Homework 7 Solutions

A. For the interpretation:

$X = \{1, 2, 3, 4, 5\}$
$\underline{\text{Ref}}(m) = 2 \qquad \underline{\text{Ref}}(n) = 4$
$\underline{\text{Ext}}(Fx) = \{1, 2, 3\} \qquad \underline{\text{Ext}}(Gx) = \{4\} \qquad \underline{\text{Ext}}(Hx) = \{5\}$

1. $(x)(-Fx \vee -Gx)$: This sentence is true relative to this interpretation, since the elements 1, 2, 3, and 5 are not in Ext(Gx) and element 4 is not in $\underline{\text{Ext}}(Fx)$ (thus, for all x, x is either not in $\underline{\text{Ext}}(Fx)$ or not in $\underline{\text{Ext}}(Gx)$).

2. $(x)((Gx\&Hx) \rightarrow Hx)$: This sentence is true relative to this interpretation, since there is no element in the domain that is in both $\underline{\text{Ext}}(Fx)$ and $\underline{\text{Ext}}(Gx)$, making the antecedent of the conditional false. Thus, the entire conditional is true.

3. $-(x)Fx \rightarrow (\exists x)(Fx\&Gx)$: This sentence is false relative to this interpretation. The sentence $-(x)Fx$ is true relative to this interpretation, since the elements 4 and 5 are not members of $\underline{\text{Ext}}(Fx)$, and thus not every element of the domain is a member of $\underline{\text{Ext}}(Fx)$. The consequent of the conditional is false relative to this interpretation, since there are no elements that are members of both $\underline{\text{Ext}}(Fx)$ and $\underline{\text{Ext}}(Gx)$. Finally, since the antecedent of the conditional is true and the consequent false, the conditional is false.

4. $-Gm\&(\exists x)(Fx\&Gn)$: This sentence is true relative to this interpretation. The first conjunct is true relative to this interpretation because the element $\underline{\text{Ref}}(m)$, 2, is not in $\underline{\text{Ext}}(Gx)$. The second conjunct is also true, because the element $\underline{\text{Ref}}(n)$, 4, is in $\underline{\text{Ext}}(Gx)$ and the elements 1, 2, and 3 are all in $\underline{\text{Ext}}(Fx)$.

5. $(\exists x)(-Fx\&-Gx)$: This sentence is true relative to this interpretation, since the element 5 is not in $\underline{\text{Ext}}(Fx)$ or in $\underline{\text{Ext}}(Gx)$.

6. $(\exists x)(Fx \rightarrow Gx)$: This sentence is true relative to this interpretation, since the elements 4 and 5 are not in $\underline{\text{Ext}}(Fx)$, making the antecedent of the conditional false, and the conditional as a whole true.

B.

1. $(x)(Fx \rightarrow Hx) \vdash (x)((Fx \vee Gx) \rightarrow Hx)$

   To see if this argument is valid, we test for the consistency of the premise and the negation of the conclusion:

   $$-(x)((Fx \vee Gx) \rightarrow Hx) = (\exists x) - ((Fx \vee Gx) \rightarrow Hx)$$

   So we have:

   $$(x)(Fx \rightarrow Hx), \quad (\exists x) - ((Fx \vee Gx) \rightarrow Hx)$$

   Using Algorithm B, we get the following instances:

   $$-((Fa \vee Ga) \rightarrow Ha), \quad Fa \rightarrow Ha.$$

   Applying Algorithm A to these instances, we construct the following truth assignment v:

   $$v(Fa) = \text{F}, \quad v(Ga) = \text{T}, \quad v(Ha) = \text{F}.$$

   Build an interpretation:

   $$X = \{1\}, \quad \underline{\text{Ext}}(Fx) = \emptyset, \quad \underline{\text{Ext}}(Gx) = \{1\}, \quad \underline{\text{Ext}}(Hx) = \emptyset.$$

   On this interpretation, the premise is true and the conclusion is false. So, the original argument is invalid.

