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CHAPTER 24

NO REQUIREMENT OF RELEVANCE

JOHN P. BURGESS

1. Relevantism: Rejecting the "Paradoxes" of Classical Logic

Classical geometry defines a conic section to be the intersection of a plane with a (double) cone. This definition obliges the geometer to recognize such degenerate conics as the hyperbola consisting of two crossed lines (the intersection of the cone with a vertical plane through its apex) or the circle consisting of a single point (the intersection of the cone with a horizontal plane through its apex).

Classic logic defines entailment to hold between a premise (or set of premises) and a conclusion if and only if their logical form guarantees that the either the premise (or at least one element of the set of premises) is false, or the conclusion is true. The definition obliges the logician to recognize certain degenerate entailments. A premise (or set of premises) that is contradictory in the sense that its logical form guarantees that it is false (or that at least one element of the set of is false) entails any conclusion: *ex falso quodlibet*. And a conclusion that is tautologous in the sense that its logical form guarantees that it is true is entailed by any premise (or set of premises): *ex quolibet verum*.

The commitment of classical logic to these principles has frequently been attacked by indignant critics who denounce the degenerate cases of entailment as "paradoxes." Among these critics there have been many with little use for any sort of formal logic; but the concern of the present survey will be with schools of anticlassical logicians who have proposed rival formal logics to the classical, avoiding the so-called paradoxes.

Since the most common single complaint of such critics is that in the degenerate cases of entailment recognized by classical logic the premise is (or the premises are) "irrelevant" to the conclusion, the critics in question will be called "relevantists." Note, however, that as they are used in the literature, "relevance logic" and "relevant logic" are trademarks—both for the same school of logic,[1] everything about which is highly contentious, beginning with whether "relevance" or "relevant" is the right label for it—and are not to be equated with "relevantism" as here defined. There have been other kinds of relevantists than "relevance" or "relevant" logicians; and there have been important contributions to "relevance" or "relevant" logic by logicians (Kit Fine, Harvey Friedman, Saul Kripke, Alasdair Urquhart) whose endorsement of relevantist criticism of classical logic has been qualified or nonexistent.

Now there would be no room for contention between classicism and relevantism if "entailment" were simply being introduced as a term previously without meaning (outside probate law), which was now simply being *stipulated* by the classicist to mean "the relation that obtains between premise and conclusion when logical form alone guarantees that either the former is false or the latter is true." There is room for disagreement only if both sides claim to be analyzing a notion of entailment that already exists, though perhaps not under that label.

And indeed classical logic's theory of entailment was largely developed in order to provide an analysis of what is meant by rigorous proof or rigorous provability in orthodox mathematics: classical logic can be viewed as an attempt to describe explicitly the implicit norms of orthodox mathematicians, and it is largely the practice of mathematicians that will be invoked in its defense. (Classical logicians do generally regard the applicability of classical logic as extending beyond mathematics as well, but are at least open to the possibility that it may require modification for extramathematical applications, since mathematics has so many features peculiar to itself: mathematicians insist that definitions should be fully precise, whereas ordinary language is full of vagueness; mathematicians insist that before a term is introduced, the existence and uniqueness of the item it is to denote must be proved, whereas in ordinary language such presuppositions often fail; and so on.)

At this point it is necessary to note a contrast between two different kinds of criticism of classical logic. It is agreed on all sides that in some sense logic is concerned with norms (of valid argument). But there is a great difference—all the difference between *is* and *ought*—between describing what the norms operative in

[1] The school in question is that represented by Anderson and Belnap (1975) and Anderson, Belnap, and Dunn (1992). These works have extensive bibliographies covering "relevantism" in all senses of the term.

a given community are, and prescribing what they ought to be. In linguistics this distinction (the distinction between descriptive and prescriptive grammar, between Chomsky and Fowler) is universally recognized. It needs to be recognized in logic also. There is a difference between *prescriptive* criticism, alleging that classical logic correctly describes unacceptable practices in the community of orthodox mathematicians, and *descriptive* criticism, alleging that classical logic endorses incorrect principles that are *not* operative in the practice of that community.

The paradigm of prescriptive criticism is the intuitionist attack on classical logic. Neither Brouwer nor any of his disciples ever suggested classical logicians were wrong in *describing* what pass for proofs in the orthodox mathematical community as often involving the inference from the double negation of a proposition to the proposition itself. The intutionists' quarrel was not with the small body of classical logicians, but with the much larger body of orthodox mathematicians, whose practices, they insisted, were in need of revision. (Classical logicians were liable to censure only if they went beyond describing orthodox mathematicians as appealing double negation elimination to *endorsing* the practice of doing so.)

The prescriptive character of the intuitionist critique is evident from the mathematical practice of intuitionist logicians themselves, in proving metatheorems about intuitionistic logic. A genuine intuitionist proving a metatheorem will take care never to appeal to double negation elimination or, more generally, to make any argumentative move that cannot be represented in *intuitionistic* logic. Or if the intuitionist logician does sometimes reason like the orthodox, it is only because there has previously been established some metatheorem to the effect that in the particular kind of case in question, a classical proof can always be replaced by an intuitionistic one. And (it ought to be needless to say) "established" here can only mean established *by intuitionistically acceptable proofs.*

By contrast with all this, relevantists have historically virtually *never* troubled themselves to check through their proofs of metatheorems to make sure that no principle they reject has been appealed to. The most natural explanation for this insouciance is that their criticism has implicitly been meant as descriptive rather than prescriptive, and that the fault of the classical logicians, according to them, has been that of endorsing unacceptable principles *that are not in practice used by orthodox mathematicians.* Also, the rhetoric attending the historically most important statements of the relevantist position—their pose as indignantly championing common sense against the monstrous "paradoxes" propounded by classical logicians—is a more explicit indication that the criticism has been intended as descriptive.

If the descriptive criticism fails—and in particular, if it turns out that relevantists themselves constantly argue in what purport to be proofs of metatheorems about their own formal systems by appeal to principles rejected in those systems—the historically earliest form of the dispute between relevantism and classicism will have been decided in the classicist's favor. The options remaining to the relevantist will then be two.

On the one hand, a relevantist might adopt a *compatibilist* position, presenting relevantist logic not as an alternative intended to supplant classical logic but, at most, as an adjunct intended to supplement it somehow. This would be to give up criticism of classical logic altogether.

On the other hand, a relevantist might become a presciptive critic, adopting a *revisionist* position, demanding, like the intuitionist, a reform of orthodox mathematical practice (though not the *same* reform the intuitionist demands). This would require in the first place a reform of their own mathematical practice, appealing henceforth only to laws of relevantist logic when proving mathematical metatheorems about their own formal systems.

