The predicate calculus is complete*

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The first thing we need to do is to precisify the inference rules UI and EE. To this end, we will use $A(c)$ to denote a sentence containing the name c, and we will use $\forall v A(v)$ to denote the sentence that results from replacing *all* occurrences of c in $A(c)$ with v and then putting the quantifier $\forall v$ on the front. (We also require that the variable v did not already occur in $A(c)$.) We use a similar convention for $\exists v A(v)$.

Universal Introduction (UI)

$$\frac{\Gamma \vdash A(c)}{\Gamma \vdash \forall v A(v)}$$

Provided that "c" does not occur in Γ.

Existential Elimination (EE)

$$\frac{\Gamma \vdash \exists v A(v) \qquad \Delta, A(c) \vdash B}{\Gamma, \Delta \vdash B}$$

Provided that "c" does not occur in Γ, Δ or B.

One would like to know that these rules are sound, i.e., they take good proofs to good proofs. Let's show soundness for UI: Suppose that $\Gamma \vdash A(c)$ is good, i.e. $\Gamma \models A(c)$. Now let $\mathcal{M}$ be an interpretation that makes all sentences in Γ true. We claim that $\mathcal{M}$ makes $\forall v A(v)$ true. Indeed, for any element a in the domain of $\mathcal{M}$, we could modify the interpretation $\mathcal{M}$ by assigning the name c to the object a. Call this modified interpretation $\mathcal{M}'$. Then $\mathcal{M}'$ still satisfies Γ since the name c doesn't occur in any of the sentences in Γ. Since $\Gamma \models A(c)$, the sentence $A(c)$ is true in $\mathcal{M}'$; that is, $a \in \mathrm{Ext}_{\mathcal{M}'}(A(v))$. Since c doesn't occur in $A(v)$, $\mathrm{Ext}_{\mathcal{M}}(A(v)) = \mathrm{Ext}_{\mathcal{M}'}(A(v))$. Therefore, $a \in \mathrm{Ext}_{\mathcal{M}}(A(v))$. Since a was an arbitrary element of the domain of $\mathcal{M}$, $\forall v A(v)$ is true in $\mathcal{M}$. Since $\mathcal{M}$ was an arbitrary interpretation satisfying Γ, it follows that $\Gamma \models \forall v A(v)$.

Exercise. Show that the rule EE is sound.

*Version 0.2, revised May 3, 2012. Please send suggestions and corrections to: hhalvors@princeton.edu

The second thing we need to do is to precisify the notion of a *sentence* of predicate logic. Suppose that we have a fixed family $\mathcal{R}$ of predicate symbols, and a fixed family $\mathcal{C}$ of constant symbols. Then we define the set Σ of sentences inductively as follows:

1. Base case: If R is an n-ary relation symbol from $\mathcal{R}$, and $c_1, \dots, c_n$ are constant symbols from $\mathcal{C}$, then $Rc_1, \dots, c_n$ is a sentence.

2. Inductive cases $(\wedge, \vee, \rightarrow, \neg)$ If A and B are sentences, then so are $A \wedge B$, $A \vee B$, $A \rightarrow B$, and $\neg A$.

3. Inductive case $(\forall, \exists)$ If $A(c)$ is a sentence that contains a constant symbol c, and if v does not occur in $A(c)$, then $\forall v A(v)$ and $\exists v A(v)$ are sentences.

OK, now we are ready to start the proof of the completeness theorem.

Theorem (Completeness of the Predicate Calculus)**.** *Let $A_1, \dots, A_n, B$ be arbitrary predicate logic sentences. If $A_1, \dots, A_n \models B$ then $A_1, \dots, A_n \vdash B$.*

First, as in the case of propositional logic, it would suffice to prove the following main lemma:

Main Lemma. *Let A be a predicate logic sentence. If $A \nvdash P \wedge \neg P$, then A is consistent, i.e. there is an interpretation that makes A true.*

How then should we prove this Main Lemma? In propositional logic, we first showed that the sentence A is provably equivalent to a sentence A^d in disjunctive normal form; then we proved the Main Lemma for sentences in disjunctive normal form. For predicate logic, not every sentence is equivalent to one in disjunctive normal form; but there is another nice, manageable form called "prenex normal form."

