3 Given that a relation is transitive, show that it is irreflexive if and only if it is asymmetric.

4 The relation of being a brother or sister of seems intuitively both transitive and irreflexive, whence by 164 it would follow that it was asymmetric, which it clearly is not. How do you explain this paradox?

5 Given that R is both symmetric and antisymmetric, prove that no two distinct things stand in the relation R to each other (i.e. $(x)(y)(x \neq y \rightarrow -Rxy)$) and further that R is transitive. Hence show that if R is reflexive, symmetric, and antisymmetric, then $(x)(y)(Rxy \leftrightarrow x = y)$. (This effectively shows that identity is the only reflexive, symmetric, and antisymmetric relation.)

6 Show that no relation can be:

(*a*) intransitive and reflexive;

(*b*) asymmetrical and non-reflexive;

(*c*) transitive, reflexive, and asymmetric;

(*d*) transitive, non-symmetric, and irreflexive.

7 (*a*) From 156–158 it can readily be seen to follow that relation R is both symmetric and asymmetric if and only if $-(\exists x)(\exists y)Rxy$. Prove correspondingly that a relation R is both transitive and intransitive if and only if $-(\exists x)(\exists y)(\exists z)(Rxy \,\&\, Ryz)$.

(*b*) Hence prove that if R is both transitive and intransitive, then R is asymmetric.

APPENDIX A

Normal Forms

It is customary to include in elementary logic courses a treatment of normal forms. Normal forms have a certain interest in connection with the truth-table method, since they provide an independent test as to whether a wff is tautologous, contingent, or inconsistent; and they are also used in certain proofs of the completeness of the propositional calculus (see, e.g., Basson and O'Connor [1]). The completeness proof given in this book (Chapter 2, Section 5), however, did not rely on normal forms, so that I have relegated an account of them to an appendix. What follows presupposes the terminology of Chapter 2, Section 3.

We begin by defining normal forms. First, by an *atom* I understand *either a propositional variable or a negated propositional variable.* Thus

$$`P\text{'}, `-P\text{'}, `Q\text{'}, `-R\text{'}, `S\text{'}$$

are all atoms, though

$$`--Q\text{'}$$

is not one. Let $A_1, \ldots, A_n$ be a list of n atoms, where n is greater than or equal to 1. Then by an *elementary disjunction* (e.d.) I understand a formula of the form

$$(A_1 \vee A_2 \vee \ldots \vee A_n).$$

Thus any list of atoms linked by '$\vee$'s counts as an elementary disjunction: for example,

$$`(P \vee Q)\text{'}$$
$$`(-P \vee Q \vee -R)\text{'}$$
$$`(-Q \vee P \vee Q \vee -S)\text{'}$$
$$`(P \vee P \vee -Q)\text{'}$$

are all elementary disjunctions. In the limiting case where $n = 1$, a single atom standing alone counts as an elementary disjunction also.

The same variable may appear both negated and non-negated in an e.d., as in the third example, and the same atom may appear more than once, as in the fourth.

Correspondingly, by an *elementary conjunction* (e.c.) I understand a formula of the form

$$(A_1 \& A_2 \& \ldots \& A_n)$$

for atoms $A_1, \ldots, A_n$, where n is greater than or equal to 1. Any e.d. becomes an e.c. if all the 'v's in it are changed to '&'s, and vice versa. Again, in the limiting case, an atom standing alone counts as an e.c.

By a *conjunctive normal form* (C.N.F.) I understand a formula of the form

$$A_1 \& A_2 \& \ldots \& A_n,$$

where $A_1, \ldots, A_n$ are elementary *disjunctions*, for n greater than or equal to 1. Thus a C.N.F. is a string of e.d.'s linked by '&'. For example,

'$(P \vee Q) \& (P \vee -Q \vee R) \& (-R \vee S)$'
'$(P \vee -P \vee Q) \& -S$'
'$(P \vee -Q) \& (P \vee -Q)$'

are all C.N.F.'s, the second of which has as its second e.d. a single atom and the third of which has as its two conjuncts the same e.d. In the limiting case where $n = 1$, a single e.d. standing alone counts as a C.N.F., so that '$(P \vee Q)$' or even 'P' alone counts as a C.N.F.

Correspondingly, by a *disjunctive normal form* (D.N.F.) I understand a formula of the form

$$A_1 \vee A_2 \vee \ldots \vee A_n,$$

where $A_1, \ldots, A_n$ are elementary *conjunctions*, for n greater than or equal to 1. Thus a D.N.F. is a string of e.c.'s linked by 'v'. Any C.N.F. becomes a D.N.F. if all 'v's are changed to '&'s and all '&'s to 'v's, and vice versa. In the limiting case where $n = 1$, a single e.c. standing alone counts as a D.N.F.