2. The Relevantist's Options

So much, for the moment, about the general character of the dispute between classicists and relevantists. As for the particulars on which relevantists differ from classicists, and among themselves, the most crucial issues can be illustrated at the level of sentential logic, and in connection with entailments where the only logical operators involved in premise(s) and conclusion are conjunction, disjunction, and negation. Thus the forms of premise(s) and conclusion alike can be represented by formulas built up from *atoms* or atomic formulas p, q, r, . . . by means of the symbols caret and wedge and tilde ($\wedge$ and $\vee$ and $\sim$) for the three logical operations just mentioned. We call these *elementary* formulas.

Most relevantists (in contrast to intuitionists, who reject the law of the excluded middle, for example) have been willing to agree, at least for the sake of argument, with the classical position as to which elementary formulas represent contradictions and which tautologies. The disagreement that will be examined below is over which elementary formulas entail which.

We write $\vdash$ for entailment. As already indicated, the primary target of criticism is the following principle:

(A) If $\alpha_1, \ldots, \alpha_m$ are contradictory, then $\alpha_1, \ldots, \alpha_m \vdash \beta$.

And the secondary target is this further principle:

(B) If β is tautologous, then $\alpha_1, \ldots, \alpha_m \vdash \beta$.

The cases most often discussed are

(A1) $p \wedge \sim p \vdash q$
(A2) $p, \sim p \vdash q$

and to a lesser extent

(B1) $p \vdash q \vee \sim q$.

But while in practice it is these cases that are primarily discussed, in principle the critics reject (A) and (B) quite generally.

Now how does the rejection of (A) and (B) square with orthodox mathematical practice? That is the issue to be considered in evaluating relevantism as a *descriptive* criticism of classical logic. On this issue, relevantists have been quick to remark that generally neither mathematicians nor other people set about trying to deduce conclusions from premises they recognize to be contradictory; nor do people generally spend much time looking for premises from which they might deduce a conclusion they recognize to be tautologous. This observation is clearly correct; what is not so clear is why it should be thought to motivate a nonclassical logic in which there is some *interesting distinction* between those conclusions that are and those that are not entailed by a given contradiction, and between those premises that do and those that don't entail a given tautology.

More important, despite the foregoing observation, it does often happen that a mathematician engaged in lengthy deductions may through some slip (say, leaving out a minus sign, or writing an inequality symbol backward) arrive at a contradiction or other absurdity without noticing the fact. And when this happens, it does often happen that the mathematician may then go on to deduce, by what appears to be quite ordinary mathematical reasoning, whatever conclusions the mathematician in question may have been seeking. This type of occurrence may be called the *Poincaré phenomenon*, after Henri Poincaré, who observed:

> The candidate often takes an immense amount of trouble to find the first false equation; but as soon as he has obtained it, it is no more than child's play for him to accumulate the most surprising results, some of which may actually be true.[2]

There is also a dual type of occurrence, where carrying out a long deduction, by quite ordinary mathematical reasoning (such as simple calculations involving rearranging terms in equations, or performing similar operations on both sides), results in obtaining what turns out at the last moment to be a tautology or other triviality. This may be called the *Quine phenomenon*, after W. V. Quine, who observed:

[2] "The New Logics," in Poincaré (1960), p. 161.

> . . . the beginner in algebra works in danger of finding that his solution-in-progress reduces to "$0 = 0$."[3]

What the Poincaré and Quine phenomena suggest is that if one rejects (A) or (B), one is going to have to reject some quite ordinary pattern of deduction, such as is used by mathematicians and others who go in for long trains of arguments. And indeed, if one wishes to reject (A), the options are quite limited, as can be seen from the following derivation, establishing (A2).

(1)	p	premise
(2)	$\sim p$	premise
(3)	$p \vee q$	from (1)
(4)	q	from (2), (3)

The principle that gives (3) from (1) is the that a disjunction is entailed by either disjunct.

(C1) $p \vdash p \vee q$
(C2) $q \vdash p \vee q$

A slightly more elaborate variant of the first of these is the following:

(C3) $p \wedge r \vdash (p \vee q) \wedge r.$

Since those who reject any one of the principles just listed tend to reject the others as well, all of them, as well as other minor variants, will be referred to below collectively as (C) or *disjunctive weakening.*

The prinicple that gives (4) from (2) and (3) is

(D1) $p \vee q, \sim p \vdash q.$

This two-premise version has a one-premise variant.

(D2) $(p \vee q) \wedge \sim p \vdash q.$

Both of these, and other minor variants, will be referred to collectively below as (D) or *disjunctive syllogism.*

The derivation (1)–(4) is usually called the *Lewis argument* after C. I. Lewis, who first brought it to the general attention of logicians in modern times, after medieval antecedents had been largely forgotten.[4] The Lewis argument seems to show that one who rejects (A) must reject either disjunctive weakening (C) or disjunctive syllogism (D). But actually there is a *third* alternative—namely, to reject the *transitivity* of entailment, the principle that if certain initial premises entail each of certain intermediate steps as conclusions, and these in turn, taken as premises, entail a certain ultimate conclusion, then the initial premises entail the ultimate conclusion.

[3] "Carnap and Logical Truth," in Quine (1966), p. 105.

[4] For the history of the argument, see Anderson and Belnap (1975), §16.1, pp. 163ff.

The issue here is most clearly seen in connection with the following variant of the Lewis argument.

(C4) $p \wedge \sim p \vdash (p \vee q) \wedge \sim p$
(D2) $(p \vee q) \wedge \sim p \vdash q$
(A1) $p \wedge \sim p \vdash q.$

Here the particular version (C4) of (C) used is an instance of the version (C3) cited earlier. The (meta-)principle needed to get from (C4) and (D2) to (A) is the simplest case of transitivity.

(E) If $\alpha \vdash \beta$ and $\beta \vdash \gamma$, then $\alpha \vdash \gamma$.

All three options have been taken up by critics in the literature: the least popular option has been to reject disjunctive weakening; the most popular has been to reject disjunctive syllogism; rejection of transitivity has been somewhere in between. The three options will be considered separately below, in order of increasing popularity.

One sometimes also meets in the literature with what at first glance appears to be a fourth option, namely, to claim there is a fallacy of equivocation in the Lewis argument. The equivocation is supposed to be between two senses of disjunction, a weaker *extensional* and a stronger *intensional* sense. It is claimed that disjunctive weakening holds for the former but not the latter, while disjunctive syllogism holds for the latter but not the former.