Definition. We say that a sentence is in *prenex normal form* just in case all of its quantifiers occur outside of the scope of its truth-functional connectives. (If such a sentence has any quantifiers, then they occur up front, and have scope to the end of the sentence.)

We are going to show that every sentence is equivalent to one in prenex normal form. For this, let's recall the ways that we can move quantifiers out to the front of a sentence.

Lemma (Swoosh Equivalences)**.** *The following equivalences are provable in our system.*

$$\vdash Q_1xFx \wedge Q_2yGy \leftrightarrow Q_1xQ_2y(Fx \wedge Gy),$$

$$\vdash Q_1xFx \vee Q_2yGy \leftrightarrow Q_1xQ_2y(Fx \vee Gy),$$

$$\vdash Q_1xFx \rightarrow Q_2yGy \leftrightarrow \overline{Q}_1xQ_2y(Fx \rightarrow Gy),$$

where Q_1 and Q_2 are arbitrary quantifiers (either $\forall$ or $\exists$), and $\overline{Q}$ is the opposite of the quantifier Q.

Proof. These equivalences are standard proofs in predicate logic. If you haven't seen them before, then it would be a good exercise to prove them yourself. □

Lemma (Prenex Normal Form Lemma)**.** *Every sentence A is provably equivalent to a sentence in prenex normal form.*

Proof. By induction on the construction of sentences.

Base case: A sentence of the form $Rc_1, \ldots, c_n$ is already in prenex normal form.

Inductive case $\neg$: Use the quantifier-negation equivalences.

Inductive cases $\wedge, \vee, \rightarrow$: The inductive cases for all of the binary connectives depend on the fact that for any two sentences A and B, there is a sentence B' provably equivalent to B such that B' shares no variables in common with A; and the (provable) swoosh equivalences allow us to bring quantifiers to the front of $A \circ B'$, where $\circ$ is either $\wedge, \vee$ or $\rightarrow$. For example, if

$$A \;=\; Q_1 x_1 \cdots Q_n x_n F(x_1, \ldots, x_n),$$

and

$$B' \;=\; Q_{n+1} x_{n+1} \cdots Q_{n+m} x_{n+m} G(x_{n+1}, \ldots, x_{n+m}),$$

where the variables $x_1, \ldots, x_{n+m}$ are all distinct, then

$$A \wedge B \;\equiv\; A \wedge B' \;\equiv\; Q_1 x_1 \cdots Q_n x_n Q_{n+1} x_{n+1} \cdots Q_{n+m} x_{n+m} (F \wedge G)(x_1, \ldots, x_{n+m}),$$

and the latter sentence is in prenex normal form.

Inductive case $\exists$: Suppose that $A(c)$ is provably equivalent to a prenex sentence B. We show that $\exists v A(v)$ is provably equivalent to a prenex sentence. We split the argument into cases: either B contains c or it doesn't. In the former case, write $B = B(c)$. Then $A(c) \vdash B(c)$ gives $A(c) \vdash \exists v B(v)$ by EI, and since $\exists v B(v)$ doesn't contain the constant symbol c, $\exists v A(v) \vdash \exists v B(v)$ by EE. Similarly, $\exists v B(v) \vdash \exists v A(v)$. Hence $\exists v A(v)$ is equivalent to the prenex sentence $\exists v B(v)$. In the other case (c not contained in B) we have $\exists v A(v) \vdash B$ by EE, and $B \vdash \exists v A(v)$ by EI. Thus, $\exists v A(v)$ is provably equivalent to the prenex sentence B.

Inductive case $\forall$: Exercise.

□

Since any predicate logic sentence is equivalent to one in prenex normal form, completeness would follow from the following version of the Main Lemma.

