PHI 312 : pset 3

1. Let Σ and Σ' be propositional signatures, and let T and T' be theories in the respective signatures. Let $f : T \to T'$ be a translation, and let $f^* : \mathrm{Mod}(T') \to \mathrm{Mod}(T)$ be the function given by

$$f^*(v)(\phi) \;=\; v(f(\phi)),$$

for all $v \in \mathrm{Mod}(T')$, and $\phi \in \mathsf{Sent}(\Sigma)$. We say that f is **essentially surjective** just in case for each sentence ψ of Σ', there is a sentence ϕ of Σ such that $T' \vdash \psi \leftrightarrow f(\phi)$.

   (a) Show that if f is essentially surjective, then f^* is injective.

   (b) Show that if f is (one half of) a homotopy equivalence, then f is essentially surjective.

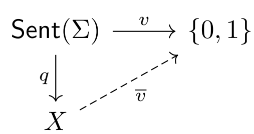
2. Let $\Sigma = \{p\}$, and define a relation R on $\mathsf{Sent}(\Sigma)$ as follows:

$$R(\phi, \psi) \quad \text{iff} \quad \vdash \phi \leftrightarrow \psi.$$

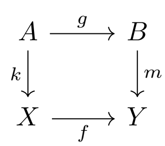
   (a) Show that R is an equivalence relation.

   (b) Show that R has exactly four equivalence classes. [Hint: two sentences are equivalent iff they have the same truth table.]

   (c) Let X be the set of equivalence classes of $\mathsf{Sent}(\Sigma)$ under the relation R, and let $q : \mathsf{Sent}(\Sigma) \to X$ be the quotient function. Show that for any valuation v of Σ, there is a unique function $\overline{v} : X \to \{0, 1\}$ that makes the following diagram commute:



3. Extra credit: Show that the pullback of a monomorphism is a monomorphism. i.e. suppose that the following diagram is a pullback and that m is monic.



Show that k is also monic.