Usually extensional disjunction is written $\vee$, and classical logic is held to give a correct account of which formulas involving it represent contradictons and which tautologies, while intensional disjunction is written $+$, and the enunciation of its laws is taken to be one of the tasks for a new, nonclassical logic. But so understood, the apparent fourth option is really just a suboption under the second option: it amounts to rejecting (D) but softening the rejection by adding a further claim to the effect that there is an acceptable principle resembling (D), namely, the principle that results from it on replacing $\vee$ with $+$.

The option of rejecting disjunctive weakening has been favored by so few relevantists that it will be given only very brief attention.[5] The only principle that

[5] This option is involved in the "analytic implication" of Parry and the "connexive implication" of McCall, for which see Anderson and Belnap (1972), §§29.6–29.8, pp. 429ff.

has been advanced in support of such a rejection is the principle that in an entailment, the subject matter of the conclusion must be contained in the subject matter of the premise. Since formulas involving distinct atoms p and q will presumably have instances in which the subject matter of what is substituted for one letter p has nothing to do with that of what is substituted for q, the formal representation of the "inclusion of subject matter" requirement is that no atom should appear in the conclusion that does not appear in the premise, which of course immediately leads to the rejection of (C).

This has seemed unacceptable even to the great majority of relevantists. Yet one may wonder *why* anyone would wish to infer a disjunction $p \vee q$ from a disjunct p, since the former in an obvious sense contains less information than the latter. One reason is a desire to subsume a particular case under a generalization. Thus in mathematics one may want to prove $(\forall n)(P(n) \vee Q(n))$, and one way to do this may be by induction, proving $P(0) \vee Q(0)$ and $P(1) \vee Q(1)$ and then proving that if the theorem holds for n and $n+1$, it holds for $n+2$. And one way to prove $P(0) \vee Q(0)$ and $P(1) \vee Q(1)$ may be to deduce the former from $P(0)$ and the latter from $Q(1)$, by disjunctive weakening. Reasoning of this kind is in fact common in mathematical proofs, and insofar as the question before us is whether classical logic gives a correct description of orthodox mathematical standards of proof, this fact allows the relevantist position rejecting disjunctive weakening to be dismissed.

3. The Option of Rejecting Transitivity: Perfectionism

The great majority of relevantists do not wish to insist that the subject matter of the conclusion must be contained in that of the premise, though they do wish to insist that the two should overlap. This overlap is a bare minimum to require in the way of "relevance" between premise and conclusion. Now this bare minimum of relevance in fact obtains in all classical entailments where the premise is not contradictory and the conclusion not tautologous, as a consequence of the Craig Interpolation Theorem. That theorem says that, except in the degenerate cases just excluded, if $\alpha \vdash \gamma$, then there is a β, called an interpolant, such that $\alpha \vdash \beta$ and $\beta \vdash \gamma$, and every atom appearing in β appears both in α and in γ.

Actually, something still stronger is true, which it will be useful for future purposes to note now. We may distinguish, among the appearances of an atom p in an elementary formula α, the *positive* from the *negative.* When α is just the atomic formula p itself, the one and only appearance of p is positive. The positive appearances of p in $\alpha \wedge \beta$ and in $\alpha \vee \beta$ are just the positive appearances in α together with the positive appearances in β; and similarly for negative appearances. Finally, the positive appearances in $\sim\alpha$ are the negative appearances in α and vice versa. Thus in the premise $(p \vee q) \wedge \sim p$ of disjunctive syllogism, the first appearance of p is positive and the second negative, while the only occurrence of q is positive. The Lyndon Interpolation Theorem tells us that in the Craig Interpolation Theorem the interpolant β may actually be so chosen that the only atoms occurring positively in β are ones appearing positively both in α and in γ, and similarly for negative appearances.

Thus classical entailment does guarantee a bare minimum and more than a bare minimum of "relevance," except in the two degenerate cases. Relevantists will not allow those exceptions, and wish to insist in *all* cases on premise and conclusion having one or more atoms in common. Atoms being also called "sentential" (or "propositional") "variables" (or "letters"), the requirment of atoms in common is often referred to as the *variable-sharing* requirement.

The classical criterion of validity—that by virtue of logical form alone, if the premise is (or the premises all are) true, then so is the conclusion—is often call *truth-preservation.* Relevantists have sometimes written in a way that might suggest to the reader that they hold a "two-factor" theory of entailment according to which what is required for a genuine entailment is just the presence of both these features, truth-preservation and variable-sharing; but in fact relevantists generally hold no such a theory. Variable-sharing all by itself (as the sole addition to truth-preservation) is not enough to satisfy relevantists, who generally reject, for instance, the following version of (A):

(A4) $(p \wedge \sim p) \vee r \vdash q \vee r$.

Here the addition of the disjunct r to both sides of (A1) results in the variable-sharing condition being fulfilled, but fundamentally we still have truth-preservation here only because the first disjunct in the premise is contradictory. What relevantists really want is not just that variable-sharing should be present alongside truth-preservation, but that it should be variable-sharing that *underlies* truth-preservation, so to speak.

One case where variable-sharing must be what underlies truth-preservation is of course where the premise is *not* a contradiction and the conclusion *not* a tautology. One cannot, however, accept as correct laws of logic only those classical entailments $\alpha \vdash \beta$ where α is noncontradictory and β nontautologous. This is because formulas and entailments involving formulas are supposed to hold in *all* instances, no matter what is put in for p and q and r and so on. On the formal level, this understanding is represented by the rule of substitution, to the effect that if

$\alpha \vdash \beta$ and α' and β' are obtained from α and β by substituting formulas $\pi_1, \ldots, \pi_k$ for atoms $p_1, \ldots, p_k$, then $\alpha' \vdash \beta'$. Thus, from the entailment $\alpha \wedge \beta \vdash \alpha$, which even relevantists accept, it follows there follows the entailment $\alpha \wedge \sim\alpha \vdash \alpha$, despite the fact that the premise in this case is contradictory. Similary, $\beta \vdash \beta \vee \sim\beta$.

One school of relevantists, however, takes examples of the kind we have just been discussing to be the *only* exceptions to the rule that the premise must not be contradictory and the conclusion must not be tautologous. To state the position more precisely, call an entailment $\alpha \vdash \beta$ *perfect*—or "as good as an argument can be on formal grounds"—if it holds classically and α is not contradictory and β is not tautologous; and call it *perfectible* if it is obtainable by substitution from a perfect entailment. The school of relevantists we have in mind take the acceptable entailments to be precisely the perfectible ones. This position involves rejection of transitivity of entailment, since in the second version of the Lewis argument above (C4) and (D2) are easily seen to be perfectible, while (A1) is not.

