Lecture 1. von Neumann Algebras

1.1. Three topologies on $\mathcal{B}(\mathcal{H})$. If $\mathcal{H}$ is a complex Hilbert space with inner product $\langle\ ,\ \rangle$, the *norm* topology on the *-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is defined by the norm: $\|x\| = \sup_{\substack{\xi\in\mathcal{H}\\ \|\xi\|\le 1}} \|x\xi\|$.

In finite dimensions x is a matrix and $\|x\|^2$ may be calculated as the largest eigenvalue of x^*x. The same is true in infinite dimensions if we replace "largest eigenvalue" by "spectral radius", where the spectral radius of an operator a is $\sup\{|\lambda| : \lambda - a \text{ is not invertible}\}$. If we consider $\mathcal{H} = L^2([0, 1], dx)$, $L^\infty([0, 1], dx)$ acts on $\mathcal{H}$ by pointwise multiplication and the norm of $f \in L^\infty$ is the essential sup of $|f|$. Thus the continuous functions $C([0, 1])$ form a norm closed subalgebra of $L^\infty([0, 1], dx)$ on $\mathcal{H}$. (Note that the choice of $([0, 1], dx)$ is inessential. The same things are true for any compact space and measure you are likely to think of in the next ten minutes.)

The *strong* topology on $\mathcal{B}(\mathcal{H})$ is that defined by the seminorms $x \mapsto \|x\xi\|$ as ξ runs through $\mathcal{H}$. Thus a sequence (or net if you must) x_n converges to x iff $x_n\xi$ converges to $x\xi$ in $\mathcal{H}$ for all $\xi \in \mathcal{H}$. The strong topology is much weaker than the norm topology. In fact we will soon see that, for the example of $L^\infty([0, 1])$ and $C([0, 1])$ acting on $L^2([0, 1])$, $C([0, 1])$ is actually strongly dense in $L^\infty([0, 1])$. To see a sequence that converges strongly without converging in norm, let x_n be the characteristic function of $[0, 1/n]$ viewed as an element of L^∞. Obviously x_n tends strongly to zero, but $\|x_n\| = 1$ for all n.

The *weak* topology on $\mathcal{B}(\mathcal{H})$ is that defined by the seminorms $x \mapsto |\langle x\xi, \eta\rangle|$ as ξ and η run through $\mathcal{H}$. The Cauchy–Schwarz inequality shows that the strong topology is stronger than the weak topology. In fact the weak topology is so weak that the unit ball of $\mathcal{B}(\mathcal{H})$ is weakly compact—which is often very useful. Probably the simplest example of a sequence of operators tending weakly but not strongly to zero is the sequence $e^{in\theta}$ in $L^\infty(S^1)$ (on $L^2(S^1)$), which by Fourier series is the same as the obvious shift operator on $\ell^2(\mathbb{Z})$.

The interplay between the above three topologies is basic to von Neumann algebras. We refrain from mentioning the many other topologies around.

1.2. von Neumann's bicommutant theorem. Let us prove the following simplified version of von Neumann's bicommutant theorem (see [vN1]). We use the following standard notation: if $S \subseteq \mathscr{B}(\mathscr{H})$ then $S' = \{x \in \mathscr{B}(\mathscr{H}) : xs = sx \text{ for all } s \in S\}$, and $S'' = (S')'$.

THEOREM. *Let* S *be a subset of* $\mathscr{B}(\mathscr{H})$ *with the following two properties:*

a) *If* $x \in S$ *then* $x^* \in S$;
b) $1 \in S$ (1 *is the identity operator on* $\mathscr{B}(\mathscr{H})$).

Then alg(S) *is strongly* (*hence weakly*) *dense in* S''. (alg(S) *is the algebra generated by* S.)

PROOF. First check that $\mathrm{alg}(S) \subseteq S''$. Now suppose $y \in S''$. What we must show is this: for any finite set $\xi_1, \dots, \xi_n$ in $\mathscr{H}$, there is an element x of alg(S) with $x\xi_i$ arbitrarily close to $y\xi_i$ for all i. Let us suppose at first that we only want to approximate one vector $y\xi$. The trick is this: let V be the closure of the vector subspace $\mathrm{alg}(S)\xi$ and let p be the operator that is orthogonal projection onto V. Clearly $aV \subseteq V$ for all $a \in S$; so by property a), $ap = pa$. Thus $yp = py$ since $y \in S''$. So $yV \subseteq V$. But by property b), $\xi \in V$ so that $y\xi \in \overline{\mathrm{alg}(S)\xi}$, which is precisely what we wanted to prove.