Main Lemma. *Let R be a sentence in prenex normal form. If $R \nvdash P \wedge \neg P$, then R is consistent, i.e. there is an interpretation that makes R true.*

The idea behind the proof is that if we tried to prove $P \wedge \neg P$ from R but *failed*, then in the process we would generate enough non-quantified sentences to tell us exactly what a universe in which R is true must look like. More specifically, suppose that we wanted to show that R does not describe a possible universe. Then we would try to use R to derive the fact that some individual c has a property F, and also to derive the fact that c has the property

$\neg F$. Or we would try to show that R implies both Scd and $\neg Scd$ for two individuals c and d.

Since R might be prefixed by many quantifiers, we would use a combination of UI and the Rule of Assumptions (intending to apply EE) to extract its specific consequences. For example, R is of the form $\forall_1 v_1 \cdots Q_n x_n A(v_1, \ldots, v_n)$, we might start by deriving the instance $Q_2 v_2 \cdots Q_n v(n) A(c_1, v_2, \ldots, v_n)$ for some constant c_1. If, on the other hand, $R = \exists_1 v_1 \cdots Q_n v_n A(v_1, \ldots, v_n)$, then we might assume the instance $Q_2 v_2 \cdots Q_n v(n) A(c_1, v_2, \ldots, v_n)$, where c_1 is a freshly chosen name (so as not to get a contradiction where none is really implied by R).

We systematize this idea by constructing two sequences $\Gamma_0, \Gamma_1, \ldots$ and $\Delta_0, \Delta_1, \ldots$, where each Γ_i and Δ_j consists of a finite list of sentences. The intuitive idea behind this construction is to generate longer and longer proofs, starting with the proof $R \vdash R$. The sentences in Γ_i will be assumptions (they go on the left hand side of $\vdash$) and the sentences in Δ_j will be conclusions (they go on the right hand side of $\vdash$).

Stage 0: Let $\Delta_0 = \{R\}$ and let $\Gamma_0 = \{R\}$.

Stage n: Suppose that sets $\Gamma_0, \ldots, \Gamma_{n-1}$ and $\Delta_0, \ldots, \Delta_{n-1}$ (all of which consist of a finite number of sentences) have been constructed. We then build Γ_n as follows: for each existential sentence $\exists v A(v)$ in Δ_{n-1}, choose the first constant c in $\{c_1, c_2, \ldots\}$ that does not occur in any sentence previously constructed, and add $A(c)$ to Γ_n. We build Δ_n as follows: for each universal sentence $\forall v A(v)$ in $\Delta_0, \ldots, \Delta_{n-1}$, and for each constant c that *does* previously occur in a constructed sentence, add $A(c)$ to Δ_n.

Notice the following fact about the sets of sentences we have constructed.

Lemma. *If* $\Gamma_0, \ldots, \Gamma_n, \Delta_0, \ldots \Delta_{n-1} \vdash B$, *where* B *is a sentence that does not contain any constant symbols, then* $\Gamma_0, \ldots, \Gamma_{n-1}, \Delta_0, \ldots, \Delta_{n-1} \vdash B$.

Proof. Let $\Gamma = \Gamma_0, \ldots, \Gamma_{n-1}$ and let $\Delta = \Delta_0, \ldots, \Delta_{n-1}$. Then the claim to be proven is:

$$\Gamma, \Delta, \Gamma_n \vdash B \quad \Longrightarrow \quad \Gamma, \Delta \vdash B.$$

Recall that Γ_n consists of a finite number of sentences, each of which is an instance of some existential sentence in Δ_{n-1}. So, we may write $A_1(d_1), \ldots, A_m(d_m)$ for the sentences in Γ_n, and $\exists v_1 A_1(v_1), \ldots, \exists v_m A_m(v_m)$ for their counterparts in Δ_{n-1}. We assume then that

$$\Gamma, \Delta, A_1(d_1), \ldots, A_m(d_m) \vdash B.$$

We may also assume that the sentences $A_i(d_i)$ are numbered by their order of introduction; thus, by stipulation, the constant d_i is not equal to d_j for $j < i$, nor does d_i occur in any of the sentences in Γ or Δ. Thus EE converts a proof $\Gamma, \Delta, A_1(d_1), \ldots, A_m(d_m) \vdash B$ to a proof

$$\Gamma, \Delta, A_1(d_1), \ldots, A_{m-1}(d_{m-1}), \exists v_m A_m(v_m) \vdash B.$$

