Antimatter

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ABSTRACT

The nature of antimatter is examined in the context of algebraic quantum field theory. It is shown that the notion of antimatter is more general than that of antiparticles. Properly speaking, then, antimatter is not matter made up of antiparticles — rather, antiparticles are particles made up of antimatter. We go on to discuss whether the notion of antimatter is itself completely general in quantum field theory. Does the matter-antimatter distinction apply to all field theoretic systems? The answer depends on which of several possible criteria we should impose on the space of physical states.

1 Introduction

Antimatter is matter made up of antiparticles, or so they say. To every fundamental particle there corresponds an antiparticle of opposite charge and otherwise identical properties. (But some neutral particles are their own antiparticles.) These facts have been known for some time — the first hint of them came when Dirac's "hole theory" of the relativistic electron predicted the existence of the positron. So they say.

All this is true enough, at a certain level of description. But if recent work in the philosophy of quantum field theory (QFT) is any indication, it must all be false at the *fundamental* level. After all, the facts about antimatter listed above are all facts about particles. And at a fundamental level, there are no particles, according to some of the best recent work in the philosophy of QFT.[1] This would seem to render the concept of antimatter irrelevant to matters of fundamental ontology. For if only particles can properly be called "anti" or not, and particles are no part of QFT's most basic ontology, it follows that the most basic things in a field theoretic universe cannot be categorized into matter and antimatter. Although "Matter comes in particle and antiparticle form. . . Particles are emergent phenomena, which emerge in domains where the underlying quantum field can be treated as approximately linear" (Wallace, 2008, p. 15).

Considered in light of some of the most important research in algebraic QFT (AQFT), these matters are not so simple. We will show in what follows that there may be a fundamental matter-antimatter distinction to be drawn in QFT. Whether there is does *not* depend on whether particles play any part in the theory's fundamental ontology. Rather, it depends on which criteria we use to determine which of the theory's mathematically well-defined states represent real possibilities, and which are surplus theoretical structure or (in the physicist's parlance) unphysical.

2 Antiparticles on the naive picture

A standard, naive picture of antimatter begins with the notion of antiparticle that emerges from quantum mechanics (QM) governed by free relativistic wave equations. The simplest

[1]See Halvorson and Clifton (2001), Fraser (forthcoming), Malament (1996) and Halvorson and Clifton (2002) for arguments to this effect. Theories which do admit particles have been put forward as empirically equivalent to QFT (Dürr et al., 2005). Such Bohmian field theories (also called "Bell-type QFTs") are beyond the scope of this work, as we are concerned solely with the interpretation of QFT's extant formalism.

of these is the Klein-Gordon equation (KGE) for a spin-zero particle[2]

$$(\Box + m^2)\phi(x) = 0, \tag{1}$$

and the simplest case of antiparticles arises when we consider its complex solutions.

A solution to the KGE can be expressed as a linear combination of plane waves

$$\varphi_k = \exp(ik_a x^a) \tag{2}$$

where the wave vector k satisfies the rest mass condition $k_a k^a = m^2$. If k_a is a future-directed vector, then φ_k is called a "positive-frequency" wave;[3] if it is past-directed, φ_k is called "negative-frequency." A linear combination of positive-frequency waves satisfying the KGE is called a "positive-frequency" solution (or a positive-frequency scalar field), and "negative-frequency" solutions are likewise defined as combinations of negative-frequency waves.

What happens to a positive-frequency solution if we take its complex conjugate in the position basis, i.e. map $\phi(x) \to \phi^*(x)$? A plane wave (2) becomes $\varphi_k^* = \exp(-ik_a x^a)$. Thus conjugating the plane wave is the same as taking k_a to $-k_a$. If k_a is future-directed, $-k_a$ is past-directed, so the complex conjugate of a positive-frequency solution is a negative-frequency solution.

Normally to find the energy of a particle with wave vector k_a in a reference frame with unit normal n_a we take the inner product $n_a k^a$. If k_a is past-directed, this gives a negative result, so it seems that negative-frequency solutions must correspond to negative-energy particles. But actually this needn't be so, if we construct the Hilbert space of KGE solutions properly. Quantum mechanically, the energy observable corresponds to the operator

$$\hat{E}\phi = \frac{\hbar}{i} n_a \nabla^a \phi. \tag{3}$$

This might seem to strictly entail that negative-frequency solutions have negative energy. But in fact, when forming a Hilbert space from the KGE solutions, we need to make a choice of *complex structure.* That is, we need to define what it is to multiply a state vector ϕ by

[2] Where the D'Alambertian $\Box = \nabla_a \nabla^a$.

[3] This because the frequency ω of a wave is proportional to the wave number k_0.

a complex number α, so that the set of solutions satisfies the axioms of a complex vector space. One possible complex structure is just to define $\alpha\phi$ in the "obvious" way as $\alpha\phi(x)$, that is, to just multiply the scalar field by the number α. But another possible choice — the right choice — is to begin by decomposing $\phi(x)$ into a positive-frequency part $\phi^+(x)$ and a negative-frequency part $\phi^-(x)$. Then the operation $\alpha\phi$ can be defined as $\alpha\phi^+(x)+\alpha^*\phi^-(x)$.[4]

If we use the correct complex structure instead of the (naively) obvious one, then for a negative-frequency state ϕ we have

$$\hat{E}\phi = \frac{\hbar}{i} n_a \nabla^a \phi = -\frac{\hbar}{i} n_a \nabla^a \phi(x), \tag{4}$$

which implies that a negative-frequency plane wave φ_{-k} has the same energy as its positive-frequency counterpart φ_k. In general, conjugate fields $\phi(x)$ and $\phi^*(x)$ will have the same (positive) energy.[5]

But not all physical quantities remain the same when we conjugate. The KGE is symmetric under the group $U(1)$ of phase transformations ($\phi(x) \to e^{i\theta}\phi(x)$); we say that $U(1)$ is an internal symmetry or *gauge group* of Klein-Gordon theory. When we derive the existence of a conserved current J from this symmetry, we find that

$$J_a(x) = \phi^*(x)\nabla_a\phi(x) - \phi(x)\nabla_a\phi^*(x). \tag{5}$$

Complex conjugation reverses the sign of $J_a(x)$, so that $\phi(x)$ and $\phi^*(x)$ would appear to carry opposite charge.[6]

All of this is relativistic QM; we haven't constructed a Klein-Gordon QFT yet. To do so we take the "one-particle" Hilbert space $\mathcal{H}$ of KGE solutions that we constructed by imposing

[4]If we impose the wrong (naively obvious) complex structure instead, we end up with a theory with no lower bound on the total energy. This is both radically empirically inadequate (since we observe ground states in nature) and contrary to rigorous axioms for quantum theories.

[5]Dirac addressed the analogous problem of interpreting negative-frequency solutions to his equation for the relativistic electron by proposing his "hole theory." This treated negative-frequency Dirac fields as negative-energy electron states, and posited that all of the negative-energy states are occupied in the ground state. An unoccupied negative-energy state will behave like a positron. This solution only works for fermions because of the exclusion principle, and cannot be applied to boson field equations like the KGE. Furthermore, the problem of negative-frequency solutions of the Dirac equation can *also* be solved by choosing the proper complex structure, so Dirac's method would seem to be outmoded.

[6]To derive a conservation law from a symmetry, one employs Noether's theorem (Ticciati, 2003, pp. 36–53).

our complex structure and build a symmetric Fock space $\mathcal{F}$ from it. From a heuristic point of view, a Fock space is needed because relativistic systems can undergo changes in particle number. Thus, we take the direct sum of all symmetric (because we're dealing with bosons) n-particle Hilbert spaces with the right complex structure:

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus S(\mathcal{H} \otimes \mathcal{H}) \oplus S(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots \tag{6}$$

where $S(\mathcal{V})$ is the symmetric subspace of a Hilbert space $\mathcal{V}$.