A consequence of these definitions is that all e.c.'s and e.d.'s are both C.N.F.'s and D.N.F.'s. For example, consider

'$P \& -Q \& R$',

an e.c. This is a C.N.F. whose three e.d.'s are each of them the limiting case of a single atom. It is also a D.N.F. in the limiting case where the number of e.c.'s is 1. As an extreme limiting case, a single atom is both a C.N.F. and a D.N.F.

I now describe a procedure for *reducing* any wff of the propositional calculus to a C.N.F. and to a D.N.F. This will consist in finding, for any wff, a C.N.F. and a D.N.F. *equivalent* to it in an appropriate sense. The steps of the procedure will all be of the same kind, namely the replacement of a part or the whole of a given formula by an equivalent formula. To this end, we note the following biconditionals, which we shall need in our work:

(1) $P \& P \leftrightarrow P$
(2) $P \vee P \leftrightarrow P$
(3) $P \& Q \leftrightarrow Q \& P$
(4) $P \vee Q \leftrightarrow Q \vee P$
(5) $P \& (Q \& R) \leftrightarrow (P \& Q) \& R$
(6) $P \vee (Q \vee R) \leftrightarrow (P \vee Q) \vee R$
(7) $--P \leftrightarrow P$
(8) $P \rightarrow Q \leftrightarrow -P \vee Q$
(9) $(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q) \& (Q \rightarrow P)$
(10) $-(P \& Q) \leftrightarrow -P \vee -Q$
(11) $-(P \vee Q) \leftrightarrow -P \& -Q$
(12) $P \vee (Q \& R) \leftrightarrow (P \vee Q) \& (P \vee R)$
(13) $P \& (Q \vee R) \leftrightarrow (P \& Q) \vee (P \& R)$.

All of (1)–(13) are provable as theorems of the propositional calculus, and are also tautologies by a truth-table test. (Most should be familiar from exercises and results in the text.) (1) and (2) are sometimes called the *laws of idempotence* for '&' and 'v'. (3) and (4) are called the *commutative laws* for '&' and 'v'. (5) and (6) are called the *associative laws* for '&' and 'v'. (7) is the *law of double negation*. (10) and (11) are forms of *de Morgan's laws*, and (12) and (13) are the *distributive laws*.

It is in virtue of the associative laws (5) and (6) that, for the purposes of truth-table computation, we permit ourselves to write 'complex conjunctions' or 'complex disjunctions', such as '$P \& Q \& -R$'

or '$-P \vee Q \vee R$', without inner brackets. Strictly, such expressions are not wffs (see the definition in Chapter 2, Section 1), and accordingly C.N.F.'s and D.N.F.'s will not in general be well-formed. But, as far as a truth-table evaluation is concerned, it makes no difference how brackets are inserted. We may compare the situation in arithmetic with respect to '$+$', which is also associative: $(x + y) + z = x + (y + z)$, so that we may write '$5 + 3 + 2$' safely without brackets. By contrast, '$-$' is *not* associative: $5 - (3 - 2) = 4$, whilst $(5 - 3) - 2 = 0$, so that '$5 - 3 - 2$' is dangerously ambiguous. Similarly, in logic '$\rightarrow$' is not associative: '$P \rightarrow (Q \rightarrow R) \leftrightarrow (P \rightarrow Q) \rightarrow R$' is not a tautology, as the reader should confirm. (Is '$\leftrightarrow$' associative or not?)

Similarly, in virtue of the commutative laws (3) and (4), the order in which the conjuncts or disjuncts of a complex conjunction or disjunction appear will not affect the truth-table evaluation. Again, '&' and 'v' resemble '$+$' in arithmetic, for which we have $x + y = y + x$; by contrast, '$-$' is not commutative ($5 - 3 = 2$, whilst $3 - 5 = -2$), nor is '$\rightarrow$' in logic, since $P \rightarrow Q \leftrightarrow Q \rightarrow P$ is not a tautology. (Is '$\leftrightarrow$' commutative?)

Further, in virtue of the idempotence laws (1) and (2), we may safely drop reduplicated conjuncts or disjuncts in a complex conjunction or disjunction without affecting the truth-table evaluation. In this respect, '&' and 'v' *differ* from '$+$', since we clearly do not have as an arithmetical truth $x + x = x$. '$\rightarrow$' is not idempotent either. (Is '$\leftrightarrow$'?)

