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Precept 5 Exercises

① $\text{graph}(f) = \{\langle x,y\rangle \in X \times Y \mid f(x) = y\}$

$= \{\langle x,y\rangle \in X \times Y \mid \langle f(x), y\rangle = \langle y,y\rangle\}$

$= \{\langle x,y\rangle \in X \times Y \mid (f \times 1_Y)\langle x,y\rangle = \delta_Y(y)\}$

Given $f \times 1_Y$ and δ_Y, the set

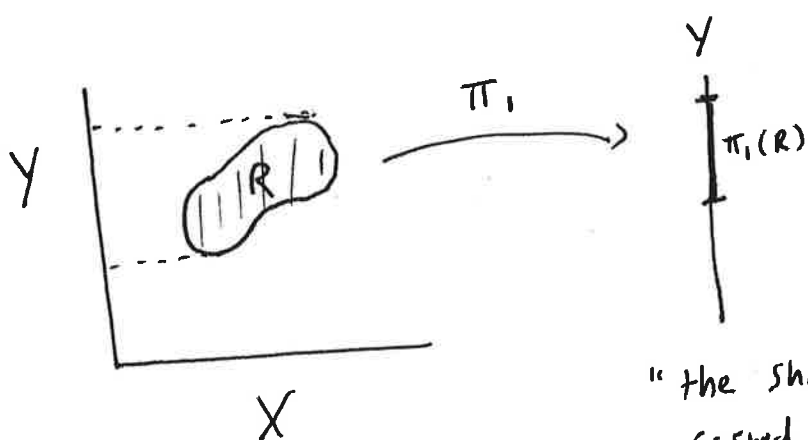
$$(X \times Y) \times_Y Y = \{\langle\langle x, y_1\rangle, y_2\rangle \mid (f \times 1_Y)\langle x, y_1\rangle = \delta_Y(y_2)\}$$

is by definition the pullback of $f \times 1_Y$ and δ_Y (see p. 11 discussion of fibered product).

But we can see this set corresponds exactly to graph (f).

(Note: in order for $(f \times 1_Y)\langle x, y_1\rangle = \delta_Y(y_2)$, we must have $y_1 = y_2$.)

② $R \subseteq X \times Y$



$\pi_1(R) =$ those $y \in Y$ which relate to some $x \in X$

"the shadow on Y casted by R"

(3) Suppose $A \subseteq B$ and $x \in f^{-1}(A)$. Then $f(x) \in A$.

Since $A \subseteq B$, $f(x) \in B$. Thus $x \in f^{-1}(B)$.

So $f^{-1}(A) \subseteq f^{-1}(B)$.

(4)
$$
\begin{aligned}
f(f^{-1}(B)) &= \{ y \in Y \mid \exists z \in f^{-1}(B).\ f(z) = y \} \\
&= \{ y \in Y \mid \exists z. (f(z) \in B \wedge f(z) = y) \} \\
&= \{ y \in Y \mid \exists z. (y = f(z) \in B) \} \\
&\subseteq \{ y \in Y \mid y \in B \} = B.
\end{aligned}
$$

Thus $f(f^{-1}(B)) \subseteq B$.

see next page $\longrightarrow$

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Exercise 5. It is not always the case that $B \subseteq f(f^{-1}(B))$.

Proof. Consider the function f from the real numbers to the real numbers defined by $f(x) = x^2$ (note that this function isn't onto: no real number multiplied by itself gives you a negative real number), and the subset $B = \{-1, 1\}$ of the real numbers. Then note that $f^{-1}(B) = \{-1, 1\}$ (recall that, by definition, $f^{-1}(B) = \{x \in dom f \mid f(x) \in B\}$, and only $f(-1)$ and $f(1)$ are in B). But then $f(f-1(B)) = \{1\}$, since the square of -1 and 1 is just 1. Hence we have that

$$B \nsubseteq f(f^{-1}(B)),$$

as we wanted.

Exercise 6. If f is surjective, then $f(f^{-1}(B)) = B$.

Proof. Let f be surjective. Since by problem 4 we have that $f(f^{-1}(B)) \subseteq B$, it suffices to show that $B \subseteq f(f^{-1}(B))$. So given an element b in B. Recall that B is a subset of the codomain of f. So b is in the codomain of f. Since f is surjective, there is an a in the domain of f such that $f(a) = b$. This means, by the definition of the preimage of a set, that $a \in f^{-1}(\{b\})$. So applying f to a gives you b, i.e., by the definition of the image of a set, $b \in f(f^{-1}(\{b\}))$.

Now, by problem 3, f^{-1} is order preserving. In particular, since $\{b\} \subseteq B$, then $f^{-1}(\{b\}) \subseteq f^{-1}(B)$. We also have that images are order preserving, i.e., that if $X \subseteq Y$ then $f(X) \subseteq f(Y)$ (quick proof: let $y \in f(X)$. So there is $x \in X$ such that $f(x) = y$. But $X \subseteq Y$, so $x \in Y$. Hence $y \in f(Y)$, as we wanted). So $f(f^{-1}(\{b\})) \subseteq f(f^{-1}(B))$. Hence $b \in f(f^{-1}(B))$, and we are done.

Exercise 7. It is *not* always the case that $f(A \cap B) = f(A) \cap f(B)$.

Proof. For consider the function $f : \{0, 1\} \to \{0\}$, and consider the sets $A = \{0\}$ and $B = \{1\}$. Then note $A \cap B = \emptyset$. But $f(A) = \{0\}$ and $f(B) = \{0\}$, so $f(A) \cap f(B) = \{0\}$. Hence $f(A \cap B) \neq f(A) \cap f(B)$.

Exercise 8. It is always the case that $f(A \cup B) = f(A) \cup f(B)$.

Proof. First we show that $f(A \cup B) \subseteq f(A) \cup f(B)$. Given $y \in f(A \cup B)$. By definition of an image, there is an $x \in A \cup B$ such that $f(x) = y$. So either $x \in A$ or $x \in B$. If $x \in A$, then $y \in f(A)$, since $f(x) = y$. So $y \in f(A) \cup f(B)$. An analogous argument holds if $x \in B$. So we are done.

Now we show that $f(A) \cup f(B) \subseteq f(A \cup B)$. Given $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then there is an $x \in A$ such that $f(x) = y$. But by $A \subseteq A \cup B$ we have $x \in A \cup B$. So $y \in f(A \cup B)$. An analogous argument holds if $y \in f(B)$. So we are done.