A state Ψ in Fock space will take the form of an ordered set

$$\Psi = [\xi, \phi_1, \phi_2, \phi_3, ...] \tag{7}$$

with ξ a complex number and ϕ_i an i-particle[7] Klein-Gordon wavefunction (i.e., an i-rank symmetric tensor on $\mathcal{H}$). From the vacuum state,

$$\Psi_0 = [1, 0, 0, ...], \tag{8}$$

we can construct a multi-particle Fock space state by introducing "creation" and "annihilation" operators. For an i-particle KG wavefunction ϕ_i and a one-particle wavefunction f, $S(f \otimes \phi_i)$, where S is the symmetrization operation, in effect composes ϕ_i with a particle of wavefunction f. Thus the creation operator $a^*(f)$ defined by

$$a^*(f)\Psi = [0, \xi f, S(f \otimes \phi_1), S(f \otimes \phi_2)...], \tag{9}$$

where S denotes symmetrization, transforms the state Ψ by adding a particle of wavefunction f. Conversely, its adjoint $a(f)$ removes a particle of wavefunction f, and so is called an annihilation operator. Now, in the complex KG Fock space we can actually define two different creation operators. The wavefunction f can be equally well represented by its Fourier transform $f(k)$. Define $\sigma^+ f(k)$ to be $f(k)$ for future-directed k, 0 else, and $\sigma^- f(k)$ to be $f(k)$ for past-directed k, 0 else. That is, σ^+ gives us the positive-frequency part of f, while σ^- gives the negative-frequency part. Then the "particle" creation operator $a^*(\sigma^+ f)$ generates a particle with a purely positive-frequency wavefunction, while the "antiparticle"

[7] Note that we do not yet discriminate between particles and antiparticles.

creation operator $a^*(\sigma^- f)$ creates a particle with a purely negative-frequency wavefunction (and therefore with opposite charge).

Taking the product $a^*(f)a(f)$ gives the self-adjoint particle occupation number operator $N(f)$, which represents how many particles are in the state f. Thus $N(\sigma^- f)$ (for instance) tells us how many (anti-)particles there are in the negative-frequency state $\sigma^- f$. Summing $N(\sigma^- f)$ over all the f's in some orthonormal basis of $\mathcal{H}$ therefore gives us an operator N^- representing the total number of antiparticles; by summing $N(\sigma^+ f)$ we can likewise construct a total particle number operator N^+. It is easy to verify that conjugating the field (transforming $\phi \to \phi^*$) switches the expectation values of N^+ and N^-.[8]

So now we have a picture of free scalar QFT involving some countable entities (negative-frequency particles) that we identify as antimatter, and some others (positive-frequency particles) we identify as normal matter. In other words, we have an example of the matter-antimatter distinction, but not yet a definition. What is it for a physical system to fall under the concept of antimatter that physicists developed in response to theoretical predictions of the sort just summarized?

The best way to begin, perhaps, is with platitudes. In our paradigm case, antimatter is governed by the same equation of motion as normal matter, and has the same mass. And of course it carries opposite charge. This last fact is of physical interest in large part because when interactions are introduced (e.g. if the system is coupled to another quantum field), it becomes possible for a system containing equal amounts of matter and antimatter (i.e. equal numbers of particles and antiparticles, on the naive picture) to evolve into a system containing none of either, without violating the conservation law (5).[9] This sort of evolution is what physicists call a particle-antiparticle annihilation event, or "pair annihilation." Likewise, without violating charge conservation, an interacting system containing no Klein-Gordon particles (i.e., one with the Klein-Gordon vacuum as a sub-system) could evolve into one containing equal numbers of particles and antiparticles — "pair creation."

So when we say that there is such a thing as antimatter, we are claiming that something like the platitudes above holds of physically possible states in QFT. Note that one of our

[8]This explication draws heavily on (Geroch, 1973), and readers seeking further details should consult these precise and highly readable notes.

[9]Of course, even an interacting system cannot evolve into one with no matter content, period – that would violate mass-energy conservation. But an interacting system could evolve into one containing no Klein-Gordon matter, i.e. one in which the Klein-Gordon vacuum Ψ_0 is a sub-system.

platitudes (that matter and antimatter carry opposite charge) depends on an ontological assumption about the nature of charge. Specifically, it requires that we can make robust sense of the notion that two charges are "opposite" properties, in a physically fundamental sense. In the case of scalar charges like the charge of Klein-Gordon particles, this requires that the sign of the charge be of absolute significance. Of course there is a simple sense in which any real number has a sign; the important distinction here is that the sign of charge must encode physically fundamental information, if it is a fundamental fact that charges have opposites. This is not true in general even of conserved quantities; for example, we assign no fundamental significance to the sign of momentum, nor is there any invariant sense in which we can ascribe "opposite momentum" to any two particles. So what is it about charge that entails that a given charge Q has a genuine opposite, $-Q$?

Of course, we are free to suppose that it is simply a brute physical fact that charges have genuine opposites. But if we can find no relevant difference in theoretical role (within QFT) between charge and those quantities which *lack* genuine opposites, such a posit would seem to have very poor epistemic support.

We will see that these questions, as well as the question of whether the notion of antimatter can be generalized beyond that of antiparticle, admit of natural and foundationally significant answers within the framework of superselection sector theory in AQFT. To explore this framework, we must now explain the important results of Doplicher et al. (1971, 1974), also called the DHR picture. Eventually we will argue on the basis of these results that a physical system counts as antimatter in virtue of standing in a certain relation (the relation of *conjugacy*) to normal matter. The question of whether the matter-antimatter distinction is fundamental then becomes the question of whether this conjugacy relation applies to fundamental physical systems.

3 The incompleteness of the naive picture

The naive textbook picture has given us a paradigm example of antimatter, but as yet no definition. One might think that a definition of 'antimatter' must have as a prerequisite a definition of 'antiparticle,' since antimatter is said to be matter made of antiparticles. If this is accurate, and if recent arguments against particles are cogent, then strictly speaking there is no antimatter. So if the naive textbook concept is committed to this assumption, it is a concept with no extension — though there may be some real systems that *approximate* antimatter in various ways.

A brief look at the no-particles arguments will make this tension explicit. These arguments come in two forms. The first, due to Wald (1994) and Halvorson and Clifton (2001), appeals to the *non-uniqueness* of particle interpretations where they are available. Even in cases like the KG field just discussed, a particle number operator can only be defined with the help of a complex structure. But there are many complex structures available; to determine which we should apply, we require a notion of which solutions possess positive frequency. The breakdown of frequencies into positive and negative depends in turn on our notion of which momentum vectors count as future-directed. But an accelerating observer defines the future-directed momenta differently from an inertial observer. Therefore each observer possesses a different complex structure, and it follows that they will ascribe different numbers of particles to the same state (e.g., according to the accelerating observer there are particles in the state that the inertial observer would call the vacuum). We may infer that the number operator does not represent an objective (invariant) physical property of field-theoretic worlds. But if there were particles, we would expect that the number of them would be an objective fact. This problem worsens in curved spacetimes, where different families of free-falling observers will generally possess inequivalent particle concepts.

The second sort of no-particles argument, due to Fraser (forthcoming), relies on the *nonexistence* of particle interpretations in physically realistic QFTs. In QFTs with interactions (non-trivial couplings between fields), there is *no* invariant way to decompose the solutions into positive- and negative-frequency modes. So no Fock space can be constructed, and no operator meets the physical criteria that we would expect of the particle number operator. Since the actual world includes interactions between fields, we may conclude that there are no particles if QFT is correct. Both of these arguments generalize straightforwardly to undermine the physical significance of the *antiparticle* number operator.

Suppose we restrict ourselves to the physically unrealistic free QFTs that do admit (non-unique) particle interpretations, and fix one such particle interpretation as the "right one." Does the textbook picture at least offer an unproblematic definition of antimatter that works in this restricted context? The textbooks can offer the beginnings of an answer, but for a complete definition we will need to supplement them with some mathematical foundations of QFT.

