \]

\[ \otimes \sum_{n=0}^{\infty} \left[ \frac{1}{n!} \beta_n \right]^{N-n} \]

\[ = \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \beta_n \right]^{N-n} \otimes |\text{In} \rangle \]

and Eq. (6.11) is therefore proved.

**APPENDIX B: RENORMALIZED OUT-STATE**

The out-state Eq. (6.11) allows us to calculate any physical characteristics of the radiation field. The stationary expectation value of an arbitrary operator \( Q \{ c^\dagger(\tau), c(\tau) \} \) is found by averaging over the out-state,

\[ Q \equiv \langle \text{Out} | Q \{ c^\dagger, c \} | \text{Out} \rangle = e^{-\hat{N}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\rho_0^{n/2} \rho_0^{m/2}}{\sqrt{mn!}} \]

\[ \times \int d\bar{\tau} \int d\bar{\tau}^* \bar{\Psi}_{\text{scatt}}^* (\bar{\tau}) \bar{\Psi}_{\text{scatt}} (\bar{\tau}) \]

\[ \times \langle 0 | e^{\sqrt{N}a^\dagger} \prod_{i=1}^{m} c(\tau_i^\prime) Q \{ c^\dagger, c \} \prod_{j=1}^{n} c(\tau_j) e^{\sqrt{N}a^\dagger} | 0 \rangle, \]

where we have substituted the explicit form of the out-state Eq. (6.11) (and also Eq. (6.10)). In such an evaluation, we need to estimate first the expression:

\[ Q \sim \langle 0 | e^{\sqrt{N}a^\dagger} \prod_{i=1}^{m} c(\tau_i^\prime) Q \{ c^\dagger, c \} \prod_{j=1}^{n} c(\tau_j) e^{\sqrt{N}a^\dagger} | 0 \rangle. \]

Here, the operators appearing in the exponents are generated by the in-component of the out-state (see out-state definition (6.11)) and correspond to transmitted plane-wave photons.

As usual in the scattering theory [19], we assume that the incident beam has a finite cross-section and that measurements of physical observables (in our case, \( Q \)) are carried outside this section. Therefore, the measured quantities do not include contributions from the transmitted components. Following the procedure outlined in [6], Eq. (B1) becomes

\[ Q \sim \langle 0 | \prod_{j=1}^{m} [c(\tau_j^\prime) + \sqrt{\rho_0} e^{i\omega_{\text{scatt}} \tau_j^\prime}] Q \{ c^\dagger, c \} \prod_{j=1}^{n} [c(\tau_j) + \sqrt{\rho_0} e^{-i\omega_{\text{scatt}} \tau_j}] | 0 \rangle, \]

(B1)

and we note that the transmitted components result in an effective shift of the scattered photon operators by the classical value proportional to the intensity of the incident light. For simplicity, we renormalize the out-state of the problem in the form[49]

\[ \langle \tilde{\text{Out}} | Q \{ c^\dagger, c \} | \text{Out} \rangle = \langle \text{Out} | Q \{ c^\dagger, c \} | \text{Out} \rangle \]

(B3)

Since the renormalization procedure does not change the expectation values of the physical observables, hereafter we drop the tilde from the out-state.

To compute expectation values, it is necessary now to open the brackets in Eq. (B2). In doing so, we assume that the \( \tau \)-coordinates of all scattered particles lie inside a finite interval between points \( -\tau_0 \) and \( +\tau_0 \), and we will set \( \tau_0 \to \infty \) only in expressions for expectation values of scattered field[50]. The out-state can be then represented as a sum of terms containing different number of scattered photons,

\[ |\text{Out} \rangle = u(2\tau_0) |0\rangle + \sum_{n=1}^{\infty} \rho_0^{n/2} \int d\bar{\tau} \prod_{j=1}^{n} \Psi_{\text{scatt}} (\tau_j) \]

\[ \times \prod_{j=1}^{n} u(\tau_j - 1) e^{i\omega_{\text{scatt}} \tau_j} c^\dagger(\tau_j) | 0 \rangle, \]

(B4)

where the function \( u(\tau) \) is found as a solution of the integral equation [6]

\[ u(\tau - \tau^\prime) = \psi(\tau - \tau^\prime) + \rho_0 \int_{-\tau_0}^{\tau_0} dz \psi(\tau - z) u(z - \tau^\prime), \]