The general case of $\xi_1, \dots, \xi_n$ involves another trick which is used all over the subject: make $\xi_1, \xi_2, \dots, \xi_n$ into a single vector on the Hilbert space $\bigoplus_{i=1}^n \mathscr{H}$. Then alg($S_i$) and y act diagonally on $\bigoplus_{i=1}^n \mathscr{H}$ and we can, after making some matrix calculations to see how commutants behave under this "amplification" of $\mathscr{H}$, repeat the previous argument with ξ replaced by $\bigoplus_{i=1}^n \xi_i$ to conclude the proof. □

NOTE. If 1 did not belong to S the theorem still applies provided one cuts down to the closed subspace of $\mathscr{H}$ that is all that S notices.

This beautiful little theorem shows that two notions, one analytic (closure in the strong topology) and one purely algebraic (being equal to one's bicommutant) are the same for *-subalgebras of $\mathscr{B}(\mathscr{H})$ containing 1. It thoroughly justifies the definition of §1.3. Note also that the theorem shows that "strongly closed" and "weakly closed" are the same thing for a *-subalgebra of $\mathscr{B}(\mathscr{H})$.

1.3. (Concrete) von Neumann algebras.

DEFINITION. If $\mathscr{H}$ is a complex Hilbert space, a *von Neumann algebra* is a *-subalgebra M of $\mathscr{B}(\mathscr{H})$ containing 1 such that either M is strongly (weakly) closed or $M = M''$. If S is a selfadjoint subset of $\mathscr{B}(\mathscr{H})$ then S'' is the von Neumann algebra generated by S.

EXAMPLES.

i) The algebra $\mathscr{B}(\mathscr{H})$ itself is certainly closed, thus a von Neumann algebra.

ii) The algebra $L^\infty([0,1], dx)$ is easily shown to be its own commutant, thus a von Neumann algebra.

iii) If G is a group and $g \mapsto u_g$ is a unitary representation of G, then the commutant $\{u_g\}'$ is a von Neumann algebra.

iv) If $\mathscr{H}$ is finite-dimensional, it is not too hard to see that a von Neumann algebra M is just a direct sum of matrix algebras corresponding to some orthogonal decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k$ so that the matrices in M will look like

$$\dim \mathscr{H}_1 \left\{ \overbrace{\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_1 \end{bmatrix}}^{\dim \mathscr{H}_1} \oplus \begin{bmatrix} x_2 & & \\ & \ddots & \\ & & x_2 \end{bmatrix} \oplus \cdots \oplus \overbrace{\begin{bmatrix} x_k & & \\ & \ddots & \\ & & x_k \end{bmatrix}}^{\dim \mathscr{H}_k} \right\} \dim \mathscr{H}_k$$

where the x_i's are matrices.

v) If M on $\mathscr{H}$ and N on $\mathscr{K}$ are von Neumann algebras there are obvious notions of direct sum $M \oplus N$ on $\mathscr{H} \oplus \mathscr{K}$ and tensor product $M \otimes N$ on $\mathscr{H} \otimes \mathscr{K}$.

We list some important facts about von Neumann algebras.

1) The set of all projections of a von Neumann algebra M forms a complete (orthomodular) lattice. M is generated by its projections since it contains the spectral projections of any selfadjoint element.

2) Abelian von Neumann algebras are completely understood. As well as example ii) above there is $\ell^\infty(\mathbb{N})$ on $\ell^2(\mathbb{N})$ and obvious reductions and combinations with example ii). One must allow some kind of "multiplicity" as can be seen in finite dimensions. But on a separable Hilbert space that is the whole story. Probably the best way to deal with the multiplicity question is to relegate it to the spectral theorem and state, as von Neumann did, the structure theorem for abelian von Neumann algebras as the fact that they are generated by a single selfadjoint operator.

3) von Neumann algebras can be abstractly characterized as C^*- algebras which are duals as Banach spaces. See [**Sa**].