According to a standard reference work, an antiparticle is defined to be

> ... a subatomic particle that has the same mass as another particle and equal but opposite values of some other property or properties. For example, the antiparticle of the electron is the positron, which has a positive charge equal in magnitude to the electron's negative charge. The antiproton has a negative charge equal to the proton's positive charge... (Isaacs, 1996, p. 15)

Clearly, this definition is not intended to be precise, because it does not answer the question of which properties are supposed to have equal but opposite values. On this question, Roger Penrose provides more detail:

> [F]or each type of particle, there is also a corresponding antiparticle for which each additive quantum number has precisely the negative of the value that it has for the original particle... (Penrose, 2005, p. 66)

So, by Penrose's account, the antiparticle is characterized by having opposite values for "additive quantum numbers," and the same values for all other quantities. In the literature, the phrase "(additive) quantum number" is typically meant to denote a superselected quantity — roughly speaking, a quantity whose value cannot change over time. Unfortunately, there is a great deal of confusion about which quantities are subject to superselection rules; and indeed, some physicists deny that there are any fundamental superselection rules (Aharonov and Susskind, 1967a,b). Thus, in order to establish the fundamentality of the antimatter concept, we will need need a principled account of which quantities are superselected. We provide such an account in Section 5. But before we discuss superselection rules, we note a couple of further conceptual difficulties in understanding antimatter in terms of "negative" values for quantities.

First, the description "the negative value of a quantity" does not always pick out an objective relation between properties. To take a ridiculously simplified example, suppose that we arbitrarily set the center of the universe in Princeton, NJ. Then Philadelphia is the "anticity" of New York, because the vector from Princeton to Philadelphia is the negative of the vector from Princeton to New York. But this notion of "anticity" depends on an arbitrary choice of a center of the universe — had we made Hoboken the center of the universe, then Philadelphia would not have been the anticity of New York. Surely, the relation of being the antiparticle is supposed to be objective in the sense that it does not depend on some arbitrary choice of origin.

In fact, for many physical quantities, the representation via real numbers carries surplus structure; and, in particular, the property denoted by zero has no privileged status, nor is there any interesting relationship between an object that has the value r and an object that has the negative value $-r$. For example, an ice cube at $-2\,^{\circ}$F bears no particularly interesting relationship to an ice cube at $2\,^{\circ}$F. What we need, then, is some explanation for why superselected quantities have an objective notion of "negative" that can underwrite the antimatter concept.

But before we explain why superselected quantities have objective "negative" values, we need to clarify what "negative" means — because it will not always be as simple as applying a minus sign to a real number. For example, the possible values for the isospin of a particle are half integers: $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$ (see Sternberg 1994, p. 181; Weinberg 2005, p. 123). So, what is the negative, or opposite, of an isospin quantum number? Of course, anyone acquainted with this quantity knows that a particle and its antiparticle have the *same* isospin: e.g. both the proton and antiproton have isospin $\frac{1}{2}$. So, the isospin quantum numbers do come equipped with a notion of the negative, or opposite; but this notion does not coincide with the additive inverse of the corresonding half integer. We will thus need to probe more deeply in order to find a principled method for determining the inverse of a charge quantum number.

One prima facie tempting proposal is to suppose that quantum numbers come equipped with group structure — i.e. there is an intrinsic notion of the neutral value, and also an intrinsic notion of the inverse of a value. (In some groups, e.g. $\mathbb{Z}_2$, every element is its own inverse.) But the example of isospin again shows that this idea is too simplistic. Indeed, if the isospin quantum numbers were a group, then the value $\frac{1}{2}$ should be its own inverse (since an isospin $\frac{1}{2}$ particle is its own antiparticle). But it is not true, *simpliciter*, that the combination of two particles of isospin $\frac{1}{2}$ is a particle of isospin 0. Rather, two isospin $\frac{1}{2}$ particles can combine to form particles of isospin 0 or 1. Therefore, not all superselected quantities carry grouplike structure.

If we remain within the naive picture, then there are insuperable obstacles to identifying necessary and sufficient conditions for a quantity to be reversed (or preserved) by the transformation from matter to antimatter. In order to make further progress, we will need some background in group representations and superselection theory. This will lead to a picture of quantum numbers not just as free-floating physical quantities, but as labels for representations of a gauge group.

4 Group representation magic

An antiparticle has opposite electric charge from its corresponding particle, but the two particles have the same isospin. Why is one quantity inverted, but not the other? In fact, the particle and antiparticle have "conjugate" values for *all* superselected quantities, but the definition of the conjugate value depends on the nature of the underlying gauge group.

To see the relation between conjugation and the gauge group, we begin with the simpler case of electric charge. Electric charge is simpler because the corresponding gauge group $U(1)$, the unit circle of complex numbers, is abelian. (Recall that with the topology inherited from $\mathbb{C}$, the group $U(1)$ is also a compact topological space, so we call it a compact topological group.) What are the possible values of quantized electric charge? We know that the answer should be $\mathbb{Z}$, the integers. We claim now that the answer, in general, is:

Group Duality (DUAL): *The charge quantum numbers for a system with abelian gauge group G are elements of the dual group $\chi(G)$. The binary group operation on $\chi(G)$ corresponds to a physical operation of "adding" charges; the identity element $\mathbf{1} \in \chi(G)$ corresponds to the "neutral" charge; and the inverse γ^{-1} corresponds to the "opposite" charge.*

DUAL says not only that the cardinality of the set of quantum numbers is fixed by G, but that the quantum numbers come equipped with group structure. We postpone our attempt to give a physical motivation for DUAL. For now, we explain the concept of a dual group, and show how to generalize DUAL to the crucial case of nonabelian gauge groups.

Let G be a topological abelian group. The *dual group* $\chi(G)$ of G consists of continuous homomorphisms of G into the multiplicative group of complex numbers of unit modulus. The binary group operation "$\circ$" on $\chi(G)$ is defined by pointwise multiplication

$$(\gamma_1 \circ \gamma_2)(g) = \gamma_1(g)\gamma_2(g), \qquad g \in G, \tag{10}$$

and we equip $\chi(G)$ with the topology of uniform convergence. It is then obvious that the map $\mathbf{1} \in \chi(G)$ defined by

$$\mathbf{1}(g) = 1, \qquad g \in G, \tag{11}$$

is the identity element of $\chi(G)$, and for each $\gamma \in \chi(G)$, the map $\overline{\gamma}$ defined by pointwise complex conjugation

$$\overline{\gamma}(g) = \overline{\gamma(g)}, \qquad g \in G, \tag{12}$$

is an inverse for γ. Therefore, $\chi(G)$ is also a topological abelian group.

DUAL gives the right result in the case of electric charge, where the gauge group $G \cong U(1)$. In this case the dual group $\chi(G)$ is isomorphic to $\mathbb{Z}$, the additive group of integers (Folland, 1995, p. 89). Furthermore, DUAL provides a mathematical explanation for the quantization of charge: if the group G is compact (as we expect of gauge groups), then the dual group $\chi(G)$ is discrete (Folland, 1995, Proposition 4.4).

To summarize, given a topological abelian group G, there is a naturally related group, $\chi(G)$; and if G is the gauge group then $\chi(G)$ gives (in all known cases) the correct answer for the set of quantum numbers as well as for the group structure on this set. But what is the physical explanation for the correctness of this mathematical recipe? As yet, we have no physical explanation for why the algorithm DUAL works. And to further complicate the situation, this recipe does not work — without modification — for the case where the gauge group G is nonabelian.

When the gauge group G is nonabelian, the dual group recipe $G \mapsto \chi(G)$ does not yield the correct quantum numbers. For example, the isospin gauge group is $SU(2)$, but there is only one continuous homomorphism of $SU(2)$ into complex numbers — viz. the trivial homomorphism that maps everything to 1 — and so the dual group of $SU(2)$ is the trivial (one element) group. In the case of isospin, DUAL gives a radically incorrect account of the quantum numbers.

