Phil 312: Solutions to PS4.

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Problem 1. Let Σ be a propositional signature, and let X be the set of valuations of Σ. Then there is no surjection $f : \Sigma \to X$.

It suffices to show that, for an arbitrary function $f : \Sigma \to X$, f is not surjective, i.e., that there is a valuation $v \in X$ which is not in the range of f. Our method of proof will be similar to that of Cantor's proof (see course website for a handout with this proof). That is, let us define this v as follows: for any propositional constant $p \in \Sigma$,

$$v(p) = \begin{cases} 1 & \text{if } f(p)(p) = 0 \\ 0 & \text{if } f(p)(p) = 1. \end{cases}$$

Then v is guaranteed to differ with $f(p)$ in at least one value, namely $p \in \Sigma$. Since functions (and hence valuations) are individuated by their values, v is distinct from every $f(p)$, and so isn't in the range of f, as we wanted.

Problem 2. Let X be a set, and let $a \in X$. If X is uncountably infinite then so is $X \setminus \{a\}$.

We proceed by contradiction, i.e., assume that $X \setminus \{a\}$ is not uncountably infinite. Then it is countable, i.e., there is an injective and surjective function $f : X \setminus \{a\} \to \omega$. Now, as Prof. Halvorson alluded to in lecture, the set of natural numbers is equinumerous to the set of natural numbers minus one element, i.e., $\omega \approx \omega \setminus \{0\}$. For we can "shift all elements in ω to the right " by means of the following injective and surjective function $k : \omega \to \omega \setminus \{0\}$:

$$k(n) = n + 1.$$

It follows that $X \setminus \{a\}$ is equinumerous to $\omega \setminus \{0\}$ (to see this: compose f with k. The resulting function $k \circ f : X \setminus \{a\} \to \omega \setminus \{0\}$ is injective and surjective). So now, consider the following function $h : X \to \omega$, defined, for any $x \in X$, as

$$h(x) = \begin{cases} (k \circ f)(x) & \text{if } x \in X \setminus \{a\}. \\ 0 & \text{if } x = a. \end{cases}$$

Since $k \circ f$ is injective and surjective, so is h. Hence $X \approx \omega$. But this contradicts our assumption that X is countably infinite, and we are done.

(Intuitively: if $X \setminus \{a\}$ were countably infinite, we could fit it into $\omega \setminus \{0\}$ but then there's no reason we couldn't fit X into ω, since we can just assign a to 0).

Problem 3. Let Σ be a countably infinite propositional signature. Let X is the set of valuations of Σ; $C_\phi \subseteq X$ is the set of valuations v such that $v(\phi) = 1$.

i We want to show that (i) if $\theta \vDash \phi$ then $C_\theta \subseteq C_\phi$ and (ii) if $C_\theta \subseteq C_\phi$ then $\theta \vDash \phi$. (i) Suppose $\theta \vDash \phi$, and given an arbitrary $v \in C_\theta$. By definition of C_θ, $v(\theta) = 1$. But by definition of semantic entailment, $v(\phi) = 1$. So $v \in C_\phi$. (ii) Suppose $C_\theta \subseteq C_\phi$, and given a valuation $v \in X$ such that $v(\theta) = 1$. Then $v \in C_\theta$, and so $v \in C_\phi$. Hence $v(\phi) = 1$. By definition of semantic entailment, $\theta \vDash \phi$, as we wanted.

ii We want to show that C_p and $C_{\neg p}$ are uncountably infinite. Since Σ is countably infinite and the set of natural numbers is equinumerous to itself plus an extra element, we can write Σ as $\{p, p_0, p_1, p_2, \ldots\}$ (effectively, we are just "pulling p to the front" if it's not already there, and assigning natural numbers to all other propositional constants). It suffices to show that if $f : \omega \to C_p$ and $g : \omega \to C_{\neg p}$, then neither f or g are surjective. That is, it suffices to construct $v_0 \in X$ and $v_1 \in X$ which are not in the range of f and g, respectively. As before, we use the method of proof which Cantor used: we want to make sure that v_0 and v_1 disagree with all $f(n)$ and $g(n)$, respectively, in at least one value.