2. $(\exists x)(Fx \rightarrow Gx), (\exists x)Fx \vee (\exists x)Gx \vdash (\exists x)Fx \rightarrow (\exists x)Gx$

   If we let

   $$P = (\exists x)(Fx \rightarrow Gx), \quad Q = (\exists x)Fx, \quad R = (\exists x)Gx,$$

then the argument takes the form:

$$P, Q \vee R \vdash Q \rightarrow R$$

A counterexample to this form is given by the truth assignment:

$$v(P) = \mathrm{T}, \quad v(Q) = \mathrm{T}, \quad v(R) = \mathrm{F}.$$

Thus, the original argument is invalid if there is an interpretation that satisfies the following sentences:

$$(\exists x)(Fx \rightarrow Gx), \quad (\exists x)Fx, \quad -(\exists x)Gx = (x) - Gx.$$

To search for an interpretation that satisfies these sentences, we use Algorithm B. It yields the following instances:

$$Fa \rightarrow Ga, \quad Fb, \quad -Ga, \quad -Gb.$$

These instances are consistent, as witnessed by the truth assignment:

$$v(Fa) = \mathrm{F}, \quad v(Fb) = \mathrm{T}, \quad v(Ga) = \mathrm{F}, \quad v(Gb) = \mathrm{F}.$$

Algorithm B tells us that the original argument is invalid, and that a witnessing interpretation can be read off of this truth assignment, namely:

$$X = \{1, 2\}, \quad \underline{\mathrm{Ext}}(Fx) = \{2\}, \quad \underline{\mathrm{Ext}}(Gx) = \emptyset.$$

3. $(x)Fx \leftrightarrow (x)Gx \vdash (x)(Fx \leftrightarrow Gx)$

Short answer: Relative to the interpretation

$$X = \{1, 2\}, \quad \underline{\mathrm{Ext}}(Fx) = \{1\}, \quad \underline{\mathrm{Ext}}(Gx) = \{2\},$$

the premise is true and the conclusion is false. Hence, the argument is invalid.

Long answer: To see if this argument is valid, we test for the consistency of the premise and the negation of the conclusion:

$$-(x)(Fx \leftrightarrow Gx) = (\exists x) - (Fx \leftrightarrow Gx)$$

In order to apply Algorithm C, we assign elementary sentence letters to the simple monadic sentences:

$$P = (x)Fx, \qquad Q = (x)Gx, \qquad R = (\exists x) - (Fx \leftrightarrow Gx)$$

and we have to test the consistency of:

$$(P \leftrightarrow Q)\&R. \tag{1}$$

Transform the latter sentence into DNF:

$$(R\& - P\& - Q) \vee (R\&Q\&P).$$

We test the first disjunct for consistency. Plugging back in the quantified sentences (from equation 1), and moving negations inside quantifiers, we get:

$$(\exists x) - (Fx \leftrightarrow Gx), \quad (\exists x) - Fx, \quad (\exists x) - Gx.$$

Algorithm B yields the instances:

$$-(Fa \leftrightarrow Ga), \quad -Fb, \quad -Gc.$$

The following truth assignment satisfies these sentences simultaneously:

$$v(Fa) = \mathrm{T}, \quad v(Ga) = \mathrm{F}, \quad v(Fb) = \mathrm{F}, \quad v(Gc) = \mathrm{F}.$$

So we build an interpretation:

$$X = \{1, 2, 3\}, \quad \underline{\mathrm{Ext}}(Fx) = \{1\}, \quad \underline{\mathrm{Ext}}(Gx) = \emptyset.$$

Since one of the disjuncts is consistent, the entire sentence is also consistent. Thus, the premise and the negation of the conclusion are consistent, so the original argument is invalid.

C.

1. Invalid. For example: DoQ $= \{1\}, \underline{\text{Ext}}(Fx) = \emptyset, \underline{\text{Ext}}(Gx) = \{1\}, \underline{\text{Ext}}(Hx) = \{1\}$.
2. Valid. [Either give a proof, go through the steps of Algorithm C, or give an informal argument (as in Budolfson's handout).]
3. Valid. [Either give a proof, go through the steps of Algorithm C, or give an informal argument (as in Budolfson's handout).]