The Category of Theories

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1 Basics

Definition. We let **Th** denote the category whose objects are propositional theories, and whose arrows are translations between theories. We say that two translations $f, g : T \rightrightarrows T'$ are equal, written $f = g$, just in case $T' \vdash f(\phi) \leftrightarrow g(\phi)$ for every $\phi \in \mathsf{Sent}(\Sigma)$. [Note well: equality between translations is weaker than set-theoretic equality.]

Definition. We say that a translation $f : T \to T'$ is **conservative** just in case: for any $\phi \in \mathsf{Sent}(\Sigma)$, if $T' \vdash f(\phi)$ then $T \vdash \phi$.

Proposition 1.1. *A translation $f : T \to T'$ is conservative if and only if f is a monomorphism in the category* **Th**.

Proof. Suppose first that f is conservative, and let $g, h : T'' \to T$ be translations such that $f \circ g = f \circ h$. That is, $T' \vdash fg(\phi) \leftrightarrow fh(\phi)$ for every sentence ϕ of Σ''. Since f is conservative, $T \vdash g(\phi) \leftrightarrow h(\phi)$ for every sentence ϕ of Σ''. Thus, $g = h$, and f is a monomorphism in **Th**.

Conversely, suppose that f is a monomorphism in the category **Th**. Let ϕ be a Σ sentence such that $T' \vdash f(\phi)$. Thus, $T' \vdash f(\phi) \leftrightarrow f(\psi)$, where ψ is any Σ sentence such that $T \vdash \psi$. Now let T'' be the empty theory in signature $\Sigma'' = \{p\}$. Define $g : \Sigma'' \to \mathsf{Sent}(\Sigma)$ by $g(p) = \phi$, and define $h : \Sigma'' \to \mathsf{Sent}(\Sigma)$ by $h(p) = \psi$. It's easy to see then that $f \circ g = f \circ h$. Since f is monic, $g = h$, which means that $T \vdash g(p) \leftrightarrow h(p)$. Therefore, $T \vdash \phi$, and f is conservative. □

Definition. We say that $f : T \to T'$ is **essentially surjective** just in case for any sentence ϕ of Σ', there is a sentence ψ of Σ such that $T' \vdash \phi \leftrightarrow f(\psi)$. (Sometimes we use the abbreviation "eso" for essentially surjective.)

Proposition 1.2. *If $f : T \to T'$ is essentially surjective, then f is an epimorphism in* **Th**.

Proof. Suppose that $f : T \to T'$ is eso. Let $g, h : T' \rightrightarrows T''$ such that $g \circ f = h \circ f$. Let ϕ be an arbitrary Σ' sentence. Since f is eso, there is a sentence ψ of Σ such that $T' \vdash \phi \leftrightarrow f(\psi)$. But then $T'' \vdash g(\phi) \leftrightarrow h(\phi)$. Since ϕ was arbitrary, $g = h$. Therefore, f is an epimorphism. □

What about the converse of this proposition? Are all epimorphisms in **Th** essentially surjective? The answer is Yes, but the result is not easy to prove. We'll prove it later on, by means of the correspondence that we establish between theories, Boolean algebras, and Stone spaces.

Proposition 1.3. *Let $f : T \to T'$ be a translation. If f is conservative and essentially surjective, then f is a homotopy equivalence.*

Proof. Let $p \in \Sigma'$. Since f is eso, there is some $\phi_p \in \mathsf{Sent}(\Sigma)$ such that $T' \vdash p \leftrightarrow f(\phi_p)$. Define a reconstrual $g : \Sigma' \to \mathsf{Sent}(\Sigma)$ by setting $g(p) = \phi_p$. As usual, g extends naturally to a function from $\mathsf{Sent}(\Sigma')$ to $\mathsf{Sent}(\Sigma)$, and it immediately follows that $T' \vdash \psi \leftrightarrow fg(\psi)$, for every sentence ψ of Σ'.

We claim now that g is a translation from T' to T. Suppose that $T' \vdash \psi$. Since $T' \vdash \psi \leftrightarrow fg(\psi)$, it follows that $T' \vdash fg(\psi)$. Since f is conservative, $T \vdash g(\psi)$. Thus, for all sentences ψ of Σ', if $T' \vdash \psi$ then $T \vdash g(\psi)$, which means that $g : T' \to T$ is a translation. By the previous paragraph, $1_{T'} \simeq fg$.

It remains to show that $1_T \simeq gf$. Let ϕ be an arbitrary sentence of Σ. Since f is conservative, it will suffice to show that $T' \vdash f(\phi) \leftrightarrow fgf(\phi)$. But by the previous paragraph, $T' \vdash \psi \leftrightarrow fg(\psi)$ for all sentences ψ of Σ'. Therefore, $1_T \simeq gf$, and f is a homotopy equivalence. □

Before proceeding, let's remind ourselves of some of the motivations for these technical investigations.

The category **Sets** is, without a doubt, extremely useful. However, a person who is familiar with **Sets** might have developed some intuitions that could be misleading when applied to other categories. For example, in **Sets**, if there are injections $f : X \to Y$ and $g : Y \to X$, then there is a bijection between X and Y. Thus, it's tempting to think, for example, that if there are embeddings $f : T \to T'$ and $g : T' \to T$ of theories, then T and T' are equivalent. [Here an embedding between theories is a monomorphism in **Th**, i.e. a conservative translation.] Similarly, in **Sets**, if there is an injection $f : X \to Y$ and a surjection $g : X \to Y$, then there is a bijection between X and Y. However, in **Th** the analogous result fails to hold.

Technical Aside. For those familiar with the category **Vect** of vector spaces: **Vect** is similar to **Sets** in that mutually embeddable vector spaces are isomorphic. That is, if $f : V \to W$ and $g : W \to V$ are monomorphisms (i.e. injective linear maps), then V and W have the same dimension, hence are isomorphic.

The categories **Sets** and **Vect** share in common the feature that the objects can be classified by cardinal numbers. In the case of sets, if $|X| = |Y|$, then $X \cong Y$. In the case of vector spaces, if $\dim(V) = \dim(W)$, then $V \cong W$.

Proposition 1.4. *Let $f : T \to T'$ be a translation. If $f^* : M(T') \to M(T)$ is surjective, then f is conservative.*

Proof. Suppose that f^* is surjective, and suppose that ϕ is a sentence of Σ such that $T \nvdash \phi$. Then there is a $v \in M(T)$ such that $v(\phi) = 0$.[1] Since f^* is surjective, there is a $w \in M(T')$ such that $f^*(w) = v$. But then

$$w(f(\phi)) = f^*w(\phi) = v(\phi) = 0,$$

from which it follows that $T' \nvdash f(\phi)$. Therefore, f is conservative. □

Example. Let $\Sigma = \{p_0, p_1, \dots\}$, and let T be the empty theory in Σ. Let $\Sigma' = \{q_0, q_1, \dots\}$, and let T' be the theory with axioms $q_0 \to q_i$, for $i = 0, 1, \dots$. We already know that T and T' are not equivalent. We will now show that there are embeddings $f : T \to T'$ and $g : T' \to T$.

Define $f : \Sigma \to \mathsf{Sent}(\Sigma')$ by $f(p_i) = q_{i+1}$. Since T is the empty theory, f is a translation. Then for any valuation v of Σ', we have

$$f^*v(p_i) = v(f(p_i)) = v(q_{i+1}).$$

Furthermore, for any sequence of zeros and ones, there is a valuation v of Σ' that assigns that sequence to $q_1, q_2, \dots$. Thus, f^* is surjective, and f is conservative.

Now define $g : \Sigma' \to \mathsf{Sent}(\Sigma)$ by setting $g(q_i) = p_0 \vee p_i$. Since $T \vdash p_0 \vee p_0 \to p_0 \vee p_i$, it follows that g is a translation. Furthermore, for any valuation v of Σ, we have

$$g^*v(q_i) = v(g(q_i)) = v(p_0 \vee p_i).$$

Recall that $M(T')$ splits into two parts: (1) a singleton set containing the valuation z where $z(q_i) = 1$ for all i, and (2) the infinitely many other valuations

[1] Here we have invoked the completeness theorem, but we haven't proven it yet. Note that our proof of the completeness theorem (page **??**) does not cite this result, or any that depend on it.

which assign 0 to q_0. Clearly, $z = g^*v$, where v is any valuation such that $v(p_0) = 1$. Furthermore, for any valuation w of Σ' such that $w(p_0) = 0$, we have $w = g^*v$, where $v(p_i) = w(q_i)$. Therefore, g^* is surjective, and g is conservative.

Exercise. In the example above: show that f and g are not essentially surjective.

Example. Let T and T' be as in the previous example. Now we'll show that there are essentially surjective (eso) translations $k : T \to T'$ and $h : T' \to T$. The first is easy: the translation $k(p_i) = q_i$ is obviously eso. For the second, define $h(q_0) = \perp$, where $\perp$ is some contradiction, and define $h(q_i) = p_{i-1}$ for $i > 0$.

Technical questions about theories

1. Does **Th** have the **Cantor-Bernstein property**? That is, if there are monomorphisms $f : T \to T'$ and $g : T' \to T$, then is there an isomorphism $h : T \to T'$?

2. Is **Th** balanced, in the sense that if $f : T \to T'$ is both a monomorphism and an epimorphism, then f is an isomorphism?

3. If there is both a monomorphism $f : T \to T'$ and an epimorphism $g : T' \to T$, then are T and T' homotopy equivalent?

4. (Quine and Goodman, "Elimination of extra-logical postulates.") Can any theory be made true by definition? That is, can T be embedded into a theory T' that has no axioms?

5. If theories have the same number of models, then are they equivalent? If not, then can we determine whether T and T' are equivalent by inspecting $M(T)$ and $M(T')$?

6. How many theories (up to isomorphism) are there with n models?

7. (Supervenience implies Reduction) Suppose that the truth value of a sentence ψ **supervenes** on the truth value of some other sentences $\phi_1, \dots, \phi_n$, i.e., for any valuations v, w of the propositional constants occurring in $\phi_1, \dots, \phi_n, \psi$, if $v(\phi_i) = w(\phi_i)$, for $i = 1, \dots, n$, then $v(\psi) = w(\psi)$. Does it follow then that $\vdash \psi \leftrightarrow \theta$, where θ contains only the propositional constants that occur in $\phi_1, \dots, \phi_n$? (The answer is Yes, as shown by **Beth's theorem**, **??**.)

8. Suppose that $f : T \to T'$ is conservative. Suppose also that every model of T extends uniquely to a model of T'. Does it follow that $T \cong T'$?

9. Suppose that T and T' are consistent in the sense that there is no sentence θ in $\Sigma \cap \Sigma'$ such that $T \vdash \theta$ and $T' \vdash \neg\theta$. Is there a unified theory T'' which extends both T and T'? (The answer is Yes, as shown by **Robinson's theorem**, **??**.)

Philosophical questions about theories

1. What does it mean for one theory to be **reducible** to another? Can we explicate this notion in terms of a certain sort of translation between the relevant theories?

   Some philosophers have claimed that the reduction relation ought to be treated semantically, rather than syntactically. In other words, they would have us consider functions from $M(T')$ to $M(T)$, rather than translations from T to T'. In light of the Stone duality theorem proved below, it appears that syntactic and semantic approaches are equivalent to each other.

2. Consider various formally definable notions of theoretical equivalence. What are the advantages and disadvantages of the various notions? Is homotopy equivalence too liberal? Is it too conservative?

3. Many more to come ...

2 Boolean algebras

Definition. A **Boolean algebra** is a set B together with a unary operation $\neg$, two binary operations $\wedge$ and $\vee$, and designated elements $0 \in B$ and $1 \in B$, which satisfy the following equations:

1. Top and Bottom
   $a \wedge 1 = a \vee 0 = a$

2. Idempotence
   $a \wedge a = a \vee a = a$

3. De Morgan's rules
   $\neg(a \wedge b) = \neg a \vee \neg b, \quad \neg(a \vee b) = \neg a \wedge \neg b$

4. Commutativity
   $a \wedge b = b \wedge a, \quad a \vee b = b \vee a$

5. Associativity
   $(a \wedge b) \wedge c = a \wedge (b \wedge c), \quad (a \vee b) \vee c = a \vee (b \vee c)$

6. Distribution
   $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

7. Excluded Middle
   $a \wedge \neg a = 0, \quad a \vee \neg a = 1$

Here we are implicitly universally quantifying over a, b, c.

Example. Let 2 denote the unique Boolean algebra with two elements $\emptyset$ and 1. We can think of 2 as the powerset of a one-element set 1, where $\wedge$ is intersection, $\vee$ is union, and $\neg$ is complement.

