Intermediate Logic: Precept 5

Definition. Let $\text{graph}(f) = \{\langle x, y\rangle \in X \times Y \mid f(x) = y\}$.

Definition. Suppose that $f : X \to Y$, and that $A \subseteq X$. Then we let $f(A) = \{y \in Y \mid \exists x \in A. f(x) = y\}$. This definition is tantamount to the following biconditional:

$$y \in f(A) \quad \text{iff} \quad \exists x(x \in A \wedge f(x) = y)$$

Definition. If $B \subseteq Y$, then we let $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. This definition is tantamount to the following biconditional:

$$x \in f^{-1}(B) \quad \text{iff} \quad f(x) \in B$$

Exercises

1. Show that $\text{graph}(f)$ is (isomorphic to) the pullback of the diagonal $\delta_Y : Y \to Y \times Y$ along the function $f \times 1_Y : X \times Y \to Y \times Y$.
2. Let R be a relation on $X \times Y$. Consider the image $\pi_1(R)$ of R under the projection $\pi_1 : X \times Y \to Y$. Draw a picture, and explain the intuitive meaning of $\pi_1(R)$.
3. Show that f^{-1} is order preserving: if $A \subseteq B$ then $f^{-1}(A) \subseteq f^{-1}(B)$.
4. Show that $f(f^{-1}(B)) \subseteq B$.
5. Show that it is not always the case that $B \subseteq f(f^{-1}(B))$. [Hint: Choose a function f that is not surjective.]
6. Show that if f is surjective, then $f(f^{-1}(B)) = B$.
7. Is it the case that $f(A \cap B) = f(A) \cap f(B)$?
8. Is it the case that $f(A \cup B) = f(A) \cup f(B)$?