1.4. Factors. The center $Z(M)$ of a von Neumann algebra is abelian. So by fact 2 of §1.3 we know everything about it. In finite dimensions it would be a direct sum of copies of $\mathbb{C}$, one for each summand in the decomposition of example 4 of §1.3. In general, using the spectral theory, von Neumann defined ([**vN2**]) (in the separable situation) a notion of "direct integral" of Hilbert spaces $\int_X^\oplus \mathscr{H}(\lambda)d\mu(\lambda)$ so that, for instance, for

$L^\infty([0,1])$ on $L^2([0,1])$ the corresponding decomposition of $L^2([0,1])$ would be $\int_{[0,1]}^{\oplus} \mathscr{H}(\lambda)\, dx(\lambda)$ where $\mathscr{H}(\lambda) \equiv \mathbb{C}$. The whole algebra M respects this decomposition and we end up with a notion of direct integral of von Neumann algebras: $M = \int_X^{\oplus} M(\lambda)\, d\lambda$ on $\int_X^{\oplus} \mathscr{H}(\lambda)\, d\lambda$, the whole decomposition being essentially unique. The individual $M(\lambda)$'s will have trivial center (to get a feel for this, work it out in finite dimensions). Thus any von Neumann algebra is the direct integral of ones with trivial center.

Although the technical details of this theory are rather messy, and it can usually be avoided by "global" methods, the direct integral decomposition is tremendously helpful in trying to visualize a von Neumann algebra on a basic level. Of course one does not get any further than the $M(\lambda)$'s with trivial center.

DEFINITION. A von Neumann algebra M whose center is just the scalar multiples of the identity is called a *factor*.

EXAMPLES.

a) $\mathscr{B}(\mathscr{H})$ is a factor.
b) In finite dimensions a factor will always be of the form $\mathscr{B}(\mathscr{H}) \otimes \mathbb{C}\,\mathrm{id}$ on $\mathscr{H} \otimes \mathscr{K}$. This is also true in infinite dimensions provided the factor is isomorphic, as an abstract algebra, to some $\mathscr{B}(\mathscr{H})$.

Example b) explains the name "factor"—such factors correspond to tensor product factorizations of the Hilbert space. The remarkable fact, discovered by Murray and von Neumann in their works **[MvN1, 2, 3]** is that not all factors are like this, and indeed, as we shall see, it is not very difficult to construct examples.

1.5. Examples of factors. a) Let Γ be a discrete group (e.g., the free group on two generators) all of whose conjugacy classes are infinite, except that of the identity (we will call such groups i.c.c.). Let $\gamma \to u_\gamma$ denote the left-regular representation of Γ on $l^2(\Gamma)$. As matrices on $l^2(\Gamma)$ with respect to the obvious basis indexed by $\gamma \in \Gamma$, the u_γ, and hence all elements of $\mathrm{alg}(\{u_g\})$ are of the form $x_{\gamma,\nu} = f(\gamma^{-1}\nu)$ (forgive me if the inverse is in the wrong place) for some function f of finite support on Γ. The same is true for weak limits of such operators except that f will no longer be of finite support. However, applying the operator to the basis element for the identity we see that f is in ℓ^2. It is thus convenient and accurate to write elements of $M = \{u_\gamma\}''$ as sums $\sum_{\gamma\in\Gamma} f(\gamma)u_\gamma$ where $f \in \ell^2$ (although not all l^2 functions define elements of M). The sense of convergence of the sum will be clear later on. In any case, in order that $\sum_{\gamma\in\Gamma} f(\gamma)u_\gamma$ belong to the center of M, it must commute with u_ν for all ν, which implies $f(\nu\gamma\nu^{-1}) = f(\gamma)$, i.e., f is constant on conjugacy classes. But f is in ℓ^2 and all nontrivial conjugacy classes are infinite. Thus the support of f is the identity so that M is a factor. Call it $\mathrm{vN}(\Gamma)$.

