Notes on the Completeness Theorem

Lemma (Weakening). Let Γ, Δ be finite sets of sentences. If $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash A$.

Proof: Given a proof of A from the sentences in Γ, we can construct a proof of A from the sentences in Δ by inserting the appropriate assumptions, and then using &I and &E repeatedly. □

Definition: For each valuation v and sentence A, let A^v be A if $v(A) = T$, and let A^v be $-A$ if $v(A) = F$.

Definition: For each valuation v and wff A, let

$$\Gamma(v, A) = \{B^v : B \text{ is an atomic sentence occurring in } A\}.$$

Lemma. For each wff A, and valuation v, we have $\Gamma(v, A) \vdash A^v$.

Proof: We need to show that for every wff A,

$$(v)[\Gamma(v, A) \vdash A^v]. \tag{1}$$

We prove this by induction on the construction of wffs.

<u>Base Case (Atomic Sentence):</u> We need to show that if A is an atomic sentence then Formula (1) holds for A. But if A is an atomic sentence then $\Gamma(v, A) = \{A^v\}$, and we have $\{A^v\} \vdash A^v$ by the Rule of Assumptions.

<u>Inductive Case ($-$):</u> We must establish the following conditional: If Formula (1) holds for A then it also holds for $-A$.

Suppose that Formula (1) holds for A. Let v be an arbitrary valuation. If $v(-A) = T$ then $v(A) = F$ and the induction hypothesis yields $\Gamma(v, A) \vdash -A$. But $\Gamma(v, -A) = \Gamma(v, A)$, and so $\Gamma(v, -A) \vdash (-A)^v$. If $v(-A) = F$ then $v(A) = T$ and the induction hypothesis yields $\Gamma(v, A) \vdash A$. Applying the inference rule DN, we obtain $\Gamma(v, A) \vdash - - A$, and so $\Gamma(v, -A) \vdash (-A)^v$.

<u>Inductive Case (&):</u> We must establish the following conditional: If Formula (1) holds for A and B, then it also holds for $A \& B$. For this, note that $\Gamma(v, A \& B) = \Gamma(v, A) \cup \Gamma(v, B)$.

Suppose that Formula (1) holds for A and B. Let v be a valuation. Then either $v(A \& B) = T$ or $v(A \& B) = F$. We consider these two cases in turn. If $v(A \& B) = T$ then $v(A) = T =$

$v(B)$, and so $A^v = A$, $B^v = B$, and $(A\&B)^v = A\&B$. Since Formula (1) holds for A and B, we have

$$\Gamma(v, A) \vdash A, \qquad \Gamma(v, B) \vdash B.$$

By &-Introduction, we obtain

$$\Gamma(v, A) \cup \Gamma(v, B) \vdash A\&B,$$

and replacing with equalities, we obtain

$$\Gamma(v, A\&B) \vdash (A\&B)^v.$$

If $v(A\&B) = F$, then either $v(A) = F$ or $v(B) = F$. If $v(A) = F$, then $A^v = -A$, and since Formula (1) holds for A we have

$$\Gamma(v, A) \vdash -A.$$

Then $\vee$-Introduction gives

$$\Gamma(v, A) \vdash -A \vee -B,$$

and SI(De Morgan's) gives

$$\Gamma(v, A) \vdash -(A\&B).$$

By Weakening, $\Gamma(v, A\&B) \vdash (A\&B)^v$. A similar argument shows that if $v(B) = F$ then $\Gamma(v, A\&B) \vdash (A\&B)^v$. Therefore, in both cases [when $v(A\&B) = T$ and when $v(A\&B) = F$], $\Gamma(v, A\&B) \vdash (A\&B)^v$, i.e. Formula (1) holds for $A\&B$.

<u>Inductive Case ($\vee$):</u> We need to establish the conditional: If Formula (1) holds for A, B, then it also holds for $A \vee B$.

Let v be a valuation. Then either $v(A \vee B) = T$ or $v(A \vee B) = F$. In the former case, either $v(A) = T$ or $v(B) = T$. If $v(A) = T$, then the inductive hypothesis yields

$$\Gamma(v, A) \vdash A,$$

weakening yields

$$\Gamma(v, A \vee B) \vdash A,$$

and $\vee$-Introduction yields

$$\Gamma(v, A \vee B) \vdash A \vee B.$$

A similar argument shows that if $v(B) = T$, then

$$\Gamma(v, A \vee B) \vdash A \vee B.$$

If $v(A \vee B) = F$ then $v(A) = F$ and $v(B) = F$. The inductive hypothesis then yields

$$\Gamma(v, A) \vdash -A, \qquad \Gamma(v, B) \vdash -B.$$

By &-Introduction, we obtain

$$\Gamma(v, A) \cup \Gamma(v, B) \vdash -A \& -B,$$

and by SI(De Morgan's), we obtain

$$\Gamma(v, A) \cup \Gamma(v, B) \vdash -(A \vee B).$$

Since $\Gamma(v, A \vee B) = \Gamma(v, A) \cup \Gamma(v, B)$ and $v(A \vee B) = F$, we have

$$\Gamma(v, A \vee B) \vdash (A \vee B)^v$$

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<u>Inductive Case ($\rightarrow$):</u> We need to establish the conditional: If Formula (1) holds for A and B then it also holds for $A \rightarrow B$.