Define v_0 as, $v(p) = 1$ (we have to add this condition, since $v \in C_p$) and, for any other $p_i \in \Sigma$,

$$v_0(p_i) = \begin{cases} 1 & \text{if } f(i)(p_i) = 0 \\ 0 & \text{if } f(i)(p_i) = 1 \end{cases}$$

Note that v_0 disagrees with all $f(n)$ on at least one value, namely, on p_n. So v_0 is not in the range of f, and we are done: C_p is uncountably infinite. Using an analogous argument, the following definition of v_1 yields that $C_{\neg p}$ is uncountably infinite: $v_0(p) = 1$, and for any other p_i,

$$v_1(p_i) = \begin{cases} 1 & \text{if } g(i)(p_i) = 0 \\ 0 & \text{if } g(i)(p_i) = 1 \end{cases}$$

iii Suppose that θ is the sentence $\gamma_1 \wedge \cdots \wedge \gamma_n$, where each γ_i is either a propositional constant or a negated propositional constant, and no propositional constant appears twice in θ. We want to show that C_θ is uncountably infinite. This proof is very similar to the last one. Let $p_1, p_2, \ldots, p_n$ be the n distinct propositional constants appearing in θ (negated or not). Since Σ is countably infinite and the set of natural numbers is equinumerous to itself plus n extra elements (since n is finite), we can write Σ as $\{p_0, p_1, p_2, \ldots, p_n, q_0, q_1, q_2, \ldots\}$ (effectively, we are just "pulling the p_i to the front" if they're not already there, and assigning natural numbers to all other propositional constants).

(Here's a proof that ω is equinumerous to itself plus a finite number $a_0, a_1, a_2 \ldots, a_n$ of elements. Consider the following injective and surjective function $h : \omega \to \omega \cup \{a_0, a_1, \ldots, a_n\}$: $h(a_i) = i$, and $h(i) = i + n$.)

Now, as before, it suffices to show that if $f : \omega \to C_\phi$ then f is not surjective. That is, it suffices to construct $v \in X$ which is not in the range of f. So define v in the following way: $v(p_i) = f(0)(p_i)$ (so that v agrees with all $f(n)$ on the values they assign to $p_0, p_1, p_2, \ldots, p_n$: this is required for $v \in C_\phi$) and, for any other propositional constant q_i:

$$v(q_i) = \begin{cases} 1 & \text{if } f(i)(q_i) = 0 \\ 0 & \text{if } f(i)(q_i) = 1 \end{cases}$$

Once again, this guarantees that v disagrees with all $f(n)$ in at least one value, namely, on the value $f(n)(q_n)$. Thus C_ϕ is uncountably infinite.

iv For any sentence ϕ, we want to show that either C_ϕ is empty or uncountably infinite. We have that either (i) ϕ is inconsistent (i.e., $\phi \vdash \bot$) or (ii) ϕ is inconsistent (i.e., $\phi \nvdash \bot$). (i) If ϕ is inconsistent, then $\phi \vdash p \wedge \neg p$, for some propositional constant p. By soundness, $\phi \vDash p \wedge \neg p$. So, by problem 3a, $C_\phi \subseteq C_{p \wedge \neg p}$. But there are no valuations which make $p \wedge \neg p$ true. Hence $C_{p \wedge \neg p}$ is the empty set. So C_ϕ is empty too, and we are done.

(ii) So suppose that ϕ is consistent. By the hint, there is a sentence θ such that $\theta \vDash \phi$, and where θ is of the form $\gamma_1 \wedge \cdots \wedge \gamma_n$ for each γ_i either a propositional constant or a negated propositional constant, and no propositional constant appears twice. Then by problem 3a, $C_\theta \subseteq C_\phi$. But by problem 3c C_θ is uncountably infinite. Hence C_ϕ is also uncountably infinite, as we wanted.