Note that 2 looks just like the truth-value set Ω. Indeed, Ω is equipped with operations $\wedge, \vee$ and $\neg$ that make it into a Boolean algebra.

Example. Let F denote the unique Boolean algebra with four elements. We can think of F as the powerset of a two-element set, where $\wedge$ is intersection, $\vee$ is union, and $\neg$ is complement.

Let $\Sigma = \{p\}$. Define an equivalence relation $\simeq$ on sentences of Σ by $\phi \simeq \psi$ just in case $\vdash \phi \leftrightarrow \psi$. The resulting set of equivalence classes naturally carries the structure of a Boolean algebra with four elements.

We now derive some basic consequences from the axioms. The first two results are called the **absorption laws**.

1. $a \wedge (a \vee b) = a$

$$a \wedge (a \vee b) = (a \vee 0) \wedge (a \vee b) = a \vee (0 \wedge b) = a \vee 0 = a.$$

2. $a \vee (a \wedge b) = a$

$$a \vee (a \wedge b) = (a \wedge 1) \vee (a \wedge b) = a \wedge (1 \vee b) = a \wedge 1 = a.$$

3. $a \vee 1 = 1$

$$a \vee 1 = a \vee (a \vee \neg a) = a \vee \neg a = 1.$$

4. $a \wedge 0 = 0$

$$a \wedge 0 = a \wedge (a \wedge \neg a) = a \wedge \neg a = 0.$$

Definition. If B is a Boolean algebra and $a, b \in B$, we write $a \leq b$ when $a \wedge b = a$.

Since $a \wedge 1 = a$, it follows that $a \leq 1$, for all $a \in B$. Since $a \wedge 0 = 0$, it follows that $0 \leq a$, for all $a \in B$. Now we will show that $\leq$ is a partial order, i.e. reflexive, transitive, and asymmetric.

Proposition 2.1. *The relation $\leq$ on a Boolean algebra B is a partial order.*

Proof. (Reflexive) Since $a \wedge a = a$, it follows that $a \leq a$.

(Transitive) Suppose that $a \wedge b = a$ and $b \wedge c = b$. Then

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a,$$

which means that $a \leq c$.

(Asymmetric) Suppose that $a \wedge b = a$ and $b \wedge a = b$. By commutativity of $\wedge$, it follows that $a = b$. $\square$

We now show how $\leq$ interacts with $\wedge, \vee$, and $\neg$. In particular, we show that if $\leq$ is thought of as implication, then $\wedge$ behaves like conjunction, $\vee$ behaves like disjunction, $\neg$ behaves like negation, 1 behaves like a tautology, and 0 behaves like a contradiction.

Proposition 2.2. *$c \leq a \wedge b$ iff $c \leq a$ and $c \leq b$.*

Proof. Since $a \wedge (a \wedge b) = a \wedge b$, it follows that $a \wedge b \leq a$. By similar reasoning, $a \wedge b \leq b$. Thus if $c \leq a \wedge b$, then transitivity of $\leq$ entails that both $c \leq a$ and $c \leq b$.

Now suppose that $c \leq a$ and $c \leq b$. That is, $c \wedge a = c$ and $c \wedge b = c$. Then $c \wedge (a \wedge b) = (c \wedge a) \wedge (c \wedge b) = c \wedge c = c$. Therefore $c \leq a \wedge b$. □

Notice that $\leq$ and $\wedge$ interact precisely as implication and conjunction interact in propositional logic. The elimination rule says that $a \wedge b$ implies a and b. Hence, if c implies $a \wedge b$, then c implies a and b. The introduction rule says that a and b imply $a \wedge b$. Hence if c implies a and b, then c implies $a \wedge b$.

Proposition 2.3. *$a \leq c$ and $b \leq c$ iff $a \vee b \leq c$*

Proof. Suppose first that $a \leq c$ and $b \leq c$. Then

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = a \vee b.$$

Therefore $a \vee b \leq c$.

Suppose now that $a \vee b \leq c$. By the absorption law, $a \wedge (a \vee b) = a$, which implies that $a \leq a \vee b$. By transitivity $a \leq c$. Similarly, $b \leq a \vee b$, and by transitivity, $b \leq c$. □

Now we show that the connectives $\wedge$ and $\vee$ are monotonic.

Proposition 2.4. *If $a \leq b$ then $a \wedge c \leq b \wedge c$, for any $c \in B$.*

Proof.

$$(a \wedge c) \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c.$$

□

Proposition 2.5. *If $a \leq b$ then $a \vee c \leq b \vee c$, for any $c \in B$.*

Proof.

$$(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c = a \vee c.$$

□

Proposition 2.6. *If $a \wedge b = a$ and $a \vee b = a$ then $a = b$.*

Proof. $a \wedge b = a$ means that $a \leq b$. We now claim that $a \vee b = a$ iff $b \wedge a = b$ iff $b \leq a$. Indeed, if $a \vee b = a$ then

$$b \wedge a = b \wedge (a \vee b) = (0 \vee b) \wedge (a \vee b) = (0 \wedge a) \vee b = b.$$

Conversely, if $b \wedge a = b$, then

$$a \vee b = a \vee (a \wedge b) = (a \wedge 1) \vee (a \wedge b) = a \wedge (1 \vee b) = a.$$

Thus, if $a \wedge b = a$ and $a \vee b = a$, then $a \leq b$ and $b \leq a$. By asymmetry of $\leq$, it follows that $a = b$. □

We now show that $\neg a$ is the unique complement of a in B.

Proposition 2.7. *If $a \wedge b = 0$ and $a \vee b = 1$ then $b = \neg a$.*

Proof. Since $b \vee a = 1$, we have

$$b = b \vee 0 = b \vee (a \wedge \neg a) = (b \vee a) \wedge (b \vee \neg a) = b \vee \neg a.$$

Since $b \wedge a = 0$, we also have

$$b = b \wedge 1 = b \wedge (a \vee \neg a) = (b \wedge a) \vee (b \wedge \neg a) = b \wedge \neg a.$$

By the preceding proposition, $b = \neg a$. □

Proposition 2.8. $\neg 1 = 0$.

Proof. We have $1 \wedge 0 = 0$ and $1 \vee 0 = 1$. By the preceding proposition, $0 = \neg 1$. □

Proposition 2.9. *If $a \leq b$ then $\neg b \leq \neg a$.*

Proof. Suppose that $a \leq b$, which means that $a \wedge b = a$, and equivalently, $a \vee b = b$. Thus, $\neg a \wedge \neg b = \neg(a \vee b) = \neg b$, which means that $\neg b \leq \neg a$. □

Proposition 2.10. $\neg\neg a = a$.

Proof. We have $\neg a \vee \neg\neg a = 1$ and $\neg a \wedge \neg\neg a = 1$. By Proposition 2.7, it follows that $\neg\neg a = a$. □

Definition. Let A and B be Boolean algebras. A **homomorphism** is a map $\phi : A \to B$ such that $\phi(0) = 0$, $\phi(1) = 1$, and for all $a, b \in A$, $\phi(\neg a) = \neg\phi(a)$, $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ and $\phi(a \vee b) = \phi(a) \vee \phi(b)$.

It is easy to see that if $\phi : A \to B$ and $\psi : B \to C$ are homomorphisms, then $\psi \circ \phi : A \to C$ is also a homomorphism. Moreover, $1_A : A \to A$ is a homomorphism, and composition of homomorphisms is associative.

Definition. We let **Bool** denote the category whose objects are Boolean algebras, and whose arrows are homomorphisms of Boolean algebras.

Since **Bool** is a category, we have notions of **monomorphisms**, **epimorphisms**, **isomorphisms**, etc.. Once again, it is easy to see that an injective homomorphism is a monomorphism, and a surjective homomorphism is an epimorphism.

Proposition 2.11. *Monomorphisms in* **Bool** *are injective.*

Proof. Let $f : A \to B$ be a monomorphism, and let $a, b \in A$. Let F denote the Boolean algebra with four elements, and let p denote one of the two elements in F that is neither 0 nor 1. Define $\hat{a} : F \to A$ by $\hat{a}(p) = a$, and define $\hat{b} : F \to A$ by $\hat{b}(p) = b$. It is easy to see that $\hat{a}$ and $\hat{b}$ are uniquely defined by these conditions, and that they are Boolean homomorphisms. Suppose now that $f(a) = f(b)$. Then $f\hat{a} = f\hat{b}$, and since f is a monomorphism, $\hat{a} = \hat{b}$, and therefore $a = b$. Therefore f is injective. □

It is also true that epimorphisms in **Bool** are surjective. However, proving that fact is no easy task. We will return to it later in the chapter.

Proposition 2.12. *If $f : A \to B$ is a homomorphism of Boolean algebras, then $a \leq b$ only if $f(a) \leq f(b)$.*

Proof. $a \leq b$ means that $a \wedge b = a$. Thus,

$$f(a) \wedge f(b) = f(a \wedge b) = f(a),$$

which means that $f(a) \leq f(b)$. □

Definition. A homomorphism $\phi : B \to 2$ is called a **state** of B.

3 Equivalent categories

We now have two categories on the table: the category **Th** of theories, and the category **Bool** of Boolean algebras. Our next main goal is to show that these categories are **structurally identical**. But what do we mean by this? What we mean is that they are **equivalent categories**. In order to explain what that means, we need a few more definitions.

Definition. Suppose that $\mathbf{C}$ and $\mathbf{D}$ are categories. We let $\mathbf{C}_0$ denote the objects of $\mathbf{C}$, and we let $\mathbf{C}_1$ denote the arrows of $\mathbf{C}$. A (covariant) **functor** $F : \mathbf{C} \to \mathbf{D}$ consists of a pair of maps: $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$, and $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ with the following properties:

1. F_0 and F_1 are compatible in the sense that if $f : X \to Y$ in $\mathbf{C}$,then $F_1(f) : F_0(X) \to F_0(Y)$ in $\mathbf{D}$.
2. F_1 preserves identities and composition in the following sense: $F_1(1_X) = 1_{F_0(X)}$, and $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

When no confusion can result, we simply use F in place of F_0 and F_1.

Note. There is also a notion of a **contravariant functor**, where F_1 reverses the direction of arrows: if $f : X \to Y$ in $\mathbf{C}$, then $F_1(f) : F_0(Y) \to F_0(X)$ in $\mathbf{D}$. Contravariant functors will be especially useful for examining the relation between a theory and its set of models. We've already seen that a translation $f : T \to T'$ induces a function $f^* : M(T') \to M(T)$. In Section **??**, we will see that $f \mapsto f^*$ is part of a contravariant functor.

Example. For any category $\mathbf{C}$, there is a functor $1_{\mathbf{C}}$ that acts as the identity on both objects and arrows. That is, for any object X of $\mathbf{C}$, $1_{\mathbf{C}}(X) = X$. And for any arrow f of $\mathbf{C}$, $1_{\mathbf{C}}(f) = f$.

Definition. Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ be functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of a family $\{\eta_X : F(X) \to G(X) \mid X \in \mathbf{C}_0\}$ of arrows in $\mathbf{D}$, such that for any arrow $f : X \to Y$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

Definition. A natural transformation $\eta : F \Rightarrow G$ is said to be a **natural isomorphism** just in case each arrow $\eta_X : F(X) \to G(X)$ is an isomorphism. In this case, we write $F \cong G$.

Definition. Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. We say that F and G are a **categorical equivalence** just in case $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$.

4 Propositional theories are Boolean algebras

In this section, we show that there is a one-to-one correspondence between theories (in propositional logic) and Boolean algebras. We first need some preliminaries.

Definition. Let Σ be a propositional signature (i.e. a set), let B be a Boolean algebra, and let $f : \Sigma \to B$ be an arbitrary function. [Here we use $\cap, \cup$ and $-$ for the Boolean operations, in order to avoid confusion with the logical connectives $\wedge, \vee$ and $\neg$.] Then f naturally extends to a map $f : \mathsf{Sent}(\Sigma) \to B$ as follows:

1. $f(\phi \wedge \psi) = f(\phi) \cap f(\psi)$;
2. $f(\phi \vee \psi) = f(\phi) \cup f(\psi)$;
3. $f(\neg\phi) = -f(\phi)$.

Now let T be a theory in Σ. We say that f is an **interpretation** of T in B just in case: for all sentences ϕ, if $T \vdash \phi$ then $f(\phi) = 1$.

Definition. Let $f : T \to B$ be an interpretation. We say that:

1. f is **conservative** just in case: for all sentences ϕ, if $f(\phi) = 1$ then $T \vdash \phi$.