But a different, related algorithm does work for isospin. Let $\mathcal{C} = \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$ denote the set of isospin quantum numbers. We define a binary operation "$\otimes$" on $\mathcal{C}$ to represent the composition of charges (isospins), so that $X \otimes Y$ is a system composed of charges X and Y. Similarly, we define a binary operation "$\oplus$" to represent a mixture of possible charges, so that $X \oplus Y$ is a system which may have either charge X or charge Y; the theory doesn't tell us which. The charge $0 \in \mathcal{C}$ is the privileged neutral quantum number in the sense that

$$X \otimes 0 \;=\; X \;=\; 0 \otimes X, \tag{13}$$

for all X. However, a composite $X \otimes Y$ is typically not itself a quantum number, i.e. is not an element of $\mathcal{C}$. For example, $\frac{1}{2} \otimes \frac{1}{2}$ cannot be identified with any particular element of $\mathcal{C}$; rather,

$$\tfrac{1}{2} \otimes \tfrac{1}{2} \quad = \quad 0 \oplus 1. \tag{14}$$

The general formula for composing isospin quantum numbers is given by the Clebsch-Gordan formula:

$$X \otimes Y \;=\; \bigoplus_{Z=|X-Y|}^{X+Y} Z \tag{15}$$

(see Sternberg, 1994, p. 184), where the direct sum runs from $|X - Y|$ to $|X + Z|$ in increments of 1. The operation "$\otimes$" is sometimes given a dynamical interpretation: e.g., when two particles with quantum number $\frac{1}{2}$ "collide", then they annihilate to produce particles with quantum numbers in the set $\{0, 1\}$. But this cannot be a strictly accurate understanding of the formalism, which can after all be used to model free as well as interacting systems. Instead, we should understand it as representing a relationship between the charges of component systems and the charge of the composite system they form. This implies that the charges of the component systems do not uniquely determine the charge of the composite system; a system composed of two $\frac{1}{2}$ charges may have either charge 0 or 1, depending on other (non-charge) features of the component systems. Then, because charge quantum numbers are conserved by dynamical evolution, we can infer that any interaction will result in a system with one of these charges.

The physicists' magic recipe (Eq. 15) makes spectacular predictions about systems whose gauge group is $SU(2)$. In fact, this recipe and the related recipe for general $SU(n)$ are the abstract backbone of the standard model of particle physics. The operation "$\otimes$" is interpreted by physicists as the composition of charges, but the space of charges is not a group under "$\otimes$". So the notion of charges as elements of a group does not survive the transition from abelian to nonabelian symmetries.

We wish now to find some rationale for the apparently magical recipe (Eq. 15) for composing isospin quantum numbers. The first step in this explanation — which we take up in the remainder of this section — is to show that the recipe follows from group representation theory. The second step in the explanation — which we take up in the following section — is to show that representations of the gauge group correspond to superselection sectors of the quantum field theory.

Recall that Group Duality (DUAL) tells us that for a system with an abelian gauge group G, the quantum numbers have the structure of a group, in particular the dual group $\chi(G)$. The relationship between G and its dual $\chi(G)$, known as Pontryagin duality, does not generalize straightforwardly for arbitrary compact groups.

In order to generalize DUAL, we need to move from group theory into category theory. A *category* is given by a class of objects (e.g. $A, B, C, \dots$) and a class of arrows or morphisms $(f, g, h, \dots)$ that relate ordered pairs of objects. When A and B are related by arrow f, we write $f : A \to B$.[10] An important sort of relation between categories is given by *functors*. A covariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a mapping that takes each object A of $\mathcal{C}$ and returns an object $F(A)$ of $\mathcal{D}$, and another mapping that takes each arrow $f : A \to B$ in $\mathcal{C}$ and returns an arrow $F(f) : F(A) \to F(B)$ of $\mathcal{D}$. The arrow mapping is required to preserve composition $[F(f \circ g) = F(f) \circ F(g)]$ and identity arrows $[F(1_A) = 1_{F(A)}]$. A contravariant functor is just like a covariant functor except that it reverses the direction of arrows [if $f : A \to B$ then $F(f) : F(B) \to F(A)$].

The notion of duality in DUAL has a natural category-theoretic expression. To make this clear, let's define the necessary terms in category language. Recall the group-theoretic definition, for any topological abelian group G, of its dual $\chi(G)$. In category-theoretic language, we have a mapping χ on the objects in the category **AbTop** of topological abelian groups. This object map naturally extends to a contravariant endofunctor: for each group homomorphism $s : G \to K$, define a corresponding group homomorphism $\chi(s) : \chi(K) \to \chi(G)$ by setting

$$\chi(s)(\gamma) = \gamma \circ s, \qquad \gamma \in \chi(K). \tag{16}$$

Obviously, $\chi(s \circ t) = \chi(t) \circ \chi(s)$, and so χ is a contravariant functor. In fact, χ^2 is naturally isomorphic[11] to the identity functor on **AbTop**; in particular, for each object G of **AbTop**,

[10]For any two morphisms $f : A \to B$ and $g : B \to C$, a category must also contain a third *composite* arrow, $g \circ f : A \to C$, and composition is required to be associative. For each object A there is also required to be an identity arrow $1_A : A \to A$ such that $1_A \circ f = f$ for all $f : B \to A$, and $g \circ 1_A = g$ for all $g : A \to C$.

[11]Given two functors F, G from category $\mathcal{C}$ to category $\mathcal{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ is a

there is an isomorphism $\alpha_G : \chi^2(G) \to G$. This fact gives the precise sense in which $\chi(G)$ is a 'dual object' of G (see Folland, 1995; Roeder, 1971, 1974)

But this form of duality does not extend to compact nonabelian groups. The major difficulty with attempted generalizations is that there does not seem to be *any* way to construct a contravariant functor χ on the category of compact groups, such that χ^2 is naturally isomorphic to the identity functor. So from mathematical considerations alone we have reason to suspect that quantum numbers might not generally carry grouplike structure.

In order to generalize DUAL to arbitrary compact groups, we need a more sophisticated notion of the dual of a group. Could DUAL be a special case of a rule that also applies to nonabelian groups? For $G = U(1)$, for example, we know that the integers parametrize the continuous homomorphisms from G to complex numbers, and so they form the dual group $\chi(G)$. But the integers *also* parametrize the irreducible unitary representations of G, which are given in a Hilbert space by the phase transformations $\pi_z(\theta) = e^{iz\theta}$, for $\theta \in G$ and $z \in \mathbb{Z}$.[12] More generally, the dual group of an abelian group G is part of (viz. the irreducible elements of) the category $\mathrm{Rep}(G)$ of the Hilbert space representations of G. So we can try generalizing DUAL as

Group Duality 2 (DUAL$_2$): *For a system with compact gauge group G, the quantum numbers have the structure of the category* $\mathrm{Rep}(G)$*, whose objects are unitary representations of G on finite-dimensional Hilbert spaces and whose arrows are intertwiners between these representations.*

Thanks to the pioneering work of Tannaka, and more recent developments by Deligne, Doplicher, and Roberts, we now know the reason why there is no *group* that is naturally dual to a compact nonabelian group. In short, a nonabelian group G does have a dual, but the dual is not a group; it is the category $\mathrm{Rep}(G)$.

Before explaining at length why DUAL$_2$ is true, we should emphasize its importance. DUAL$_2$ has much to teach us about the nature of antimatter. We're looking for a notion of

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collection of arrows

$$\{\alpha_X : F(X) \to G(X) \mid X \text{ is an object of } \mathcal{C}\},$$

such that if $f : X \to Y$ then $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$. We say that α is a *natural isomorphism* just in case each α_X is an isomorphism.

[12] A unitary representation of a group G is a pair (H, π) where H is a Hilbert space and π is a homomorphism of G into the group of unitary operators on H.

"opposite" that applies to additive quantum numbers, so that we can explain why matter and antimatter systems take on opposite values for these numbers. We've seen that the group-theoretic notion of opposite (the inverse) is insufficient, since not all quantum numbers form groups. But $\mathrm{Rep}(G)$ includes a more general notion of opposite: representations of G always possess so-called *conjugates*. Indeed, for each Hilbert space H and basis $\{e_i\}$, there is an antiunitary mapping J defined by setting $J(\sum_i c_i e_i) = \bar{c}_i e_i$. Then given a representation (H, π) of G, we can define another representation $\overline{\pi}$ on H by setting $\overline{\pi}(g) = J^{-1}\pi(g)J$, for all $g \in G$. In the case where the representation π is one-dimensional, i.e. a homomorphism of G into $U(1)$, the conjugate $\overline{\pi}$ is simply the map that assigns the conjugate scalar.

