Phil 312: Solutions to PS2.

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Problem 1. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is a monomorphism, then f is a monomorphism.

Assume that $g \circ f$ is a monomorphism. We want to show that f is a monomorphism too. Recall that this means that, for any two functions $h, k : Z \to X$, if $f \circ h = f \circ k$, then $h = k$. Thus, it suffices to show that, given any such functions h, k, we can conclude that $h = k$. By assumption, then, we have that $f \circ h = f \circ k$. Compose both sides of this equation with g on the left. Then we get

$$g \circ (f \circ h) = g \circ (f \circ k).$$

By associativity of $\circ$ on both sides, this gives us that $(g \circ f) \circ h = (g \circ f) \circ k$. But $g \circ f$ is a monomorphism. Hence, $h = k$, as we wanted.

Problem 2. Let $f : X \to Y$ be a function, and let $\delta_Y : Y \to Y \times Y$ be the diagonal map. Then $\delta_Y \circ f = \langle f, f \rangle$.

Let $\pi_0, \pi_1 : Y \times Y \to Y$ be the two projections of the product $Y \times Y$ given by Axiom 2. Then, by the definition of δ_Y, we have both that (i) $\pi_0 \circ \delta_Y = 1_Y$ and (ii) $\pi_1 \circ \delta_Y = 1_Y$. Compose both sides of both equations with f on the right to get (i) $(\pi_0 \circ \delta_Y) \circ f = 1_Y \circ f$ and (ii) $(\pi_1 \circ \delta_Y) \circ f = 1_Y \circ f$. But by the definition of the identity 1_Y we get (i) $(\pi_0 \circ \delta_Y) \circ f = f$ and (ii) $(\pi_1 \circ \delta_Y) \circ f = f$. In turn, by associativity of $\circ$, this gives us the following two equations:

$$\pi_0 \circ (\delta_Y \circ f) = f$$

$$\pi_1 \circ (\delta_Y \circ f) = f$$

These should look familiar: they are the equations which define the function $\langle f, f \rangle$ which we obtain from Axiom 2 (if we plug in 'Y' for 'X' and 'f' for 'g' in the original statement of the axiom). Note to that this axiom states that such function is unique. Hence we get that $\delta_Y = \langle f, f \rangle$, as we wanted.

Problem 3. If $f : X \to Y$ is surjective, then f is an epimorphism.

Suppose that $f : X \to Y$ is surjective (remember that this means that, for any $y \in Y$, there is an $x \in X$ such as $f \circ x = y$). We want to prove that f is an epimorphism, i.e., that for any two functions $h, k : Y \to Z$, if $h \circ f = k \circ f$ then $h = k$. Thus, it suffices to show that, given any such functions h, k, we can conclude that $h = k$.

To prove that $h = k$, recall that 1 is a separator for **Sets**, i.e., if for all $y \in Y$ we have that $h \circ y = k \circ y$, then $h = k$. So it suffices to show this. So given an element $y \in Y$. By f being a surjection, there is an element $x \in X$ such that $f \circ x = y$. Now, composing both sides of this equation with h on the left gives us (i) $h \circ (f \circ x) = h \circ y$; whereas composing both sides with k on the left gives us (ii) $k \circ (f \circ x) = k \circ y$. By associativity of $\circ$, these two equations give us:

$$(h \circ f) \circ x = h \circ y$$

$$(k \circ f) \circ x = k \circ y$$

Now, recall that by assumption $h \circ f = k \circ f$. So the left sides of these two equations are identical. Hence $h \circ y = k \circ y$. This is true for all $y \in Y$, and such $h = k$ as we wanted.

Problem 4. We want to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Since sets are defined by their elements, it suffices to show that a set x is an element of $A \cap (B \cup C)$ if and only if it is also an element of $(A \cap B) \cup (A \cap C)$. So we proceed by showing this.

By definition of $\cap$, $x \in A \cap (B \cup C)$ iff $x \in A$ *and* $x \in (B \cup C)$. In turn, by definition of $\cup$, this is the case iff

$$(\star) \quad x \in A \quad \text{and} \quad (x \in B \quad \text{or} \quad x \in C)\,.$$

But 'and' distributes over 'or' (i.e., if we have that $p \wedge (q \vee r)$, then we also have that $(p \wedge q) \vee (p \wedge r)$, and vice versa). So we get that $(\star)$ holds iff $(\star\star)$ does:

$$(\star\star) \quad (x \in A \quad \text{and} \quad x \in B) \quad \text{or} \quad (x \in A \quad \text{and} \quad x \in C)\,.$$

Finally, by definition of $\cap$ and $\cup$, we get that $(\star\star)$ holds iff $x \in (A \cap B) \cup (A \cap C)$, as we wanted.