2. f **surjective** just in case: for each $a \in B$, there is a $\phi \in \mathsf{Sent}(\Sigma)$ such that $f(\phi) = a$.

Lemma 4.1. *Let $f : T \to B$ be an interpretation. Then the following are equivalent:*

1. *f is conservative.*

2. *For any $\phi, \psi \in \mathsf{Sent}(\Sigma)$, if $f(\phi) = f(\psi)$ then $T \vdash \phi \leftrightarrow \psi$.*

Proof. Note first that $f(\phi) = f(\psi)$ if and only if $f(\phi \leftrightarrow \psi) = 1$. Suppose then that f is conservative. If $f(\phi) = f(\psi)$ then $f(\phi \leftrightarrow \psi) = 1$, and hence $T \vdash \phi \leftrightarrow \psi$. Suppose now that (2) holds. If $f(\phi) = 1$, then $f(\phi) = f(\phi \vee \neg\phi)$, and hence $T \vdash (\phi \vee \neg\phi) \leftrightarrow \phi$. Therefore $T \vdash \phi$, and f is conservative. □

Lemma 4.2. *If $f : T \to B$ is an interpretation, and $g : B \to A$ is a homomorphism, then $g \circ f$ is an interpretation.*

Proof. This is almost obvious. □

Lemma 4.3. *If $f : T \to B$ is an interpretation, and $g : T' \to T$ is a translation, then $f \circ g : T' \to B$ is an interpretation.*

Proof. This is almost obvious. □

Lemma 4.4. *Suppose that T is a theory, and $e : T \to B$ is a surjective interpretation. If $f, g : B \rightrightarrows A$ are homomorphisms such that $fe = ge$, then $f = g$.*

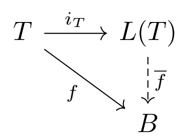
Proof. Suppose that $fe = ge$, and let $a \in B$. Since e is surjective, there is a $\phi \in \mathsf{Sent}(\Sigma)$ such that $e(\phi) = a$. Thus, $f(a) = fe(\phi) = ge(\phi) = g(a)$. Since a was arbitrary, $f = g$. □

Let T' and T be theories, and let $f, g : T' \rightrightarrows T$ be translations. Recall that we defined identity between translations as follows: $f = g$ if and only if $T \vdash f(\phi) \leftrightarrow g(\phi)$ for all $\phi \in \mathsf{Sent}(\Sigma')$.

Lemma 4.5. *Suppose that $m : T \to B$ is a conservative interpretation. If $f, g : T' \rightrightarrows T$ are translations such that $mf = mg$, then $f = g$.*

Proof. Let $\phi \in \mathsf{Sent}(\Sigma')$, where Σ' is the signature of T'. Then $mf(\phi) = mg(\phi)$. Since m is conservative, $T \vdash f(\phi) \leftrightarrow g(\phi)$. Since this holds for all sentences, it follows that $f = g$. □

Proposition 4.6. *For each theory T, there is a Boolean algebra $L(T)$, and a conservative, surjective interpretation $i_T : T \to L(T)$ such that for any Boolean algebra B, and interpretation $f : T \to B$, there is a unique homomorphism $\overline{f} : L(T) \to B$ such that $\overline{f} i_T = f$.*



We define an equivalence relation $\equiv$ on the sentences of Σ:

$$\phi \equiv \psi \quad \text{iff} \quad T \vDash \phi \leftrightarrow \psi,$$

and we let

$$E_\phi \;:=\; \{\psi \mid \phi \equiv \psi\}.$$

Finally, let

$$L(T) \;:=\; \{E_\phi \mid \phi \in \mathsf{Sent}(\Sigma)\}.$$

We now equip $L(T)$ with the structure of a Boolean algebra. To this end, we need the following facts, which correspond to easy proofs in propositional logic.

Fact 4.7. *If $E_\phi = E_{\phi'}$ and $E_\psi = E_{\psi'}$, then:*

1. $E_{\phi\wedge\psi} = E_{\phi'\wedge\psi'}$;
2. $E_{\phi\vee\psi} = E_{\phi'\vee\psi'}$;
3. $E_{\neg\phi} = E_{\neg\phi'}$.

We then define a unary operation $-$ on $L(T)$ by:

$$-E_\phi := E_{\neg\phi},$$

and we define two binary operations on $L(T)$ by:

$$E_\phi \cap E_\psi := E_{\phi\wedge\psi}, \qquad E_\phi \cup E_\psi := E_{\phi\vee\psi}.$$

Finally, let ϕ be an arbitrary Σ sentence, and let $0 = E_{\phi\wedge\neg\phi}$ and $1 = E_{\phi\vee\neg\phi}$. The proof that $\langle L(T), \cap, \cup, -, 0, 1\rangle$ is a Boolean algebra requires a series of straightforward verifications. For example, let's show that $1 \cap E_\psi = E_\psi$, for all sentences ψ. Recall that $1 = E_{\phi\vee\neg\phi}$ for some arbitrarily chosen sentence ϕ. Thus,

$$1 \cap E_\psi \;=\; E_{\phi\vee\neg\phi} \cap E_\psi \;=\; E_{(\phi\vee\neg\phi)\wedge\psi}.$$

Moreover, $T \vdash \psi \leftrightarrow ((\phi \vee \neg\phi) \wedge \psi)$, from which it follows that $E_{(\phi\vee\neg\phi)\wedge\psi} = E_\psi$. Therefore, $1 \cap E_\psi = E_\psi$.

Consider now the function $i_T : \Sigma \to L(T)$ given by $i_T(\phi) = E_\phi$, and its natural extension to $\mathsf{Sent}(\Sigma)$. A quick inductive argument, using the definition of the Boolean operations on $L(T)$, shows that $i_T(\phi) = E_\phi$ for all $\phi \in \mathsf{Sent}(\Sigma)$. The following shows that i_T is a conservative interpretation of T in $L(T)$.

Proposition 4.8. *$T \vdash \phi$ if and only if $i_T(\phi) = 1$.*

Proof. $T \vdash \phi$ iff $T \vdash (\psi \vee \neg\psi) \leftrightarrow \phi$ iff $i_T(\phi) = E_\phi = E_{\psi\vee\neg\psi} = 1$. □

Since $i_T(\phi) = E_\phi$, the interpretation i_T is also surjective.

Proposition 4.9. *Let B be a Boolean algebra, and let $f : T \to B$ be an interpretation. Then there is a unique homomorphism $\overline{f} : L(T) \to B$ such that $\overline{f} i_T = f$.*

Proof. If $E_\phi = E_\psi$, then $T \vdash \phi \leftrightarrow \psi$, and so $f(\phi) = f(\psi)$. Thus, we may define $\hat{f}(E_\phi) = f(\phi)$. It is straightforward to verify that $\hat{f}$ is a Boolean homomorphism, and it is clearly unique. □

Definition. The Boolean algebra $L(T)$ is called the **Lindenbaum algebra** of T.

Proposition 4.10. *Let B be a Boolean algebra. There is a theory T_B and a conservative, surjective interpretation $e_B : T_B \to B$ such that for any theory T, and interpretation $f : T \to B$, there is a unique interpretation $\overline{f} : T \to T_B$ such that $e_B\overline{f} = f$.*

$$\begin{array}{ccc} T_B & \xrightarrow{e_B} & B \\ {\scriptstyle \overline{f}}\uparrow & \nearrow_{f} & \\ T & & \end{array}$$

Proof. Let $\Sigma_B = B$ be a signature. (Recall that a propositional signature is just a set, where each element represents an elementary proposition.) We define $e_B : \Sigma_B \to B$ as the identity, and use the symbol e_B also for its extension to $\mathsf{Sent}(\Sigma_B)$. We define a theory T_B on Σ_B by: $T_B \vdash \phi$ if and only if $e_B(\phi) = 1$. Thus, $e_B : T_B \to B$ is automatically a conservative interpretation of T_B in B.

Now let T be some theory in signature Σ, and let $f : T \to B$ be an interpretation. Since $\Sigma_B = B$, f automatically gives rise to a reconstrual $f : \Sigma \to \Sigma_B$, which we will rename $\overline{f}$ for clarity. And since e_B is just the identity on $B = \Sigma_B$, we have $f = e_B\overline{f}$.

Finally, to see that $\overline{f} : T \to T_B$ is a translation, suppose that $T \vdash \phi$. Since f is an interpretation of T_B, $f(\phi) = 1$, which means that $e_B(\overline{f}(\phi)) = 1$. Since e_B is conservative, $T_B \vdash \overline{f}(\phi)$. Therefore, $\overline{f}$ is a translation. □

We have shown that each propositional theory T corresponds to a Boolean algebra $L(T)$, and each Boolean algebra B corresponds to a propositional theory T_B. We will now show that these correspondences are functorial. First we show that a morphism $f : B \to A$ in **Bool** naturally gives rise to a morphism $T(f) : T_B \to T_A$ in **Th**. Indeed, consider the following diagram:

$$\begin{array}{ccc} T_B & \xrightarrow{T(f)} & T_A \\ \downarrow{\scriptstyle e_B} & & \downarrow{\scriptstyle e_A} \\ B & \xrightarrow{f} & A \end{array}$$

Since fe_B is an interpretation of T_B in A, Prop. 4.10 entails that there is a unique translation $T(f) : T_B \to T_A$ such that $e_A T(f) = fe_B$. The uniqueness clause also entails that T commutes with composition of morphisms, and maps identity morphisms to identity morphisms. Thus, $T : \mathbf{Bool} \to \mathbf{Th}$ is a functor.

Let's consider this translation $T(f) : T_B \to T_A$ more concretely. First of all, recall that translations from T_B to T_A are actually equivalence classes of maps from Σ_B to $\mathsf{Sent}(\Sigma_A)$. Thus, there's no sense to the question, "which function is $T(f)$?" However, there's a natural choice of a representative function. Indeed, consider f itself as a function from $\Sigma_B = B$ to $\Sigma_A = A$. Then, for $x \in \Sigma_B = B$, we have

$$(e_A \circ T(f))(x) = e_A(f(x)) = f(x) = f(e_B(x)),$$

since e_A is the identity on Σ_A, and e_B is the identity on Σ_B. In other words, $T(f)$ is the equivalence class of f itself. [But recall that translations, while initially defined on the signature Σ_B, extend naturally to all elements of $\mathsf{Sent}(\Sigma_B)$. From this point of view, $T(f)$ has a larger domain than f.]

A similar construction can be used to define the functor $L : \mathbf{Th} \to \mathbf{Bool}$. In particular, let $f : T \to T'$ be a morphism in $\mathbf{Th}$, and consider the following diagram:

$$\begin{array}{ccc} T & \xrightarrow{\;f\;} & T' \\ \downarrow{\scriptstyle i_T} & & \downarrow{\scriptstyle i_{T'}} \\ L(T) & \overset{L(f)}{\dashrightarrow} & L(T') \end{array}$$

Since $i_{T'} f$ is an interpretation of T in $L(T')$, Prop. 4.6 entails that there is a unique homomorphism $L(f) : L(T) \to L(T')$ such that $L(f) i_T = i_{T'} f$.

More explicitly,

$$L(f)(E_\phi) \;=\; L(f)(i_T(\phi)) \;=\; i_{T'} f(\phi) \;=\; E_{f(\phi)}.$$

Recall, however, that identity of arrows in $\mathbf{Th}$ is *not* identity of the corresponding functions, in the set-theoretic sense. Rather, $f \simeq g$ just in case $T' \vdash f(\phi) \leftrightarrow g(\phi)$, for all $\phi \in \mathsf{Sent}(\Sigma)$. Thus, we must verify that if $f \simeq g$ in $\mathbf{Th}$, then $L(f) = L(g)$. Indeed, since $i_{T'}$ is an interpretation of T', we have $i_{T'}(f(\phi)) = i_{T'}(g(\phi))$; and since the diagram above commutes, $L(f) \circ i_T = L(g) \circ i_T$. Since i_T is surjective, $L(f) = L(g)$. Thus, $f \simeq g$ only if $L(f) = L(g)$. Finally, the uniqueness clause in Prop. 4.6 entails that L commutes with composition, and maps identities to identities. Therefore, $L : \mathbf{Th} \to \mathbf{Bool}$ is a functor.

We will soon show that the functor $L : \mathbf{Th} \to \mathbf{Bool}$ is an equivalence of categories, from which it follows that L preserves all categorically-definable properties. For example, a translation $f : T \to T'$ is monic if and only if $L(f) : L(T) \to L(T')$ is monic, etc.. However, it may be illuminating to prove some such facts directly.