In reduction to normal forms, we are seeking, for a given wff, a normal form which will be *equivalent to the wff under a truth-table test*: i.e. for any given assignment of truth-values to the variables of the wff, the normal form shall have the same truth-value as the original wff, so that the biconditional of the wff and its normal form will be a tautology (compare the definition of equivalence in Chapter 3, Section 3). Hence, in virtue of (1)–(6), in the search for a normal form we allow ourselves to take brackets out of, rearrange the order of items in, and delete reduplicated items from, complex conjunctions and disjunctions, whenever such moves suit our purpose. Further, in virtue of (7), we shall allow ourselves freely to drop '$--$' whenever we wish, since such a move cannot affect the truth-table evaluation either.

In fact, all our moves in the transformation of a wff into a normal

form are of the same kind: we replace a part (sometimes the whole) of a given formula by some formula equivalent to it under a truth-table test. And the only equivalences we use are (1)–(13) or substitution-instances of them. It should be clear that no move of this kind will affect the truth-value evaluation, so that the normal form which emerges at the end of the process will be equivalent to the given wff, as is desired. But for the record we state, though do not prove, the principle of replacement which guides our work:

(R) If A is equivalent to B, and D results from C by replacing some occurrence of A in C by B, then C is equivalent to D, for any formulae A, B, C, D.

It is in fact this principle, together with (1)–(7), that justifies the moves we have already agreed to make (dropping of '$--$', rearrangement of the order of conjuncts in a conjunction, etc.).

The reductions of a wff to a C.N.F. and a D.N.F. may conveniently be divided into two stages, the first of which is common to both reductions.

Stage I. The objective here is to obtain a formula with the following three properties: (i) '$--$' nowhere appears; (ii) the only connectives that appear are '$-$', '&', or 'v'; (iii) '$-$' appears only before propositional variables, nowhere before a bracket (in fact, of course, (iii) implies (i)).

As to (i), this requirement is simply satisfied by dropping '$--$' in virtue of (7). Our first step, therefore, will be to eliminate any occurrences of '$\rightarrow$' or '$\leftrightarrow$' in the given wff. This can be effected in virtue of (8) and (9), which permit us to replace any conditional by a disjunction the first disjunct of which is the negation of its antecedent and the second disjunct of which is its consequent, and any biconditional by a certain conjunction of two conditionals, each of which in turn can be eliminated in the manner just described.

When this step is complete, we shall have a formula satisfying condition (ii). However, it may be the case that '$-$' appears in this formula outside a bracket, so that (iii) is not satisfied. Still, the main connective inside the bracket can now only be either '&' or 'v', so that the formula has as a part either something of the form

$$-(\ldots \& \ldots)$$

or something of the form

$$-(\ldots \vee \ldots).$$

If the first case arises, we can apply (10) and replace the part by a disjunction whose disjuncts are the negations of the original conjuncts in the bracket; if the second case arises, we can similarly apply (11) and replace the negated disjunction by a conjunction with negated conjuncts. As a result of these moves, it may still be the case that ' — ' appears before a bracket. But it should be clear that by *repeating* moves of this kind, we shall eventually work the ' — 's into the formula in such a way that they appear, if at all, only before variables, so that sooner or later condition (iii) will also be met, and Stage I will be complete.

The procedures for C.N.F. and for D.N.F. diverge after this point. Accordingly, we describe each separately.

Stage II (*a*) (continuation for C.N.F.). If, at the end of Stage I, we have not already obtained a C.N.F., this can be for one reason only, as the definition of C.N.F. makes plain: at one or more places there must be an occurrence of ' & ' which is subordinate to some occurrence of ' v '. For if this were not so, given conditions (i)–(iii) of Stage I, we should in fact have a C.N.F. Hence (though this may involve a rearrangement of disjuncts) some part of the given formula is of the form

$$\ldots \vee (\ldots \& \ldots).$$

By using (12), this can be replaced by a formula of the form

$$(\ldots \vee \ldots) \& (\ldots \vee \ldots),$$

in which the ' v ' has moved into subordinate position and the ' & ' into subordinating position. Of course, it may be that this new ' & ' is still subordinate to some other occurrence of ' v '; but in that case we can reapply the same procedure. Eventually, using only the equivalence (12) or its substitution-instances, we can bring all the ' & 's into subordinating positions and relegate all ' v 's into subordinate positions. The result will be a C.N.F. (How can we be sure that all the conjuncts of the C.N.F. will be *elementary* disjunctions?)