Since $\mathrm{Rep}(G)$ has an intrinsic notion of conjugates, we can use this to define the "opposite" of a quantum number. In the following section, we will see that each element of $\mathrm{Rep}(G)$ also corresponds to a family (folium) of states, which allows us to define antimatter as those states associated with the representation conjugate to that of matter states. Unlike the naive picture, this definition makes no appeal to the notion of *particle*, and indeed it applies to many states that lack particle interpretations. Along the way we'll show why additive quantum numbers are always conserved.

But this is somewhat premature. At this point all we have is a rule (DUAL_2) that takes as input a QFT's internal symmetry group and outputs its charge quantum numbers. Why does DUAL_2 succeed?

5 What makes the magic work?

Group representation theory is like the magician's hat of elementary particle physics. Once the symmetry group is fixed, we need only consult our local group representation theorist in order to obtain a complete classification of elementary particles, based on their charge quantum numbers. We have seen that DUAL_2 is the key step in this process, but not why it works. Understanding the success of DUAL_2 requires a grasp of superselection rules. DUAL_2 is a natural consequence of the fact that, in AQFT, the physical property of a charge (or additive quantum number, or *superselection sector*) corresponds to a representation of the gauge group, i.e. an element of $\mathrm{Rep}(G)$. Furthermore, superselection theory also provides a natural explanation of why all additive quantum numbers are conserved, since it is dynamically impossible for a state to change sectors. To see why, read on as we expound the details.

5.1 Superselection rules

We begin by recalling how the formalism of C^*-algebras makes precise the idea of a superselection rule between quantum states. (For a detailed exposition, see Earman, forthcoming). Roughly speaking, a *superselection rule* prohibits superposing two given pure states. This effectively tells us that the state vectors for a system are not contained in a single Hilbert space (all elements of which can be superposed); rather, the states are contained in a collection of two or more disjoint Hilbert spaces.

Recall that a C^*-algebra A is an algebra (i.e. it has both addition and multiplication operations) over the complex numbers (i.e. there is a product cx for $c \in \mathbb{C}$ and $x \in A$) that has an antilinear involution $x \mapsto x^*$, and a norm $\|\cdot\| : A \to \mathbb{R}^+$ relative to which

$$\|xy\| \leq \|x\| \cdot \|y\|, \quad \text{and} \quad \|x^*x\| = \|x\|^2,$$

for all $x, y \in A$. It is also assumed that A is complete relative to this norm (i.e. all Cauchy sequences converge), and that A has a multiplicative identity 1. The motivating examples of C^*-algebras are algebras of $n \times n$ matrices over complex numbers or, more generally, the algebra $B(H)$ of bounded linear operators on a Hilbert space H.

We call a positive, trace 1 operator on H a *state* on $B(H)$, since such a density operator can be understood as an assignment of expectation values to observables (self-adjoint operators) acting on H. More generally, if A is a C^*-algebra then a *state* on A is a linear map $\omega : A \to \mathbb{C}$ such that $\omega(x^*x) \geq 0$ for all $x \in A$, and $\omega(1) = 1$. A state ω on A is said to be *pure* if $\omega = a\rho + (1-a)\sigma$, with $a \in (0,1)$ and ρ, σ states of A, entails that $\rho = \sigma = \omega$. The standard gloss on this formalism is that if observables in A represent physical quantities, then a pure state on A represents a physical possibility. A non-pure (mixed) state represents an ignorance measure over possibilities.

The basic physical idea behind superselection rules is that the states of a system fall into equivalence classes: $\{[\omega] : \omega \text{ is a state of } A\}$. Within each equivalence class, or *sector*, the pure states can be superposed to give another pure state. However, a 'superselection rule' forbids the superposition of states from different equivalence classes. In order for this to work, the relevant equivalence relation must be

Same sector. If ω and ρ are states of A, then we say that $\omega \sim \rho$ just in case there is a unitary operator $u \in A$ such that $\omega(u^* x u) = \rho(x)$ for all $x \in A$. (Recall that u is unitary iff $u^* u = 1 = u u^*$.)

That is, two states are in the same sector just in case there is a unitary mapping between them.

Trivially, this explains why no state can ever change sectors. Quantum dynamics is unitary, so if ω can change into ρ then a unitary mapping must exist. We say that sectors are 'dynamical islands' which no state can ever leave.

Just like groups, C^*-algebras have Hilbert space representations. We can use this fact, combined with a beautiful result of Gelfand, Naimark and Segal, to determine when two states are in the same sector. A representation of a C^*-algebra A is a pair (H, π) where H is a Hilbert space, and π is a $*$-homomorphism [an algebra homomorphism such that $\pi(x^*) = \pi(x)^*$] of A into $B(H)$. A representation (H, π) of A is said to be *irreducible* just in case no non-trivial subspaces of H are invariant under $\pi(A)$.

GNS Theorem. For each state ω of A, there is a representation (H_ω, π_ω) of A, and a vector $\Omega \in H_\omega$ such that $\omega(x) = \langle \Omega, \pi_\omega(x)\Omega \rangle$, for all $x \in A$, and the vectors $\{\pi_\omega(x)\Omega : x \in A\}$ are dense in H_ω. This representation is unique in the sense that for any other representation (H, π) satisfying the previous two conditions, there is a unitary operator $u : H_\omega \to H$ such that $u\pi_\omega(x) = \pi(x)u$, for all x in A.

The theorem says, in short, that every state on A has a unique "home" Hilbert space representation of A. Using it, we can show that $\omega \sim \rho$ just in case there is a vector φ in the GNS Hilbert space H_ω for ω such that $\rho(x) = \langle \varphi, \pi_\omega(x)\varphi \rangle$, for all $x \in A$. Thus $\omega \sim \rho$ tells us, roughly, that ω and ρ are "vectors in the same Hilbert space."

Now, $\omega \sim \rho$ iff there is a unitary operator $u : \pi_\omega \to \pi_\rho$.[13] Thus, the superselection sectors of states correspond to unitary equivalence classes of representations of A. In other

[13] *Proof:* If $\omega \sim \rho$ then there is a unitary operator $v \in A$ such that $\omega(a) = \rho(v^* a v)$ for all $a \in A$. But the vector $\pi_\omega(v)\Omega_\omega$ is cyclic in H_ω for $\pi_\omega(A)$, and

$$\rho(a) = \langle \pi_\omega(v)\Omega_\omega, \pi_\omega(a)\pi_\omega(v)\Omega_\omega \rangle,$$

for all $a \in A$. By the uniqueness of the GNS representation, it follows that (H_ω, π_ω) and (H_ρ, π_ρ) are unitarily equivalent.

Conversely, suppose that there is a unitary operator $u : H_\omega \to H_\rho$ such that $u\pi_\omega(a) = \pi_\rho(a)u$ for all words, for a system with observable algebra A, the 'charge quantum numbers' are names for isomorphism classes of objects in the category $\mathrm{Rep}(A)$ of representations of A. $\mathrm{Rep}(A)$'s objects are Hilbert space representations of M, and the arrows from (H, π) to (H', π') are given by bounded linear operators from H to H' such that $v\pi(x) = \pi'(x)v$, for all x in A.

According to the algebraic formalism, quantum numbers label dynamically isolated islands of states, and hence conserved properties of physical objects. But if we remain at this level of abstraction, then the quantum numbers have very little structure — not enough to support the sorts of explanations provided by elementary particle physics. In particular, the category $\mathrm{Rep}(A)$ does *not* have a tensor product, and so cannot support the notion of composing superselection sectors or quantum numbers (which we've seen is needed in the case of isospin). Indeed, consider how we might try to define the tensor product $\pi \otimes \pi'$ of two representations (H, π) and (H', π') of a C^*-algebra A. It would seem natural to use the tensor product $H \otimes H'$ of the Hilbert spaces. But the mapping $A \ni x \mapsto \pi(x) \otimes \pi'(x)$ is *not* linear, and so is not a representation. Other attempts to define the tensor product of representations also end in failure.