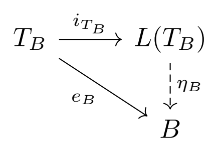
Proposition 4.11. *Let $f : T \to T'$ be a translation. Then f is conservative if and only if $L(f)$ is injective.*

Proof. Suppose first that f is conservative. Let $E_\phi, E_\psi \in L(T)$ such that $L(f)(E_\phi) = L(f)(E_\psi)$. Using the definition of $L(f)$, we have $E_{f(\phi)} = E_{f(\psi)}$, which means that $T' \vdash f(\phi) \leftrightarrow f(\psi)$. Since f is conservative, $T \vdash \phi \leftrightarrow \psi$, from which $E_\phi = E_\psi$. Therefore, $L(f)$ is injective.

Suppose now that $L(f)$ is injective. Let ϕ be a Σ sentence such that $T' \vdash f(\phi)$. Since $f(\top) = \top$, we have $T' \vdash f(\top) \leftrightarrow f(\phi)$, which means that $L(f)(E_\top) = L(f)(E_\phi)$. Since $L(f)$ is injective, $E_\top = E_\phi$, from which $T \vdash \phi$. Therefore, f is conservative. □

Proposition 4.12. *For any Boolean algebra B, there is a natural isomorphism $\eta_B : B \to L(T_B)$.*

Proof. Let $e_B : T_B \to B$ be the interpretation from Prop. 4.10, and let $i_{T_B} : T_B \to L(T_B)$ be the interpretation from Prop. 4.6. Consider the following diagram:



By Prop. 4.6, there is a unique homomorphism $\eta_B : L(T_B) \to B$ such that $e_B = \eta_B i_{T_B}$. Since e_B is the identity on Σ_B,

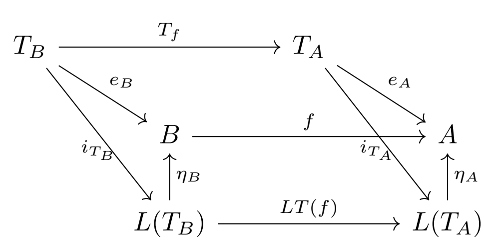
$$\eta_B(E_x) = \eta_B i_{T_B}(x) = e_B(x) = x,$$

for any $x \in B$. Thus, if η_B has an inverse, it must be given by the map $x \mapsto E_x$. We claim that this map is a Boolean homomorphism. To see this, recall that $\Sigma_B = B$. Moreover, for $x, y \in B$, the Boolean meet $x \cap y$ is again an element of B, hence an element of the signature Σ_B. By the defintion of T_B, we have $T_B \vdash (x \cap y) \leftrightarrow (x \wedge y)$, where the $\wedge$ symbol on the right is conjunction in $\mathsf{Sent}(\Sigma_B)$. Thus,

$$E_{x \cap y} = E_{x \wedge y} = E_x \cap E_y.$$

A similar argument shows that $E_{-x} = -E_x$. Therefore, $x \mapsto E_x$ is a Boolean homomorphism, and η_B is an isomorphism.

It remains to show that η_B is natural in B. Consider the following diagram:



The top square commutes by the definition of the functor T. The triangles on the left and right commute by the definition of η. And the outmost square commutes by the definition of the functor L. Thus we have

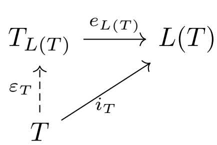
$$\begin{aligned} f \circ \eta_B \circ i_{T_B} &= f \circ e_B \\ &= e_A \circ T_f \\ &= \eta_A \circ i_{T_A} \circ T_f \\ &= \eta_A \circ LT(f) \circ i_{T_B}. \end{aligned}$$

Since i_{T_B} is surjective, it follows that $f \circ \eta_B = \eta_A \circ LT(f)$, and therefore η is a natural transformation. □

Discussion. Consider the algebra $L(T_B)$, which we have just proved is isomorphic to B. This result is hardly surprising. For any $x, y \in \Sigma_B$, we have $T_B \vdash x \leftrightarrow y$ if and only if $x = e_B(x) = e_B(y) = y$. Thus, the equivalence class E_x contains x and no other element from Σ_B. [That's why $\eta_B(E_x) = x$ makes sense.] We also know that for every $\phi \in \mathsf{Sent}(\Sigma_B)$, there is an $x \in \Sigma_B = B$ such that $T_B \vdash x \leftrightarrow \phi$. In particular, $T_B \vdash e_B(\phi) \leftrightarrow \phi$. Thus, $E_\phi = E_x$, and there is a natural bijection between elements of $L(T_B)$ and elements of B.

Proposition 4.13. *For any theory T, there is a natural isomorphism $\varepsilon_T : T \to T_{L(T)}$.*

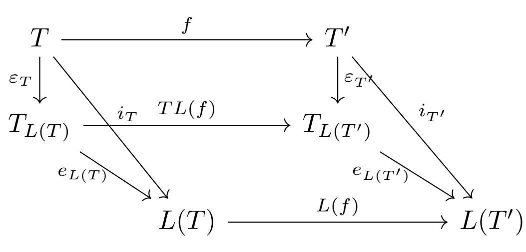
Proof. Consider the following diagram:



By Prop. 4.10, there is a unique interpretation $\varepsilon_T : T \to T_{L(T)}$ such that $e_{L(T)}\varepsilon_T = i_T$. We claim that ε_T is an isomorphism. To see that ε_T is conservative, suppose that $T_{L(T)} \vdash \varepsilon_T(\phi)$. Since $e_{L(T)}$ is an interpretation, $e_{L(T)}\varepsilon_T(\phi) = 1$ and hence $i_T(\phi) = 1$. Since i_T is conservative, $T \vdash \phi$. Therefore ε_T is conservative.

To see that ε_T is essentially surjective, suppose that $\psi \in \mathsf{Sent}(\Sigma_{L(T)})$. Since i_T is surjective, there is a $\phi \in \mathsf{Sent}(\Sigma)$ such that $i_T(\phi) = e_{L(T)}(\psi)$. Thus, $e_{L(T)}(\varepsilon_T(\phi)) = e_{L(T)}(\psi)$. Since $e_{L(T)}$ is conservative, $T_{L(T)} \vdash \varepsilon_T(\phi) \leftrightarrow \psi$. Therefore, ε_T is essentially surjective.

It remains to show that ε_T is natural in T. Consider the following diagram:



The triangles on the left and the right commute by the definition of ε. The top square commutes by the definition of L, and the bottom square commutes by the definition of T. Thus, we have

$$\begin{aligned} e_{L(T')} \circ \varepsilon_{T'} \circ f &= i_{T'} \circ f \\ &= L(f) \circ i_T \\ &= L(f) \circ e_{L(T)} \circ \varepsilon_T \\ &= e_{L(T')} \circ TL(f) \circ \varepsilon_T . \end{aligned}$$

Since $e_{L(T')}$ is conservative, $\varepsilon_{T'} \circ f = TL(f) \circ \varepsilon_T$. Therefore ε_T is natural in T. □

Discussion. Recall that ε_T doesn't denote a unique function; it denotes an equivalence class of functions. One representative of this equivalence class is the function $\varepsilon_T : \Sigma \to \Sigma_{L(T)}$ given by $\varepsilon_T(p) = E_p$. In this case, a straightforward inductive argument shows that $T_{L(T)} \vdash E_\phi \leftrightarrow \varepsilon_T(\phi)$, for all $\phi \in \mathsf{Sent}(\Sigma)$.

We know that ε_T has an inverse, which itself is an equivalence class of functions from $\Sigma_{L(T)}$ to $\mathsf{Sent}(\Sigma)$. We can define a representative f of this equivalence class by choosing, for each $E \in \Sigma_{L(T)} = L(T)$, some $\phi \in E$, and setting $f(E) = \phi$. Another straightforward argument shows that if we made a different set of choices, the resulting function f' would be equivalent to f, i.e. it would correspond to the same translation from $T_{L(T)}$ to T.

Based on these definitions, $f\varepsilon_T(p) = f(E_p)$ is some $\phi \in E_p$, i.e. some ϕ such that $T \vdash p \leftrightarrow \phi$. Thus, $f\varepsilon_T \cong 1_T$. Similarly, $f(E_\phi) = \psi$, for some $\psi \in E_\phi$, and hence ε_T ...

Since there are natural isomorphisms $\varepsilon : 1_{\mathbf{Th}} \Rightarrow TL$ and $\eta : 1_{\mathbf{Bool}} \Rightarrow LT$, we have the following result:

Lindenbaum Theorem

The categories **Th** and **Bool** are equivalent.

5 Boolean algebras again

The Lindenbaum Theorem would deliver everything we wanted — if we had a perfectly clear understanding of the category **Bool**. However, there remain questions about **Bool**. For example, are all epimorphisms in **Bool** surjections? In order to shed even more light on **Bool**, and hence on **Th**, we will show that **Bool** is dual to a certain category of topological spaces. This famous result is called the **Stone Duality Theorem**. But in order to prove it, we need to collect a few more facts about Boolean algebras.

Definition. Let B be a Boolean algebra. A subset $F \subseteq B$ is said to be a **filter** just in case:

1. If $a, b \in F$ then $a \wedge b \in F$;
2. If $a \in F$ and $a \leq b$ then $b \in F$.

If, in addition, $F \neq B$, then we say that F is a **proper filter**. We say that F is an **ultrafilter** just in case F is maximal among proper filters, i.e. if $F \subseteq F'$ where F' is a proper filter, then $F = F'$.

Discussion. Consider the Boolean algebra B as a theory. Then a filter $F \subseteq B$ can be thought of as supplying an update of information. The first condition says that if we learn a and b, then we've learned $a \wedge b$. The second condition says that if we learn a, and $a \leq b$, then we've learned b. In particular, an ultrafilter supplies maximal information.

Exercise. Let F be a filter. Show that F is proper if and only if $0 \notin F$.

Definition. Let $F \subseteq B$ be a filter, and $a \in B$. We say that a is **compatible** with F just in case $a \wedge x \neq 0$ for all $x \in F$.

Lemma 5.1. *Let $F \subseteq B$ be a proper filter, and let $a \in B$. Then either a or $\neg a$ is compatible with F.*

Proof. Suppose for reductio ad absurdum that neither a nor $\neg a$ is compatible with F. That is, there is an $x \in F$ such that $x \wedge a = 0$, and there is a $y \in F$ such that $y \wedge \neg a = 0$. Then

$$x \wedge y = (x \wedge y) \wedge (a \vee \neg a) = (x \wedge y \wedge a) \vee (x \wedge y \wedge \neg a) = 0.$$

Since $x, y \in F$, it follows that $0 = x \wedge y \in F$, contradicting the assumption that F is proper. Therefore either a or $\neg a$ is compatible with F. $\square$

Proposition 5.2. *Let F be a proper filter on B. Then the following are equivalent:*

1. *F is an ultrafilter.*
2. *For all $a \in B$, either $a \in F$ or $\neg a \in F$.*
3. *For all $a, b \in B$, if $a \vee b \in F$ then either $a \in F$ or $b \in F$.*

Proof. ($1 \Rightarrow 2$) Suppose that F is an ultrafilter. By Lemma 5.1, either a or $\neg a$ is compatible with F. Suppose first that a is compatible with F. Then the set

$$F' = \{y : x \wedge a \leq y, \text{ some } x \in F\},$$

is a proper filter that contains F and a. Since F is an ultrafilter, $F' = F$, and hence $a \in F$. By symmetry, if $\neg a$ is compatible with F, then $\neg a \in F$.

(2 ⇒ 3) Suppose that $a \vee b \in F$. By 2, either $a \in F$ or $\neg a \in F$. If $\neg a \in F$, then $\neg a \wedge (a \vee b) \in F$. But $\neg a \wedge (a \vee b) \leq b$, and so $b \in F$.

(3 ⇒ 1) Suppose that F' is a filter that contains F, and let $a \in F' - F$. Since $a \vee \neg a = 1 \in F$, it follows from (3) that $\neg a \in F$. But then $0 = a \wedge \neg a \in F'$, that is $F' = B$. Therefore F is an ultrafilter. □

Proposition 5.3. *There is a bijective correspondence between ultrafilters in B and homomorphisms from B into 2. In particular, for any homomorphism $f : B \to 2$, the subset $f^{-1}(1)$ is an ultrafilter in B.*

Proof. Let U be an ultrafilter on B. Define $f : B \to 2$ by setting $f(a) = 1$ iff $a \in U$. Then

$$\begin{aligned} f(a \wedge b) = 1 \quad & \text{iff} \quad a \wedge b \in U \\ & \text{iff} \quad a \in U \,\text{and}\, b \in U \\ & \text{iff} \quad f(a) = 1 \text{ and } f(b) = 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} f(\neg a) = 1 \quad & \text{iff} \quad \neg a \in U \\ & \text{iff} \quad a \notin U \\ & \text{iff} \quad f(a) = 0. \end{aligned}$$

Therefore f is a homomorphism.