There is some question of how this perfectionist criterion should be extended to arguments with multiple premises, and in the literature arguments with multiple conclusions have also been considered. Classically, a *sequent*

$$\alpha_1, \ldots, \alpha_m \vdash \beta_1, \ldots, \beta_n$$

is taken to hold if logical form guarantees that either at least one of the premises α_i is false or at least one of the conclusions β_i is true. Note that the multiple premises are being taken conjointly, and the conclusions alternatively. That is, classically the sequent holds if the simple one-premise, one-conclusion entailment

$$\alpha_1 \wedge \cdots \wedge \alpha_m \vdash \beta_1 \vee \cdots \vee \beta_n$$

holds.

These definitions are to be understood in such a way that, as a special case, if the set of conclusions β is *empty*, then

$$\alpha_1 \wedge \cdots \wedge \alpha_m \vdash \emptyset$$

holds if and only if the α_i are contradictory taken conjointly, and in particular

$$\alpha \vdash \emptyset$$

holds if α is contradictory. Similarly,

$$\emptyset \vdash \beta$$

holds if β is tautologous, and when

$$\emptyset \vdash \beta_1 \vee \cdots \vee \beta_m,$$

we may say that the β_j are tautologous (taken alternatively).

Now one way to extend the one-premise, one-conclusion version of the perfectionist criterion would be to take

$$\alpha_1, \ldots, \alpha_m \vdash \beta_1, \ldots, \beta_n$$

to be perfectible and acceptable if and only if

$$\alpha_1 \wedge \ldots \wedge \alpha_m \vdash \beta_1 \vee \cdots \vee \beta_n$$

is. But there is a different, and in some respects more natural, way to carry out the extension. If we look again at the above definitions, we see that

$$\alpha \vdash \beta$$

is perfect if and only if it holds classically while

$$\alpha \vdash \emptyset$$

and

$$\emptyset \vdash \beta$$

do not. Generalizing, we may take

$$\alpha_1, \cdots, \alpha_m \vdash \beta_1, \cdots, \beta_n$$

to be perfect if it holds classically, but no longer does so if the left-hand side is replaced by any proper subset of the α_i or the right-hand side by any proper subset of the β_j. We may then define the sequent to be perfectible if it is obtainable by substitution from a sequent that is perfect in the sense just indicated. This is Tennant's perfectionist logic, at the sentential level (see chapter 23 above).

Tennant's logic is decidable, at the sentential level. For note, first, that if a formula involves n sentence and connective and punctuation signs altogether, then any formula of which it is a substitution instance involves no more than n such signs. Note further that if a formula involves no more than m such signs and no more than k sentence letters, the formula is a substitution instance of a formula involving no more than m such signs and involving no sentence letters beyond the first k. Note further that there are only finitely many formulas having no more than m signs and no sentence letters beyond the first k. Note finally that this observation generalizes to sets of formulas. So for given sets of α_i and β_j, in seeking to determine whether the sequent having the former as premises and the latter as conclusions holds only finitely many pairs of other sets of sentences need to be evaluated classically (which may be done by truth tables or otherwise).

The main advantage of working with sequents is that though transitivity fails for perfectionist logic in the sense that

$$\alpha_1, \ldots, \alpha_m \vdash \beta_1, \ldots, \beta_n$$

and

$$\beta_1, \ldots, \beta_n \vdash \gamma_1, \ldots, \gamma_k$$

may be perfectible while

$$\alpha_1, \ldots, \alpha_m \vdash \gamma_1, \ldots, \gamma_k$$

is not, nonetheless one can develop (meta-)rules for passing from *sequents* taken as "premises" to a *sequent* taken as "conclusion," and the operation of these (meta-)rules is transitive. One has transitivity "one level up," so to speak. This makes it possible to represent perfectionist logic as a deductive system comparable to the deductive system for classical logic when the latter is developed in "sequent calculus" or Gentzen style.

Now in every branch of mathematics, axioms are assumed, various combinations of them are used to obtain lemmas, various combinations of axioms and lemmas are used to obtain theorems, various combinations of axioms and lemmas and theorems are used to obtain corollaries, and all the lemmas and theorems and corollaries are taken to follow from the axiom set initially assumed, as well as from any larger axiom set subsequently assumed. These are the facts that show that mathematicians treat entailment as transitive. And if we are concerned only with the question of whether classical logic correctly *describes* orthodox mathematical practice, any position rejecting transitivity would have to be rejected for this reason alone.

Nonetheless, a revisionist version of perfectionism, according to which, though mathematicians in practice *do* give "proofs" relying on transitivity, they in principle *ought not* to do so, might be thought to have certain attractions in view of some formal results of Tennant's. For Tennant seems to show that though orthodox mathematicians make moves in "proofs" not directly representable by sequent-calculus derivations in his system, if one is concerned only with prov*ability* and not directly with *proof*, there is no great loss.[6] It is not quite true that every classical "proof" can be replaced by a perfectionist "proof," leaving the class of conclusions provable from given premises unchanged; but something quite close to this, and quite attractive, does hold.

Let us try to describe Tennant's result more precisely. For a given set

$$A = \alpha_1, \ldots, \alpha_m$$

of premises and a given conclusion β, a mathematician's proof of a theorem whose logical form is represented by β from a set of premises whose logical form is represented by A can fairly routinely be represented by a derivation of $A \vdash \beta$ in a sequent-calculus version of classical logic with the *Cut* rule (a version of transitivity). By the Cut Elimination Theorem of Gentzen, the uses of Cut can in principle be eliminated. Then Tennant shows how to transform a Cut-free classical derivation of $A \vdash \beta$ into a derivation in his perfectionist system.

The result this two-step transformation is not always a derivation in the perfectionist system showing that $A \vdash \beta$ holds perfectionistically. This could or can fail in two situations. If A were inconsistent, $A \vdash \beta$ might fail perfectionistically, and if it did, the result of the two-step transformation would be a derivation showing that $A \vdash$. If the original hypotheses are not all needed, $A \vdash \beta$ may fail perfectionistically, and if it does, the result of the two-step transformation will be a derivation showing that $A' \vdash \beta$ for some proper subset A' of A. Since ultimately the mathematician is interested in proving results from consistent axioms and from the fewest of those possible, Tennant's result seems to say that nothing, or next to nothing, is lost by switching from classical logic to perfectionist logic: where a classical entailment is rejected, perfectionism supplies an improvement.

There is, however, a difficulty here. Transitivity, as noted above, is involved whenever a lemma is deduced from axioms and a theorem then deduced from the lemma, and claimed to follow from the axioms, and is thus ubiquitous in ordinary mathematical proofs. When these proofs are formalized in sequent calculus, instances of transitivity become instances of Cut. Obtaining perfectionist replacements for the proofs would require first eliminating Cut. But in general, Cut can be eliminated only at the cost of making proofs infeasibly long.[7] So even though *in principle* almost anything classically provable will be perfectionistically provable or else something even better will be, *in practice* the proofs mathematicians actually give not only *do* fail to adhere to the restrictions of perfectionism, but *must* do so if they are to be kept to a humanly comprehensible length.