In order to give the quantum numbers additional structure, we must place additional physical constraints on our algebra of observables. The obvious place to look is special relativity, since relativistic QFT ought to share its symmetries. To implement this, we'll need to associate our physical quantities (operators) with regions of Minkowski spacetime:

1) Assign to each double cone O a unital C^*-algebra $A(O)$, representing the observable quantities localized within O. We require that if $O_1 \subseteq O_2$ then there is an injection $i_{1,2} : A(O_1) \to A(O_2)$, and so the mapping $O \mapsto A(O)$ is a "net" of algebras. Since the double cones of Minkowski spacetime are directed under inclusion, there is an inductive limit C^*-algebra A generated by the $A(O)$.

Since the theory is supposed to be relativistic, we assume that spacelike-separated observables

$a \in A$. Thus,

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega \rangle = \langle u\Omega_\omega, u\pi_\omega(a)\Omega_\omega \rangle = \langle u\Omega_\omega, \pi_\rho(a)u\Omega_\omega \rangle,$$

for all $a \in A$. Since ρ is pure, the representation (H_ω, π_ω) is irreducible, and it follows that there is a unitary operator v in A such that $\pi_\rho(v)\Omega_\rho = u\Omega_\omega$. Clearly then

$$\rho(v^* a v) = \langle u\Omega_\omega, \pi_\rho(a)u\Omega_\omega \rangle = \omega(a),$$

for all $a \in A$, and therefore $\rho \sim \omega$.

are causally independent:

2) *Microcausality:* For self-adjoint $a_1 \in A(O_1), a_2 \in A(O_2)$ and O_1, O_2 spacelike separated, a_1 and a_2 commute.

We ensure covariance under the symmetries of relativity by insisting that

3) $g \mapsto \alpha_g$ is a representation of some group $\mathcal{G}$ of symmetries of Minkowski spacetime in the group $\mathrm{Aut}(A)$ of automorphisms of the C^*-algebra A. Furthermore, $\alpha_g(A(O)) = A(g(O))$ for each region O and symmetry g. We typically assume only covariance under the translation group of Minkowski spacetime. We will explicitly note when we need to assume covariance under the Euclidean group, or even under the Poincaré group.

4) The preferred vacuum state ω_0 is invariant under all symmetries:

$$\omega_0(\alpha_g(a)) = \omega_0(a), \qquad \forall a \in A, \forall g \in \mathcal{G}.$$

These four conditions, taken as axioms, constrain the models of 'algebraic quantum field theory' (see Haag, 1996). Unfortunately they don't yet provide enough structure to introduce tensor products of superselection sectors.

5.2 DHR representations

The C^*-algebra A will typically have many more states than are needed in physics. A *selection criterion* is a further condition on which states are physically possible. For example, Arageorgis et al. (2003, p. 181) argue that, since physical possibilities must assign expectation values to the stress-energy tensor, only so-called *Hadamard* states are possible. Even if they are correct, this may not be the only necessary condition. Which selection criteria are needed to give a plausible space of possibilities is thus a vexed question. That said, selection criteria can be very useful even in the absence of solid justification. By "pretending" that the physical possibilities are limited by a given criterion, we can develop a physical concept (such as additive quantum number) that covers at least *some* of the possibilities, and which can then hopefully be generalized to include all of them.

Proceeding in this spirit, the most extensively investigated criterion is that proposed by Doplicher et al. (1969a):

DHR selection criterion: Let (H_0, π_0) be the GNS representation induced by the privileged vacuum state ω_0 of A. A representation (H, π) of A is *DHR* iff (1) for each double cone O, the representations $\pi_0|_{A(O')}$ and $\pi|_{A(O')}$ are unitarily equivalent; and (2) (H, π) possesses finite statistics, that is, a finite-dimensional representation of the permutation group. Here O' is the spacelike complement of O, and $A(O')$ is the C^*-algebra generated by $A(O_1)$ with O_1 a double cone spacelike separated from O.

The requirement of finite statistics is quite weak, since the standard Bose and Fermi representations of the permutation group are both *one*-dimensional, and hence trivially satisfy finite statistics. In fact, allowing finite statistics is liberal in the sense that it also permits — but does not require — the existence of systems with parastatistics. So, all known physical systems meet the finite statistics requirement.

The DHR states (the physically possible states according to the DHR criterion) are elements of the folia of DHR representations. The intuitive idea is that the DHR states are those that look identical to the vacuum state, except possibly in some bounded region of spacetime. It is obvious that this criterion is too stringent to count as a necessary condition for physical possibility. Charged states in electromagnetism, for instance, differ from the vacuum at infinity due to Gauss' law. However, the DHR criterion is the only proposal for which we currently have a body of worked-out mathematical results. Even for the slightly more liberal Buchholz-Fredenhagen criterion (Buchholz and Fredenhagen, 1982), we still lack a full understanding of the category of superselection sectors. Thus, we will begin by considering only possibilities meeting the DHR criterion.

Since A is a C^*-algebra, the collection of *all* of its representations are objects of a category, $\mathrm{Rep}(A)$, whose arrows are intertwiners between representations. The DHR representations of A form a sub-category $DHR(A)$ of $\mathrm{Rep}(A)$, and this category has tensor products. [A category with tensor products is called a *tensor category.* See (Halvorson and Müger, 2007) for details.] Indeed, it can be shown that a representation (H, π) of A is DHR just in case there is a particular sort (i.e. "localized" and "transportable") of endomorphism $\rho : A \to A$ such that (H, π) is unitarily equivalent to $(H_0, \pi_0 \circ \rho)$, where (H_0, π_0) is the vacuum representation. Furthermore, given two DHR representations, corresponding to two such endomorphisms ρ_1 and ρ_2, it can be shown that $\rho_1 \circ \rho_2$ also corresponds to a DHR representation. This construction gives us a notion of the tensor product of DHR representations — just what we need for our additive quantum numbers. Finally the representations of our algebra have enough mathematical structure to represent the physical behavior we set out to describe.

We have a set of physical possibilities (the DHR states) which fall into natural families: superselection sectors (DHR representations). Since states cannot change sectors, and sectors (like quantum numbers) possess a tensor product, the sectors can be taken to correspond to additive quantum numbers. But we still need to explain DUAL_2. Why do the additive quantum numbers have the same structure as the category $\mathrm{Rep}(G)$ of representations of the gauge group? Since additive quantum numbers are just sectors, and the sectors form the structure $DHR(A)$, we can explain this by showing that $\mathrm{Rep}(G)$ and $DHR(A)$ must be equivalent categories.[14]

But we're getting ahead of ourselves. We haven't yet explained what it is for an AQFT to possess a global internal symmetry given by a gauge group G. Once that's out of the way, we can go about justifying DUAL_2. And then, at last, *antimatter* will appear.

5.3 Gauge groups and the Doplicher-Roberts reconstruction

The *Doplicher-Roberts reconstruction theorem* is a remarkable result, and essential to understanding DUAL_2. It establishes that given an AQFT system, described in terms of its algebra of observables A, we can derive the global gauge group G which leaves that system invariant. We can then show that the irreducible representations of G are isomorphic to the DHR representations of the observable algebra — exactly what we need to explain DUAL_2.

The definition of an AQFT system is given purely in terms of an algebra of observables A and a mapping $O \mapsto A(O)$ from bounded regions of spacetime to subalgebras of A. The self-adjoint elements of A are supposed to represent measurable (at least in principle) physical quantities, which take on values within the bounded regions O (which is why we have the net mapping $O \mapsto A(O)$). We might wonder what it is for such a theory so defined to have a gauge group G, since normally all of a theory's measurable quantities are left unchanged by its internal symmetries. Thus every element of A should be left unchanged by G — so in what nontrivial sense is there a symmetry at all?

