PHI 312 : pset 1

Definition. Let Σ and Σ' be propositional signatures. A **reconstrual** from Σ to Σ' is an assignment of elements in Σ to sentences of Σ'.

A reconstrual f extends naturally to a function $\overline{f} : \mathsf{Sent}(\Sigma) \to \mathsf{Sent}(\Sigma')$, as follows:

1. For p in Σ, $\overline{f}(p) = f(p)$.
2. For any sentence ϕ, $\overline{f}(\neg\phi) = \neg\overline{f}(\phi)$.
3. For any sentences ϕ and ψ, $\overline{f}(\phi \circ \psi) = \overline{f}(\phi) \circ \overline{f}(\psi)$, where $\circ$ stands for an arbitrary binary connective.

When no confusion can result, we use f for $\overline{f}$.

Substitution Theorem. *For any reconstrual $f : \Sigma \to \Sigma'$, if $\phi \vdash \psi$ then $f(\phi) \vdash f(\psi)$.*

Definition. Let T be a theory in Σ, let T' be a theory in Σ', and let $f : \Sigma \to \Sigma'$ be a reconstrual. We say that f is a **translation** or **interpretation** of T into T', written $f : T \to T'$, just in case:

$$T \vdash \phi \quad \Longrightarrow \quad T' \vdash f(\phi).$$

Definition (equality of translations)**.** Let T and T' be theories, and let both f and g be translations from T to T'. We write $f \simeq g$ just in case $T' \vdash f(p) \leftrightarrow g(p)$ for each atomic sentence p in Σ.

Definition. For each theory T, the identity translation $1_T : T \to T$ is given by the identity reconstrual on Σ. If $f : T \to T'$ and $g : T' \to T$ are translations, we let gf denote the translation from T to T given by $(gf)(p) = g(f(p))$, for each atomic sentence p of Σ. Theories T and T' are said to be **homotopy equivalent** (or simply **equivalent**) just in case there are translations $f : T \to T'$ and $g : T' \to T$ such that $gf \simeq 1_T$ and $fg \simeq 1_{T'}$.

Exercises

For the following exercises, you may assume the soundness and completeness theorems.

Let $\Sigma = \{p_0, p_1, \dots\}$. Let T be the empty theory in Σ, i.e. $T \vdash \phi$ if and only if ϕ is a tautology. Let T' be the theory with axioms $p_0 \vdash p_i$, for $i = 1, 2, \dots$. In these exercises, you will show that T and T' are not equivalent theories.

1. Let ϕ be a **contingent** sentence, i.e. there is a valuation v of Σ such that $v(\phi) = 1$, and there is a valuation w of Σ such that $w(\phi) = 0$. Show that there is another contingent sentence ψ such that $\psi \vdash \phi$ but $\phi \nvdash \psi$. (Hint: by definition, a sentence ϕ can contain only finitely many of the propositional constants $p_0, p_1, \dots$.)

2. Let ϕ be a sentence. Show that if $T' \vdash \phi \to p_0$, then either $T' \vdash \neg\phi$, or $T' \vdash p_0 \to \phi$. (Hint: show that there is only one model of T' that makes p_0 true.)

3. The previous problem shows that p_0 is an **atom** for the theory T'. Show that there are no atoms for T. (Note: an atom for T is, by definition, a contingent sentence.)

4. Show that there isn't a pair of translations $g : T \to T'$ and $f : T' \to T$ such that $gf \simeq 1_{T'}$ and $fg \simeq 1_T$. (Hint: Show that if f and g existed, then $f(p_0)$ would be an atom.)