One may see quickly that this factor is not as in example b) of §1.4 by observing that the linear function $\mathrm{tr}(\sum f(\gamma)u_\gamma) = f(\text{identity})$ has the property $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and is not identically zero. It is simple to show that no such function exists on $\mathscr{B}(\mathscr{H})$ unless $\dim \mathscr{H} < \infty$.

b) The previous example was an example of a very general construction called the *crossed product*, where one begins with a von Neumann algebra N on $\mathscr{H}$ and a group Γ acting by automorphisms on N (in example a), $N = \mathbb{C}$) and one forms a von Neumann algebra $M = N \rtimes \Gamma$ (on $\mathscr{H} \otimes \ell^2(\Gamma)$) generated by $u_\gamma = \mathrm{id} \otimes u_\gamma$ and an action of N on $\mathscr{H} \otimes \ell^2(\Gamma)$. All elements of $N \rtimes \Gamma$ can be represented as sums $\sum_{\gamma \in \Gamma} x_\gamma u_\gamma$, $x_\gamma \in N$, and $u_\gamma x u_\gamma^{-1} = \gamma(x)$ (the action of γ on x) for $x \in N$. It is then trivial to show that the following conditions together suffice to imply that $N \rtimes \Gamma$ is a factor.

(i) The action of Γ is "free", i.e., $xy = y\gamma(x)$ for all $x \in N$ implies $y = 0$ or $\gamma = 1$.

(ii) The algebra of fixed points for Γ is a factor.

Crossed products may also be formed by continuous (locally compact) groups, but they are algebraically less transparent.

c) Let us give an important example of the previous construction. The group will be $\mathbb{Z}$ and N will be $L^\infty(S^1)$. The generator of $\mathbb{Z}$ will act by an irrational rotation. As in example a) there is a trace functional on $L^\infty(S^1) \rtimes \mathbb{Z}$ given on $\sum_{n \in \mathbb{Z}} f_n u_n$ by $\int_{S^1} f_0(\theta)\, d\theta$. This example can obviously be varied by replacing $\mathbb{Z}$ by any discrete group and S^1 by any finite measure space, provided the group action preserves the measure and is free and ergodic. It was recognized very early on that in this situation the crossed product algebra depends only on the *equivalence relation* defined on the measure space by the orbits of the group action, indeed that it is possible to define the crossed product algebra given only the measure space and the equivalence relation (with countable equivalence classes). For the definitive treatment see [FM].

d) *The G.N.S. construction* provides an elementary but useful way to pass from a *-algebra which is not necessarily complete to a von Neumann algebra. The necessary data are a *-algebra A and $\varphi : A \to \mathbb{C}$ with $\varphi(a^*a) \geq 0$. One then forms a Hilbert space by defining a not necessarily definite inner product on A by $\langle a, b\rangle = \varphi(b^*a)$. The Hilbert space $\mathscr{H}_\varphi$ is then the completion of the quotient of A by the kernel of this form. Under favorable circumstances (such as if A is a C^*-algebra), A will act on $\mathscr{H}_\varphi$ by left multiplication. This representation of A is called the G.N.S. representation. The von Neumann algebra generated by the image of A in this representation should be thought of as a *completion* of A with respect to φ. In general, it is difficult to say if the G.N.S completion is a factor or not. One often meets surprises where A has trivial center but its completion does not.

To illustrate the procedure let us give an alternative way of constructing the example 1.5 a). On the group algebra $\mathbb{C}\Gamma$ of finite sums $\sum_\gamma c_\gamma \gamma$ one defines $\gamma^* = \gamma^{-1}$ and $\mathrm{tr}(\sum_\gamma c_\gamma \gamma) = c_{\mathrm{id}}$. It is clear that the G.N.S. Hilbert space $\mathscr{H}_{\mathrm{tr}}$ is naturally the same as $l^2(\Gamma)$ and the G.N.S. representation of $\mathbb{C}\Gamma$ on it is just the linear extension of the left-regular representation.

Of more interest, especially for these lectures, are the following examples. Let (A_n, φ_n) be an increasing union of finite-dimensional von Neumann algebras together with compatible linear functionals φ_n (i.e., $\varphi_{n+1}|_{A_n} = \varphi_n$). Then the union or inductive limit of the A_n's is a *-algebra and the φ_n's define a φ on it for which the G.N.S. construction works perfectly. One obtains many interesting factors in this way. The simplest nontrivial example occurs when $A_n = \bigotimes_{i=1}^n M_2(\mathbb{C})$, the inclusion of A_n in A_{n+1} is $a \hookrightarrow \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ and φ_n is the trace, normalized so that $\varphi_n(\mathrm{id}) = 1$. Once again the G.N.S. completion of A admits a trace so is not $\mathscr{B}(\mathscr{H})$. We will have occasion to examine many more examples in the course of these lectures.