[6] The interpretation of Tennant's results, and the question of whether they merely seem to show that there is no great loss or really do so, are unsurprisingly not uncontroversial. My evaluation will appear below.

[7] See "Don't Eliminate Cut," in Boolos (1998), pp. 365–369.

The perfectionist might reply that in practice mathematicians could go on working as they do now, since Tennant's work shows that in principle everything, or almost everything, they are doing could be justified from a perfectionist standpoint. The trouble with this response is that it relies on a theorem—Tennant's result described above—for which only a classically and not a perfectionistically acceptable "proof" has been given. How far can a logician who *professes* to hold that perfectionism is the correct criterion of valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about perfectionist logic itself, be regarded as *sincere* or *serious* in objecting to classical logic? This question will be left to the reader to ponder, as we turn to consideration of the one remaining relevantist option, rejection of disjunctive syllogism.

4. The Option of Rejecting Disjunctive Syllogism: Stringency

The very simplest elementary entailments are perhaps those of the form

$$(\sim)p_1 \wedge \ldots \wedge (\sim)p_m \vdash (\sim)q_1 \vee \ldots \vee (\sim)q_n,$$

where the premise is a conjunction, and the conclusion a disjunction, of plain or negated atomic formulas or sentence letters. These may be called *proto-elementary*. Classically, a proto-elementary entailment holds in three cases:

(I) Some atom occurs with the same sign in both premise and conclusion.
(II) Some atom occurs both plain and negated in the premise.
(III) Some atom occurs both plain and negated in the conclusion.

Here (I) means that either some atom occurs positively or plain in both, or that some atom appears negatively or negated in both. Perfectionist logic does not recognize cases (II) and (III), and takes (I) to be a necessary and sufficient condition for the entailment to hold. The major school rejecting Disjunctive Syllogism, "relevance" or "relevant" logic, adheres to the same criterion for proto-elementary entailments, but arrives at different results about more general elementary entailments, including Disjunctive Syllogism.

Since the roots of "relevance" or "relevant" logic are traced back by the logicians of that school itself to Wilhelm Ackermann,[8] we may sidestep the "relevance"/"relevant" terminological problem by adopting (a suitable English translation of) Ackermann's label for the kind of logic in question. Ackermann, writing in German,

[8] See the dedication to Anderson and Belnap (1972).

called his system *strenge Implikation.* Now though this was intended as a translation of Lewis's label "strict implication," Ackermann's system is very different from any of those considered by Lewis. So in translating his German back into English, we may use a different word than "strict," namely, "stringent."

The first (historically earliest) formulation of the stringency criterion for general elementary entailments was based on an alternative to truth tables as a method for testing classical elementary entailment. The classical method has four steps:

(i) Repeatedly apply the double negation and De Morgan laws of classical sentential logic, which stringent logicians accept, to replace subformulas

$$\begin{aligned}&\sim\sim\alpha \text{ by } \alpha\\&\sim(\alpha \wedge \beta) \text{ by } \sim\alpha \vee \sim\beta\\&\sim(\alpha \vee \beta) \text{ by } \sim\alpha \wedge \sim\beta\end{aligned}$$

so as to reduce premise and conclusion to formulas built up from plain and negated atoms by conjunction and disjunction.

(ii) Similarly apply the distributive laws of classical sentential logic, which stringent logicians also accept, to replace subformulas

$$\begin{aligned}&\alpha \wedge (\beta \vee \gamma) \text{ by } (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\\&(\alpha \vee \beta) \wedge \gamma \text{ by } (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)\end{aligned}$$

in the premise and subformulas

$$\begin{aligned}&\alpha \vee (\beta \wedge \gamma) \text{ by } (\alpha \vee \beta) \wedge (\alpha \vee \gamma)\\&(\alpha \wedge \beta) \vee \gamma \text{ by } (\alpha \vee \gamma) \wedge (\beta \vee \gamma)\end{aligned}$$

in the conclusion, so as to reduce the premise to the form

$$\alpha_1 \vee \alpha_2 \vee \ldots$$

and the conclusion to the form

$$\beta_1 \wedge \beta_2 \wedge \ldots$$

where each α_i is a conjunction of, and each β_j is a disjunction of plain and negated atomic formulas.

(iii) Apply the laws of disjunction elimination and conjunction introduction of classical sentential logic, which stringent logicians also accept, to reduce the question of whether the premise entails the conclusion to the question of whether each α_i entails each β_j, a question of proto-elementary entailment.

(iv) Apply the classical criteria (I), (II), (III) to answer these questions of proto-elementary entailment.

Stringent logic's departure from classicism occurs only at the last step here, where instead of (iv) one would have

(iv*) Apply the criterion (I) to answer the questions of proto-elementary entailment.

To illustrate the process, consider Disjunctive Syllogism

(D2) $(p \vee q) \wedge \sim p \vdash q$.

At step (i) there is nothing to do. At step (ii) we get

$$(p \wedge \sim p) \vee (q \wedge \sim p) \vdash q$$

At step (iii) we get two entailments to consider:

$$p \wedge \sim p \vdash q$$
$$q \wedge \sim p \vdash q.$$

The latter is acceptable by (I), but the former, though acceptable classically by (II), is rejected by stringent logicians; so (D2) is rejected.

For another example, consider the dual to Disjunctive Syllogism

(D2*) $\sim(p \wedge q) \wedge p \vdash \sim q$.

At step (i) we get

$$(\sim p \vee \sim q) \wedge p \vdash \sim q.$$

At step (ii) we get

$$(\sim p \wedge p) \vee (\sim q \wedge p) \vdash \sim q.$$

At step (iii) we get

$$\sim p \wedge p \vdash \sim q$$
$$\sim q \wedge p \vdash \sim q.$$

And again the latter holds stringently, but the former holds only classically, not stringently; so (D2*) is rejected.

Now it seems that Disjunctive Syllogism (D2), and for that matter its dual (D2*), are used ubiquitously in mathematics—and not just in mathematics, either.[9] If this is so, then stringency fails as a *descriptive* criticism of classical logic. However, at this point we must take note of an option alluded to earlier, that of claiming that the apparent instances of (D2) in mathematical proofs are really instances of something else:

(D2†) $(p + q) \wedge \sim p \vdash q$,

where $+$ is an *intensional* disjunction, stronger than mere $\vee$.