Now suppose that $f : B \to 2$ is a homomorphism, and let $U = f^{-1}(1)$. Since $f(a) = 1$ and $f(b) = 1$ only if $f(a \wedge b) = 1$, it follows that U is closed under conjunction. Since $a \leq b$ only if $f(a) \leq f(b)$, it follows that U is closed under implication. Finally, since $f(a) = 0$ iff $f(\neg a) = 1$, it follows that $a \notin U$ iff $\neg a \in U$. □

Definition. For $a, b \in B$, define

$$a \to b \;:=\; \neg a \vee b,$$

and define

$$a \leftrightarrow b \;:=\; (a \to b) \wedge (b \to a).$$

It's straightforward to check that $\to$ behaves like the conditional from propositional logic. The next lemma gives a Boolean algebra version of modus ponens.

Lemma 5.4. *Let F be a filter. If $a \to b \in F$ and $a \in F$ then $b \in F$.*

Proof. Suppose that $\neg a \vee b = a \to b \in F$ and $a \in F$. We then compute:

$$b = b \vee 0 = b \vee (a \wedge \neg a) = (a \vee b) \wedge (\neg a \vee b).$$

Since $a \in F$ and $a \leq a \vee b$, we have $a \vee b \in F$. Since F is a filter, $b \in F$. □

Exercises:

1. Let B be a Boolean algebra, and let $a, b, c \in B$. Show that the following hold:
   (a) $(a \to b) = 1$ iff $a \leq b$
   (b) $(a \wedge b) \leq c$ iff $a \leq (b \to c)$
   (c) $a \wedge (a \to b) \leq b$
   (d) $(a \leftrightarrow b) = (b \leftrightarrow a)$
   (e) $(a \leftrightarrow a) = 1$
   (f) $(a \leftrightarrow 1) = a$
2. Let $\mathscr{P}N$ be the powerset of the natural numbers, and let $\mathscr{U}$ be an ultrafilter on $\mathscr{P}N$. Show that if $\mathscr{U}$ contains a finite set F, then $\mathscr{U}$ contains a singleton set.

Definition. Let B be a Boolean algebra, and let R be an equivalence relation on the underlying set of B. We say that R is a **congruence** just in case R is compatible with the operations on B in the following sense: if aRa' and bRb' then $(a \wedge b)R(a' \wedge b')$, and $(a \vee b)R(a' \vee b')$, and $(\neg a)R(\neg a')$.

In a category $\mathbf{C}$ with limits (products, equalizers, pullbacks, etc.), it's possible to formula the notion of an equivalence relation in $\mathbf{C}$. Thus, in **Bool**, an equivalence relation R on B is a subalgebra R of $B \times B$ that satisfies the appropriate analogues of reflexivity, symmetry, and transitivity. Since R is a subalgebra of $B \times B$, it follows in particular that if $\langle a, b\rangle \in R$, and $\langle a', b'\rangle \in R$, then $\langle a \wedge a', b \wedge b'\rangle \in R$. Continuing this reasoning, it's not difficult to see that congruences, as defined above, are precisely the equivalence relations in the category **Bool** of Boolean algebras. Thus, in the remainder of this chapter, when we speak of an equivalence relation on a Boolean algebra B, we mean an equivalence relation in **Bool**, in other words, a congruence. (To be clear, not every equivalence relation on the set B is an equivalence relation on the Boolean algebra B.)

Now suppose that $\mathbf{C}$ is a category in which equivalence relations are definable, and let $p_0, p_1 : R \rightrightarrows B$ be an equivalence relation. [Here p_0 and p_1 are the projections of R, considered as a subobject of $B \times B$.] Then we can ask: do these two maps p_0 and p_1 have a coequalizer? That is, is there an object B/R, and a map $q : B \to B/R$, with the relevant universal property? In the case of **Bool**, a coequalizer can be constructed directly. We merely note that the Boolean operations on B can be used to induce Boolean operations on the set B/R of equivalence classes.

Definition (Quotient algebra)**.** Suppose that R is an equivalence relation on B. For each $a \in B$, let E_a denote its equivalence class, and let $B/R = \{E_a \mid a \in B\}$. We then define $E_a \wedge E_b = E_{a \wedge b}$, and similarly for $E_a \vee E_b$ and $\neg E_a$. Since R is a congruence (i.e. an equivalence relation on **Bool**), these operations are well-defined. It then follows immediately that B/R is a Boolean algebra, and the quotient map $q : B \to B/R$ is a surjective Boolean homomorphism.

Lemma 5.5. *Let $R \subseteq B \times B$ be an equivalence relation. Then $q : B \to B/R$ is the coequalizer of the projection maps $p_0 : R \to B$ and $p_1 : R \to B$. In particular, q is a regular epimorphism.*

Proof. It is obvious that $qp_0 = qp_1$. Now suppose that A is another Boolean algebra and $f : B \to A$ such that $fp_0 = fp_1$. Define $g : B/R \to A$ by setting $g(E_x) = f(x)$. Since $fp_0 = fp_1$, g is well-defined. Furthermore,

$$g(E_x \wedge E_y) = g(E_{x\wedge y}) = f(x \wedge y) = f(x) \wedge f(y) = g(E_x) \wedge g(E_y).$$

Similarly, $g(\neg E_x) = \neg g(E_x)$. Therefore g is a Boolean homomorphism. Since q is an epimorphism, g is the unique homomorphism such that $gq = f$. Therefore, $q : B \to B/R$ is the coequalizer of p_0 and p_1. □

The category **Bool** has further useful structure: there is a one-to-one correspondence between equivalence relations and filters.

Lemma 5.6. *Suppose that $R \subseteq B \times B$ is an equivalence relation. Let $F = \{a \in B \mid aR1\}$. Then F is a filter, and $R = \{\langle a, b\rangle \in B \times B \mid a \leftrightarrow b \in F\}$.*

Proof. Suppose that $a, b \in F$. That is, $aR1$ and $bR1$. Since R is a congruence, $(a\wedge b)R(1\wedge 1)$ and therefore $(a\wedge b)R1$. That is, $a\wedge b \in F$. Now suppose that x is an arbitrary element of B such that $a \leq x$. That is, $x \vee a = x$. Since R is a congruence, $(x \vee a)R(x \vee 1)$ and so $(x \vee a)R1$, from which it follows that $xR1$. Therefore $x \in F$, and F is a filter.

Now suppose that aRb. Since R is reflexive, $(a\vee\neg a)R1$, and thus $(b\vee\neg a)R1$. Similarly $(a \vee \neg b)R1$, and therefore $(a \leftrightarrow b)R1$. That is, $a \leftrightarrow b \in F$. □

Lemma 5.7. *Suppose that F is a filter on B. Let $R = \{\langle a, b\rangle \in B \times B \mid a \leftrightarrow b \in F\}$. Then R is an equivalence relation, and $F = \{a \in B \mid aR1\}$.*

Proof. Showing that R is an equivalence relation requires several straightforward verifications. For example, $a \leftrightarrow a = 1$, and $1 \in F$, therefore aRa. We leave the remaining verifications to the reader.

Now suppose that $a \in F$. Since $a = (a \leftrightarrow 1)$, it follows that $a \leftrightarrow 1 \in F$, which means that $aR1$. □

Definition (Quotient algebra)**.** Let F be a filter on B. Given the correspondence between filters and equivalence relations, we write B/F for the corresponding algebra of equivalence classes.

Proposition 5.8. *Let F be a proper filter on B. Then B/F is a two-element Boolean algebra if and only if F is an ultrafilter.*

Proof. Suppose first that $B/F \cong 2$. That is, for any $a \in B$, either $a \leftrightarrow 1 \in F$ or $a \leftrightarrow 0 \in F$. But $a \leftrightarrow 1 = a$ and $a \leftrightarrow 0 = \neg a$. Therefore, either $a \in F$ or $\neg a \in F$, and F is an ultrafilter.

Suppose now that F is an ultrafilter. Then for any $a \in B$, either $a \in F$ or $\neg a \in F$. In the former case, $a \leftrightarrow 1 \in F$. In the latter case, $a \leftrightarrow 0 \in F$. Therefore, $B/F \cong 2$. □

Exercise. (This exercise presupposes knowledge of measure theory.) Let Σ be the Boolean algebra of Borel subsets of $[0,1]$, and let μ be Lebesgue measure on $[0,1]$. Let $\mathscr{F} = \{S \in \Sigma \mid \mu(S) = 1\}$.

1. Show that $\mathscr{F}$ is a filter.

2. Describe the equivalence relation on Σ corresponding to $\mathscr{F}$.

According to our motivating analogy, a Boolean algebra B is like a theory, and a homorphism $\phi : B \to 2$ is like a model of this theory. We say that the algebra B is **syntactically consistent** just in case $0 \neq 1$. (In fact, we defined Boolean algebras so as to require syntactic consistency.) We say that the algebra B is **semantically consistent** just in case there is a homomorphism $\phi : B \to 2$. Then semantic consistency clearly implies syntactic consistency. But does syntactic consistency imply semantic consistency?

It's at this point that we have to invoke a powerful theorem — or, perhaps more accurately, a powerful set-theoretic axiom. In short, if we use the axiom of choice, or some equivalent such as Zorn's lemma, then we can prove that every syntactically consistent Boolean algebra is semantically consistent. However, we do not actually need the full power of the Axiom of Choice. As set-theorists know, the Boolean ultrafilter axiom ("UF" for short) is strictly weaker than the axiom of choice.[2]

Proposition 5.9. *The following are equivalent:*

1. ***Boolean ultrafilter axiom (UF)*** *For any Boolean algebra B, there is a homomorphism $f : B \to 2$.*

2. *For any Boolean algebra B, and proper filter $F \subseteq B$, there is a homomorphism $f : B \to 2$ such that $f(a) = 1$ when $a \in F$.*

3. *For any Boolean algebra B, if $a, b \in B$ such that $a \neq b$, then there is a homomorphism $f : B \to 2$ such that $f(a) \neq f(b)$.*

4. *For any Boolean algebra B, if $\phi(a) = 1$ for all $\phi : B \to 2$, then $a = 1$.*

5. *For any two Boolean algebras A, B, and homomorphisms $f, g : A \rightrightarrows B$, if $\phi f = \phi g$ for all $\phi : B \to 2$, then $f = g$.*

Proof. ($1 \Rightarrow 2$) Suppose that F is a proper filter in B. Then there is a homomorphism $q : B \to B/F$ such that $q(a) = 1$ for all $a \in F$. By UF, there is a homomorphism $\phi : B/F \to 2$. Therefore, $\phi \circ q : B \to 2$ is a homomorphism such that $(\phi \circ q)(a) = 1$ for all $a \in F$.

($1 \Rightarrow 3$) Suppose that $a, b \in B$ with $a \neq b$. Then either $\neg a \wedge b \neq 0$ or $a \wedge \neg b \neq 0$. Without loss of generality, we assume that $\neg a \wedge b \neq 0$. In this case,

[2] Algebraists often invoke an equivalent axiom called the Boolean Prime Ideal Theorem (BPI). It's often called a "Theorem" because it can be proven from the Axiom of Choice, or equivalently, Zorn's Lemma. But BPI can also be taken as an axiom, in which case it's strictly weaker than AC.

the filter F generated by $\neg a \wedge b$ is proper. By UF, there is a homomorphism $\phi : B \to 2$ such that $\phi(x) = 1$ when $x \in F$. In particular, $\phi(\neg a \wedge b) = 1$. But then $\phi(a) = 0$ and $\phi(b) = 1$.

($2 \Rightarrow 4$) Suppose that $\phi(a) = 1$ for all $\phi : B \to 2$. Now let F be the filter generated by $\neg a$. If F is proper, then by (2), there is a $\phi : B \to 2$ such that $\phi(\neg a) = 1$, a contradiction. Thus, $F = B$, which implies that $\neg a = 0$ and $a = 1$.

($4 \Rightarrow 5$) Let $f, g : A \to B$ be homomorphisms, and suppose that for all $\phi : B \to 2$, $\phi f = \phi g$. That is, for each $a \in A$, $\phi(f(a)) = \phi(g(a))$. But then $\phi(f(a) \leftrightarrow g(a)) = 1$ for all $\phi : B \to 2$. By (4), $f(a) \leftrightarrow g(a) = 1$, and therefore $f(a) = g(a)$.

