Phil 312: Solutions to PS1.

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Problem 1. Given a contingent sentence ϕ. We want to show that there is another contingent sentence ψ such that $\psi \vdash \phi$ and $\phi \nvdash \psi$.

Now, sentences are *finite* strings of symbols. So, in particular, ϕ contains a finite number of propositional constants in Σ. But Σ contains an infinite number of propositional constants. Hence we can fix some propositional constant p in Σ which does not appear in ϕ.

Then consider the sentence ψ given by: $\phi \wedge p$. To show that this sentence is as desired, we need to prove three things:

i $\phi \wedge p$ is also contingent.

ii $\phi \wedge p \vdash \phi$.

iii $\phi \nvdash \phi \wedge p$.

Proof of (i). We know that ϕ is contingent. Hence there are valuations v_1, v_2 such that $v_1(\phi) = 1$ and $v_2(\phi) = 0$. Note that v_2 is such that $v_2(\phi \wedge p) = 0$, by the truth-conditions of '$\wedge$'. So we only need a valuation that makes $\phi \wedge p$ true. Define the following valuation w_1: for any q in Σ,

$$w_1(q) = \begin{cases} v_1(q) & \text{if } q \text{ occurs in } \phi. \\ 1 & \text{for every other } q \text{ in } \Sigma \text{ (including } p\text{).} \end{cases}$$

Intuitively speaking, w_1 is a copy of v_1 over the propositional constants contained in ϕ; and w_1 makes all the others true. So, in particular (since all that matters to the truth-value of ϕ are the propositional constants contained in it), we have that $w_1(\phi) = 1$ and $w_1(p) = 1$, i.e., by the truth-conditions of '$\wedge$', $w_1(\phi \wedge p) = 1$, as we wanted.

Proof of (ii). Follows easily by the rule of '$\wedge$'-elimination (i.e., for any two sentences γ and δ, $\gamma \wedge \delta \vdash \gamma$).

Proof of (iii). Consider the valuation w_2 defined by:

$$w_2(q) = \begin{cases} v_1(q) & \text{if } q \text{ occurs in } \phi. \\ 0 & \text{for every other } q \text{ in } \Sigma \text{ (including } p\text{).} \end{cases}$$

As in (i), $w_2(\phi) = 1$. But we also have, by definition of w_2, that $w_2(p) = 0$. So $w_2(\phi \wedge p) = 0$. This valuation shows that $\phi \nvDash \phi \wedge p$. By soundness, this gives us that $\phi \nvdash \phi \wedge p$, as we wanted. □

Problem 2. Let ϕ be a sentence of Σ. We want to show that if $T' \vdash \phi \to p_0$, then either $T' \vdash \neg\phi$ or $T' \vdash p_0 \to \phi$.

Before we show this, it will be useful to prove the hint given to you in the problem set:
Claim 1. There is only one model of T' which makes p_0 true.
Proof. Given one such model v. By definition $v(p_0) = 1$. Now, we have that $p_0 \vdash p_i$ is an axiom of T', for i a natural number. Hence, by soundness, we also have that $p_0 \vDash p_i$. By the definition of semantic entailment, this means that $v(p_i) = 1$ for any natural number i. This defines v over the entire signature Σ, and hence v is unique.

Now we can return to prove our main claim. Recall that when you want to prove a conditional $P \to Q$ you need to do a conditional proof, i.e., to show that *assuming* P, Q follows. So let us assume that $T' \vdash \phi \to p_0$. We proceed by contradiction, i.e., suppose that it is *not* the case that either $T' \vdash \neg\phi$ or $T' \vdash p_0 \to \phi$. (If you don't like proofs by contradiction, afterwards I present an alternate solution as given to me by our other preceptor, Alex Meehan). In other words, suppose both (i) and (ii):

i $T' \nvdash \neg\phi$

ii $T' \nvdash p_0 \to \phi$.

If we can show that this leads to a contradiction, then we will be done. By completeness, (i) and (ii) give us that $T' \nvDash \neg\phi$ and that $T' \nvDash p_0 \to \phi$. (By the definition of semantic entailment), the former fact means that there is a model v of T' such that $v(\neg\phi) = 0$, i.e., $v(\phi) = 1$; the latter fact that there is a model w of T' such that $w(p_0 \to \phi) = 0$ (note that, so far, we have no guarantee that v and w are the same valuation). Now, by the truth-conditions of '$\to$', we have that $w(p_0) = 1$ and $w(\phi) = 0$. But since w is a model of T' which makes p_0 true, then, by Claim 1:

($\star$) w is the valuation that makes all propositional constants in Σ true, and $w(\phi) = 0$.