The claim that "or" often means something stronger than $\vee$ has been advanced many times on many different grounds. And certainly, whenever a mathematician or whoever seriously wants to argue by Disjunctive Syllogism from "p or q" and "not p" to q, the reasoner will be in a position to say *something* stronger than just "$p \vee q$." Namely, something like, "$p \vee q$ and my present grounds for this assertion are not just the knowledge or assumption that p or the knowledge or assumption that q." For on the one hand, if the reasoner already knows q, there is no need to infer q by Disjunctive Syllogism or in any other way; while on the other hand, if the reasoner is assuming p to get $p \vee q$, presumably the same reasoner will not also at the same time want to be assuming $\sim p$, the other premise needed for an application of Disjunctive Syllogism.

But note that the "something extra" over and above $p \vee q$ is *purely subjective* and may consist of nothing but a *state of ignorance* on the part of the mathematician or whomever. The reasoner may have established $p \vee q$ yesterday by inferring it from q, and yet today recall only that $p \vee q$ has been established, and not how. Such a reasoner can still say, "My *present* grounds for $p \vee q$ are not simply knowledge that p or knowledge that q," for the reasoner's grounds are, rather, an imperfect memory of having *somehow* established $p \vee q$.

Indeed, the reasoner may even remember that it was *either* by inference from p *or* by inference from q, without recalling which. If the reasoner is now satisfied that "not p," then the inference by Disjunctive Syllogism may be a way of *recovering* the knowledge that has been forgotten, since then the inference must have been from q and not from p.

In mathematics, a later mathematician may start from the case $P(0) \vee Q(0)$ of an earlier mathematician's theorem that $\forall n(P(n) \vee Q(n))$ and establish $\sim P(0)$, and so establish $Q(0)$. And the later mathematician may do so without remembering (and, indeed, without ever having known) what the earlier mathematician's

[9] For a compendium of examples, due to E. M. Curley and others, see Burgess (1981).

proof of $\forall n(P(n) \vee Q(n))$ was. (Indeed, that proof may even have been an induction starting by inferring $P(0) \vee Q(0)$ from $Q(0)$. In that case, the later mathematician's inference that $Q(0)$ is not new to mathematics, but only to him- or herself; but that does not make the inference invalid.)

This issue has been extensively discussed in the literature, and the "intensional disjunction" option for defending stringency as a descriptive criticism of classical logic has, it seems, been pretty well given up.[10] The basic implausibility of this strategy ought to have been apparent from the fact that those who use it are committed not only to rejecting Disjunctive Syllogism (D2) but also to rejecting the dual version (D2*); and to defend the latter rejection along the same lines as the former, one would need to make the dual claim that there are both an extensional $\wedge$ and an intensional $\cdot$ conjunction, with the latter *weaker* than the former. But whatever plausibility may attach to the suggestion that there is a sense of "or" stronger than $\vee$, in which p does not entail "p or q," none at all attaches to the dual suggestion that there is a sense of "and" weaker than $\wedge$, in which "p and q" does not entail p.

5. "By Faith Unfaithful Kept Falsely True"

There remain to be considered some historically more recent rationales for stringency, based on certain equivalent formulations of the stringency criterion that are technically more convenient to work with than the original version given in the preceding section. Suppose the formulas of concern involve atoms $p_1, p_2, p_3, \ldots$. Let us then introduce new atoms $q_1, q_2, q_3, \ldots$ and $r_1, r_2, r_3, \ldots$. For elementary α involving only p atoms, let α^* (respectively $^*\alpha$) be the result of replacing each positive occurrence of p_i with the corresponding q_i (respectively r_i) and each negative occurrence with the corresponding r_i (respectively q_i). The second formulation of the stringency criterion is: *To test if elementary α stringently entails elementary β, test if α^* classically entails β^** (or equivalently, *if $^*\alpha$ classically entails $^*\beta$*; or, equivalently and more symmetrically, *if both*).

The second formulation of the criterion can be obtained from the first by looking at matters from from a classical point of view. From that point of view, ignoring (II) and (III) when testing for proto-elementary entailment amounts to

[10] For the later views of the originators of the option, see Anderson, Belnap, and Dunn (1992), §80.4, pp. 502ff.

forgetting that p and $\sim p$ are an atom and the negation of the *same* atom, and treating them as if they were just one atom q and the negation $\sim r$ of some *other* atom. More formally, performing step (iv*) in the first criterion amounts to first performing the *-substitution of the second formulation and then performing the step (iv) of evaluating the proto-elementary entailment resulting by the *classical* criterion (because the *-substitution prevents cases (II) and (III), which distinguish (iv) from (iv*) from arising).

But now it is readily checked that each of the laws used at steps (i), (ii), and (iii) is such that the result of first applying the law and then making the *-substitution or first making the *-substitution and then applying the law is the same. For instance, for double negation one has $(\sim\sim\alpha)^* = \sim\sim(\alpha^*)$. So instead of performing (i), (ii), (iii), *, (iv), one could perform (i), (ii), *, (iii), (iv) or (i), *, (ii), (iii), (iv) or *, (i), (ii), (iii), (iv). Since any other test for classical entailment would yield the same result as the test consisting in performing steps (i), (ii), (iii), (iv), the whole amounts simply to performing the *-substitution and testing *by one method or another* for classical entailment, which is the second formulation of the criterion. (This remains so whether "the *-substitution" means substituting q-atoms for positive and r-atoms for negative occurrences, or the reverse. Hence the parenthetical clause at the end of the second formulation.)

The second formulation shows that if α stringently entails β, then α perfectibly entails β. For if α stringently entails β, then α^* classically entails β^*; and since a formula like α^* or β^*, in which no atom occurs both positively or negatively, cannot be unsatisfiable or tautologous, because it can always be valued true (respectively, false) by so valuing the atoms that occur positively (respectively, negatively) and valuing oppositely the other atoms, it follows that α^* perfectly entails β^*; and since α and β are obtainable by substitution, namely of each p_i for both q_i and r_i, from α^* and β^*, it follows that α perfectibly entails β.

The second formulation shows that substitution holds for stringent entailment. For suppose α stringently entails β, so that α^* classically entails β^*, and consider the result of substituting a formula π for an atom p in α and β to obtain γ and δ, respectively. A little thought shows that the result of performing the *-substition to obtain γ^* and δ^* is the same as the result of substituting π^* for q and $^*\pi$ for r in α^* and β^*, and hence γ^* classically entails δ^*, and γ stringently entails δ.

The second formulation also shows that transitivity of entailment or hypothetical syllogism holds stringently. For if α stringently entails β, which stringently entails γ, then α^* classically entails β^*, which classically entails γ^*; thus α^* classically entails γ^* and α stringently entails γ.