To define the notion of a gauge group, we need to expand the formalism to include unobservable, non-gauge-invariant structure. This structure is given by a *field algebra* F. A field algebra is built like an algebra of observables — in particular, it has a local subalgebra

[14]Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$.

$F(O)$ for every open region O — but it need not satisfy microcausality. A field algebra is meant to signify a collection of theoretical quantities, the elements of F, which are assigned values by the states but are not necessarily measurable, or covariant under the theory's internal symmetries. An AQFT then possesses an internal symmetry given by a gauge group G just in case its algebra of observables, A, is given by the gauge-invariant part of some field algebra F. That is,

$$A(O) = \{a \in F(O) : g^{-1}ag = a, \forall g \in G\}$$

for all open regions O.[15]

For all we've shown so far, an AQFT given by A may have no field algebra, or it may have many. If so, there is no such thing as *the* gauge group for the theory, and DUAL_2 becomes nonsense. This is where the DR theorem comes in: it establishes that a given observable algebra A possesses a unique distinguished field algebra F and gauge group G. The following was first proved by Doplicher and Roberts (1990); a simpler proof appears in the appendix to Halvorson and Müger (2007).

DR Reconstruction Theorem. Let A be an algebra of observables satisfying the axioms of AQFT and ω_0 a vacuum state on A. Then there exists a unique (up to unitary equivalence) complete, normal field algebra F and gauge group G such that A is the G-invariant subalgebra of F.

All that's left is to show that $\text{Rep}(G)$ and $DHR(A)$ are isomorphic categories. The field algebra F acts irreducibly on a Hilbert space H, but the subalgebra $A \subset F$ of observables typically leaves non-trivial subspaces of H invariant. In this case, H decomposes into a direct sum of superselection sectors

$$H = H_1 \oplus H_2 \oplus \cdots ,$$

[15] More precisely, a *field system with gauge group* G consists of a net $O \mapsto F(O)$ of von Neumann algebras acting on some Hilbert space H, a privileged vacuum vector Ω in H, and and also a compact gauge group G acting (via unitary operators) on H. It is required that the gauge transformations act internally, that is $g^{-1}F(O)g = F(O)$ for each double cone O and for each $g \in G$, and leave the vacuum invariant: $g\Omega = \Omega$ for all $g \in G$. There are some additional technical conditions that we can safely ignore at present — e.g. the requirement of normal (Bose-Fermi) commutation relations between operators localized in spacelike separated regions. See (Halvorson and Müger, 2007, p. 808)

where A leaves each sector H_i globally invariant, and also the gauge group G leaves each sector H_i globally invariant. (That is, if $a \in A$ and $\psi \in H_i$, then $a\psi \in H_i$, and similarly for $g \in G$.) It then follows that, for each subspace H_i, the restriction of the observable algebra A to H_i is a representation of A. To be precise, we define

$$\pi_i(a) = ap_i, \qquad \forall a \in A,$$

where p_i is the orthogonal projection onto H_i. Then each (H_i, π_i) is a DHR representation of A. Furthermore, the restriction of the action of the gauge group G to H_i is a unitary representation of the gauge group; indeed, it is equivalent to a direct sum of irreducible representations of G, all with the same character. Thus, each sector H_i yields simultaneously a DHR representation of A and a representation of the gauge group G, so we have a nice one-to-one correspondence between objects of the category $DHR(A)$ of DHR representations and objects the category $\mathrm{Rep}(G)$ of representations of G. This correspondence is functorial (Halvorson and Müger, 2007, pp. 808-815), and since quantum numbers are just labels for DHR representations (i.e. for superselection sectors), DUAL$_2$ follows.

At last all the formal machinery is in place to define the relation of *conjugacy* that holds between matter and antimatter.

6 A quite general notion of antimatter

The naive textbook presentation has it that, at the fundamental level, a particle and its antiparticle counterpart take on opposite values for all additive quantum numbers. We have seen that in realistic QFTs there are no particles, and that additive quantum numbers are just labels for superselection sectors, so really this definition is not given in physically fundamental terms at all. We will show in this section that the real definition of antimatter is as follows:

> *A matter system and its antimatter counterpart are given by states in conjugate superselection sectors.*

This definition applies at least to all states that satisfy the Buchholz-Fredenhagen (BF) superselection criterion (Buchholz and Fredenhagen, 1982). It is more general than the textbook definition, since all massive free particle states satisfy the BF selection criterion, and some BF sectors have conjugates but no particle interpretation. If we accept the BF criterion, the superselection sectors are all elements of category $\mathcal{C}$ that is provably equivalent to the category $\mathrm{Rep}(G)$ of representations of a compact group. The notion of *conjugacy* employed in our definition is a relation between elements of $\mathrm{Rep}(G)$. Irreducible representations of a compact group always possess unique conjugates; therefore, so do sectors.

When a system's gauge group is abelian, its sectors have the structure of a group, so for any two sectors (charge quantum numbers) X and Y there is a product $X, Y \mapsto X \circ Y$, and an inverse $X \mapsto \overline{X}$. In this special case the definition of conjugate is obvious. But in general, the product of sectors is a tensor product in a category: $X, Y \mapsto X \otimes Y$. We cannot expect that the "conjugate" $\overline{X}$ of a sector will always satisfy the defining equation $\overline{X} \otimes X = X \otimes \overline{X} = \mathbf{1}$ for group inverses.

What are we looking for in a notion of conjugation for sectors? It must explain antimatter behavior — that is, the possibility of pair annihilation. This means it must be possible for a system composed of states $\omega_X, \omega_{\overline{X}}$ from X and its conjugate sector $\overline{X}$ to evolve into an element of the zero-charge vacuum sector, which we'll call V. Since the composite state lives in the tensor product of these sectors, $X \otimes \overline{X}$, it must be physically possible for a state in this tensor product to end up in V. Since it's impossible for states to change sectors, this means that V must be a part (that is, a subrepresentation) of $X \otimes \overline{X}$.

In $DHR(A)$, the vacuum representation is always given by the identity object of the category; i.e. $V = \mathbf{1}$. For categories like $DHR(A)$, if there is a monomorphism from A to B, then B is either A or the direct sum of A with some other objects, and is therefore a subrepresentation of A. Setting $B = X \otimes \overline{X}$ and $A = \mathbf{1}$, the conjugacy relation must ensure that $X \otimes \overline{X} = \mathbf{1} \oplus$ (other representations). Thus it must ensure the existence of a monomorphism from $\mathbf{1}$ to $X \otimes \overline{X}$. Since conjugacy should be a symmetric relation, we must require the same for $\overline{X} \otimes X$. Thus we define conjugacy as follows (Longo and Roberts, 1997):

Definition. Let $\mathcal{C}$ be a tensor $*$-category and let X be an object of $\mathcal{C}$. A *conjugate* of X is a triple $(\overline{X}, r, \overline{r})$ where $\overline{X}$ is an object of $\mathcal{C}$, and $r : \mathbf{1} \to \overline{X} \otimes X$ and $\overline{r} : \mathbf{1} \to X \otimes \overline{X}$ are arrows satisfying the 'conjugate equations'

$$1_X \otimes r^* \circ \overline{r} \otimes 1_X = 1_X, \tag{17}$$

$$1_{\overline{X}} \otimes \overline{r}^* \circ r \otimes 1_{\overline{X}} = 1_{\overline{X}}. \tag{18}$$

If every non-zero object of the category $\mathcal{C}$ has a conjugate then we say that $\mathcal{C}$ has conjugates.

If $(\overline{X}, r, \overline{r})$ and $(\overline{X}', r', \overline{r}')$ both are conjugates of X then one easily verifies that $1_{\overline{X}'} \otimes \overline{r}^* \circ r' \otimes 1_{\overline{X}} : \overline{X} \to \overline{X}'$ is unitary. Thus conjugates, if they exist, are unique up to unitary equivalence.