1.6. Comparison of projections.

DEFINITION. If M is a von Neumann algebra on $\mathscr{H}$ and p and q are projections in M, we say that $p \precsim q$ if there is an operator $u \in M$ with $uu^* = p$ and $u^*uq = u^*u$ (or equivalently, u^*u is a projection onto a subspace of $\mathscr{H}$ contained in $q\mathscr{H}$, written $u^*u \le q$). We say that p and q are equivalent $(p \sim q)$ if there is u in M with $uu^* = p$, $u^*u = q$. It is true that $p \sim q$ if $p \precsim q$ and $q \precsim p$ (see [**MvN1**]).

The point is that the operator u must be in M so that the notion of comparison depends heavily on M. If $M = \mathscr{B}(\mathscr{H})$ it is trivial that two projections are equivalent if and only if their images have the same dimension. Thus the idea was born that equivalence classes of projections represent an abstract notion of dimension for an arbitrary factor. The first result confirming this is the following.

THEOREM. *If M is a factor and p, q are projections in M then either $p \precsim q$ or $q \precsim p$.*

A proof may be found in [**MvN1**]. The result and the proof are quite natural if one considers the analogy with ergodic theory, p and q corresponding to measurable subsets of a measure space on which a group is acting. Ergodicity corresponds to being a factor and $p \precsim q$ means that the set p admits a (countable) partition into subsets p_i for each of which there is an element g_i of the group with $g_i(p_i) \subseteq q$ and $g_i(p_i) \cap g_j(p_j) = \varnothing$ for $i \neq j$. In fact I have heard tell, but never followed up the references, that the whole theory of comparison of projections, including the type I, II, III classification of §1.7, was done in the ergodic theory context by E. Hopf in [**Hop**], before [**MvN1**].

1.7. Types I, II_1, II_∞, and III factors. So far the only way we have distinguished between factors was by the existence or otherwise of a trace function. The comparison of projections will give a related but more precise tool.

We begin by calling a projection q *finite* if $p \leq q$, $p \sim q$ implies $p = q$ and *infinite* if there is a $p \sim q$ with $p \lneqq q$. A projection $p \neq 0$ is called *minimal* if it dominates no other projection in M other than 0.

DEFINITION. A factor M is of type I, II_1, II_∞, or III according to the following mutually exclusive conditions:

type I: M has a minimal projection;

type II_1: M has no minimal projections and every projection is finite;

type II_∞: M has no minimal projections but it has both finite and infinite projections;

type III: M has no finite projections except 0.

It is easy to see that if M has a trace tr with $\mathrm{tr}(x^*x) > 0$ for $x \neq 0$ then it is finite-dimensional or of type II_1. It is also fairly easy to prove that if M is of type I it is like $\mathscr{B}(\mathscr{H}) \otimes \mathrm{id}$ on $\mathscr{H} \otimes \mathscr{K}$, and that any type II_∞ factor is a tensor project of a II_1 and a type I factor.

Murray and von Neumann showed in **[MvN1]** that if M is a factor there is an essentially unique "dimension function" d: projections of $M \to [0, \infty]$ subject to

(i) $d(0) = 0$,
(ii) $d(\sum_{i=1}^\infty p_i) = \sum_{i=1}^\infty d(p_i)$ if $p_i \perp p_j$ for $i \neq j$;
(iii) $d(p) = d(q)$ if $p \sim q$.

It follows that $d(p) = d(q) \Rightarrow p \sim q$ and that d may always be normalized so that its range is as follows:

type I: $\{0, 1, 2, \ldots, n\}$ with $n = \infty$ possible;

type II_1: $[0, 1] =$ the whole unit interval;

type II_∞: $[0, \infty]$;

type III: $\{0, 1\}$.