(B5a)
while the function \( v(\tau) \) is given by

\[
v(\tau) = 1 + \rho_0 \int_{-\tau_0}^{\tau} d\tau' u(\tau - \tau'). \tag{B5b}
\]

The Fourier transforms of these functions are found to be

\[
u(\omega) = \frac{\psi(\omega)}{1 - \rho_0 \psi(\omega)} = \frac{\gamma}{\omega [h(\omega + \omega_L) + i\gamma/2] - \gamma \rho_0}, \tag{B6a}
\]

\[
v(\omega) = \frac{i}{\omega + i 0} [1 + \rho_0 u(\omega)] \times i [h(\omega + \omega_L) + i\gamma/2] \frac{\gamma}{\omega [h(\omega + \omega_L) + i\gamma/2] - \gamma \rho_0}. \tag{B6b}
\]

As we will show in the next section, the wave function \( v(\tau) \) does not affect expectation values of the scattered field, and one can set, for simplicity, \( v(2\tau_0) = 0 \). It is also convenient to omit hereafter the exponential factors \( \exp(i\omega_L \tau) \), assuming that all the frequencies in the problem are measured from the laser frequency \( \omega_L \). Then, we finally obtain

\[
|\text{Out} \rangle = \sum_{n=1}^{\infty} (\rho_0)^{n/2} \int_{-\infty}^{\infty} d\tau v(\tau_n + \tau_0) \times \prod_{j=1}^{n} u(\tau_{j-1} - \tau_j) c^\dagger(\tau_j) |0\rangle. \tag{B7}
\]

The equations (B6) and (B7) solve completely the multiparticle scattering problem in frequency dependent photonic reservoirs.

**APPENDIX C: CALCULATION OF THE SPECTRAL DENSITY OF THE SCATTERED FIELD**

The simplifications made concerning the function \( v(\tau) \) do not affect expectation values in the case of steady-state resonance fluorescence (\( \tau_0 \to \infty \)),

\[
Q = \frac{\langle \text{Out}|Q\{c^\dagger, c\}|\text{Out}\rangle}{\langle \text{Out}|\text{Out}\rangle}, \tag{C1}
\]

but they can change essentially the norm of the out-state. Therefore, we need first to evaluate

\[
\mathcal{N} = \langle \text{Out}|\text{Out} \rangle = \sum_{n=1}^{\infty} \rho_0^n \int d\tau V(\tau_n + \tau_0) \times \prod_{j=1}^{n} U(\tau_{j-1} - \tau_j), \tag{C2}
\]

where we introduced \( U(\tau) \equiv |u(\tau)|^2 \) and \( V(\tau) \equiv |v(\tau)|^2 \). In deriving Eq. (C2) we took into account that, because the wave function \( u(\tau) \) vanishes for \( \tau < 0 \), only direct pairings of the photon operators, \( \{0|c(\tau_j)^c\dagger(\tau_j)|0\} \) with \( j = j' \), have a non-vanishing contribution (the contributions of indirect pairings with \( j \neq j' \) containing the functions \( u(\tau) \) with negative arguments vanish).

The Fourier images of the functions \( U(\tau) \) and \( V(\tau) \) are given by

\[
U(\omega) = \int_{-\infty}^{\infty} d\tau U(\tau) e^{i\omega\tau} = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} u^*(\omega' - \omega) u(\omega'). \tag{C3}
\]

\[
V(\omega) = \int_{-\infty}^{\infty} d\tau V(\tau) e^{i\omega\tau} = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} v^*(\omega' - \omega) v(\omega'). \tag{C4}
\]

Equation (C2) yields

\[
\mathcal{N} = \int \frac{d\omega}{2\pi} V(\omega) H(\omega) e^{-i\omega\tau_0}, \tag{C5a}
\]

where

\[
H(\omega) = \sum_{n=1}^{\infty} \rho_0^n [U(\omega)]^n = \frac{\rho_0 U(\omega)}{1 - \rho_0 U(\omega)}. \tag{C5b}
\]

Since the norm of the out-state is canceled in the expressions for expectation values, we do not need to do any further computations in Eqs. (C5). However, the spectral density of the scattered field will depend on the factor

\[
\mathcal{H} = \int \frac{d\omega}{2\pi} H(\omega) e^{-i\omega\tau_0}. \tag{C6}
\]