Stage II (*b*) (continuation for D.N.F.). By entirely similar considerations, we can see that, if the result of Stage I is not already a D.N.F., there must be in it at least one occurrence of ' v ' subordinate to an ' & '. Hence some part of it, perhaps after rearrangement of conjuncts, is of the form

$$\ldots \& (\ldots \vee \ldots).$$

By using (13), this can be replaced by a formula of the form

$$(\ldots \& \ldots) \vee (\ldots \& \ldots),$$

and, by repeating steps of this kind, we can bring all ' v 's into subordinating positions and all ' & 's into subordinate positions. The result will be a D.N.F.

Let us illustrate these procedures for a wff of medium complexity

(i) $S \rightarrow -((P \rightarrow Q) \rightarrow R)$.

Embarking on Stage I, we apply (8) to transform the three conditionals into disjunctions. This yields

(ii) $-S \vee -(-(-P \vee Q) \vee R)$.

This satisfies the second condition of Stage I, but there are still two ' —'s outside brackets. We apply (11) to the second disjunct of (ii) to obtain

(iii) $-S \vee (--(-P \vee Q) \& -R)$,

or, dropping a double negation,

(iv) $-S \vee ((-P \vee Q) \& -R)$.

Stage I is now clearly complete, but the result is neither a C.N.F. nor a D.N.F. We set out, therefore, on Stage II (*a*). Observe that, in (iv), the sole occurrence of ' & ' is subordinate to the main connective ' v '. We may, therefore, apply (12) to (iv) as a whole, and obtain

(v) $(-S \vee (-P \vee Q)) \& (-S \vee -R)$.

Within the first conjunct of (v) the brackets are needless by (6), and we have as a C.N.F.

(vi) $(-S \vee -P \vee Q) \& (-S \vee -R)$.

We now revert to (iv), and start Stage II (*b*). The first disjunct ' —S ' of (iv) will do, of course, as an e.c. for our desired D.N.F., and we need only concentrate on the ' v ' which is subordinate ' & '. Rearranging the conjuncts, we obtain

(vii) $-S \vee (-R \& (-P \vee Q))$,

whence, applying (13) to the second disjunct,

(viii) $-S \vee ((-R \& -P) \vee (-R \& Q))$.

This is in effect a D.N.F., and we may drop a pair of brackets to obtain

(ix) $-S \vee (-R \& -P) \vee (-R \& Q)$.

In actual practice, Stage II is usually the more formidable, though the principles involved are very simple. This is because an application of the distributive laws doubles the number of brackets, and the resulting formula may be almost twice the length of the original. Students who do normal form work must not be daunted by this, and also need to pay very close attention to the bracketing of their formulae. It should also be remembered that, in Stage II, there is often a choice as to where one begins on subordinate occurrences of 'v's or '&'s. As a result, different C.N.F.'s and D.N.F.'s for the same original wff may be obtained—there is no *unique* C.N.F. or D.N.F. for a given formula.

In order to put C.N.F.'s and D.N.F.'s to some use, we first state some obvious equivalences. Let T be *any* tautology, and I *any* inconsistency; then the following are tautologous:

(14) $T \vee P \leftrightarrow T$

(15) $T \,\&\, P \leftrightarrow P$

(16) $I \vee P \leftrightarrow P$

(17) $I \,\&\, P \leftrightarrow I$.

By (14), a disjunction with a tautologous disjunct is itself a tautology, and by (17) a conjunction with an inconsistent conjunct is itself an inconsistency. By (15), a conjunction with a tautologous conjunct is equivalent to the other conjunct (and so the former may be deleted as far as a truth-table evaluation goes), and by (16) a disjunction with an inconsistent disjunct is equivalent to the other disjunct (and so the former may be deleted as far as a truth-table evaluation goes).

It follows from (15) that (α) a complex conjunction is tautologous if and only if each of its conjuncts is tautologous, and from (16) that (β) a complex disjunction is inconsistent if and only if each of its disjuncts is inconsistent. From (α) we may infer that an *elementary* conjunction can never be tautologous, for no atom can be tautologous; similarly, no *elementary* disjunction can be inconsistent. However, we can establish

(γ) An e.d. is tautologous if and only if it has among constituent atoms a propositional variable and the negation of the same variable.