It should be clear at this stage that examples 1.5 a) and c) are both type II_1 factors. One can "see" the dimension function on projections in example 1.5 c) by examining the abelian subalgebra $L^\infty(S^1)$. A projection in here is the characteristic function of some set and its dimension is its (normalized) Haar measure. Thus continuous dimensionality is not a mysterious phenomenon at all.

In these examples the dimension function comes from a trace and in [MvN2] it is shown that any II_1 factor has a unique normalized trace extending its dimension function.

1.8. Standard form for II_1 factors. A II_1 factor M, considered as an abstract complex *-algebra, possesses a (unique) trace tr which is a state of the kind for which the G.N.S. construction may be performed. The resulting Hilbert space completion of M for the inner product $\langle a, b\rangle = \mathrm{tr}(b^*a)$ is denoted $L^2(M, \mathrm{tr})$ or often $L^2(M)$. The reason for this notation is that if M were $L^\infty(X, \mu)$ for some nice probability space (X, μ) and if $\mathrm{tr}(f)$ were $\int_X f\,d\mu$ then $L^2(M)$ would be the Hilbert space $L^2(X, \mu)$. Indeed, there is a highly developed theory of "noncommutative L^p spaces" for II_1 factors, where the p-norm $\|x\|_p$ of $x \in M$ is defined to be $(\mathrm{tr}(|x|^p))^{\frac{1}{p}}$; see [Di2], [N].

The continuity properties of tr are such that the algebra of operators on $L^2(M)$ defined by left multiplication by M is already weakly closed so that M acts on $L^2(M)$ as a von Neumann algebra. This action is called the *standard form* of M. Since $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, right multiplication by elements of M also extends to give bounded operators on $L^2(M)$ to give an action of the opposite von Neumann algebra M^{opp} on $L^2(M)$. The situation is completely symmetric and $M' = M^{\mathrm{opp}}$. It is important to see that the symmetry between the left and right operations of M is implemented by a conjugate linear isometry $J : L^2(M) \to L^2(M)$, which is simply the extension to $L^2(M)$ of the map $x \to x^*$ defined on M. It is a trivial calculation that, if $\xi \in L^2(M)$ then $\xi x = Jx^*J\xi$ so that $JMJ = M'$. For full details see [Di1].

One must think of the standard form as being nothing but the left regular representation of M.

1.9. The fundamental group of a II_1 factor. The continuous dimensionality of II_1 factors makes them look somewhat homogeneous. If q is a nonzero projection in a II_1 factor M it follows from basic theory that qMq is also a II_1 factor, which one might guess to be isomorphic to M. Further thought shows that there is no good reason for thinking this, though notice that this isomorphism property depends only on the trace (hence equivalence class) of q. The fundamental group encodes the set of all traces of projections q for which $M \cong qMq$.

The best way to define the fundamental group of M is to consider the type II_∞ factor $M \otimes \mathcal{B}(\mathcal{H})$ of infinite matrices over M. This $M \otimes \mathcal{B}(\mathcal{H})$ has an infinite trace tr and an automorphism α of $M \otimes \mathcal{B}(\mathcal{H})$ may multiply tr by a positive real constant λ. (There are no minimal projections to normalize tr by and $\mathrm{tr}(1) = \infty$.) The set $\{\lambda | \lambda \in \mathbb{R}$ and there is $\alpha \in \mathrm{Aut}(M \otimes \mathcal{B}(\mathcal{H}))$ with $\mathrm{tr} \circ \alpha = \lambda\, \mathrm{tr}\}$ is obviously a group and is called the fundamental group of M.

We see that if $\lambda < 1$ is in the fundamental group and p is a minimal projection of $\mathcal{B}(\mathcal{H})$ then $(1 \otimes p)M \otimes \mathcal{B}(\mathcal{H})(1 \otimes p) \cong M$, and if α is an automorphism with $\mathrm{tr} \circ \alpha = \lambda\, \mathrm{tr}$ then $\alpha(1 \otimes p) < p$. But then $\alpha(1 \otimes p)$ is of the form $q \otimes p$ for $q \in M$, $\mathrm{tr}(q) = \lambda$. But then

$M \cong (1\otimes p)M\otimes\mathcal{B}(\mathcal{H})(1\otimes p) \cong (q\otimes p)M\otimes\mathcal{B}(\mathcal{H})(q\otimes p) \cong qMq$. Conversely, given an isomorphism $\theta : M \to qMq$ one may construct an automorphism of $M \otimes \mathcal{B}(\mathcal{H})$ which scales tr by $\mathrm{tr}(q)$.