At this point it is possible to make various "cosmetic" changes in the presentation of the second stringency criterion. A *Routley–Routley valuation* or *Sylvan–Plumwood valuation* or *double valuation* is a pair $(V^*, {}^*V)$ of assignments to atoms of truth-values, 1 for true and 0 for false, extended to compound formulas by the following recursion equations.

$$(V^*(\sim\alpha), {}^*V(\sim\alpha)) = (1 - {}^*V(\alpha),\ 1 - V^*(\alpha))$$
$$(V^*(\alpha\wedge\beta), {}^*V(\alpha\wedge\beta)) = (\min(V^*(\alpha),\ V^*(\beta)),\ \min({}^*V(\alpha),\ {}^*V(\beta)))$$
$$(V^*(\alpha\vee\beta), {}^*V(\alpha\vee\beta)) = (\max(V^*(\alpha),\ V^*(\beta)),\ \max({}^*V(\alpha),\ {}^*V(\beta)))$$

Then β is a (*right*) *double consequence* of α if $V^*(\alpha) \leq V^*(\beta)$ in all double valuations (and a *left double consequence* if ${}^*V(\alpha) \leq {}^*V(\beta)$, and a *symmetrical double consequence* if both). A third formulation of the stringency criterion is: *To test if elementary* α *stringently entails elementary* β, *test if* β *is a (right) double consequence of* α (or equivalently, *a left double consequence*, or equivalently *a symmetrical double consequence*).

The third formulation of the criterion can be obtained from the second by camouflaging the role of the q-atoms and r-atoms. The second formulation says to test whether for all valuations V of the q-atoms and r-atoms, extended classically to compound formulas, it is the case that $V(\alpha^*) \leq V(\beta^*)$. Take a valuation V of q-atoms and r-atoms and a pair $(V^*, {}^*V)$ of valuations of p-atoms to *correspond* if for all p_i one has $V^*(p_i) = V(p_i^*) = V(q_i)$ and ${}^*V(p_i) = V({}^*p_i) = V(r_i)$. Observe, first, that every V corresponds to exactly one $(V^*, {}^*V)$ and vice versa. Observe, second, that one will have $V^*(\alpha) = V(\alpha^*)$ and ${}^*V(\alpha) = V({}^*\alpha)$ not just for atoms but for all elementary α if one extends the pair $(V^*, {}^*V)$ to molecules by the above recursion equations for $\sim$, $\wedge$, and $\vee$, which are nonclassical for $\sim$ because it reverses positive and negative occurrences, and classical for $\wedge$ and $\vee$ because they preserve positive and negative occurrences. These observations together establish the equivalence of the second and third formulations of the criterion (with the parenthetical clause at the end of the second corresponding to the parenthetical clause at the end of the third).

A fourth formulation can be obtained by sensationalizing the third. A *dialectical valuation* is an assignment V to atoms of a subset of the set $\{0,1\}$ of classical truth-values, which may be $\{1\}$ or *true and unfalse*, or may be $\{0\}$ or *false and untrue*, or may be $\varnothing$ or *untrue and unfalse*, or may be $\{0,1\}$ or *true and false*, extended to molecules by the following recursion conditions:[11]

$0 \in V(\sim\alpha)$ iff $1 \in V(\alpha)$
$1 \in V(\sim\alpha)$ iff $0 \in V(\alpha)$
$0 \in V(\alpha\wedge\beta)$ iff $0 \in V(\alpha)$ or $0 \in V(\beta)$
$1 \in V(\alpha\wedge\beta)$ iff $1 \in V(\alpha)$ and $1 \in V(\beta)$

[11] The particular labels for the subsets given here represent only one of several options or "plans." One could, for instance, call the four values *true*, *false*, *gap*, and *glut*.

$0 \in V(\alpha\vee\beta)$ iff $0 \in V(\alpha)$ and $0 \in V(\beta)$
$1 \in V(\alpha\vee\beta)$ iff $1 \in V(\alpha)$ or $1 \in V(\beta)$

Then β is a (*right*) *dialectical consequence* of α if for any dialectical valuation, if 1 is among the values of α, then 1 is among the values of β (and a *left dialectical consequence* if whenever 0 is among the values of β, then 0 is among the values of α; and a *symmetrical dialectical consequence* if both). A fourth formulation of the stringency criterion is: *To test if elementary* α *stringently entails elementary* β, *test if* β *is a (right) dialectical consequence of* α (or equivalently, *a left dialectical consequence*, or equivalently *a symmetrical dialectical consequence*).

The fourth formulation can be related to the third as follows. Take a dialectical valuation V and a double valuation $(V^*, {}^*V)$ to *correspond* if and only if for all atomic α one has $1 \in V(\alpha)$ if and only if $V^*(\alpha) = 1$ and $0 \notin V(\alpha)$ if and only if ${}^*V(\alpha) = 1$, so that V^* says whether something is true or untrue, and ${}^*V$ whether something is unfalse or false. Then it can be observed that every dialectical valuation corresponds to exactly one double valuation and vice versa. And, comparing the two sets of recursion equations, it can be observed that the condition on atomic α defining correspondence actually holds for *all* α. This suffices to establish the equivalence of the third and fourth formulations (with the parenthetical clause at the end of third corresponding to the parenthetical clause at the end of the fourth).

Further ringing of changes is possible, but with the dialectical valuations we have come as far as we need to for present purposes. Note that with the last formulation, the notion that there is some subject-matter overlap requirement *additional to* truth-preservation has wholly dropped out, and entailment has become just truth-preservation for the enlarged set of four rather than the classical set of two truth-values. (By contrast, it can be shown that perfectionist logic is not a finite-valued logic.) This is one reason why some feel it inappropriate to call stringent logic "relevance logic" or "the logical of relevance." On the other hand, attractive as it may be to relevantists to call their logic "relevant logic" and classical logic an "irrelevant logic," this terminology is objectionable in the same way as would be a proposal to call Brouwer's logic "intuitive logic" and classical logic "counterintuitive logic".

Various rationales for stringency based on the alternate formulations have been advanced by various logicians with varying degrees of commitment, all based on the notion that something can be both true and false. This notion is understood by moderates in a *figurative* or imaginary sense, as "epistemic" truth according to a database or information corpus or belief system or whatnot; it is understood by radicals in a *literal* or real sense as "metaphysical" truth-in-the-world. In a terminology introduced earlier, to rely on the moderate's notion as a rationale would in effect amount to adopting a compatibilist position (though in actual fact the primary exposition of the moderate position was not accompanied by a retraction of the authors' previous criticisms of classical logic); while to rely on the radical's notion as a rationale would in effect amount to adopting a revisionist position (though in actual fact, enunciations of this view have not been followed by much work on revising "proofs" of metatheorems about "relevance" or "relevant" logics that rely on Disjunctive Syllogism).