Repeating this process for each of the sentences $A_i(d_i)$, there is a proof

$$\Gamma, \Delta, \exists v_1 A(v_1), \ldots, \exists v_m A(v_m) \vdash B.$$

Since each of the sentences $\exists v_i A(v_i)$ is in Δ_{n-1}, and hence in Δ, it follows that $\Gamma, \Delta \vdash B$ $\square$

Lemma. *Let B be a sentence that does not contain any constant symbols. If*

$$\Gamma_0, \ldots, \Gamma_n, \Delta_0, \ldots, \Delta_n \vdash B,$$

then $R \vdash B$.

Proof. Suppose that there is a proof

$$\Gamma_0, \ldots, \Gamma_n, \Delta_0, \ldots, \Delta_n \vdash B.$$

Since each sentence in Δ_n follows from a sentence in $\Delta_0, \ldots, \Delta_{n-1}$ by UE, there is a proof

$$\Gamma_0, \ldots, \Gamma_n, \Delta_0, \ldots, \Delta_{n-1} \vdash B.$$

By the previous Lemma,

$$\Gamma_0, \ldots, \Gamma_{n-1}, \Delta_0, \ldots, \Delta_{n-1} \vdash B.$$

Now repeating the previous two steps, we arrive eventually at

$$\Gamma_0, \Delta_0 \vdash B.$$

But $\Gamma_0 = \Delta_0 = \{R\}$. Therefore $R \vdash B$. □

It follows immediately from the previous lemma that if $R \nvdash P \wedge \neg P$, then for all numbers n,

$$\Gamma_0, \ldots, \Gamma_n, \Delta_0, \ldots, \Delta_n \nvdash P \wedge \neg P.$$

Of course, if there is no proof of $P \wedge \neg P$ from a set of sentences, then there is no proof of $P \wedge \neg P$ from a smaller set of sentences. So if Θ_n is the set of sentences in $\Gamma_0, \ldots, \Gamma_n, \Delta_0, \ldots, \Delta_n$ that contain no quantifiers, then $R \nvdash P \wedge \neg P$ implies that $\Theta_n \nvdash P \wedge \neg P$. By the completeness of the propositional calculus, Θ_n is truth-functionally consistent. We want now to conclude that the entire set $\bigcup_{i=1}^{\infty} \Theta_i$ is consistent (and so we can build a universe out of it!). That claim is true, but not obviously so — we need to prove it.

Theorem (Compactness of Propositional Logic)**.** *Let $\Theta_0, \Theta_1, \ldots$ be finite sets of sentences without quantifiers such that $\Theta_0 \subseteq \Theta_1 \subseteq \Theta_2 \subseteq \cdots$. If each Θ_i is truth-functionally consistent, then $\bigcup_{i=1}^{\infty} \Theta_i$ is truth-functionally consistent.*

Proof. For each i, let S_i be the set of truth valuations that make all sentences in Θ_i true. By assumption, each S_i is non-empty; and clearly $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. We now define inductively a sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ of truth values with the following feature:

($\star$) For each i and j in $\mathbb{N}$, there is a $v \in S_j$ such that $v(P_i) = \varepsilon_i$,

which will ensure that the truth valuation $v(P_i) = \varepsilon_i$ satisfies $\bigcup_j \Theta_j$.

Base case ($i = 0$) We first define ε_0 such that for each $j \in \mathbb{N}$, there is a $v \in S_j$ such that $v(P_0) = \varepsilon_0$. Now, either for all j, there is $v \in S_j$ such that $v(P_0) = F$, or there is a j such that for all $v \in S_j$, $v(P_0) = T$. In the first case, let $\varepsilon_0 = F$. In the second case, let $\varepsilon_0 = T$.

In the second case, for each $k \geq j$, all truth valuations v in S_k assign T to P_0. In either case, then, for all j, there is a $v \in S_j$ such that $v(P_0) = \varepsilon_0$.