For suppose an e.d. does contain a variable, say 'P', and also '$-P$', the negation of that variable. Then, by rearranging the atoms if necessary, we can bring it into the form

$$P \vee -P \vee \ldots,$$

whence, by (14), it is tautologous. Conversely, if it lacks as atoms any variable together with the negation of the same variable, we can find an assignment of truth-values that makes each atom false (namely, for variables that appear negated, the value T, and for variables that appear non-negated, the value F), and so the whole e.d. false. Similarly, using (17), we can show

(δ) An e.c. is inconsistent if and only if it has among its constituent atoms a propositional variable and the negation of the same variable.

From (α) and (γ) we infer that a C.N.F. is tautologous if and only if all its e.d.'s are tautologous, i.e. if and only if *every e.d. in it has amongst its constituent atoms a propositional variable and the negation of the same variable.* That is to say, we can 'read off' from a C.N.F. of a wff whether it is tautologous or not. For example, from (vi) above we can infer that (i) is not tautologous, since (vi) has at least one e.d. lacking a variable and the negation of the same variable (in fact *both* e.d.'s are like that in this case).

Similarly, from (β) and (δ) we infer that a D.N.F. is inconsistent if and only if all its e.c.'s are inconsistent, i.e. if and only if *every e.c. in it has amongst its constituent atoms a propositional variable and the negation of the same variable.* For example, from (ix) above we infer that (i) is not inconsistent, since (ix) has at least one e.c. lacking a variable and the negation of the same variable. In fact, we can now conclude that (i), being neither tautologous nor inconsistent, is contingent. Considering the second conjunct of (vi), we see that the assignment $S = T$, $R = T$ makes (i) false; and considering the first disjunct of (ix), we see that the assignment $S = F$ makes (i) true. In general, for any wff, we can tell very simply from any C.N.F. equivalent to it whether it is tautologous or not and from any D.N.F. equivalent to it whether it is inconsistent or not, so that reduction to C.N.F. and D.N.F. provides a test as to whether a wff is tautologous, contingent, or inconsistent which is independent of the truth-table test. Reduction to normal form may save us the labour of a truth-table test—at the cost of labour of a different kind.

A certain interest attaches to normal forms of a special kind, which are sometimes called *canonical*. A *canonical conjunctive normal form* (C.C.N.F.) is a C.N.F. in which every propositional variable that occurs (negated or unnegated) in some e.d. occurs (negated or unnegated) in all e.d.'s in the C.N.F. Correspondingly, a *canonical disjunctive normal form* (C.D.N.F.) is a D.N.F. in which every propositional variable that occurs (negated or unnegated) in some e.c. occurs (negated or unnegated) in all e.c.'s in the D.N.F. For example (vi) above is not canonical, since 'R' appears in the second e.d. but not in the first, and (ix) is not canonical, since 'S' appears in neither the second nor the third e.c. On the other hand

$$`(P \vee -Q \vee -R) \,\&\, (Q \vee -P \vee R)'$$

is a C.C.N.F., and the interchange of '&' and 'v' in it yields a C.D.N.F.

In order to obtain canonical normal forms from non-canonical ones, we note first the two equivalences

(18) $P \longleftrightarrow (P \vee Q) \,\&\, (P \vee -Q)$

(19) $P \longleftrightarrow (P \,\&\, Q) \vee (P \,\&\, -Q)$.

If, now, a given C.N.F. is not a C.C.N.F., there must be some variable occurring in some e.d. in the given form which does not occur in all e.d.'s in the form. We may use (18) to replace any e.d. lacking a given variable by a pair of e.d.'s each containing that variable as well as the other atoms of the given e.d.; in one member of the pair it appears unnegated, and negated in the other. For example, wishing to transform (vi) into a C.C.N.F., we should replace the first conjunct by

$$`(-S \vee -P \vee Q \vee R) \,\&\, (-S \vee -P \vee Q \vee -R)',$$

to which it is equivalent by (18), thus obtaining two e.d.'s in which all four variables of (vi) appear. (There would then remain the task of transforming the second conjunct of (vi) into e.d.'s containing all four variables.) Thus repeated use of (18) will transform a C.N.F. that is not canonical into an equivalent C.C.N.F. In an entirely similar way, we can, using (19), transform any D.N.F. that is not canonical into a C.D.N.F.