($5 \Rightarrow 3$) Let B be a Boolean algebra, and $a, b \in B$. Suppose that $\phi(a) = \phi(b)$ for all $\phi : B \to 2$. Let F be the four element Boolean algebra, with generator p. Then there is a homomorphism $\hat{a} : F \to B$ such that $\hat{a}(p) = a$, and a homomorphism $\hat{b} : F \to B$ such that $\hat{b}(p) = b$. Thus, $\phi\hat{a} = \phi\hat{b}$ for all $\phi : B \to 2$. By (5), $\hat{a} = \hat{b}$, and therefore $a = b$.

($3 \Rightarrow 1$) Let B be an arbitrary Boolean algebra. Since $0 \neq 1$, (3) implies that there is a homomorphism $\phi : B \to 2$. □

We are finally in a position to prove the completeness of the propositional calculus. The following result assumes the Boolean ultrafilter axiom (UF).

Completeness Theorem

If $T \vDash \phi$ then $T \vdash \phi$.

Proof. Suppose that $T \nvdash \phi$. Then in the Lindenbaum algebra $L(T)$, we have $E_\phi \neq 1$. In this case, there is a homomorphism $h : L(T) \to 2$ such that $h(E_\phi) = 0$. Hence, $h \circ i_T$ is a model of T such that $(h \circ i_T)(\phi) = h(E_\phi) = 0$. Therefore, $T \nvDash \phi$. □

Exercise. Let $\mathscr{P}N$ be the powerset of the natural numbers. We say that a subset E of N is **cofinite** just in case $N \backslash E$ is finite. Let $\mathscr{F} \subseteq \mathscr{P}N$ be the set of cofinite subsets of N.

1. Show that $\mathscr{F}$ is a filter.

2. Show that there are infinitely many ultrafilters containing $\mathscr{F}$.

6 Stone spaces

Philosophers like to talk about the, "space of possible worlds." Logicians and mathematicians like to talk about, "sets of models." Physicists like to talk about, "models of a theory." It's all pretty much the same thing, at least from an abstract point of view. But if we're going to undertake an exact study of the space of possible worlds, then we need to make a proposal about what structure this space carries. But what do I mean by "structure"? Isn't the space of possible worlds just a bare set? Let me explain a couple of reasons why we might want to think of the space of possible worlds as a structured set, and in particular, as a topological space.

Suppose that there are infinitely many possible worlds, which we represent by elements of a set X. As philosophers are wont to do, we then represent **propositions** by subsets of X. But should we think that all $2^{|X|}$ subsets of X correspond to genuine propositions? What would warrant such a claim?

There is another reason to worry about this approach. For a person with training in set theory, it is not difficult to build a collection $C_1, C_2, \dots$ of subsets of X with the following features: (i) each C_i is non-empty, (ii) $C_{i+1} \subseteq C_i$ for all i, and (iii) $\bigcap_i C_i$ is empty. Intuitively speaking, $\{C_i \mid i \in \mathbb{N}\}$ is a family of propositions that are individually consistent (since non-empty), that are becoming more and more specific, and yet there is no world in X that makes all C_i true. Why not? It seems that X is missing some worlds! Indeed, here's a description of a new world w that does not belong to X: for each proposition ϕ, let ϕ be true in w if and only if $\phi \cap C_i$ is nonempty for all i. It's not difficult to see that w is in fact a truth valuation on the set of all propositions, i.e. it is a possible world. But w is not represented by a point in X. What we have here is a mismatch between the set X of worlds, and the set of propositions describing these worlds.

The idea behind logical topology is that not all subsets of X correspond to propositions. A designation of a topology on X is tantamount to saying which subsets of X correspond to propositions. However, the original motivation for the study of topology comes from geometry (and analysis), not from logic. Recall high school mathematics, where you learned that a continuous function is one where you don't have to lift your pencil from the paper in order to draw the graph. If your high school class was really good, or if you studied calculus in college, then you will have learned that there is a more rigorous definition of a continuous function — a definition involving epsilons and deltas. In the early 20th century, it was realized that the essence of continuity is even more abstract than epsilons and deltas would suggest: all we need is a notion of nearness of points, which we can capture in terms of a notion of a neighborhood of a point. The idea then is that a function $f : X \to Y$ is continuous at a point x just in case for any neighborhood V of $f(x)$, there is some neighborhood U of x such that $f(U) \subseteq V$. Intuitively speaking, f preserves closeness of points.

Notice, however, that if X is an arbitrary set, then it's not obvious what "closeness" means. To be able to talk about closeness of points in X, we need specify which subsets of X count as the neighborhoods of points. Thus, a

topology on X is a set of subsets of X that satisfies certain conditions.

Definition. A **topological space** is a set X and a family $\mathscr{F}$ of subsets of X satisfying the following conditions:

1. $\emptyset \in \mathscr{F}$ and $X \in \mathscr{F}$;
2. If $U, V \in \mathscr{F}$ then $U \cap V \in \mathscr{F}$;
3. If $\mathscr{F}_0$ is a subfamily of $\mathscr{F}$, then $\bigcup_{U \in \mathscr{F}_0} U \in \mathscr{F}$.

The sets in $\mathscr{F}$ are called **open subsets** of the space $(X, \mathscr{F})$. If $p \in U$ with U an open subset, we say that U is a **neighborhood** of p.

There are many familiar examples of topological spaces. In many cases, however, we only know the open sets indirectly, by means of certain nice open sets. For example, in the case of the real numbers, not every open subset is an interval. However, every open subset is a union of intervals. In that case, we call the open intervals in $\mathbb{R}$ a **basis** for the topology.

Proposition 6.1. *Let $\mathscr{B}$ be a family of subsets of X with the property that if $U, V \in \mathscr{B}$ then $U \cap V \in \mathscr{B}$. Then there is a unique smallest topology $\mathscr{F}$ on X containing $\mathscr{B}$.*

Proof. Let $\mathscr{F}$ be the collection obtained by taking all unions of sets in $\mathscr{B}$, and then taking finite intersections of the resulting collection. Clearly $\mathscr{F}$ is a topology on X, and any topology on X containing $\mathscr{B}$ also contains $\mathscr{F}$. □

Definition. If $\mathscr{B}$ is a family of subsets of X that is closed under intersection, and if $\mathscr{F}$ is the topology generated by $\mathscr{B}$, then we say that $\mathscr{B}$ is a **basis** for $\mathscr{F}$.

Proposition 6.2. *Let $(X, \mathscr{F})$ be a topological space. Let $\mathscr{F}_0$ be a subfamily of $\mathscr{F}$ with the following properties: (1) $\mathscr{F}_0$ is closed under finite intersections, and (2) for each $x \in X$ and $U \in \mathscr{F}_0$ with $x \in U$, there is a $V \in \mathscr{F}_0$ such that $x \in V \subseteq U$. Then $\mathscr{F}_0$ is a basis for the topology $\mathscr{F}$.*

Proof. We need only show that each $U \in \mathscr{F}$ is a union of elements in $\mathscr{F}_0$. And that follows immediately from the fact that if $x \in U$, then there is $V \in \mathscr{F}_0$ with $x \in V \subseteq U$. □

Definition. Let X be a topological space. A subset C of X is called **closed** just in case $C = X \backslash U$ for some open subset U of X. The intersection of closed sets is closed. Hence, for each subset E of X, there is a unique smallest closed set $\overline{E}$ containing E, namely the intersection of all closed supersets of E. We call $\overline{E}$ the **closure** of E.

Proposition 6.3. *Let $p \in X$ and let $S \subseteq X$. Then $p \in \overline{S}$ if and only if every open neighborhood U of p has nonempty intersection with S.*

Proof. Exercise. □

Definition. Let S be a subset of X. We say that S is **dense** in X just in case $\overline{S} = X$.

Definition. Let $E \subseteq X$. We say that p is a **limit point** of E just in case for each open neighborhood U of p, $U \cap E$ contains some point besides p. We let E' denote the set of all limit points of E.

Lemma 6.4. $E' \subseteq \overline{E}$.

Proof. Let $p \in E'$, and let C be a closed set containing E. If $p \in X \backslash C$, then p is contained in an open set that has empty intersection with E. Thus, $p \in C$. Since C was an arbitrary closed superset of E, it follows that $p \in \overline{E}$. □

Proposition 6.5. $\overline{E} = E \cup E'$

Proof. The previous lemma gives $E' \subseteq \overline{E}$. Thus, $E \cup E' \subseteq \overline{E}$.

Suppose now that $p \notin E$ and $p \notin E'$. Then there is an open neighborhood U of p such that $U \cap E$ is empty. Then $E \subseteq X \backslash U$, and since $X \backslash U$ is closed, $\overline{E} \subseteq X \backslash U$. Therefore $p \notin \overline{E}$.

□

Definition. A topological space X is said to be:

- T_1 or **Frechet** just in case all singleton subsets are closed.
- T_2 or **Hausdorff** just in case for any $x, y \in X$ if $x \neq y$ then there are disjoint open neighborhoods of x and y.
- T_3 or **regular** just in case for each $x \in X$, and for each closed $C \subseteq X$ such that $x \notin C$, there are open neighborhoods U of x, and V of C, such that $U \cap V = \emptyset$.
- T_4 or **normal** just in case any two disjoint closed subsets of X can be separated by disjoint open sets.

Clearly we have the implications

$$(T_1 + T_4) \Rightarrow (T_1 + T_3) \Rightarrow T_2 \Rightarrow T_1$$

A discrete space satisfies all of the separation axioms. A non-trivial indiscrete space satisfies none of the separation axioms. A useful heuristic here is that the stronger the separation axiom, the closer the space is to discrete. In this book, most of the spaces we consider are very close to discrete in a precise sense we will describe below.

Exercises.

1. Show that X is regular iff for each $x \in X$ and open neighborhood U of x, there is an open neighborhood V of x such that $\overline{V} \subseteq U$.
2. Show that if $E \subseteq F$ then $\overline{E} \subseteq \overline{F}$.

3. Show that $\overline{\overline{E}} = \overline{E}$.

4. Show that the intersection of two topologies is a topology.

5. Show that the infinite distributive law holds:

$$U \cap \left(\bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} (U \cap V_i).$$

6. Show that a space X is Hausdorff if and only if the diagonal $\Delta = \{\langle x, x \rangle : x \in X\}$ is closed in the product topology on $X \times X$. [Oops: you cannot solve this exercise until you know what the product topology is!]

Definition. Let $S \subseteq X$. A family $\mathscr{C}$ of open subsets of X is said to **cover** S just in case $S \subseteq \bigcup_{U \in \mathscr{C}} U$. We say that S is **compact** just in case for every open cover $\mathscr{C}$ of S, there is a finite subcollection $\mathscr{C}_0$ of $\mathscr{C}$ that also covers S. We say that the space X is compact just in case it's compact as a subset of itself.

Definition. A collection $\mathscr{C}$ of subsets of X is said to satisfy the **finite intersection property** if for every finite subcollection $C_1, \dots, C_n$ of $\mathscr{C}$, the intersection $C_1 \cap \dots \cap C_n$ is nonempty.

Discussion. Suppose that X is the space of possible worlds, so that we can think of subsets of X as propositions. If $A \cap B$ is nonempty, then the propositions A and B are consistent, i.e. there is a world in which they are both true. Thus, a collection $\mathscr{C}$ of propositions has the finite intersection property just in case it is finitely consistent.

Recall that compactness of propositional logic states that if a set $\mathscr{C}$ of propositions is finitely consistent, then $\mathscr{C}$ is consistent. The terminology here is no accident; a topological space is compact just in case finite consistency entails consistency.

Proposition 6.6. *A space X is compact if and only if for every collection $\mathscr{C}$ of closed subsets of X, if $\mathscr{C}$ satisfies the finite intersection property, then $\bigcap \mathscr{C}$ is nonempty.*

Proof. ($\Rightarrow$) Assume first that X is compact, and let $\mathscr{C}$ be a family of closed subsets of X. We will show that if $\mathscr{C}$ satisfies the finite intersection property, then the intersection of all sets in $\mathscr{C}$ is nonempty. Assume the negation of the consequent, i.e. that $\bigcap_{C \in \mathscr{C}} C$ is empty. Let $\mathscr{C}' = \{C' : C \in \mathscr{C}\}$, where $C' = X \backslash C$ is the complement of C in X. [Warning: this notation can be confusing. Previously I used E' to denote the set of limit points of E. This C' has nothing to do with limit points.] Each C' is open, and

$$\left(\bigcup_{C \in \mathscr{C}} C' \right)' = \bigcap_{C \in \mathscr{C}} C,$$

which is empty. It follows then that $\mathscr{C}'$ is an open cover of X. Since X is compact, there is a finite subcover $\mathscr{C}_0'$ of $\mathscr{C}'$. If we let $\mathscr{C}_0$ be the complements of sets in $\mathscr{C}_0'$, then $\mathscr{C}_0$ is a finite collection of sets in $\mathscr{C}$ whose intersection is empty. Therefore $\mathscr{C}$ does not satisfy the finite intersection property.