Suppose that Formula (1) holds for A and B. Let v be a valuation. If $v(A \rightarrow B) = T$ then either $v(A) = F$ or $v(B) = T$. If $v(A) = F$ then the induction hypothesis yields $\Gamma(v, A) \vdash -A$, SI(Negative Paradox) yields $\Gamma(v, A) \vdash A \rightarrow B$, and the Weakening Lemma yields $\Gamma(v, A \rightarrow B) \vdash A \rightarrow B$. If $v(B) = T$ then the induction hypothesis yields $\Gamma(v, B) \vdash B$, SI(Positive Paradox) yields $\Gamma(v, B) \vdash A \rightarrow B$, and the Weakening Lemma yields $\Gamma(v, A \rightarrow B) \vdash A \rightarrow B$.

If $v(A \rightarrow B) = F$ then $v(A) = T$ and $v(B) = F$. Then the induction hypothesis yields

$$\Gamma(v, A) \vdash A, \qquad \Gamma(v, B) \vdash -B,$$

and &-Introduction yields

$$\Gamma(v, A \rightarrow B) \vdash A \& -B.$$

By SI(Material Implication),

$$\Gamma(v, A \rightarrow B) \vdash -(A \rightarrow B).$$

□

Definition: If v is a valuation and A is a wff, we let $C(v, A)$ denote the conjunction of all sentences in $\Gamma(v, A)$.

For example, if $A = P \rightarrow (-Q \vee R)$ and v is the valuation such that $v(P) = v(Q) = T$ and $v(R) = F$, then

$$C(v, A) = P \& Q \& -R.$$

Note that a previous lemma shows that

$$\Gamma(v, A) \vdash A^v,$$

for any valuation v, and wff A. Using &-Elimination, it follows that

$$C(v, A) \vdash A^v,$$

for any valuation v, and wff A.

Theorem (Weak Completeness). If A is a tautology then $\vdash A$.

Proof. Let $P_1, \ldots, P_n$ be the atomic sentences that occur in A. We show first, using induction on n, that the sentence

$$(P_1 \& \cdots \& P_n) \vee (P_1 \& \cdots \& - P_n) \vee \cdots \vee (-P_1 \& \cdots \& - P_n) \tag{2}$$

can be proven without any dependencies.

Base Case ($n = 1$): This is just the tautology $P_1 \vee -P_1$.

Inductive Case: We show that if the result is true for n, then it is true for $n + 1$. So, suppose that a proof is given of Sentence (2). Now we know that we can obtain a proof of $P_{n+1} \vee -P_{n+1}$. So, using &-Introduction, we have a proof

$$[(P_1 \& \cdots \& P_n) \vee (P_1 \& \cdots \& - P_n) \vee \cdots \vee (-P_1 \& \cdots \& - P_n)] \& (P_{n+1} \vee -P_{n+1}).$$

Using SI(Distribution), we obtain the result.

Let A be an arbitrary tautology, and let v be a valuation. Then $A^v = A$, and by Lemma, $\Gamma(v, A) \vdash A$. Using &-Elimination if necessary, it follows that $C(v, A) \vdash A$. By Lemma X,

$$\vdash C(v_1, A) \vee \cdots \vee C(v_{2^n}, A).$$

Thus, by $\vee$-Elimination, $\vdash A$. □

Theorem (Completeness). If $A_1, \ldots, A_n \models B$ then $A_1, \ldots, A_n \vdash B$.

Proof: Suppose that $A_1, \ldots, A_n \models B$. By truth tables, $\models (A_1 \& \cdots \& A_n) \rightarrow B$. By Weak Completeness, $\vdash (A_1 \& \cdots \& A_n) \rightarrow B$. Thus, if $A_1, \ldots, A_n$ occur as assumptions on lines 1 through n, then &-Introduction yields $A_1 \& \cdots \& A_n$ depending on $1, \ldots, n$. By SI, we can insert $(A_1 \& \cdots \& A_n) \rightarrow B$ with no dependencies, and then MPP yields B, depending on $1, \ldots, n$. That is, $A_1, \ldots, A_n \vdash B$. □