Notes on the Soundness Theorem

Definition: A (propositional logic) *valuation* is an assignment of truth values to sentences that obeys the functional relationships given by the truth tables.

Definition: Let Γ be a set of sentences, and let v be a valuation. We say that v *satisfies* Γ just in case $(A)(A \in \Gamma \Rightarrow v(A) = T)$. In words: v assigns T to all sentences in Γ.

Definition: Let Γ be a set of sentences, and let A be a sentence. Then Γ semantically implies A, written $\Gamma \models A$ just in case any valuation that satisfies Γ also assigns T to A. That is, $(v)(v \text{ satisfies } \Gamma \Rightarrow v(A) = T)$.

Lemma (Expansion): If $\Delta \models A$, and $\Delta \subseteq \Gamma$, then $\Gamma \models A$.

(*Note: The following proof mixes formal — Lemmon style proof — and informal methods.*)

1	(1) $\Delta \models A$	PREMISE
2	(2) $\Delta \subseteq \Gamma$	PREMISE
3	(3) v satisfies Γ	Assumption
2	(4) $(B)(B \in \Delta \Rightarrow B \in \Gamma)$	2 Defn of $\subseteq$
5	(5) $B \in \Delta$	Assumption
2	(6) $B \in \Delta \Rightarrow B \in \Gamma$	4 UE
2,5	(7) $B \in \Gamma$	7,6 MPP
3	(8) $(B)(B \in \Gamma \Rightarrow v(B) = T)$	3 Defn of "satisfies"
3	(9) $B \in \Gamma \Rightarrow v(B) = T$	8 UE
2,3,5	(10) $v(B) = T$	9,7 MPP
2,3	(11) $B \in \Delta \Rightarrow v(B) = T$	3,10 CP
2,3	(12) $(B)(B \in \Delta \Rightarrow v(B) = T$	11 UI
2,3	(13) v satisfies Δ	12 Defn of "satisfies"
1,2,3	(14) $v(A) = T$	12,1 Defn of $\models$
1,2	(15) v satisfies $\Gamma \Rightarrow v(A) = T$	3,14 CP
1,2	(16) $(v)(v \text{ satisfies } \Gamma \Rightarrow v(A) = T)$	15 UI
1,2	(17) $\Gamma \models A$	16 Defn of $\models$

Lemma (Transitivity): If $\Gamma \models A$ for all A in Δ, and $\Delta \models B$, then $\Gamma \models B$.

1	(1) $(A)[A \in \Delta \Rightarrow \Gamma \text{ implies } A]$	PREMISE
2	(2) Δ implies B	PREMISE
3	(3) Suppose v satisfies Γ.	Assumption
1	(4) $A \in \Delta \Rightarrow \Gamma$ implies A	1 UE
5	(5) $A \in \Delta$	Assumption
1,5	(6) Γ implies A	4,5 MPP
1,3,5	(7) $v(A) = T$	3,6 Defn of "implies"

1,3	(8) $A \in \Delta \Rightarrow v(A) = T$	5,7 CP
1,3	(9) $(A)(A \in \Delta \Rightarrow v(A) = T)$	8 UI (Note: A doesn't occur free in 1,3)
1,3	(10) v satisfies Δ	9 Defn of "satisfies"
1,2,3	(11) $v(B) = T$	2,10 Defn of "implies"
1,2	(12) v satisfies $\Gamma \Rightarrow v(B) = T$	3,11 CP
1,2	(13) $(v)[v$ satisfies $\Gamma \Rightarrow v(B) = T]$	12 UI (Note: v doesn't occur free in 1,2)
1,2	(14) Γ implies B	13 Defn of "implies"

Lemma (ST1): Let A_i, A_j, A_k be the sentences that occur on lines i, j, k of a proof. If line k results from lines i, j by application of a Stage 1 rule of inference, then $\{A_i, A_j\} \models A_k$.

Proof: Examine the truth tables for the connectives. □

Theorem (Soundness). Let

$D(n) \quad (n)\ A_n$

be a line of a correctly written proof. Then $\underline{D}(n) \models A_n$, where $\underline{D}(n)$ is the set of formulas on the dependency lines $D(n)$.

Proof: We use induction on the construction of proof lines. So, we have one base case (Rule of Assumptions), and inductive cases corresponding to each of the other inference rules.

Base Case: The rule of assumptions yields lines of the form:

$n \quad (n)\ A$ Assumption

In this case we have $\underline{D}(n) = \{A\}$, and so we need only note that $\{A\} \models A$.

Inductive Step (Stage 1 Rules): Suppose that line k results from lines i, j via some Stage 1 rule, and suppose that lines i and j are good. We need to show that line k is good.

Saying that lines i and j are good means $\underline{D}(i) \models A_i$ and $\underline{D}(j) \models A_j$. By the Expansion Lemma, $\underline{D}(i) \cup \underline{D}(j) \models A_i$ and $\underline{D}(i) \cup \underline{D}(j) \models A_j$. By the Stage 1 Lemma, $\{A_i, A_j\} \models A_k$. Thus, by the Transitivity Lemma, $\underline{D}(i) \cup \underline{D}(j) \models A_k$. However, $\underline{D}(k) = \underline{D}(i) \cup \underline{D}(j)$, since Stage 1 rules aggregate dependency numbers. Therefore, $\underline{D}(k) \models A_k$, which means that line k is good.

Inductive Step (CP): Suppose that line k results from lines i, j via CP, and suppose that lines i and j are good. We need to show that line k is good.

For CP to be applicable, it must be the case that line i is an assumption; so, $\underline{D}(i) = \{A_i\}$, where A_i is the sentence occurring on line i. It must also be the case that $A_k = A_i \rightarrow A_j$.

Furthermore, it must be the case that $\underline{D}(k) = \underline{D}(j) - \underline{D}(i)$. Now let v be a valuation that satisfies $\underline{D}(k)$. Then either $v(A_i) = F$ or $v(A_i) = T$. We consider these two cases in turn, and show that in each case, $v(A_i \rightarrow A_j) = T$.

If $v(A_i) = F$, then truth tables immediately yields $v(A_i \rightarrow A_j) = T$. If $v(A_i) = T$, then v satisfies $\underline{D}(j)$. Indeed, we assumed that v satisfies $\underline{D}(k) = \underline{D}(j) - \underline{D}(i) = \underline{D}(j) - \{A_i\}$. But now we also know that $v(A_i) = T$, and so v makes all the sentences in $\underline{D}(j)$ true. But then since line j is good, $\underline{D}(j) \models A_j$, and so $v(A_j) = T$. Therefore by truth tables, $v(A_i \rightarrow A_j) = T$. So, any valuation v that satisfies $\underline{D}(k)$ also assigns T to A_k, which means that line k is good.

Inductive Step (RAA): Suppose that line k results from lines i and j via RAA, and suppose that lines i and j are good. We need to show that line k is good.

For RAA to be applicable, it must be the case that line i is an assumption, that A_j is a contradiction, that $\underline{D}(k) = \underline{D}(j) - \{A_i\}$, and that $A_k = -A_i$. Since line j is assumed to be good, we have $\underline{D}(j) \models A_j$, where A_j is a sentence which is assigned false by all valuations. Thus, $\underline{D}(j)$ is not satisfied by any valuation.

Let v be a valuation that satisfies $\underline{D}(k)$. If $v(A_i) = T$ then v satisfies $\underline{D}(j)$, which is impossible. So, $v(A_i) = F$, and $v(A_k) = v(-A_i) = T$. Therefore $\underline{D}(k) \models A_k$, which means that line k is good.

Inductive Step ($\vee$-Elimination): Homework Assignment.

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