($\Leftarrow$) Assume now that X is not compact. In particular, suppose that $\mathscr{U}$ is an open cover with no finite subcover. Let $\mathscr{C} = \{X\backslash U \mid U \in \mathscr{U}\}$. For any finite subcollection $X\backslash U_1, \ldots, X\backslash U_n$ of $\mathscr{C}$, we have

$$U_1 \cup \cdots \cup U_n \neq X,$$

and hence

$$(X\backslash U_1) \cap \cdots \cap (X\backslash U_n) \neq \emptyset.$$

Thus, $\mathscr{C}$ has the fip. Nonetheless, since $\mathscr{U}$ covers X, the intersection of all sets in $\mathscr{C}$ is empty. ☐

Proposition 6.7. *In a compact space, closed subsets are compact.*

Proof. Let $\mathscr{C}$ be an open cover of S, and consider the cover $\mathscr{C}' = \mathscr{C} \cup \{X\backslash S\}$ of X. Since X is compact, there is a finite subcover $\mathscr{C}_0$ of $\mathscr{C}'$. Removing $X\backslash S$ from $\mathscr{C}_0$ gives a finite subcover of the original cover $\mathscr{C}$ of S. ☐

Proposition 6.8. *Suppose that X is compact, and let U be an open set in X. Let $\{F_i\}_{i\in I}$ be a family of closed subsets of X such that $\bigcap_{i\in I} F_i \subseteq U$. Then there is a finite subset J of I such that $\bigcap_{i\in J} F_i \subseteq U$.*

Proof. Let $C = X\backslash U$, which is closed. Thus, the hypotheses of the proposition say that the family $\mathscr{C} := \{C\} \cup \{F_i : i \in I\}$ has empty intersection. Since X is compact, $\mathscr{C}$ also fails to have the finite intersection property. That is, there are $i_1, \ldots, i_k \in I$ such that $C \cap F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset$. Therefore $F_{i_1} \cap \cdots \cap F_{i_k} \subseteq U$. ☐

Proposition 6.9. *If X is compact Hausdorff, then X is regular.*

Proof. Let $x \in X$, and let $C \subseteq X$ be closed. For each $y \in C$, let U_y be an open neighborhood of x, and V_y an open neighborhood of y such that $U_y \cap V_y = \emptyset$. The V_y form an open cover of C. Since C is closed and X is compact, C is compact. Hence there is a finite subcollection $V_{y_1}, \ldots, V_{y_n}$ that cover C. But then $U = \cap_{i=1}^n U_{y_i}$ is an open neighborhood of x, and $V = \cup_{i=1}^n V_{y_i}$ is an open neighborhood of C, such that $U \cap V = \emptyset$. Therefore X is regular. ☐

Proposition 6.10. *In Hausdorff spaces, compact subsets are closed.*

Proof. Let p be a point of X that is not in K. Since X is Hausdorff, for each $x \in K$, there are open neighborhoods U_x of x and V_x of p such that $U_x \cap V_x = \emptyset$. The family $\{U_x : x \in K\}$ covers K. Since K is compact, it is covered by a finite subcollection $U_{x_1}, \ldots, U_{x_n}$. But then $\cap_{i=1}^n V_{x_i}$ is an open neighborhood of p that is disjoint from K. It follows that $X - K$ is open, and K is closed. ☐

Definition. Let X, Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** just in case for each open subset U of Y, $f^{-1}(U)$ is an open subset of X.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be the function that is constantly zero on $(-\infty, 0)$, and 1 on $[0, \infty)$. Then f is not continuous: $f^{-1}(\frac{1}{2}, \frac{3}{2}) = [0, \infty)$, which is not open.

In the exercises, you will show that a function f is continuous if and only if $f^{-1}(C)$ is closed whenever C is closed. Thus, in particular, if C is a clopen subset of Y, then $f^{-1}(C)$ is a clopen subset of X.

Proposition 6.11. *Let* **Top** *consist of the class of topological spaces and continuous maps between them. For* $X \xrightarrow{f} Y \xrightarrow{g} Z$*, define* $g \circ f$ *to be the composition of* g *and* f*. Then* **Top** *is a category.*

Proof. It needs to be confirmed that if f and g are continuous, then $g \circ f$ is continuous. We leave this to the exercises. Since composition is associative, **Top** is a category. □

Proposition 6.12. *Suppose that* $f : X \to Y$ *is continuous. If* K *is compact in* X*, then* $f(K)$ *is compact in* Y*.*

Proof. Let $\mathscr{G}$ be a collection of open subsets of Y that covers $f(K)$. Let

$$\mathscr{G}' = \{f^{-1}(U) : U \in \mathscr{G}\}.$$

When $\mathscr{G}'$ is an open cover of K. Since K is compact, $\mathscr{G}'$ has a finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$. But then $U_1, \dots, U_n$ is a finite subcover of $\mathscr{G}$. □

We remind the reader of the category theoretic definitions:

- f is a **monomorphism** just in case $fh = fk$ implies $h = k$.
- f is an **epimorphism** just in case $hf = kf$ implies $h = k$.
- f is an **isomorphism** just in case there is a $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

For historical reasons, isomorphisms in **Top** are usually called **homeomorphisms**. It is easy to show that a continuous map $f : X \to Y$ is monic if and only if f is injective. It is also true that $f : X \to Y$ is epi if and only if f is surjective (but the proof is somewhat subtle). In contrast, a continuous bijection is not necessarily an isomorphism in **Top**. For example, if we let X be a two element set with the discrete topology, and Y be a two element set with the indiscrete topology, then any bijection $f : X \to Y$ is continuous, but is not an isomorphism.

Exercise.

1. Show that if f and g are continuous, then $g \circ f$ is continuous.

2. Suppose that $f : X \to Y$ is a surjection. Show that if E is dense in X, then $f(E)$ is dense in Y.

3. Show that $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed whenever C is closed.

4. Let Y be a Hausdorff space, and let $f, g : X \to Y$ be continuous. Show that if f and g agree on a dense subset S of X, then $f = g$.

Exercise. Show that $f^{-1}(V) \subseteq U$ if and only if $V \subseteq Y \backslash f(X \backslash U)$.

Definition. A continuous mapping $f : X \to Y$ is said to be **closed** just in case for every closed set $C \subseteq X$ the image $f(C)$ is closed in Y. Similarly, $f : X \to Y$ is said to be **open** just in case for every open set $U \subseteq X$, the image $f(U)$ is open in Y.

Proposition 6.13. *Let $f : X \to Y$ be continuous. Then the following are equivalent.*

1. *f is closed.*
2. *For every open set $U \subseteq X$, the set $\{y \in Y \mid f^{-1}\{y\} \subseteq U\}$ is open.*
3. *For every $y \in Y$, and every neighborhood U of $f^{-1}\{y\}$, there is a neighborhood V of y such that $f^{-1}(V) \subseteq U$.*

Proof. ($2 \Leftrightarrow 3$) The equivalence of (2) and (3) is straightforward, and we leave its proof as an exercise.

($3 \Rightarrow 1$) Suppose that f satisfies condition (3), and let C be a closed subset of X. To show that $f(C)$ closed, assume that $y \in Y \backslash f(C)$. Then $f^{-1}\{y\} \subseteq X \backslash C$. Since $X \backslash C$ is open, there is a neighborhood V of y such that $f^{-1}(V) \subseteq U$. Then

$$V \subseteq Y \backslash f(X \backslash U) = Y \backslash f(C).$$

Since y was an arbitrary element of $Y \backslash f(C)$, it follows that $Y \backslash f(C)$ is open, and $f(C)$ is closed.

($1 \Rightarrow 3$) Suppose that f is closed. Let $y \in Y$, and let U be a neighborhood of $f^{-1}\{y\}$. Then $X \backslash U$ is closed, and $f(X \backslash U)$ is also closed. Let $V = Y \backslash f(X \backslash U)$. Then V is an open neighborhood of y and $f^{-1}(V) \subseteq U$. □

Proposition 6.14. *Suppose that X and Y are compact Hausdorff. If $f : X \to Y$ is continuous, then f is a closed map.*

Proof. Let B be a closed subset of X. By Proposition 6.7, B is compact. By Proposition 6.12, $f(B)$ is compact. And by Proposition 6.10, $f(B)$ is closed. Therefore, f is a closed map. □

Proposition 6.15. *Suppose that X and Y are compact Hausdorff. If $f : X \to Y$ is a continuous bijection, then f is an isomorphism.*

Proof. Let $f : X \to Y$ be a continuous bijection. Thus, there is function $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. We will show that g is continuous. By Proposition 6.14, f is closed. Moreover, for any closed subset B of X, we have $g^{-1}(B) = f(B)$. Thus, g^{-1} preserves closed subsets, and hence g is continuous. □

Definition. A topological space X is said to be **totally separated** if for any $x, y \in X$, if $x \neq y$ then there is a closed and open (clopen) subset of X containing x but not y.

Definition. We say that X is a **Stone space** if X is compact and totally separated. We let **Stone** denote the full subcategory of **Top** consisting of Stone spaces. To say that **Stone** is a full subcategory means that the arrows between two Stone spaces X and Y are just the arrows between X and Y considered as topological spaces, i.e. continuous functions.

Note. Let E be a clopen subset of X. Then there is a continuous function $f : X \to \{0, 1\}$ such that $f(x) = 1$ for $x \in E$, and $f(x) = 0$ for $x \in X \backslash E$. Here we are considering $\{0, 1\}$ with the discrete topology.

Proposition 6.16. *Let X and Y be Stone spaces. If $f : X \to Y$ is an epimorphism, then f is surjective.*

Proof. Suppose that f is not surjective. Since X is compact, the image $f(X)$ is compact in Y, hence closed. Since f is not surjective, there is a $y \in Y \backslash f(X)$. Since Y is a regular space, there is a clopen neighborhood U of y such that $U \cap f(X) = \emptyset$. Define $g : Y \to \{0, 1\}$ to be constantly 0. Define $h : Y \to \{0, 1\}$ to be 1 on U, and 0 on $Y \backslash U$. Then $g \circ f = h \circ f$, but $g \neq h$. Therefore f is not an epimorphism. □

Proposition 6.17. *Let X and Y be Stone spaces. If $f : X \to Y$ is both a monomorphism and an epimorphism, then f is an isomorphism.*

Proof. By Proposition 6.16, f is surjective. Therefore, f is a continuous bijection. By Proposition 6.15, f is an isomorphism. □

7 Stone duality

In this section we show that the category **Bool** is dual to a certain category of topological spaces, namely the category **Stone** of Stone spaces. To say that categories are "dual" means that the first is equivalent to the mirror image of the second.

Definition. We say that categories **C** and **D** are **dual** just in case there are contravariant functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $GF \cong 1_{\mathbf{C}}$ and

$FG \cong 1_{\mathbf{D}}$. To see that this definition makes sense, note that if F and G are contravariant functors, then GF and FG are covariant functors. If $\mathbf{C}$ and $\mathbf{D}$ are dual, we write $\mathbf{C} \cong \mathbf{D}^{op}$, to indicate that $\mathbf{C}$ is equivalent to the opposite category of $\mathbf{D}$, i.e. the category that has the same objects as $\mathbf{D}$, but arrows running in the opposite direction.

The functor from Bool to Stone

We now define a contravariant functor $S : \mathbf{Bool} \to \mathbf{Stone}$. For reasons that will become clear later, the functor S is sometimes called the **semantic functor**.

Consider the set $\hom(B, 2)$ of 2-valued homomorphisms of the Boolean algebra B. For each $a \in B$, define

$$C_a \;=\; \{\phi \in \hom(B,2) \mid \phi(a) = 1\}.$$

Clearly, the family $\{C_a \mid a \in B\}$ forms a basis for a topology on $\hom(B, 2)$. We let $S(B)$ denote the resulting topological space. Note that $S(B)$ has a basis of clopen sets. Thus, if $S(B)$ is compact, then $S(B)$ is a Stone space.