1.10. Type III factors. Although we will not spend much time on them in the lectures, it seems that this rapid survey would be absurd and misleading if we did not discuss, in the same freewheeling spirit we have established, the structure of type III factors.

Murray and von Neumann obviously considered them pathological and many problems were solved for a long time in all cases except type III. The technical problem one runs into is that if one considers the G.N.S. construction for a faithful state $\varphi : M \to \mathbb{C}$ for a type III factor M (φ weakly continuous), the mapping $* : M \to M$, which would be an isometry if φ were a trace, does not extend to an operator on the Hilbert space completion $\mathcal{H}_\varphi$ of M. It was Tomita who first used the unbounded operator S defined by $*$. One needs to extend its domain so that it is a closed operator and then one may consider the *polar decomposition* $S = J\Delta^{1/2}$ where Δ is a positive operator and J is a conjugate linear isometry. What Tomita saw and what was ultimately proved (see **[Ta1]**) was that J may be used in place of $*$. In particular, $JMJ = M'$. But it is also true that $\Delta^{it}M\Delta^{-it} = M$ (for $t \in \mathbb{R}$); so one gets a one-parameter automorphism group σ_t^φ, the modular group, straight from the state φ! Connes showed in **[Co1]** that σ_t^φ only depends on φ up to inner automorphisms so that the group $T(M) = \{t|\sigma_t^\varphi \text{ is inner}\}$ is an invariant of M itself. There are still many questions of interest about $T(M)$, but another invariant $S(M)$, defined by Connes to be the intersection of the spectra of the Δ's obtained by the above process, (minus zero) letting φ vary, is necessarily a closed multiplicative subgroup of $\mathbb{R}^+$ and hence one may classify type III factors into III_λ, $\lambda \in [0, 1]$, by:

III_0: $S(M) = \{1\}$;

III_λ, $0 < \lambda < 1$: $S(M) = \{\lambda^n | n \in \mathbb{R}\}$;

III_1: $S(M) = \mathbb{R}^+$.

To obtain an example of all this, one may look at $\bigotimes_{i=1}^\infty M_2(\mathbb{C})$ and consider on it the state φ_λ, for $0 < \lambda < 1$, given by

$$\varphi_\lambda(x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \otimes \cdots) = \prod_{j=1}^n \mathrm{trace}\left(\frac{1}{1+\lambda}\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix} x_j\right).$$

The von Neumann algebras coming from the G.N.S. construction are the Powers factors R_λ. They were shown to be mutually nonisomorphic type III factors by Powers in **[Pow]**. The operators Δ and J can be handled by finite-dimensional calculations and one may show that the factors are of type III_λ. The modular group is just conjugation by $\bigotimes_{j=1}^\infty \exp(i\left(\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right))$.

An alternative construction of III_λ factors is to take an automorphism α of a II_∞ factor M, scaling the trace by λ, and forming the crossed product

$M \rtimes \mathbb{Z}$, $\mathbb{Z}$ acting via α. Connes showed that all III_λ factors, $0 < \lambda < 1$, arise in this way and all III_0 factors also, if one replaces M by a II_∞ nonfactor with α acting ergodically on its center.

There is no such general discrete decomposition for III_1 factors but Takesaki showed in **[Ta2]** that any III_1 factor is of the form $M \rtimes \mathbb{R}$ where M is a II_∞ factor and $\mathbb{R}$ acts so as to scale the trace nontrivially.

Thus, in some sense, type III factors are reduced to type II factors and their automorphism groups.

1.11. Hyperfiniteness, R. A famous question in algebra is this: is a discrete group Γ determined up to isomorphism by the isomorphism class of the integral group ring $\mathbb{Z}\Gamma$? If we change from $\mathbb{Z}$ to $\mathbb{C}$ the answer is clearly no (witness $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$) and we might be led to ask the question: to what extent does the von Neumann algebra completion $\mathrm{vN}(\Gamma)$ (see §1.5) remember Γ? There seem to be two answers to this question depending on what kind of group Γ is: not very much, and completely. The second possibility has been raised by Connes (though not yet proved), working by analogy with the Mostow rigidity result **[Mos]**, for discrete cofinite subgroups of semisimple Lie groups of real rank ≥ 2. We shall concentrate on the other extreme. We will see that as soon as a group Γ is a union of finite subgroups (and is i.c.c.) then the II_1 factor $\mathrm{vN}(\Gamma)$ is independent of Γ.