Inductive step ($i = n+1$) Suppose that $\varepsilon_0, \ldots, \varepsilon_n$ have been defined such that: ($\star$) for each $i = 1, \ldots, n$ and for each $j \in \mathbb{N}$, there is a $v \in S_j$ such that $v(P_i) = \varepsilon_i$. We now define ε_n so that ($\star$) is also true for $i = 0, 1, \ldots, n+1$. For each $j \in \mathbb{N}$, let

$$T_j = S_j \cap \{v : v(P_i) = \varepsilon_i, \text{ for } i = 1, \ldots, n\}.$$

That is, a truth valuation v is in T_j just in case it satisfies Θ_j and $v(P_i) = \varepsilon_i$ for $i = 1, \ldots, n$. Since ($\star$) is true for $\varepsilon_0, \ldots, \varepsilon_n$, each T_j is non-empty; and clearly $T_0 \supseteq T_1 \supseteq T_2 \cdots$. If for all $j \in \mathbb{N}$, there is $v \in T_j$ such that $v(P_{n+1}) = F$, then let $\varepsilon_{n+1} = F$. If there is a $j \in \mathbb{N}$ such that for all $v \in T_j$, $v(P_{n+1}) = T$, then let $\varepsilon_{n+1} = T$. As before, in this second case, all S_j will contain a v such that $v(P_{n+1}) = T = \varepsilon_{n+1}$. Thus, ($\star$) is true for $i = 0, 1, \ldots, n+1$. This completes the construction of the sequence $\varepsilon_0, \varepsilon_1, \ldots,$ and we have verified that ($\star$) is true for each $i \in \mathbb{N}$.

Now define a truth valuation v by setting $v(P_i) = \varepsilon_i$ for each atomic sentence P_i. We claim that $v(A) = T$ for all $A \in \bigcup_i \Theta_i$. Indeed, if $A \in \bigcup_i \Theta_i$, then $A \in \Theta_k$, for some finite k. Let $\{P_0, \ldots, P_m\}$ contain all the atomic sentences in A. By ($\star$), there is $w \in S_k$ such that $w(P_i) = \varepsilon_i$ for $i = 0, 1, \ldots, m$. Moreover, since $A \in \Theta_k$ and $w \in S_k$, $w(A) = T$. But v and w assign the same truth values to the atomic sentences occurring in A, hence $v(A) = w(A) = T$.

□

With the compactness theorem (for propositional logic) in hand, we can now show that there is an interpretation $\mathcal{M}$ in which R is true. Let v be a truth-valuation that assigns T to each sentence in $\bigcup_i \Theta_i$, and define a predicate logic interpretation $\mathcal{M}$ by taking the domain of quantification to consist of constant symbols that occur in $\bigcup_i \Theta_i$, and define the extension of a predicate symbol S by

$$\langle d_1, \ldots, d_n \rangle \in \text{Ext}_{\mathcal{M}}(S) \qquad \text{iff} \qquad v(Sd_1, \ldots, d_n) = T.$$

To finish the argument, we must now show that R is true in $\mathcal{M}$. To do so, we argue by induction. Let $\Gamma = (\bigcup_i \Gamma_i) \cup (\bigcup_j \Delta_j)$. Obviously $\mathcal{M}$ makes every sentence in Γ with zero quantifiers true. We show that if $\mathcal{M}$ makes every sentence in Γ with n quantifiers true, then it makes every sentence in Γ with $n+1$ quantifiers true.

So let A be a sentence in Γ with $n+1$ quantifiers. Let i be the stage at which A first appears. If A is universal, then writing $A = \forall v B(v)$, every sentence of the form $B(c)$ with c a constant occurring in Γ, will eventually be generated in Γ. Since $B(c)$ has only n-quantifiers, it is true in $\mathcal{M}$. Thus, $B(c)$ is true for every c in the domain of quantification; hence $\forall v B(v)$ is true in $\mathcal{M}$. If, on the other hand, A is existential, then $A = \exists v B(v)$, and $B(c)$ is generated for some constant c. By assumption, $\mathcal{M}$ makes $B(c)$ true, and so it makes $\exists v B(v)$ true. Since R has a finite number of quantifiers, $\mathcal{M}$ makes R true, as was to be shown.