Lemma 7.1. *If B is a Boolean algebra, then $S(B)$ is a Stone space.*

Proof. Let $\mathscr{B} = \{C_a \mid a \in B\}$ denote the chosen basis for the topology on $S(B)$. To show that $S(B)$ is compact, it will suffice to show that for any subfamily $\mathscr{C}$ of $\mathscr{B}$, if $\mathscr{C}$ has the finite intersection property, then $\bigcap \mathscr{C}$ is nonempty. Now let F be the set of $b \in B$ such that

$$C_{a_1} \cap \cdots \cap C_{a_n} \subseteq C_b,$$

for some $C_{a_1}, \ldots, C_{a_n} \in \mathscr{C}$. Since $\mathscr{C}$ has the finite intersection property, F is a filter in B. Thus, UF entails that F is contained in an ultrafilter U. This ultrafilter U corresponds to a $\phi : B \to 2$, and we have $\phi(a) = 1$ whenever $C_a \in \mathscr{C}$. In other words, $\phi \in C_a$, whenever $C_a \in \mathscr{C}$. Therefore, $\bigcap \mathscr{C}$ is nonempty, and $S(B)$ is compact. □

Let $f : A \to B$ be a homomorphism, and let $S(f) : S(B) \to S(A)$ be given by $S(f) = \hom(f, 2)$; that is,

$$S(f)(\phi) \;=\; \phi \circ f, \qquad \forall \phi \in S(B).$$

We claim now that $S(f)$ is a continuous map. Indeed, for any basic open subset C_a of $S(A)$, we have

$$S(f)^{-1}(C_a) \;=\; \{\phi \in S(B) \mid \phi(f(a)) = 1\} \;=\; C_{f(a)}. \tag{1}$$

It is straightforward to verify that $S(1_A) = 1_{S(A)}$, and that $S(g \circ f) = S(f) \circ S(f)$. Therefore, $S : \mathbf{Bool} \to \mathbf{Stone}$ is a contravariant functor.

The functor from Stone to Bool

Let X be a Stone space. Then the set $K(X)$ of clopen subsets of X is a Boolean algebra, and is a basis for the topology on X. We now show that K is the object part of a contravariant functor $K : \mathbf{Stone} \to \mathbf{Bool}$. For reasons that will become clear later, K is sometimes called the **syntactic functor**.

Indeed, if X, Y are Stone spaces, and $f : X \to Y$ is continuous, then for each clopen subset U of Y, $f^{-1}(U)$ is a clopen subset of X. Moreover, f^{-1} preserves union, intersection, and complement of subsets; thus $f^{-1} : K(Y) \to K(X)$ is a Boolean homomorphism. We define the mapping K on arrows by $K(f) = f^{-1}$. Obviously, $K(1_X) = 1_{K(X)}$, and $K(g \circ f) = K(f) \circ K(g)$. Therefore K is a contravariant functor.

Now we will show that KS is naturally isomorphic to the identity on **Bool**, and SK is naturally isomorphic to the identity on **Stone**. For each Boolean algebra B, define $\eta_B : B \to KS(B)$ by

$$\eta_B(a) \;=\; C_a \;=\; \{\phi \in S(B) \mid \phi(a) = 1\}.$$

Lemma 7.2. *The map $\eta_B : B \to KS(B)$ is an isomorphism of Boolean algebras.*

Proof. We first verify that $a \mapsto C_a$ is a Boolean homomorphism. For $a, b \in B$, we have

$$\begin{array}{rcl} C_{a \wedge b} & = & \{\phi \mid \phi(a \wedge b) = 1\} \\ & = & \{\phi \mid \phi(a) = 1 \text{ and } \phi(b) = 1\} \\ & = & C_a \wedge C_b. \end{array}$$

A similar calculation shows that $C_{\neg a} = X \backslash C_a$. Therefore, $a \mapsto C_a$ is a Boolean homomorphism.

To show that $a \mapsto C_a$ is injective, it will suffice to show that $C_a = \emptyset$ only if $a = 0$. In other words, it will suffice to show that for each $a \in B$, if $a \neq 0$ then there is some $\phi : B \to 2$ such that $\phi(a) = 1$. Thus, the result follows from UF.

Finally, to see that η_B is surjective, let U be a clopen subset of $S(B)$. Since U is open, $U = \bigcup_{a \in I} C_a$, for some subset I of B. Since U is closed in the compact space $G(B)$, it follows that U is compact. Thus, there is a finite subset F of B such that $U = \bigcup_{a \in F} C_a$. And since $a \mapsto C_a$ is a Boolean homomorphism, $\bigcup_{a \in F} C_a = C_b$, where $b = \bigvee_{a \in F} a$. Therefore, η_B is surjective. $\square$

Lemma 7.3. *The family of maps $\{\eta_A : A \to KS(A)\}$ is natural in A.*

Proof. Suppose that A and B are Boolean algebras, and that $f : A \to B$ is a Boolean homomorphism. Consider the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ KS(A) & \xrightarrow{KS(f)} & KS(B) \end{array}$$

For $a \in A$, we have $\eta_B(f(a)) = C_{f(a)}$, and $\eta_A(a) = C_a$. Furthermore,

$$KS(f)(C_a) \;=\; S(f)^{-1}(C_a) \;=\; C_{f(a)},$$

by Eqn. 1. Therefore, the diagram commutes, and η is a natural transformation. □

Now we define a natural isomorphism $\theta : 1_{\mathbf{S}} \Rightarrow SK$. For a Stone space X, $K(X)$ is the Boolean algebra of clopen subsets of X, and $SK(X)$ is the Stone space of $K(X)$. For each point $\phi \in X$, let $\hat{\phi} : K(X) \to 2$ be defined by

$$\hat{\phi}(C) = \begin{cases} 1 & \phi \in C, \\ 0 & \phi \notin C. \end{cases}$$

It's straightforward to verify that $\hat{\phi}$ is a Boolean homomorphism. We define $\theta_X : X \to SK(X)$ by $\theta_X(\phi) = \hat{\phi}$.

Lemma 7.4. *The map $\theta_X : X \to SK(X)$ is a homeomorphism of Stone spaces.*

Proof. It will suffice to show that θ_X is bijective and continuous. (Do you remember why? Hint: Stone spaces are compact Hausdorff.) To see that θ_X is injective, suppose that ϕ and ψ are distinct elements of X. Since X is a Stone space, there is a clopen set U of X such that $\phi \in U$ and $\psi \notin U$. But then $\hat{\phi} \neq \hat{\psi}$. Thus, θ_X is injective.

To see that θ_X is surjective, let $h : K(X) \to 2$ be a Boolean homomorphism. Let

$$\mathscr{C} \;=\; \{C \in K(X) \mid h(C) = 1\}.$$

In particular $X \in \mathscr{C}$; and since h is a homomorphism, $\mathscr{C}$ has the finite intersection property. Since X is compact, $\bigcap \mathscr{C}$ is nonempty. Let ϕ be a point in $\bigcap \mathscr{C}$. Then for any $C \in K(X)$, if $h(C) = 1$, then $C \in \mathscr{C}$ and $\phi \in C$, from which it follows that $\hat{\phi}(C) = 1$. Similarly, if $h(C) = 0$ then $X \backslash C \in \mathscr{C}$, and $\hat{\phi}(C) = 0$. Thus, $\theta_X(\phi) = \hat{\phi} = h$, and θ_X is surjective.

To see that θ_X is continuous, note that each basic open subset of $SK(X)$ is of the form

$$\hat{C} \;=\; \{h : K(X) \to 2 \mid h(C) = 1\},$$

for some $C \in K(X)$. Moreover, for any $\phi \in X$, we have $\hat{\phi} \in \hat{C}$ iff $\hat{\phi}(C) = 1$ iff $\phi \in C$. Therefore,

$$\theta_X^{-1}(\hat{C}) \;=\; \{\phi \in X \mid \hat{\phi}(C) = 1\} \;=\; C.$$

Therefore, θ_X is continuous. □

Lemma 7.5. *The family of maps $\{\theta_X : X \to SK(X)\}$ is natural in X.*

Proof. Let X, Y be Stone spaces, and let $f : X \to Y$ be continuous. Consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \theta_X} & & \downarrow{\scriptstyle \theta_Y} \\ SK(X) & \xrightarrow{SK(f)} & SK(Y) \end{array}$$

For arbitrary $\phi \in X$, we have $(\theta_Y \circ f)(\phi) = \widehat{f(\phi)}$. Furthermore,

$$SK(f) \;=\; \hom(K(f), 2) \;=\; \hom(f^{-1}, 2)$$

In other words, for a homomorphism $h : K(X) \to 2$, we have

$$SK(f)(h) \;=\; h \circ f^{-1}.$$

In particular, $SK(f)(\hat{\phi}) = \hat{\phi} \circ f^{-1}$. For any $C \in K(Y)$, we have

$$(\hat{\phi} \circ f^{-1})(C) \;=\; \begin{cases} 1 & f(\phi) \in C, \\ 0 & f(\phi) \notin C. \end{cases}$$

That is, $\hat{\phi} \circ f^{-1} = \widehat{f(\phi)}$. Therefore, the diagram commutes, and θ is a natural isomorphism. □

This completes the proof that K and S are quasi-inverse, and yields the famous theorem:

Stone Duality Theorem

The categories **Stone** and **Bool** are dual to each other. In particular, any Boolean algebra B is isomorphic to the field of clopen subsets of its state space $S(B)$.

Proposition 7.6. *Let $A \subseteq B$, and $a \in B$. Then the following are equivalent:*

1. *For any states f and g of B, if $f|_A = g|_A$ then $f(a) = g(a)$.*
2. *If h is a state of A, then any two extensions of h to B agree on a.*
3. *$a \in A$.*

Proof. Since every state of A can be extended to a state of B, (1) and (2) are obviously equivalent. Furthermore, (3) obviously implies (1). Thus, we only need to show that (1) implies (3).

Let $m : A \to B$ be the inclusion of A in B, and let $s : S(B) \to S(A)$ be the corresponding surjection of states. We need to show that $C_a = s^{-1}(U)$ for some clopen subset U of $S(A)$.

By (1), for any $x \in S(A)$, either $s^{-1}\{x\} \subseteq C_a$ or $s^{-1}\{x\} \subseteq C_{\neg a}$. By Proposition 6.14, s is a closed map. Since C_a is open, Proposition 6.13 entails that the sets

$$U = \{x \in S(B) \mid s^{-1}\{x\} \subseteq C_a\}, \quad \text{and} \quad V = \{x \in S(B) \mid s^{-1}\{x\} \subseteq C_{\neg a}\},$$

are open. Since $U = S(A)\backslash V$, it follows that U is clopen. Finally, it's clear that $s^{-1}(U) = C_a$. □

Proposition 7.7. *In* **Bool***, epimorphisms are surjective.*

Proof. Suppose that $f : A \to B$ is not surjective. Then $f(A)$ is a proper subalgebra of B. By Proposition 7.6, there are states g, h of B such that $g \neq h$, but $g|_{f(A)} = h|_{f(A)}$. In other words, $g \circ f = h \circ f$, and f is not an epimorphism. □

8 Discussion

Combining the previous two theorems, we have the following equivalences:

Th $\cong$ **Bool** $\cong$ **Stone**^{op}.

We will now exploit these equivalences to explore the structure of the category of theories.

Further reading

The most in-depth book on Stone duality is Johnstone, *Stone Spaces*. More rudimentary treatments can be found in Halmos, *Logic as Algebra*, and in Cori and Lascar, *Mathematical Logic*.

Category theory

1. F. Borceux. *Handbook of Categorical Algebra*, Vol I.

2. S. Mac Lane. *Categories for the Working Mathematician.*

3. J. van Oosten. *Basic Category Theory.* `http://www.staff.science.uu.nl/~ooste110/syllabi/catsmoeder.pdf`

Boolean algebras

1. S. Koppelberg, "General theory of Boolean algebras" J.D. Monk (ed.) R. Bonnet (ed.), *Handbook of Boolean algebras*, volume 3, North-Holland (1989)
2. S. Givant and P. Halmos. *Introduction to Boolean Algebras.*
3. R. Sikorski. *Boolean algebras.*
4. Bell and Slomson.
5. Jech. *Set Theory.*
6. W. Just and M. Weese. *Discovering Modern Set Theory*, Vol II. (Chapter 25)
7. P. Dwinger. *Introduction to Boolean Algebras.*

Topology

There are many excellent textbooks on point-set topology. We recommend especially the following:

1. R. Engelking. *General Topology.*
2. S. Willard. *General Topology.*

Stone spaces are treated, albeit briefly, in Bell and Machover, *A Course in Mathematical Logic*. Proper maps are treated in Bourbaki, *General Topology*, and in Escardó, "Intersections of compactly many open sets are open." Profinite spaces are treated in Ribes and Zallesski, *Profinite Groups.*