We shall say that a von Neumann algebra M is *hyperfinite* (not wonderful terminology but it seems to have some primitive appeal) if there is an increasing sequence A_n of finite-dimensional von Neumann subalgebras of M whose union is weakly dense in M.

It is a fundamental theorem of Murray and von Neumann that there is, up to abstract algebraic isomorphism, a unique hyperfinite II_1 factor which we shall denote by the letter R. This is proved by a cutting and rebuilding argument which is nowadays considered standard technical machinery.

Let us reconsider the examples of §1.5 in the light of the above theorem. If Γ is the group S_∞ of finite permutations of $\mathbb{N}$, $\mathrm{vN}(\Gamma)$ is obviously hyperfinite, so $\cong R$. It follows from deep results of Connes (see §1.12) that $\mathrm{vN}(\Gamma)$ is hyperfinite as soon as Γ is amenable, i.e., there is a left-invariant mean on $l^\infty(\Gamma)$. It is not obvious but true that the II_1 factor $L^\infty(S^1) \rtimes \mathbb{Z}$ is also hyperfinite. In fact, the crossed product of $L^\infty(X, \mu)$ by $\mathbb{Z}$ is always hyperfinite. Much more generally, by the results of **[Co6]**, $M \rtimes \Gamma$ is hyperfinite as soon as M is and Γ is amenable. The infinite tensor product algebra of §1.5 d) is obviously hyperfinite.

Are there nonhyperfinite II_1 factors? Let us point out a special feature of R using, say, the $\mathrm{vN}(S_\infty)$ model. Let S_i be the transposition $(i\ i+1)$, permutation of $\mathbb{N}$. If we use the 2-norm $\|x\|_2 = \sqrt{\mathrm{tr}(x^*x)}$, it is clear that for any $y \in \mathbb{C}S_\infty$, $[S_i, y] = 0$ for large i, so that for any $y \in \mathrm{vN}(S_\infty)$, $\lim_{i\to\infty} \|[S_i, y]\|_2 = 0$. On the other hand, $\mathrm{tr}(S_i) = 0$; so S_i

stays well away from the center of $\mathrm{vN}(S_\infty)$. Such a (norm bounded) sequence is called a central sequence (exercise: prove that all central sequences are trivial in finite dimensions). Thus R has nontrivial central sequences. On the other hand, if Γ is the free group F_2 with generators a and b, it is not hard to show that there is a constant K such that, for $y \in \mathrm{vN}(F_2)$, $\|y - \mathrm{tr}(y)1\|_2 < K \max\{\|[a, y]\|_2, \|[b, y]\|_2\}$. Thus any central sequence in $\mathrm{vN}(F_2)$ is trivial. Hence $\mathrm{vN}(F_2) \ncong R$.

So how many II_1 factors are there? More than you care to think about. With one glorious exception due to Connes (using Kazdan's property T; see **[Co2]**), all constructions of many many II_1 factors ultimately rely on a clever manipulation of central sequences. If there are none, and property T is not around, we remain totally in the dark. It is shameful but true that we do not know if $\mathrm{vN}(F_2) \cong \mathrm{vN}(F_3)$! Nor do we know if the fundamental group of $\mathrm{vN}(F_2)$ contains a single element different from 1.*

In fact all hyperfinite factors are known. The type I case is trivial, the II_1 case is given by the uniqueness result of Murray and von Neumann. Connes showed that there is only one hyperfinite factor in the cases II_∞, III_λ, $0 < \lambda < 1$, and that hyperfinite III_0 factors are classified by ergodic transformations, using work of Krieger (**[Kr]**). Haagerup proved uniqueness of the hyperfinite III_1 factor, in **[Ha]**.

*But for F_∞ it appears that the fundamental group is $\mathbb{R}$ according to recent results of Voiculescu and Radulescu.