The moderate idea is the basis for a suggestion about the potential utility of relevantism. The background is as follows. In ancient history, logic arose in a context of public debate in which argument was used at least as much for purposes of refutation as for purposes of demonstration: The recognition that a previously accepted premise entails some previously not accepted conclusion was taken to be grounds for ceasing to accept the premise at least as often as it was taken to be grounds for beginning to accept the conclusion. (This fact doubtless facilitated the recognition that logical validity of an argument is not simply a matter of truth or falsehood of premise or conclusion.) The critical faculty of being able to use the recognition of logical relationships thus in the reverse direction, like the more basic critical faculty of being able to reject some of what one is told, is an essential feature of natural intelligence. But it is a feature that "artificial intelligence" may not find it technically feasible to imitate in the foreseeable future.

In designing a computer to answer questions by inference from a database of things it has been told, allowance may have to be made for the fact that perhaps the computer cannot in the present state of the art be designed to reject any of its database (even when same item has been *both* said to be true *and* said to be false), or to use recognition of classical logical entailment relations between an accepted premise and an unaccepted conclusion as grounds for rejecting the premise (even when the conclusion has no overlap of subject matter with the premises, so that by the classical interpolation theorem, unless the conclusion is tautologous—in which case it is deducible even without the premise—the premise must be inconsistent). The computer, so to speak, may not be able to exhibit anything worthy of being called artificial intelligence, but only a kind of artificial bureaucratic stupidity. If so, it may well be appropriate *not* to allow such a computer to recognize all classical entailments, and it may even be appropriate to allow such a computer to recognize all and only stringent entailments.[12]

The classical response to this suggestion has a positive aspect, the acknowledgment that thus nonliterally interpreted stringent logic may well be useful in the manner described: that is for specialists in computer science to say. It also has a negative aspect, the insistence that the entailment relations *it would be appropriate to allow a computer with limitations of the kind described to recognize* are not the

[12] See Anderson, Belnap, and Dunn (1992), §81, pp. 506ff., though it is a lot easier to see why we would not want the computer to be able to infer anything whatsoever from a contradiction than why we would not want the computer to be able to infer a tautology from anything whatsoever.

only entailment relations that hold, and that the possible usefulness of stringent logic for the kind of application in question in no way sustains any relevantist criticism of classical logic.

The radical idea is that classical logic fails for the simplest reason, namely, that it is not truth-preserving, because there are cases where a premise of form $\alpha \wedge {\sim}\alpha$ is true, while not every conclusion β is true.[13] Tradition is scoured for examples of truths of the form $\alpha \wedge {\sim}\alpha$: Epimenides ("What I am now saying is false"), Heraclitus ("The way up and the way down are the same"), Zeno ("The arrow is both in motion and at rest"), Athanasius ("God is both one and three"), Leibniz ("Infinitesimals are both distinct from zero and less than anything greater than zero"), Kant ("Space and time are infinite in extent and divisibility, and space and time are finite in extent and divisibility"), Hegel-Marx-Engels-Lenin-Stalin-Mao ("There is contradiction in all things"), Meinong ("The round square is as surely round as it is square"), Russell ("The set of non-self-elements is a self-element and a non-self-element"), Bohr ("Waves are particles, and vice versa"), and others are canvassed.

The classical response, when there has been one—for some principles are so fundamental that there can hardly be non-question-begging argument about them, and many would hold that the law of noncontradiction is one of these[14]—has most often been that of Wellington ("If you can believe that, you can believe anything").

[13] Proponents of this idea—by some called "dialethism"—almost invariably claim that while adoption of a logic in which a contradiciton does not imply everything—by many called "paraconsistent"—enables one to deal with these inconsistent situations, somehow adoption of such a logic does *not* prevent one from using classical logic in consistent situations, so that adopting such a logic involves a pure benefit, without the enormous costs that would appear to be involved in giving up the ubiquitous Disjunctive Syllogism. The incoherence of this claim has been pointed out by Anderson, Belnap, and Dunn (1992, §80.4.2, p. 503):

> One might think as follows. The point of relevantism is to take seriously the threat of contradiction. But there is in this vicinity . . . no real such threat. So here it is OK to use the d.s. . . . That *sounds* OK; but is it? After all, we suppose that "here there is no threat of contradiction" is to be construed as an added premiss. But a little thought shows that *no* such added premiss should permit the relvantist to use the d.s., for a very simple reason: as we said, avoidance of the d.s. was bound up with the threat of contradiction, and one thing that is clear is that *adding* premisses cannot possibly *reduce* that threat.

The context of these remarks is a discussion of what relevantists should make of the fact (urgently brought to their attention by Saul Kripke) that the proof of a major metatheorem about their systems employs the forbidden disjunctive syllogism.

[14] Thus Lewis, "Logic for Equivocators," in (1998), p. 101, writes:

> The reason we should reject this proposal is simple. No truth does have, and no truth could have, a true negation. Nothing is, and nothing could

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be literally true and false. This we know for certain, and *a priori*, and without any exception for especially perplexing subject matters. The radical case for relevance should be dismissed just because the hypothesis it requires us to entertain is inconsistent.

That may seem dogmatic. And it is: I am affirming the very thesis that [Richard Sylvan né] Routley and [Graham] Priest have called into question and—contrary to the rules of debate—I decline to defend it. Further, I concede that it is indefensible against their challenge. They have called so much into question that I have no foothold on undisputed ground. So much the worse for the demand that philosophers always must be ready to defend their theses under the rules of debate.

CHAPTER 25

HIGHER-ORDER LOGIC

STEWART SHAPIRO

> . . . our definition of the [standard second-order] consequences of a system of postulates . . . can be seen to be not essentially different from [that] required for the . . . treatment of classical mathematics. . . . It is true that the non-effective notion of consequence, as we have introduced it . . . presupposes a certain absolute notion of ALL propositional functions of individuals. But this is presupposed also in classical mathematics, especially classical analysis. . . .
>
> —Church [1956, 326n]

Formal languages typically have variables that (purport to) range over some objects. These are called *first-order variables*, and the collection of objects they range over is called the "domain of discourse." *Second-order variables* range over properties, classes, relations, or functions of the items in the domain of discourse. *Third-order variables* range over properties, or relations, or functions on the items in the range of the second-order variables. And so on. A formal language is *first-order* if it contains first-order variables, and no others. A language is *second-order* if it contains first-order and second-order variables, and no others. And on it goes. A language is *higher-order* if it is at least second-order. Second-order logic is the logic of second-order languages, and higher-order logic is the logic of higher-order languages. To make a bad joke, if we could agree on what logic is, we would then be about done with our topic of higher-order logic. Stay tuned.