Now, recall that we had assumed that $T' \vdash \phi \to p_0$. By soundness, this means that $T' \vDash \phi \to p_0$, that is, every model of T' makes $\phi \to p_0$ true. So, in particular, $v(\phi \to p_0) = 1$, i.e., (by the truth-conditions of '$\to$') either $v(\phi) = 0$ or $v(p_0) = 1$. By definition of v, however, the former is impossible, since $v(\phi) = 1$. Hence it must be that $v(p_0) = 1$. But, by Claim 1, this gives us that:

($\star\star$) v is the valuation that makes all propositional constants in Σ true, and $v(\phi) = 1$.

($\star$) and ($\star\star$) contradict each other (since $w = v$ and yet $w(\phi) \neq v(\phi)$), and we are done.

Now, here's Alex Meehan's other proof (which assumes that we have already proved the hint):

Suppose $T' \vdash \phi \to p_0$. By soundness $T' \vDash \phi \to p_0$. Thus for every valuation v which is a model of T', either $v(\neg\phi) = 1$ or $v(p_0) = 1$. The hint shows that there's exactly one model of T' – call it v^* – that evaluates p_0 to 1. The rest of the models of T' must therefore evaluate p_0 to 0 and thus $\neg\phi$ to 1. So to summarize, we have

$$V = \{v : v \text{ is a model of } T' \}$$
$$W = \{v : v \in V \text{ but } v \neq v^*\}$$

where $v^*(p_0) = 1$ and for every $v \in W$, $v(\neg\phi) = 1$ and $v(p_0) = 0$. It follows from this last fact that for all $v \in W$, $v(\neg p_0) = 1$ and hence $v(p_0 \to \phi) = 1$.

Our goal is to show that either (1) for all $v \in V$, $v(\neg\phi) = 1$, or (2) for all $v \in V$, $v(p_0 \to \phi) = 1$. (Note that this suffices: if (1) holds, then by the completeness theorem we get that $T' \vdash \neg\phi$, and if (2) holds then by the completeness theorem we get that $T' \vdash p_0 \to \phi$.)

We consider two exhaustive and mutually exclusive cases:
(a) $v^*(\neg\phi) = 1$, or
(b) $v^*(\phi) = 1$
If (a) holds, then (1) follows immediately (since we also have that for every $v \in W$, $v(\neg\phi) = 1$). If (b) holds, then (2) follows immediately since $v^*(p_0 \to \phi) = 1$ (and we also have that for all $v \in W$, $v(p_0 \to \phi) = 1$). So in both cases (a) and (b), we get either (1) or (2), as desired.

Problem 3. We need to show that there are no atoms in T.

((Recall the definition of an atom in a theory: ϕ is an atom in a theory T iff (1) it is a contingent sentence in T, i.e., $T \nvdash \phi$ and $T \nvdash \neg\phi$ and (2) for each sentence ψ of T, if $T \vdash \psi \to \phi$, then either $T \nvdash \neg\phi$ or $T \vdash \phi \to \psi$.))

Given a sentence ϕ of Σ. It is either a tautology (such that for any valuation v, $v(\phi) = 1$) or not. If it is then, by definition of T, $T \vdash \phi$ and so ϕ is not contingent. Otherwise, there exists a valuation v such that $v(\phi) = 0$. But this valuation is a model for T (it is vacuously true that if a sentence is in T then v makes it true), and so it shows that $T \nvDash \phi$, by the definition of semantic entailment. By soundness, it follows that $T \nvdash \phi$, and so ϕ is not contingent. So either way ϕ is not contigent, and thus not an atom, as we wanted.

Problem 4. We want to show that there isn't a pair of translations $g : T \to T'$ and $f : T' \to T$ such that $gf \simeq 1_{T'}$ and $fg \simeq 1_T$. (I.e., show that T and T' are not equivalent).

First, a word on the definition of equivalence at play in this problem. Suppose you have theories R (with signature Σ, set of sentences Δ) and R' (with signature Σ', set of sentences Δ'). We have have given the following definition: R and R' are **equivalent** iff there are two maps $F : R \to R'$ and $G : R' \to R$ such that $FG \simeq 1_{R'}$ and $GF \simeq 1_R$, that is, such that we have that the following two facts hold for any Σ sentence ϕ and any Σ' sentence ψ:

$$T \vdash \phi \leftrightarrow G(F(\phi))$$

$$T' \vdash \psi \leftrightarrow F(G(\psi))$$

Why have we chosen this definition? To see the intuitive idea behind it, consider the following scenario. Suppose you don't know any Spanish and yet I explain to you Einstein's relativity theory first *in my native language, Spanish* (this would be theory R). Luckily, you have a translator friend, Freddy, who is perfectly fluent in Spanish – he is translating for you every sentence I utter into a sentence about relativity theory *in English*. You look at the transcript that your friend has produced (this would be R'). How can you be sure that this is a good translation of what I said, i.e., that whatever you are reading in English really corresponds to what I originally said in Spanish? You come up with this idea: you ask your other translator friend, Gabby, to translate Freddy's transcript back into Spanish. Then you ask me to read the new transcript. It contains everything I said, sentence by sentence!! (More formally speaking, each sentence in the transcript is either the original sentence I uttered or a sentence that is logically equivalent to it). In other words, the composition of Freddy's translation and then Gabby's is equal to the identity translation – it just takes me back to the same sentences from which I started (or to sentences which are logically equivalent to them).