Making use of the explicit expression of the function \( u(\omega) \), we find

\[
U(\omega = 0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} u^*(\omega') u(\omega') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i \omega - i 0} [u^*(\omega') - u(\omega')] = \frac{1}{\rho_0}, \tag{C7a}
\]

for an arbitrary function \( h(\lambda) \). This shows that the function \( H(\omega) \) has a pole at the point \( \omega = 0 \) i.e., \( 1 - \rho_0 U(\omega = 0) = 0 \). Therefore, for \( \omega \to 0 \) the denominator of the function \( H(\omega) \) can be represented as

\[
1 - \rho_0 U(\omega) = -i(\omega + i 0) F(\omega), \tag{C7b}
\]

where \( F(\omega = 0) \neq 0 \). As \( \tau_0 \to \infty \), the value of \( \mathcal{H} \) is determined only by the contribution of the pole at \( \omega = 0 \), and we obtain

\[
\mathcal{H} = \frac{1}{F_0}, \tag{C8}
\]

where \( F_0 = F(\omega = 0) \). By expanding in series the function \( U(\omega) \) in the vicinity of \( \omega = 0 \), the normalization factor \( F_0 \) becomes

\[
F_0 = i \rho_0 \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} U^{(1)}(\omega'), \tag{C9}
\]
with \(U^{(1)}(\omega')\) given by
\[
U^{(1)}(\omega') = u(\omega') \left[ \frac{du^*(\omega)}{d\omega} \right]_{\omega=\omega'}.
\] (C10)

We now compute the correlator
\[
G(\tau, \tau') = \frac{1}{N} \langle \text{Out}|c^\dagger(\tau)c(\tau')|\text{Out} \rangle,
\] (C11)
whose Fourier image over the difference of coordinates gives the spectral density of the scattered field per unit of time, and where the |Out state is given by Eq. (B7).

Apart from pairings of the “internal” operators involved into the out-state, Eq. (C11) contains also two pairings between the “external” operators \(c(\tau')\) and \(c^\dagger(\tau)\), and internal operators, \(\langle 0|c(\tau')c^\dagger(\tau)|0 \rangle = \delta(\tau-\tau')\) and \(\langle 0|c(\tau')c^\dagger(\tau)|0 \rangle = \delta(\tau-\tau)\). There are two different types of contributions corresponding to the external pairings. The first type contains only one of two external pairings, \(l \neq j\), while in the second one both external pairings are involved, \(l = j\). The correlator \(G(\tau, \tau')\) is then represented as \(G(\tau, \tau') = G^{(1)}(\tau, \tau') + G^{(2)}(\tau, \tau')\). We demonstrate this fact for the case of a small number of scattered particles. For example, in the case of \(n = 4\), we have:

\[
G^{(1)}_{n=4}(\tau, \tau') = \frac{\rho_0^4}{N} \int d\tau_1 d\tau_2 d\tau_3 V(\tau_3 + \tau_0)
\]
\[
\times \Gamma^{(1)}(\tau|\tau_3, \tau_2) \Gamma^{(1)}(\tau|\tau_2, \tau_1) U(\tau_0 - \tau_1),
\] (C12)

where
\[
\Gamma^{(1)}(\tau|\tau_3, \tau_2) = u(\tau' - \tau_3) u^*(\tau_2 - \tau') u^*(\tau_2 - \tau_3),
\] (C13a)
\[
\Gamma^{(1)}(\tau|\tau_2, \tau_1) = u^*(\tau - \tau_2) u^*(\tau_1 - \tau) u(\tau_1 - \tau_2).
\] (C13b)

Adding the contributions from all diagrams containing the contribution of the first type, we find in the Fourier representation
\[
G^{(1)}(\tau - \tau') = \frac{\rho_0^3}{F_0} \int d\omega \frac{\Gamma^{(1)}(\omega) \Gamma^{(1)*}(-\omega)}{1 - \rho_0 U(\omega)} e^{i\omega(\tau - \tau')},
\] (C14a)

where
\[
\Gamma^{(1)}(\omega) = \int \frac{d\omega'}{2\pi} u^*(\omega') u(\omega') u(\omega + \omega').
\] (C14b)

Here, the term \(F_0^{-1}\) is generated by the summation of all diagrams between the points \(\tau\) and \(\tau_0\), whereas the sum of diagrams between the points \(-\tau_0\) and \(\tau'\) gives the norm of the out-state \(N\), which cancels the norm in the denominator of Eq. (C12). The summation of contributions between the points \(\tau'\) and \(\tau\) results in the denominator \(1 - \rho_0 U(\omega)\) in Eq. (C14a).