Given a canonical normal form, we may simplify it by (i) deleting any repetitions of atoms occurring in any e.d. or e.c. in the form, (ii) deleting any repetitions of e.d.'s or e.c.'s in the form, (iii) in the case of a C.N.F., deleting any tautologous e.d.'s in virtue of (15), and, in the case of a D.N.F., deleting any inconsistent e.c.'s in virtue of (16). In virtue of (iii), the normal form may vanish altogether, and will do if the C.N.F. is tautologous or the D.N.F. inconsistent. In that case, we agree to write '$P \vee -P$' and '$P \,\&\, -P$' respectively. Let us call the result of these manoeuvres a *distinguished* (conjunctive or disjunctive) normal form. Then it can be shown that, for each wff, its distinguished (conjunctive or disjunctive) normal form is *unique*, apart from variations in the order of atoms in e.d.'s or e.c.'s and in the order of the e.d.'s or e.c.'s themselves. Moreover, these forms bear a close relation to the truth-table for the given wff, in that the truth-table can be read off from either of them and they can be read off from it.

This is perhaps best shown by an example. Let us suppose a wff A contains the three variables 'P', 'Q', 'R', and that when subjected to a truth-table it yields the following column under its main connective:

P	*Q*	*R*	A
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Selecting the assignments for which A comes out true, we may write down corresponding e.c.'s in which a variable appears unnegated if it takes the value T in the assignment and negated if it takes the value F. Thus, corresponding to lines 1, 4, and 6 of the above table, we have

$$`(P \,\&\, Q \,\&\, R)'$$
$$`(P \,\&\, -Q \,\&\, -R)'$$
$$`(-P \,\&\, Q \,\&\, -R)'.$$

Forming the disjunction of these three, we obtain a distinguished D.N.F. which, it is fairly evident, is equivalent to the given wff A. Conversely, given a distinguished D.N.F. for a certain wff, each e.c. in it determines an assignment of values to its variables for which the wff is true, and the truth-table for it can be written down. Thus the distinguished D.N.F. of a wff embodies in symbolic form the outcome of a truth-table test on that wff.

Similar remarks can be made about the distinguished C.N.F. Pursuing the above example, corresponding to each assignment for which A comes out false, we may write down e.d.'s in which a variable appears *negated* if it takes the value T in the assignment and *unnegated* if it takes the value F. Thus, corresponding to lines 2, 3, 5, 7, and 8 of the test, we obtain

'$-P \vee -Q \vee R$'
'$-P \vee Q \vee -R$'
'$P \vee -Q \vee -R$'
'$P \vee Q \vee -R$'
'$P \vee Q \vee R$'.

Forming the conjunction of these, we obtain a distinguished C.N.F. which is equivalent to the given wff (the reader should test this for himself). Conversely, given a distinguished C.N.F. for a certain wff, each e.d. in it determines an assignment of values to its variables for which the wff is *false* and the truth-table for it can be written down. The distinguished C.N.F., like the distinguished D.N.F., is the symbolic embodiment of a truth-table test.

It is only fair to point out to students that, if they are simply called upon to obtain a normal form equivalent to a given wff, the *quickest* way is usually to perform a truth-table test and read off a normal form from that; the only merit possessed by the burdensome procedures for obtaining normal forms described in this appendix is that they are *independent* of such a test.

APPENDIX B

The Elementary Theory of Classes

This appendix is intended to give readers a foretaste of what they are likely to find if they pursue logic beyond the confines of this book. Not that anything in it is particularly difficult: indeed, in its first stages the theory of classes is no harder than, and bears a close resemblance to, the propositional calculus, as we shall see. But in its upper stages it raises problems of great interest concerning the foundations of mathematics which, at the date of writing, remain unsolved.

It is not possible to give a precise definition of what a class is. Intuitively, a class is a collection of entities of any kind, and we come to know classes typically in one of two ways: either we are given a *list* of their members, or we are given a *condition* for membership of the class. Let us consider these two ways in turn.

Given a list of things, say Tom, Dick, and Harry, we may consider the class which has just those things as members. Let us agree to name this class by enclosing the names in the list in curly brackets. Thus

'{Tom, Dick, Harry}'

shall be our name for the class whose members are just Tom, Dick, and Harry. Similarly {0,1,2,3,4,5,6,7,8,9} shall be the class whose members are the first ten natural numbers. It is normal to abbreviate 'is a member of' to the Greek letter 'ε'. Then true propositions will be

Dick ε {Tom, Dick, Harry}
$3 \in \{0,1,2,3,4,5\}$
$(3 + 3) \in \{6,7,8,9\}$.

We more commonly, however, determine classes by stating conditions for membership of them: thus we speak of the class of inhabitants of London, the class of even numbers, the class of female chiropodists, etc. We use this device for describing classes