You have pretty good reason to believe, then, that the English and Spanish versions of what I said are equivalent – that they say the same thing. But you want to make sure, so you grab the English transcript of what I said and make Gabby translate it into Spanish, and then ask Freddy to translate it back into English. (This is an important point where the analogy breaks down – we don't require both that $FG \simeq 1_{R'}$ and $GF \simeq 1_R$ to "make sure" that both theories are equivalent, but rather to make sure that we are not leaving any sentences out of our translations. Instead of forcing the analogy, I decided to leave it as it is so that it is still helpful in understanding the intuitive idea). Once again, you notice that the sentences in the final transcript are either identical to the sentences in the original English transcript or are logically equivalent to them. You can be sure, then, that there was no translating error: both the Spanish and English versions of what I have said to you are *equivalent*. With this intuitive idea, look back at the formalism and make sure you understand it!

Back to our problem. The main idea at play here is that if a sentence ϕ is an atom of a theory R, then if R is equivalent to another theory R' then the latter theory will also have an atom corresponding to ϕ. In slogan form, "atoms are preserved under equivalence." Since this point holds for any two theories (and since it is a pretty interesting result in its own right), I will prove it for our two theories R and R' as defined above and then apply the result to our problem.

Claim. Given theories R, R' and translations $F : R \to R'$, $G : R' \to R$ which show that they are equivalent. If α is an atom of T, then $F(\alpha)$ is an atom of T'.

Proof. First, let's unpack some definitions. That F and G are translations means that, for any Σ sentence ϕ and any Σ' sentence ψ:

$$(\star) \quad R \vdash \phi \;\Rightarrow\; R' \vdash F(\phi)$$

$$(\star\star)\quad R' \vdash \psi \quad \Rightarrow \quad R \vdash G(\psi)$$

Also, by $FG \simeq 1_{R'}$ and $GF \simeq 1_R$, we get

$$\text{(A)}\quad R \vdash \phi \leftrightarrow G(F(\phi))$$

$$\text{(B)}\quad R' \vdash \psi \leftrightarrow F(G(\psi))$$

Now we are ready. Suppose that α is an atom of R. To show that $F(\alpha)$ is an atom of R' we need to show both (i) and (ii):

i $F(\alpha)$ is contingent in R'.

ii For any Σ sentence ϕ, if $R' \vdash \phi \to F(\alpha)$, then either $R' \vdash \neg\phi$ or $R' \vdash F(\alpha) \to \phi$.

Proof of (i). Suppose not, i.e., either (a) $R' \vdash F(\alpha)$ or (b) $R' \vdash \neg F(\alpha)$. If (a), then by $(\star\star)$ we get that $R \vdash G(\neg F(\alpha))$, i.e., $R \vdash \neg G(F(\alpha))$ (since translations distribute over connectives). But, by (A) and elementary logic, we can substitute α for $G(F(\alpha))$. So we get $R \vdash \neg\alpha$. This is impossible, since α is contingent. Case (b) leads in a similar way to a contradiction.

Proof of (ii). Suppose that $R' \vdash \phi \to F(\alpha)$. Then, by $(\star\star)$, $R \vdash G(\phi \to F(\alpha))$. But translations distribute over connectives, i.e., $G(\phi \to F(\alpha))$ is $G(\phi) \to G(F(\alpha))$. So $R \vdash G(\phi) \to G(F(\alpha))$. Now, by (A) (and elementary logic), we can substitute any occurrence of $G(F(\alpha))$ with an occurrence of α in any formula. So we get that $R \vdash G(\phi) \to \alpha$. Now, α is an atom. So, by the definition of an atom, this means that either (a) $R \vdash \neg G(\phi)$ or (b) $R \vdash \alpha \to G(\phi)$.

If (a), then, by $(\star)$, $R' \vdash F(G(\neg\phi))$. But, by (B), $\neg\phi$ is substitutable for $F(G(\neg\phi))$. So $R' \vdash \neg\phi$, and we are done.

If (b), then, by $(\star)$, $R' \vdash F(\alpha \to G(\phi))$, i.e., $R' \vdash F(\alpha) \to F(G(\phi))$. But, by (B), ϕ is substitutable for $F(G(\phi))$. So $R' \vdash \phi$, and we are done with the proof of the Claim.

(It is crucial to note that, despite all the formalism, we needed only to follow the definitions plus a few facts to prove the claim!) Finally, we can show what we wanted, i.e., that f and g as stated in the problem don't exist. For suppose they did. Then, since p_0 is an atom in T', then $f(p_0)$ would be an atom in T, by the Claim. But T has no atoms. Hence we are done!