Equation (C14a) is valid only for \(\tau > \tau'\). By repeating the calculations for the case \(\tau < \tau'\), we find the total expression for \(G^{(1)}(\tau - \tau')\) valid for an arbitrary difference of the coordinates,
\[
G^{(1)}(\tau - \tau') = \frac{\rho_0^3}{F_0} \int d\omega \frac{\Gamma^{(1)}(\omega) \Gamma^{(1)*}(-\omega)}{1 - \rho_0 U(\omega)} e^{i\omega(\tau - \tau')} + \frac{\Gamma^{(1)}(-\omega) \Gamma^{(1)*}(\omega)}{1 - \rho_0 U(-\omega)} e^{-i\omega(\tau - \tau')},
\] (C15)

Let us consider now the contributions of the second type, which contain both of the external pairings. In the case of \(n = 3\), we obtain
\[
G^{(2)}_{n=3}(\tau, \tau') = \frac{\rho_0^3}{N} \int d\tau_1 d\tau_2 V(\tau_2 + \tau_0)
\]
\[
\times \Gamma^{(2)}(\tau, \tau', |\tau_2, \tau_1) U(\tau_0 - \tau_1),
\] (C16a)

where
\[
\Gamma^{(2)}(\tau, \tau', |\tau_2, \tau_1) = u(\tau' - \tau_2) u^*(\tau - \tau_2)
\]
\[
\times u(\tau_1 - \tau') u^*(\tau_1 - \tau).
\] (C16b)

Summarizing all second type contributions that contain \(\Gamma^{(2)}\), we find
\[
G^{(2)}(\tau - \tau') = \frac{\rho_0^3}{F_0} \int d\omega \frac{\Gamma^{(2)}(\omega) e^{-i\omega(\tau - \tau')}}{2\pi},
\] (C17a)

where
\[
\Gamma^{(2)}(\omega) = \int d\omega' \frac{du'(\omega') u(\omega') u^*(\omega + \omega') u(\omega + \omega).}{2\pi}
\] (C17b)

The spectral density of scattered field is then found to be
\[
G(\omega) = \int_{-\infty}^{\infty} d\tau \Gamma(\tau)e^{i\omega\tau} = \frac{\rho_0^3}{F_0} \frac{\Gamma^{(2)}(\omega)}{1 - \rho_0 U(\omega)}
\]
\[
+ \frac{\Gamma^{(1)}(\omega) \Gamma^{(1)*}(-\omega) + \Gamma^{(1)}(-\omega) \Gamma^{(1)*}(\omega)}{1 - \rho_0 U(-\omega)}.
\] (C18)


[46] For example, the photonic DOS can change by a factor of 100 on a frequency interval $\Delta \nu = 10^{-6} \cdot \nu_c$, Here $\nu_c$ is the band-edge frequency and $n$ is a coefficient determined by the structure length (empirically[9], $n = L/5 + 1$, where $L$ is the number of unit cells in the sample).

[47] The time interval is chosen to be $\Delta t = 0.95\Delta/(c\nu^3)$. These parameters are within the stability bound [42]. For light propagating in vacuum, more than 20 grid points per wavelength, i.e., $\omega \leq 0.5(2\pi/a)$ if $\Delta = 0.1a$, will lead to an intrinsic grid velocity anisotropy to be less than 0.2%. The numerical wave propagating in vacuum will grow exponentially when $\Delta t > \Delta/(c\nu^3)$.

[48] As shown in Appendix C, the correlation function $G(r)$ can be represented as a sum of two contributions, corresponding to different pairings of the operators appearing in the evaluation of the spectral density of the scattered field (see Sec. C).

[49] The renormalization procedure presented here is necessary, since one of our goals is to compare the results predicted by the multi-photon scattering theory to the results obtained using the more traditional optical Bloch equations approach (OBE). In the OBE approach (see [33] for a detailed review of OBE picture of the resonance fluorescence) one eliminates the transmitted field (this elimination proceeds in a straightforward manner, because the incident field is assumed to be a classical variable), and evaluates only the fluorescent spectrum scattered by the atom.

[50] This approach allows us to avoid computational artifacts usually encountered in multi-particle scattering theories [11], caused by the non-commutation of limit and integration operations in Eq. (B1).