Do conjugates exist in the relevant category, namely the category of sectors? Recall first that in the category $\mathrm{Rep}(G)$ of representations of a compact group G, the conjugate of (H, π) is defined by

$$\overline{\pi}(g) = J^{-1}\pi(g)J, \qquad \forall g \in G, \tag{19}$$

where J is an antiunitary operator on H. In this case, a linear map $r : \mathbf{1} \to \overline{\pi} \otimes \pi$ can be defined by setting $r(1) = \sum_i Je_i \otimes e_i$, and then extending linearly. Similarly, the arrow $\overline{r} : \mathbf{1} \to \pi \otimes \overline{\pi}$ is defined by setting $\overline{r}(1) = \sum_i e_i \otimes Je_i$. Some elementary linear algebra then shows that $(\overline{\pi}, r, \overline{r})$ satisfies the conjugate equations, and so $\mathrm{Rep}(G)$ has conjugates.

In the case of the category $DHR(A)$ of superselection sectors, the existence of a conjugate sector is guaranteed for any sector that can be reached from the vacuum by application of field operators (Doplicher et al., 1969b). So, if a field net $O \mapsto F(O)$ is given *a priori*, then every sector has a conjugate. Furthermore, even if only the observable net $O \mapsto A(O)$ is given *a priori*, a sector has a conjugate iff it has finite statistics (Doplicher et al., 1971), and the existence of a conjugate is also independently guaranteed for any sector with a mass gap (Fredenhagen, 1981). Indeed, proving the existence of conjugate sectors is a key step in the Doplicher-Roberts reconstruction, which shows that the category of sectors (i.e. the category $DHR(A)$) is equivalent to the category of representations of the gauge group (i.e. the category $\mathrm{Rep}(G)$).

We've ended up with a rather orderly picture. Any state ω meeting the DHR condition lives in a DHR representation. Every DHR representation has a unique conjugate. And every state in the conjugate representation is conjugate to ω. Thus for any "matter" state we might choose, if it is DHR we have a whole representation full of "antimatter" states which can annihilate it while conserving all additive quantum numbers (that is, without changing sectors) if the two states are composed.

We are now in a position to challenge some assumptions of the naive picture. Most importantly, we can show that the concept of antimatter is not confined solely to particle systems. Nothing about our definition of conjugate rules out non-particle systems — but can we show that there are QFT systems, with no particles, to which it applies?

We can. By the plausible argument of Fraser (forthcoming), no interacting QFT admits a particle interpretation. One theory that falls under Fraser's purview is the Yukawa interaction between charged fermions and neutral bosons, used to describe the strong force as it acts between mesons and nucleons. Summers (1982) has shown that the two-dimensional version of this theory (Yukawa$_2$, one of the few interacting QFTs which has been proven to exist) satisfies the DHR condition. So a state of the Yukawa$_2$ theory is a clear example of a state with no particle interpretation, but which possesses conjugates — therefore, *antimatter*.

As noted in the Introduction, Wallace (2008) has claimed this is impossible.[16] For Wallace, the existence of antimatter requires a particle interpretation, and so antimatter only exists in free QFT. This may seem strange even in the absence of our results, since antimatter is supposed to explain pair creation and annihilation events which can only occur in interacting theories. Wallace might hold that his antimatter concept applies *approximately* in the asymptotic scattering limit, and can therefore do the needed explanatory work without applying exactly. But we find it much more satisfying to suppose that it is exactly true that matter-antimatter annihilation events can occur in interacting QFT — and this is what we have shown, using the machinery of DHR.

The restrictiveness of the DHR criterion is, we grant, an outstanding limitation for our antimatter concept. Since charged states in electrodynamics are globally, as well as locally, inequivalent to the vacuum, we cannot at present prove that these states possess conjugates. That is a project for future research. The existence of conjugates has already been shown for QFTs (in four spacetime dimensions) meeting the less stringent Buchholz-Fredenhagen condition, which requires equivalence to the vacuum outside one spacelike cone (Doplicher and Roberts, 1990, pp. 75–85). DHR superselection theory has also been generalized to the case of curved spacetimes (see Brunetti and Ruzzi, 2007), allowing us to define antimatter in yet another arena where particle interpretations fail. Since the *nonexistence* of conjugates has only been proven for systems with infinite statistics, which no known physical systems

[16]Wallace uses 'antimatter' to describe a narrower set of cases than we do — for him, a system has antimatter only if it has *nontrivial* superselection sectors. That is to say, antimatter for Wallace occurs only when a particle and its conjugate live in unitarily inequivalent sectors; he does not count self-conjugate systems as possessing antimatter. This difference amounts to a mere choice of words, we think, especially since Summers' Yukawa$_2$ theory is nontrivial in Wallace's sense. But we also think our choice of words is closer to that of practicing physicists, who are happy to say that "the photon is its own antiparticle."

obey, we are optimistic that proofs of their existence can be generalized. Even if not, our main point stands: the antimatter concept does not stand or fall with the particle concept. It may (or may not) stand or fall with physically unrealistic restrictions on the space of states, like DHR, in which case there may be no antimatter in nature. But the notion of antimatter is in no way parasitic on the particle notion.

Of course, like Wallace's, our antimatter concept *also* applies to free and asymptotic scattering states. So if need be, we can co-opt Wallace's claim that the concept of antimatter applies at least approximately to non-DHR states which resemble free states. But at least our definition is strictly more general than his.

Besides the conceptual dependence of antimatter on particles, another view that has been aired in the literature (especially in Feynman's popular writings) is that matter is antimatter moving "backward in time."

> The backwards-moving electron when viewed with time moving forwards appears the same as an ordinary electron, except it's attracted to normal electrons — we say it has positive charge. . . For this reason it's called a 'positron'. The positron is a sister to the electron, and it is an example of an 'anti-particle'.
>
> This phenomenon is general. Every particle in Nature has an amplitude to move backwards in time, and therefore has an anti-particle. (Feynman, 1985, p. 98)

Feynman's thought is motivated by the behavior of antimatter in the case of free particles, in which a particle and its antiparticle have opposite *frequency*. Since negative-frequency particles have past-directed wave vectors, it appears natural to say that these particles are moving "back in time."

Is this picture borne out by our definition of conjugate? In order for this to hold, it would have to be the case that a state and its conjugate have opposite temporal orientations. This would require that, if a state ω has future-directed momentum, its conjugate state(s) must have past-directed momentum. But, as shown in Corollary 5.3 of Doplicher et al. (1974), all Poincaré covariant DHR sectors meet the spectrum condition, which requires that all their states have future-directed momentum. We suspect that Feynman's view arises from ignoring that, when the proper complex structure is applied to free particle systems, an antiparticle's wave vector and its four-momentum have *opposite* temporal orientation. So, in the standard form of free QFT as well as in all DHR sectors, both matter and antimatter systems always move "forward in time" by virtue of meeting the spectrum condition.

It remains to be seen whether an alternative (perhaps empirically equivalent) formalism can be devised on which Feynman's claim holds true, but it is straightforwardly false according to the standard formalism. Further, superselection theory provides a plausible explanation of its falsity. The relationship between matter and antimatter (conjugate sectors) arises from a physical system's global internal symmetries (its gauge group). But one would expect any relationship between a particle and its past-directed counterpart to be grounded in its external *spacetime* symmetries. Insofar as internal and external symmetries really are different in kind and not just in name, we should expect Feynman's claim to turn out false.

7 Conclusions

The dogma that antimatter is matter made up of antiparticles has been turned on its head. We have shown that the concept of antimatter is strictly more general than this naive picture would suggest, since it applies perfectly well to physical systems with no particle interpretation. Decades of careful research in AQFT have shown that all DHR states, as well as Buchholz-Fredenhagen states, possess antimatter counterparts. If these conditions together were true of all physically possible states, the distinction between matter and antimatter would be fundamental, in the sense of applying to all the fundamental constituents of the relativistic quantum world.

As it turns out, these conditions are too restrictive to include all of the physical possibilities. But there is also no known obstacle to generalizing the results of DHR even further. So for all we know, our world may be made up of matter and antimatter even at the most fundamental level of quantum field-theoretic description, the level at which we err when we claim that there are particles.

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